

CONCEPTUALIZING MATHEMATICAL AUTHORITY WITH TECHNOLOGY

by

SUSAN SEXTON STANTON

(Under the Direction of Jeremy Kilpatrick)

ABSTRACT

The purpose of this study was to identify and understand the placement of mathematical authority of four preservice elementary teachers enrolled in a geometry content course as they engaged with concepts of the course and technology in a dynamic geometry environment. Mathematical investigations were the primary pedagogical tool used in the course as the students explored concepts from geometry, measurement, and statistics and probability. Data were collected from videorecorded class sessions consisting of the participants' investigations and interactions throughout the course, three personal interviews that I conducted with each participant, and the responses from pre- and post-surveys on the participants' beliefs about mathematics and mathematics teaching and learning. Data on the participants' beliefs about technology and how those beliefs affect their allocation of mathematical authority came from the clinical portion of the interviews as they interacted with a dynamic geometry software program to explore a mathematics task. To provide an understanding of how the participants sought and used various mathematics sources and where they placed mathematical authority, I classified the participants' work method from the framework of instrumental genesis and developed a typology of authority behaviors. Collectively, the data were used to form a case study analysis of each

participant. The results of the study point to the complexity of mathematical authority and show that an individual's allocation of mathematical authority in various sources does not necessarily happen in a hierarchical fashion. Typical markers (e.g., confidence or mathematical knowledge) that may be associated with an individual's propensity to allocate mathematical authority in particular sources were not always evident for the participants of this study. Further, technology can be viewed as a source of mathematics and serve as an absolute authority in the face of mathematical conflict or uncertainty. The results of the study indicate a need for future research that identifies learning environments that not only create mathematical understanding but also successfully promote and sustain personal mathematical authority in all learning contexts—even those in technology-based environments.

INDEX WORDS: Mathematical authority, preservice elementary teachers, geometry, technology, instrumental genesis

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DEDICATION

For John and Shime.

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TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	v
LIST OF TABLES	ix
LIST OF FIGURES	x
CHAPTER	
1 INTRODUCTION	1
Personal Perspective	1
Research Perspective	2
Research Questions	6
2 LITERATURE REVIEW	7
Situating and Clarifying Mathematical Authority	8
Teaching and Learning Mathematics with Technology	14
Why Geometry?	22
Frameworks for Analyzing Authority	27
3 METHOD	38
Context	38
Participant Selection	42
Data Collection	50
Data Analysis	68
Validity	73

Limitations	73
4 RESULTS	77
Garrett	78
Skyler	112
Janina	135
Kayla	157
Cross-Case Comparison.....	177
5 DISCUSSION.....	185
Summary	186
Conclusions.....	189
Revisiting Theory.....	198
Implications.....	199
Concluding Remarks.....	203
REFERENCES	206
APPENDICES	
A HABITS OF MIND	213
B INFORMATION FORM	214
C BELIEFS CATEGORIES WITH SAMPLE BELIEFS SURVEY ITEMS	215
D GENERAL INTERVIEW QUESTIONS	216
E CRAZY CAKES	221
F EXPLORING AREA HANDOUT	222
G PORTION OF A SAMPLE LESSON GRAPH.....	223
H PORTION OF A SAMPLE CODED LESSON GRAPH SYNOPSIS	225

LIST OF TABLES

	Page
Table 1: Basic Concepts 2 Cohort Responses to the Information Form.....	43
Table 2: Basic Concepts 2 Section 2 Spring 2010 Meeting Schedule.....	45
Table 3: Authority Behaviors Whole and Small Group Setting	178
Table 4: Basic Concepts 2 Cohort Beliefs Pre- and Post-Survey Means by Category.....	183
Table 5: Participants' Beliefs Pre- and Post-Survey Results by Category	184

LIST OF FIGURES

	Page
Figure 1: Lapp's (1997) theoretical model for student perception of technological authority	28
Figure 2: The chart of essential knowledge	34
Figure 3: Typology of five work methods	35
Figure 4: Restored view of two camera angles used for recording class sessions.....	53
Figure 5: Homework assignment task.....	55
Figure 6: "Triangles" of the Triangle Dissection Paradox.....	58
Figure 7a: Oblique triangle projected onto the virtual geoboard.....	61
Figure 7b: Student's suggestion of how to create the corresponding parallelogram.....	61
Figure 8: Screenshot of the second mathematics task	62
Figure 9: Third mathematics task taken from Driscoll (2007)	63
Figure 10: Mathematical shearing homework task from Bennett, Burton, and Nelson (2009).....	65
Figure 11: Screenshot of third task with certain elements visually available.....	66
Figure 12: Screenshot of all elements of the third task.....	67
Figure 13: Skyler's written work illustrating how she would complete the figure	92
Figure 14: Janina's work indicating how she could fill in a hexagon	97
Figure 15: Garrett uses the square script to fill in the gaps	99
Figure 16a: Garrett's different parallelograms	101
Figure 16b: Garrett's decomposition and rearrangement of parts of the parallelogram.....	101
Figure 17: Skyler's work to generate a rule for a trapezoid	102

Figure 18: Screenshot of Garrett's exploration with GSP	103
Figure 19a: Garrett assigns numbers to the base and height of the two figures	105
Figure 19b: Garrett changes the numbers to variables.....	105
Figure 20a: Garrett's investigation on paper	107
Figure 20b: Garrett's work to illustrate his reasoning	107
Figure 21: Garrett's triangle used to determine his conclusions.....	109
Figure 22: Skyler's scratch work on tessellating the octagon.....	122
Figure 23: Screenshot of Skyler's attempts to rearrange the pieces of the triangle.....	125
Figure 24a: Skyler points to the two pieces that create the hole.....	127
Figure 24b: Skyler indicates the pieces in their original configuration	127
Figure 25a: Skyler indicates that the area would not be the same	130
Figure 25b: Skyler's work to illustrate her thinking	130
Figure 26: Problem investigation Janina referred to in order to explain the paradox.....	148
Figure 27: Janina's investigation on paper	151
Figure 28: Kayla's work to show how to decompose and rearrange a parallelogram.....	169
Figure 29a: Kayla's parallelogram.....	170
Figure 29b: Shaded area to count as the actual area of the parallelogram.....	170

CHAPTER ONE

INTRODUCTION

This study concerned the implications for the use of technology on the teaching and learning of mathematics. In particular, the study focused on how preservice elementary teachers enrolled in a geometry content course, with an emphasis on student mathematical learning and understanding, allocated mathematical authority and how technology affected that allocation.

Personal Perspective

Anecdotally, mathematics teachers tell stories of how students' mathematical convictions come to rest solely on the results presented by technology. My own experience of teaching mathematics can attest to this phenomenon as well. My stories include encountering students who believe above all else that the calculator is right, to those who readily have available (on their desk) two different calculators (scientific and graphing) to be ready for the type of computation required in their mathematical tasks, and to those who reach for the calculator for even the simplest of calculations (although they clearly know their mathematical facts).

One particular student will forever stand out in my mind and is one of the main reasons for my interest in conceptualizing mathematical authority with technology. He was enrolled in a college algebra class I taught over the course of one summer at a university that did not allow the use of graphing calculators in college algebra but did allow the use of scientific calculators. Regardless, this student refused to use any calculator and, to my knowledge, did not own one. I often witnessed him performing long calculations on scrap paper, convinced of his own

mathematical knowledge in a course where many students lamented the lack of their trusty graphing calculator.

Before the final exam, I advised the student that I believed he needed a nongraphing calculator with logarithmic capabilities. Logarithms were going to be assessed on the exam and having such technology might prove beneficial. On the day of the exam, he brought with him a brand new, still-in-the-package calculator with the receipt of purchase. He informed me that if he needed the calculator, he would open the package and use it. But in the event that he did not use it, he would immediately take it back to the store. He never opened the package and earned an A on the exam.

Needless to say, after a number of years of teaching, this incident has proven an anomaly. This student was not in an advanced course, nor (as a music major) was he enrolled in an intense mathematical preparation program. A myriad of questions comes to mind regarding him in relation to his peers and the scores of other students I had taught. What factors contributed to his decision to not use the technology (not just that day, but for the entire course)? Was it that he placed the ultimate authority in his own mathematical knowledge over the technology? Did he simply not trust the technology? Did he exhibit this much confidence or consider himself the authority in other areas of his learning and life? In regard to the other students I have taught, what makes so many of them overly reliant on technology—reaching for it for even the simplest of calculations?

Research Perspective

Researching Mathematical Authority

The explication of authority is not new in the educational literature. Philosophers and sociologists such as John Dewey, Michel Foucault, and Max Weber have explored ideas related

to authority and power relations in a variety of areas, including the institutions of education and society at large (Amit & Fried, 2005; Klein, 2003). Given the variations in how one might conceive of authority (depending on the context), authority “is not a simple and univocal concept” (Amit & Fried, 2005, p. 146). These variations are especially problematic when trying to flesh out issues of mathematical authority within the learning environment and research in mathematics education.

A wide body of research within the field of mathematics education mentions or alludes to issues of authority, ranging from Erlwanger’s (1973) classic article exploring the knowledge and conceptions of young Benny to present-day reform initiatives by organizations such as the National Council of Teachers of Mathematics (NCTM, 2000) that call for students to take “ownership” (p. 25) of their mathematical learning through challenging and engaging tasks and reflection and evaluation of their work. Amit and Fried (2005) note that whereas “literature exists touching on the question of authority, . . . explicit attention to the question has, on the whole, been rare in the research literature” (p. 146).

Discerning and identifying where mathematical authority lies is critical in moving the field forward, especially in regard to practitioner change. Wilson and Lloyd (2000) argue that understanding “teachers’ orientations to authority can be very powerful in helping us understand their conceptions of mathematics teaching and their abilities to operate in substantially different ways in the mathematics classroom” (p. 152). Lloyd (2004) specifically points to research that indicates “teachers’ orientations toward authority relate closely to their tendency to teach in innovative ways and to be reflective in their thinking about teaching and learning” (p. 2). Further, Amit and Fried (2005) claim that the study of issues of authority may bode well in considering “many of the important trends in mathematics education” (p. 145). They argue that

pedagogical strategies utilizing “cooperative learning” (p. 146) with movements “towards independent and individual thinking” (p. 146) and “student participation in the construction of mathematical ideas, appear to have an antiauthoritarian component, or, at least, to require the diminution of teachers’ authority” (p. 146). Amit and Fried echo the sentiments of Goos, Galbraith, Renshaw, and Geiger (2003), who also claim that students in these types of learning environments “are expected to propose and defend mathematical ideas and conjectures [and to] respond thoughtfully to the mathematical arguments of their peers [rather than] relying on the teacher as an unquestioned authority” (p. 74). Lloyd (2004) furthers this particular sentiment as she considers the balance in such a classroom: “If textbooks and teachers cease to serve as the primary sources of mathematical authority, students must play a much greater role in the development and testing of mathematical ideas” (p. 2).

Incorporating Technology

To Amit and Fried’s (2005) list of “important trends” (p. 145), I add a relatively new component of mathematics education to consider: technology. Given the complex and increasingly widespread use of technology in mathematics learning, one must consider issues of authority in how students understand and conceptualize mathematical concepts *while* using such technology. Technology could be viewed as anything a student uses as an aid in the learning of mathematics. This may include visual aids, such as drawn representations, or hands-on aids, such as manipulatives. In the context of this study, however, technology was exclusively considered as any electronic medium used in the learning and exploration of mathematics concepts and ideas (e.g., graphing and symbolic calculators or dynamic geometry environments [DGE]).

Technology, touted as a means to facilitate mathematics learning and understanding (Heid, 1997), may instead bring about unexpected perceptions of authority by students using it in

mathematical investigations. These perceptions may mirror students' other perceptions of traditional authorities such as the textbook and teacher (Lingefj rd, 2000). And although a wide body of research exists on the effects of technology (e.g., students have better dispositions toward mathematics or may lose basic computational abilities) with respect to mathematical learning, the term *effects* does not incorporate the notion of authority in the research questions pursued. It appears that, in some studies, researchers become aware of issues of authority as they are manifested in the *results* of those studies (e.g., Artigue, 2005; Hativa, 1988; Lingefj rd, 2000). In order for teachers to prepare students mathematically in an age of technological advances, mathematics educators must understand issues of authority that students either develop during or bring to technology use.

Issues Unique to Preservice Teachers

The charge to teachers to mathematically prepare their students through a variety of nonconventional means (e.g., utilizing technology) and to understand issues of mathematical authority applies to teacher educators as well. However, the journey of the teacher educator in fulfilling this quest may be quite different than that of a classroom teacher. Teacher educators work with preservice teachers who may harbor deep-seated beliefs about traditional modes of learning and teaching of mathematics and who have been quite successful in such environments. Teacher educators teaching such students may be trying to educate these individuals about teaching mathematics in ways that are counter to those beliefs (Cooney, Shealy, & Arvold, 1998).

Preservice teachers could provide a unique perspective to deepen the field's relatively shallow understanding of issues of mathematical authority. Their perceptions of mathematical authority are formed in comparison to students who are first encountering mathematical concepts

and ideas and are in the process of developing their individual perceptions of mathematical authority. In other words, preservice teachers who are learning mathematics through the lens of being a future teacher may provide a means for conceptualizing and explicating mathematical authority as compared to a classroom student exploring mathematical concepts for the first time.

Research Questions

The lack of “explicit attention” (Amit & Fried, 2005, p. 145) to issues of mathematical authority has resulted in a range of working uses of the construct (discussed in the next chapter) as well as a confusing interchangeability with other existing terminology, such as mathematical confidence. Further, the advent of technology in the learning and teaching of mathematics may have introduced issues of mathematical authority not previously considered by teachers or teacher educators. In recognition of the need to further clarify and define mathematical authority and how it is manifested in mathematics learning involving technology, the following research questions were formed to guide this study:

1. Where do preservice elementary teachers allocate mathematical authority in the context of a geometry content course using pedagogy consistent with student-centered learning?
2. How does the use of technology, while engaged with concepts of the geometry course, affect these preservice elementary teachers’ allocation of mathematical authority?

CHAPTER TWO

LITERATURE REVIEW

Until recently, attempts to understand or conceptualize mathematical authority were virtually nonexistent. Thus the literature is limited on discussions for which the construct has been explicitly explored or addressed with respect to the advancement of mathematics learning and understanding. In contrast, there exist a multitude of studies on the learning and teaching of mathematics while using various technologies. In this chapter, I provide an overview of the literature used to frame this study. I begin with an explicit focus on the construct of authority and, in particular, mathematical authority. This discussion includes various considerations of the construct in the literature and how it differs from and is connected to other constructs (e.g., mathematical confidence or self-efficacy). Next, the focus changes to the literature on teaching and learning mathematics with technology. This discussion includes: the various roles technology assumes in the mathematical learning environment; the importance of reflective behaviors and incorporating an element of anticipation while learning with technology; and a plausible teaching design for effective and successful mathematics learning in a technology-based environment. I then consider the literature on the importance and affordances of geometrical study in understanding the placement of mathematical authority with technology. Finally, I conclude this chapter with a detailed discussion of the two theoretical frameworks that helped to shape the analysis and interpretations of the results of the study: student perception of

technological authority (Lapp, 1997) and instrumental genesis (Artigue, 2002; Trouche, 2005b; Vérillon & Rabardel, 1995).

Situating and Clarifying Mathematical Authority

As discussed in Chapter 1, the explication of authority is not a new discussion in the literature and has been explored intellectually in a variety of domains outside of mathematics. Concerning the managerial sciences, Duffy (1959) remarked upon the “little material on the subject that is not speculative and rational rather than empirical” (p. 167) and noted the inability of research “to reduce concepts of authority to ones that may be experimentally tested” (p. 167). Duffy’s argument demonstrates an early concern for clarification of the construct of authority. In an effort to operationalize the construct, Duffy attempted to create a definition of the term “expressed in such a way that direct observation is at least conceptually possible” (p. 168):

Authority is the relationship that exists between individuals when one accepts the directive of another as authoritative, that is, when the individual receiving the directive weighs the consequences of accepting it against the consequences of rejecting it, and decides in favor of acceptance. The authoritative nature of the directive is confirmed when the person accepting the directive acts in accordance with it, within the confines of his understanding and ability. (p. 167)

Duffy’s argument certainly applies to the field of mathematics education for which, as noted in Chapter 1, discussion of the construct appears in the results of studies. There is relatively little literature that explicitly and directly addresses authority in an effort to understand how it may be manifested in the learning environment and, more importantly, factors that contribute to such manifestations. Although there is an implied (or taken-for-granted) perception that authority is directly related to other, more popular, constructs (e.g., beliefs, motivation, and attitude), the literature is limited in considering how authority operates and comingles with such constructs. However, the attempts of Cooney, Shealy, and Arvold (1998) to understand how the

belief structures of an individual may affect changes in his or her beliefs led to “concerns about authority” (p. 313):

Thus we see a certain tension between an orientation toward knowing based on an external authority with beliefs held nonevidentially and an orientation toward integrated knowing with beliefs held evidentially so as to promote reflection and attention to context. This tension is perhaps best conceived as a continuum in which there are intermediate phases between being close-minded, with authority dictating truth without careful evaluation, and being open-minded by virtue of attending to context. (p. 312)

The existing literature that does explicitly address issues of authority in mathematics education discusses the construct in a variety of ways for different contexts, depending on the focus of the study. Rather than perceiving the construct as a *relationship* (as in Duffy’s proposed definition), one traditional notion of authority alludes to the idea that the construct can be held or harbored by certain persons or entities. This notion is illustrated in a definition of *authority* found online: “The power to determine, adjudicate, or otherwise settle issues of disputes” (“Authority,” 2012). This notion of authority can be found in the literature as well. For example, Nathan and Knuth’s (2003) research found that, as one middle grades teacher altered her classroom practice to be less of a mathematical authority, the interactions of the students changed as well. The change engineered a new behavior from students as they “frequently shared information horizontally with one another as they worked together” (p. 198). The atmosphere of the class was conducive to discourse for which “student ideas were offered publicly for others to pick up, refute, or ignore” (p. 198). Nathan and Knuth found that, with the lack of appropriate teacher intervention, undesirable repercussions of mathematics learning and understanding resulted from the new student interactions. For example, when the students entered into a stalemate of determining definitions of certain mathematical terms, they used a whole class vote to come to a final consensus of their debates. The students were satisfied with establishing new

understandings using means that were not clearly or necessarily linked to mathematics knowledge or justification. Nathan and Knuth found that “these young students did not always have the resources to construct or verify correct mathematical ideas or conventions” (p. 197) because “there was no clear authority for students to turn to in the face of their uncertainty” (p. 198). They concluded that while students “showed they [could] fill the conversational void, [they] may not and often cannot serve as the analytic authority necessary to promote correct understanding about all of the content matters” (p. 203).

Another example of how the construct of mathematical authority may be evoked as an entity to be held and used by crucial players in the learning environment can be found in Wilson and Lloyd’s (2000) discussion of three teachers’ individual attempts to implement a reform curriculum. In their attempt to understand the teachers’ efforts, Wilson and Lloyd found that the teachers “worked to renegotiate where mathematical authority should lie—with the teacher, the students, or both” (p. 146). In their discussion, they extend the umbrella of the construct in regard to teachers and what they term *pedagogical authority*, described as the “strength of their *own* voices and conceptions for determining classroom activities and content” (p. 151). In regard to students, Wilson and Lloyd claim that as students “explore and discuss mathematical ideas, they must accept much of the responsibility for learning” (p. 151) and use “mathematical authority to make sense of situations, issues, and questions” (p. 151).

The notions of *responsibility*, *strength* and *voice* in regard to authority are echoed in other discussions and used in ways that begin to illuminate the construct as complex, consisting of various facets of learning, rather than just as an entity held by pivotal persons. For example, Anderson (2002) discussed “voice” (p. 29) as “ways that students develop their own authority and construct their own knowledge, which are representations of their voices” (p. 31). She

clarified voice as being “closely related [to], perhaps indistinguishable, from authorship” (p. 29) and chose to use the term *authorship* throughout her study rather than *voice*. Anderson found that, in the nontraditional teaching of an all-girls mathematics course, the students felt compelled to take “responsibility” (p. 127) for their learning rather than place it on their peers or their teacher. As a result of the students “authoring knowledge” (p. 127), Anderson found that they became more “confident” (p. 128) in their mathematics abilities and understanding. Further, Anderson claimed that “authorship appeared to play a role in the participants’ production of mathematical knowledge” (p. 182) and was “acquired” (p. 182) free from external sources or validation. Anderson argued that a learning environment using the notion of authoring could provide “mathematics educators with a new definition of success, one that depends on authorship and ownership of mathematics” (p. 183).

The idea of *ownership* is how Lingefjärd (2000) appears to conceptualize mathematical authority as he discussed, “The development of a student’s trust in his or her understanding and in the value of the solution to a problem leads to the important question of who owns the mathematics” (p. 22). Lingefjärd sought to understand why the preservice secondary teachers in an undergraduate mathematical modeling course tended to “[mistake] model for reality” (p. 4) and “[shift] their sources of authority during the modeling process from the mathematics to the computer” (p. 4). In explicating authority in environments where students “normally have at least two authorities they can rely on: the textbook and the teacher” (p. 21), Lingefjärd surmised that those “with a strong tendency to trust ‘the authority’ often give up their ownership of ideas and of problem-solving when questioned by the teacher” (p. 22). He discussed his surprise at how the “students would take the calculator or computer as the sole mathematical authority” (p. 23) and

found that, while using technology, his students “seemed unwilling to take full responsibility for their own learning and performance” (p. 4).

Lingefjärd’s (2000) study laid the groundwork for conceptualizing authority with technology—the kernel of the present study. In the other studies discussed above, sources of mathematics understanding, knowledge, and learning (in regard to authority) were discussed strictly in terms of the teacher or the student or the negotiation of authority from one to the other. A particularly pertinent manifestation of the construct of authority in Lingefjärd’s study is how technology came to be viewed as a *source* of mathematics knowledge. Further, his discussion of why “students sometimes shifted their sources of authority during the modeling process from mathematics to the computer” (p. 5) alludes to the idea that knowledge and understanding lie within the mathematics itself, not necessarily the teacher or student—that the mathematics can be an authority. This view of mathematics and authority is echoed in Amit and Fried’s (2005) explication of authority—particularly what they call “revised authority” (p. 151). Amit and Fried found that the eighth-grade students in their study used a variety of sources as mathematical authority and that their choices of when to use who (or what) was quite intricate. Further, they found that the “relationships of authority [interfered] with students’ ability to reflect for themselves and participate in the construction of mathematical ideas and, at the other extreme, with their ability to work collaboratively to solve problems and obtain mathematical insights” (p. 164).

These studies illustrate that issues of mathematical authority can occur at all levels and foci of mathematics education. Wilson and Lloyd (2000) focused on inservice teachers implementing a new curriculum. Anderson (2002) focused on a feminist teaching environment in an intensive summer mathematics program for high school girls. Lingefjärd (2000) focused on

preservice teachers learning mathematical modeling at the undergraduate level. And the studies of Amit and Fried (2005) and Nathan and Knuth (2003) concerned middle grades students learning mathematics, with the latter focused on teacher-initiated pedagogical change.

The notions of ownership, power, source, responsibility, strength, and voice pervading the studies indicate that an intricate web of various interrelated constructs may be at the heart of conceptualizing issues of mathematical authority. These notions suggest that there is a sociopsychological component critical to mathematics teaching, learning, and understanding. Past discussions of a psychological nature concerning mathematics learning dealt mainly with affective issues (DeBellis & Goldin, 2006; McLeod, 1992; Phillip, 2007) and suggested that related constructs such as anxiety, self-efficacy, confidence, beliefs, attitude, and motivation play crucial roles for what is deemed to be successful learning in mathematics.

Considering the web of relationships among these various constructs is critical in understanding a student's placement of mathematical authority or existing issues of mathematical authority in various contexts. Pajares and Miller (1994) provide a point of entry from which to consider the web of relations between these various constructs as they point to research that argues efficacy beliefs determine the coping skills students use to deal with their mathematics anxiety. They describe anxiety as "determined by the confidence individuals bring to a task" (p. 194) and confidence as a "conceptual forerunner to math self-efficacy" (p. 194). However, Hart (1989) defines confidence in learning mathematics as "the degree to which a person feels certain of her or his ability to learn and perform well in mathematics" (p. 243). Additionally, Lingefj rd (2000) determined that *trust* was a critical component for the confidence in the technology that the students in his study presented. It appeared that the more

the students trusted the technology, the more confidence they held in the results presented by the technology.

However, when defining confidence in learning mathematics, one must also consider the confidence students have in the various sources they seek out and use while engaging in mathematics. Thus, in contrast to Hart's discussion of confidence in learning mathematics and using Lingefj rd's link between confidence and trust, I propose that confidence in a particular source should be considered as the degree to which a person trusts that source in a given mathematical context based on his or her knowledge and understanding of that source. For the purposes of this study, a clear link is assumed to exist between confidence and placement of mathematical authority in a particular mathematics source while clearly marking confidence and mathematical authority as two distinct and separate constructs. Further, a source is considered any person or tool (e.g., a course text, the Internet, or technology) that a person views as possessing mathematics knowledge or information regardless of how that knowledge or information is used. The source becomes a mathematical authority for the individual when he or she uses that information to apply it to mathematics situations, to make sense of a concept, or to defer to regarding uncertainty of mathematical correctness. Thus, an individual can obtain information from a mathematics source without viewing the source as a mathematical authority. However, a mathematical authority is always considered a mathematics source.

Teaching and Learning Mathematics with Technology

A wide body of research has been conducted on the effects of technology in nearly every aspect of mathematics education including instruction, learning, and curriculum (Ellington, 2003; Goos, 2005; Hativa, 1998; Heid, 1997; Hembree & Dessart, 1986; Lapp, 1997; Lingefj rd, 2000; Olive & Makar, 2010). The dialogue on learning with technology has roots in the very

early introduction of computers used for programming in research and the possible effects of “the teaching machine” (Tyler, 1963, p. 123). These concerns have not dissipated over the past several decades, as the discourse is still relevant today. A few of the concerns regarding technology use in the teaching and learning of mathematics include: nonacquisition of basic skills; shortcutting the deeper understanding of concepts; students’ perceptions of the role of proof; how students come to appropriate the affordances of available technologies; the significance of the role of the teacher; and mathematical issues unique to students hidden through the use of technology (Heid, 1997; Hollebrands, Laborde, & Sträßer, 2007; Kissane, 1995; Lingefjärd, 2000; Trouche, 2005a).

Various discussions found in the research literature have offered ways to consider the roles of technology as it is manifested in the learning environment. In regard to the curriculum, Pea (1985) addresses how the “cognitive power tools” (p. 175) of technology act both as an *amplifier* and a *reorganizer* of the processes of the mind. Thus, Pea argues that the education curriculum should be adjusted to take advantage of such effects. In regard to learning, Goos et al. (2003) describe and theorize technology in terms of “four metaphors for how technology can mediate learning” (p. 74). These metaphors describe technology in regard to its *relationship to the user* as “‘master’, ‘servant’, ‘partner’, and ‘extension of self’” (p. 74). Pea (1987) discusses the potential for technology to become different tools to aid the learner and identifies them as: a conceptual fluency tool, a mathematical exploration tool, a representational tool, a tool for learning how to learn, and a tool for learning problem-solving methods. Doerr and Zagor (2000) also developed a list of tools for the role of the graphing calculator for the students in their study. They observed the students using the technology as: a computational tool, a transformational tool, a data collection and analysis tool, a visualizing tool, and a checking tool. In regard to

teaching, Heid (1997) provides four principles that should guide choices regarding the use of technology in the classroom. The first principle puts value on student-centered education and asserts that technology can make education more student-centered. The second principle puts value on giving students the experience of being a mathematician and states that technology can provide such experiences. The third principle holds that reflection enhances learning and that technology can provide a means for reflection. And the fourth principle declares that in the technology-based classroom “there is a redefinition of epistemological authority” (p. 9). Heid asserts, “As students assume a more personal authority for the development of their own knowledge, the first three principles of creating a student-centered curriculum, engaging students in real mathematical activity, and promoting reflection become increasingly essential” (p. 9).

The emphasis on *reflection* in mathematics learning is not a new concept in the literature (Cooney et al., 1998) and has various synonyms that include: *interpretation*, *analysis of results*, *comparison*, *discrimination*, and *validation*. The ability to mathematically reflect upon the results presented by varying sources of mathematics knowledge is a key component to mathematics understanding and is “not unproblematic in conventional learning environments” (Noss, 1988, p. 254). This ability not only requires a sufficient mathematics knowledge base of the topic at hand but also the ability to reason and make sense of mathematics results. In particular, Cooney et al. (1998) discuss the construct of authority as “particularly relevant to conceptualizing the notion of being reflective and adaptive” (p. 311).

Artigue (2005) furthers this argument of the necessity of reflection while using technology for the teaching and learning of mathematics in her discussion of the results of two “didactical engineering projects” (p. 231). These results contributed to the development of instrumental genesis (discussed in more detail in the following section), one of the key

theoretical frameworks used for the present study. Artigue claims that one difficulty students face while using technology comes from the necessity to *discriminate* between various forms of the results that the technology could present in contrast to what the students may actually understand mathematically. Their ability for successful discrimination comes with revealing consequences. She elaborated:

In the face of such difficulties, students tended to convince themselves easily that the machine was inevitably right. . . . The machine served as a *revelator of the fragility* of the students' mathematical knowledge and evidenced up to what point mathematical knowledge was necessary to *efficiently pilot* the machine. . . . By exposing the students to unusual expressions, the work with the machine provided experience of a certain fragility of knowledge which could have remained invisible with simpler examples and more routine tasks. (p. 246)

Lapp (1997) found similar results in his study to determine contributing “factors involved in the student’s perception of the authority of the computer and/or calculator” (p. 4). After working with technology that was deliberately programmed to provide incorrect results, the students’ comments suggested they “place a great deal of authority in the machine” (p. 3) and “are willing to ignore their own common sense and follow the lead of the calculator” (p. 3). Similarly, Lingefjärd (2000) also concluded that a lack of *validation* of the results presented by technology caused a shift in the preservice teachers’ perceptions of mathematical authority toward the technology.

Creating a vicious cycle, the use of technology may exacerbate the user’s tendency to not adequately reflect on its results. The lack of reflection can lead to behaviors as those observed in the research of Berry and Graham (2005) and Berry, Graham, and Smith (2006). These studies report on how students used graphing calculators in their mathematics learning and investigations while using Keyrecord¹ (a program that discreetly runs in the background as it

¹ Also known as Key Recorder (Sheryn, 2005), the program is available at <http://www.fi.uu.nl/wisweb/software/softwareovz.xml?language=en>.

records the keystrokes of the user). The program was installed to gain an understanding of how, and perhaps why, the students used the graphing calculator in their individual investigations. Berry et al. claimed that Keyrecord provided insights about the students' technology use in the most "natural way" (p. 304) possible. Berry and Graham reported their surprise at the behaviors of students "just [copying] the calculator screen" (p. 146) and providing "no thought to what they would have expected to see" (p. 146). Similarly, Berry et al. found that as students engaged with the technology, they "predominately used a 'trial and improvement' or 'trial and error' approach, rather than an analytic approach" (p. 303).

The use of adequate reflection upon the results presented by technology can create meaningful opportunities for students in regard to their conceptions of mathematics and mathematics learning. Doerr and Zangor (2000) found that questioning the validity of results provided by the technology of the calculator and the ability to give mathematical reasoning and justification for those results (given the limitations of the calculator) appeared to help students rely more on their own mathematical knowledge. Further, Povey and Ransom (2000) found that many of the students in their study "stressed the importance of 'understanding' the maths (through learning with paper and pen) before using technology and resisted the idea of being taught to 'press buttons' without 'understanding'" (p. 53). These students were majoring in scientific or mathematical fields of study and possessed solid mathematics backgrounds. They desired more time for reflection on their learning of the concepts and thought that the technology provided results so quickly that there was no time to "absorb and reflect" (p. 53). Povey and Ransom noted that this loss of ability to absorb and reflect upon the learning "can deprive the user of their sense of control" (p. 56). Further, Lapp (1997) observed that students with stronger mathematics backgrounds were more willing to "challenge the calculator results" (p. 7) than

those with “comparably weaker mathematics backgrounds” (p. 7). He reported that the stronger students’ mathematics backgrounds allowed “them to choose to ignore the calculator generated model on several occasions” (p. 14).

Lagrange (2005a) argues that it is not only the reflection upon results that is important for successful mathematics learning but also adequate *anticipation* or *expectation* of results. In his discussion, he stresses that mathematics pedagogy utilizing technology “cannot be thought of as simply a matter of using a machine to ease problem solving or to enhance inductive activity” (p. 80). In his experiments, Lagrange observed that the “results that the teacher expected to be amazing to students . . . did not alone create much surprise [for them]” (p. 78). He further argued that technological integration in mathematics teaching and learning is “effective only if students have a suitable preparation” (p. 77) and suggested:

The learning situation has to bring to the fore puzzling peculiarities and to challenge students’ anticipations. For instance it is often interesting to prepare examples where students’ predictions will very probably be wrong and to ask them to compare these predictions with the calculator’s answer. (p. 78)

Lapp (1997) and Artigue (2005) offer similar arguments for the element of anticipation. Among his recommendations for instruction utilizing technology, Lapp suggests that teachers “make it a point to expose students to activities which will allow them to be confronted by conflicting information given by the technology” (p. 8). However, Artigue claims that “surprise effects and the resulting motivation for understanding can only exist if there is some *expectation*” (p. 243). She further explains that “the work of forming such a prediction is only of reasonable cost if it can be partly done *mentally* and does not oblige the student to carry out very detailed calculations” (p. 243). Artigue found that the results of the didactic engineering studies revealed that “the *multiplicity of the variables*” (p. 243) can “intervene in the efficiency” (p. 243) of using

technology and that “the work of prediction can only be productive if machines are switched off” (p. 243). She further argues:

Another point is that the phase of prediction must have a real *status*. If it seems just like an introduction, the confrontation will lose much of its efficiency; the authority of the result supplied by the machine will sweep away all the previous work, whatever its coherence. (p. 274)

This sentiment leads to the pedagogical concern of whether or not technology should be taught as secondary to paper-and-pencil techniques to avoid potential costs to successful learning. However, technology use in the learning and teaching of mathematics has been shown to have positive effects for students as well as teachers. For example, studies have found that when students used a calculator during instruction or assessment, they possessed a better mathematical disposition and self-concept than students who did not use a calculator (Ellington, 2003; Hembree & Dessart, 1986; McCulloch, 2011). Further, technology can provide affordances in particular mathematics applications and content areas that were previously unavailable (Heid, 1997). These affordances include “the expansion of discrete mathematics, applied logic and algorithmics” (Trouche, 2005a, p. 11). Technology has also been heralded with the ability to “foster the reification” (Drijvers & Gravemeijer, 2005, p. 175) of conceptualized ideas.

Thus, one must consider the careful balance between learning and discovering mathematical concepts with technology versus the utilization of two completely separate techniques (i.e., paper-and-pencil and “push button”; Lagrange, 2005a, p. 73). Olive and Makar (2010) note “that it is not the technology itself that facilitates new knowledge and practice, but technology’s affordances for development of tasks and processes that forge new pathways” (p. 154). Olive and Makar argue that one can develop a “working relationship” (p. 155) with technology through “operationalizing the mathematics with the technology” (p. 155). Similarly,

the NCTM (2000) points to the affordances of technology as a tool for student use. In its Technology Principle, NCTM states that “electronic technologies—calculators and computers—are essential tools for teaching, learning and doing mathematics. . . . When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving” (p. 25). However, with this view comes great debate among various populations (e.g., educators, parents, policy-makers, and researchers) surrounding the use of technology in the educational setting and, specifically, particular techniques that can be used for successful learning.

Artigue (2005) points to a particular design, “a lever” (p. 249), shown to be “productive” (p. 249) in regard to mathematics learning with technology and consisting of the inherent and critical features of both expectation with real status and unexpected results. She lists the conditions necessary for the “effectiveness of this lever” (p. 249) as follows:

The phenomenon has to be easily identifiable; it has to arouse the students’ curiosity enough for motivation towards understanding to be provided not only by the teacher; finally, the expected mathematical work has to be accessible to the students, with their mathematical and instrumental knowledge. (p. 249)

Artigue cautions that such “conditions are not easy to satisfy” (p. 249) and clarifies that such a learning environment must contain “[scenarios] *tightly piloted* by the teacher” (p. 265). Otherwise, “an *explosion of techniques*” (p. 287) will be manifested and “remain in a relatively simple-crafted state” (p. 287) resulting in “a technical congestion” (p. 287).

Hadas, Hershkowitz, and Schwarz (2000) provide evidence for the use of such a lever as they “capitalized on surprise” (p. 148) in their research on how students used methods of proof in a dynamic geometry environment [DGE]. These students attempted “to overcome uncertainty” (p. 131) about the possibility of particular constructions. Hadas et al. explained that such a design “may trigger a need for explanation” (p. 134). Their research focused on how students developed techniques of proof, particularly deductive arguments, when faced with uncertainty. The research

illustrated how a DGE can promote such learning but did not take into account how the “authority of the computer” (Hollebrands et al., 2007, p. 182) might have affected the psychological component of the mathematical development. In a similar vein, Cooney et al. (1998) point to the construct of “*reflecting-in-action*” (p. 308) in which “reflection coexists with the experience” (p. 308). In particular, they argue, “Over time, patterns of reflection-on-action and a firm grounding in knowledge enable one to reflect in anticipation rather than in response” (p. 308). Thus, this research design (i.e., the use of the lever) was also used in the design of the present study although for a different purpose and hopefully to provide insight into how technology affects the placement of mathematical authority.

Why Geometry?

The Importance of Geometrical Study

It is important to address why studying geometry may help in understanding and fleshing out issues of mathematical authority. It is equally important, however, to discuss the importance of geometry in general. The mathematician Sir Michael Atiyah (1982) characterized geometry as “that part of mathematics in which visual thought is dominant whereas algebra is that part in which sequential thought is dominant” (p. 183). Atiyah perceived a need to remind the field of mathematics that the study and teaching of geometry should not take a back seat to other domains and asserted, “Geometry is not so much a branch of mathematics as a way of thinking that permeates all branches” (p. 183).

Atiyah is not alone on his views of geometry and geometric learning. Clements and Sarama (2011) claim, “Geometry can serve as a core—relating science and mathematics” (p. 134) and point to research in which certain geometric abilities (e.g., spatial reasoning) have been linked to achievement in regard to “proportional reasoning, judgmental application of

knowledge, concepts and properties, and managing and processing skills” (p. 135). Further, they suggest geometrical thinking as “a gateway skill to the teaching of higher-order mathematics thinking skills” (p. 135). Battista (1999) describes geometry as consisting “of ways of structuring space and analyzing the consequences of that structuring” (p. 177). Battista (1998) argues that deficiencies in geometric understanding can have consequences for the understanding of particular mathematics concepts, such as those learned in the early grades. For example, he provides an illustration of how “students’ lack of attention to spatial structuring can cause them to attain a superficial understanding of the use of multiplication in certain extremely important geometric contexts” (p. 408).

Further, the domain itself requires a unique type of reasoning and sense-making. Driscoll (2007) identified four geometric habits of mind that are important for proper geometric learning and understanding in Grades 5–10. Cuoco, Goldenberg, and Mark (1996) also identify mathematical habits of mind specifically unique to geometers.² NCTM’s (2000) discussion on the learning strand in Grade K–12 schooling furthers this perspective on geometrical learning and provides a multitude of reasons for studying geometry. These reasons identify geometry as “a natural place for the development of students’ reasoning and justification skills” (p. 41); offering “students an aspect of mathematical thinking that is different from, but connected to, the world of numbers” (p. 41); providing “a rich context for the development of mathematical reasoning, including inductive and deductive reasoning, making and validating conjectures, and classifying geometric objects” (p. 233); and as “a means of describing, analyzing, and understanding the world and seeing beauty in its structures” (p. 233).

² See Appendix A for a more detailed explanation of the habits of mind discussed by Driscoll (2007) and Cuoco et al. (1996).

Within the last 15 years, two presidents of NCTM, Glenda Lappan and Henry Kepner, have argued for the continual instructional support of the learning of geometry and geometric reasoning. Both presidents cite the nation's students' poor performance on "international comparisons in geometry" (Kepner, 2009, p. 3) as one indicator to remind educators that geometry needs the type of considerable attention that algebra and data analysis have been receiving over the past few decades. Lappan (1999) specifically challenges the field to "build a geometry strand that engages our students in this interesting and important area of mathematics throughout their school experience, from pre-K through grade 12" (paragraph 5). She argues that this challenge can be met only when "we give students the time throughout their school mathematics to explore geometry to its fullest—to play, observe, analyze, conjecture, imagine, represent, transform, create arguments, and simply experience the beauty and joy of geometry" (paragraph 9). Lappan's argument points to a critical activity that, at times, seems to have all but completely disappeared from the mathematics curriculum—the activity of play.

Play is not a new topic in educational discourse. Dewey (1916) devoted an entire chapter to play's crucial role in learning. He made a distinction between play and work—arguing that play ultimately leads to the desired effects of traditional work, specifically, "the tedium and strain of 'regular' school work" (p. 228):

Persons who play are not just doing something (pure physical movement); they are trying to do or effect something, an attitude that involves anticipatory forecasts which stimulate their present responses. . . . It is important not to confuse the psychological distinction between play and work with the economic distinction. Psychologically, the defining characteristic of play is not amusement nor aimlessness. It is the fact that the aim is thought of as more activity in the same line, without defining continuity of action in reference to results produced. Activities as they grow more complicated gain added meaning by greater attention to specific results achieved. Thus they pass gradually into work. Both are equally free and intrinsically motivated. (p. 238)

Specifically, Ahmed, Clark-Jeavons, and Oldknow (2004), detail mathematical play as “that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using formal rules of mathematics to follow any ideas to some sort of conclusion” (p. 319). They further argue that mathematical play contains “no obvious short-term goals” (p. 319), thus allowing for “complete freedom on the part of the solvers to wander over the mathematical landscape available to them” (p. 319). Arguably, there is no better arena for students to engage with mathematical play than with geometrical investigations and activities. As discussed earlier, the geometrical ideas permeating other branches of mathematics can only serve to enhance and deepen conceptualization and understanding throughout those other branches of knowledge.

Technological Infusion of Geometrical Study

The advent of DGEs, such as the Geometer’s Sketchpad [GSP], the Geometric Supposer, and GeoGebra, has witnessed an increase in the research on understanding how technology use may affect student geometric reasoning. In regard to the learning of geometry, DGEs could potentially turn “geometry classrooms into laboratories for the generation and discovery of geometric relationships” (Heid, 1997, p. 18). Learning geometry with DGEs allows for the type of geometric exploration and reasoning Lappan (1999) calls for as well as spark a “new interest in the teaching of geometry” (Hanna, 2000, p. 12). Studies have found that students learning geometry within a DGE begin to use the type of reasoning and explanation associated with the formal reasoning of proof, regardless of the level of the student (Hadas et al., 2000; Jones, 2000; Marrades & Gutiérrez, 2000). However these studies also note that educators need to incorporate the use of carefully designed mathematical investigations with technology in order to achieve such effects and to give students the ability to engage in “authentic mathematical activity” (Heid,

1997, p. 36). Students may take ownership of the activity as they see the “computer screens as ‘theirs’ . . . which they can appropriate and use for their own ends” (Noss, 1988, p. 257). Further, while in “communication with the [technological] device, the mental processes of the learner are externalized” (Hollebrands et al., 2007, p. 190). Explorations within DGEs can be extended to other mathematical domains, allowing students to explore algebraic ideas, numbers, measurements, and functions with geometric representations. There are even applications that demonstrate, dynamically, data analysis concepts using GSP.³ These various applications could aid student development of mathematical understanding across the different domains as well as within geometry itself.

The explication of geometric understanding and mental manipulation may be considered less abstract than, perhaps, the mental manipulations of algebraic equations. However, Heid (1997) argues that cognitive technologies provide educational affordances through “their degree of transparency and the extent to which they foster the externalization of mathematical representations” (p. 7). Thus the student’s understanding of geometric concepts may be more accessible for the researcher in regard to observations and interviews. Artigue (2005) argues that various DGEs “were *conceived as educational tools*” (p. 233) and therefore do not “raise the same problems” (p. 233) as technology (e.g., graphing and symbolic calculators) that “[takes] charge of all the mathematical work expected from the student” (p. 233). Thus, through the exploration of geometric concepts and ideas, existing issues and perceptions of mathematical authority may be easier to observe or draw out in a research design that utilizes a DGE.

³ A resource center of Geometer’s Sketchpad applications for different content domains is located at: http://www.dynamicgeometry.com/General_Resources/Advanced_Sketch_Gallery.html.

Frameworks for Analyzing Authority

Two types of frameworks were used to aid in the analysis for this study: student perception of technological authority (Lapp, 1997) and instrumental genesis (Artigue, 2002; Trouche, 2005b, Vérillon & Rabardel, 1995). Considered together, the frameworks address two critical aspects for considering the placement of mathematical authority in regard to technology: the dynamics of the influence of outside factors on the individual learner and key aspects necessary for and explicating a student's individual appropriation of the medium for learning.

Student Perception of Technological Authority

Lapp (1997) conceptualized what he termed *technological authority* as how students come to view the authority of technology while learning mathematics concepts and ideas in a technology-rich environment. He proposed the theoretical model shown in Figure 1 for conceptualizing student perception of technological authority and identified five crucial factors (and their respective relationships between each other) as a possible explanation for students' perception of the authority of the technology.

Lapp's model suggests that the *instructor* is an influential element in technological authority. He found that how the instructor privileges technology and stresses the importance of its results "played a role in the tendency for students to believe the computer or calculator model" (p. 6). Thus when students were faced with "an apparent contradiction" (p. 6) between a *match of representations* (i.e., the results presented by technology versus results they were expecting), they would default to comments made by the course instructor during class and not believe the technology. However, if a match of representations of results did occur, then the students "lent greater credibility to answers" (p. 6) presented by the technology. Additionally, Lapp found that a student's perception of technological authority was also influenced by how the

instructor discusses the creators of the technology at hand. Thus the *creator/human* element, based upon a student's knowledge of the origins of the technology, can affect his or her views of the credibility of the results presented by the technology. Lapp also found that the *use of repetition* from students' previous experiences of solving similar problems tended to affect those students' willingness to accept results presented by the technology and was greatly dependent on two other factors, the *instructor* (i.e., whether or not the instructor provided experiences to solve similar problems) and each student's *mathematics background* (i.e., whether or not the students had explored similar concepts in previous mathematics courses).

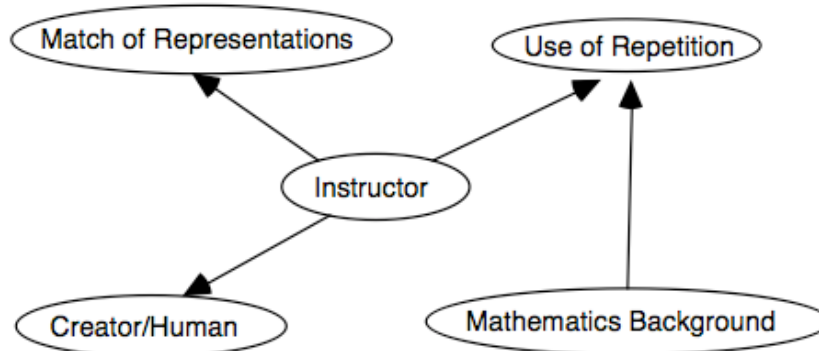


Figure 1. Lapp's (1997) theoretical model for student perception of technological authority.

Instrumental Genesis

The rationale for its development. Attempting to account for psychological aspects of cognition when considering the human evolution of technological use, Vérillon and Rabardel (1995) signify the importance of research to explore “the effects of technology and of technological change on the way we live, learn and work” (p. 77). Specifically, Vérillon and Rabardel mark the importance of instrumented activity given that students are learning and living “in present day fast changing technological contexts” (p. 96). Further, they argue that student cognition does not evolve independent of itself; learning occurs based on the tools available and

in the environment unique to that student. Thus, Vérillon and Rabardel charge researchers to consider the genesis of the instrumentation of such tools on the student's learning.

Noss (1988) claimed, "To be sure, those [technological] innovations which are successful will invariably have some success in eroding the traditional content of the syllabus" (p. 265). However, the literature indicates otherwise and points to a misalignment between society's technological drive and the actual technological use by practicing teachers (Trouche, 2005a). Different reasons exist for why teachers may not make full use of the technology available to them in order to develop or enhance their students' mathematics understanding. Such reasons include: the change in epistemological authority (Heid, 1997); the mandated curriculum and its corresponding assessments as teachers' reflections of "the position of the institution" (Lagrange, 2005b, p. 126); how "mathematical culture is implicitly linked with paper-and-pencil techniques" (p. 118); and the challenge facing teachers to "integrate [technological] techniques into [their] own understanding of the domain . . . into [their] own personality" (p. 129).

Even if a teacher chooses to use technology as part of his or her pedagogical agenda, it may be reduced to "technical aspects of the use of the software" (Kendal, Stacey, & Pierce, 2005, p. 107). Elbaz-Vincent (2005) discusses the perpetual "risk" (p. 63) of technological use in the classroom and points to the "sufficient knowledge of the [technology]" (p. 63) that teachers must first possess "if they want to 'feel in control' in the computer classroom" (p. 63). In regard to the student, Lagrange (2005a) claims that "in learning, as in mathematical sciences, understanding of concepts does not emerge spontaneously from observation, even with the help of powerful tools" (p. 74). He argues that "authors stressing the potentialities of new tools generally [tend] to underestimate the difficulties of their classroom use" (p. 75). The institutional concern that teachers play a critical role in the technology-based classroom coupled

with the pedagogical concern that technology should not be viewed by students as a “black box” (Drijvers & Gravemeijer, 2005, p. 187) form the underpinnings for the theoretical framework of instrumental genesis.

In regard to research, Zbiek, Heid, Blume, and Dick (2007) offer the benefits of utilizing the framework:

The construct of instrumental genesis is helpful to researchers in examining the role of technology in learning. It explains how technology does not have the same automatic power for all users and how its intelligent use requires both conceptual and technical knowledge. (pp. 1178–1179)

Drijvers et al. (2010) further this sentiment as they discuss instrumental genesis as “an ongoing, nontrivial and time-consuming evolution” (p. 108). They stress the issue of *nontrivial* because, “besides the artifact, the instrument also involves the techniques and mental schemes that the user develops and applies while using the artifact” (p. 108), and it is in these mental schemes that “technical and conceptual aspects are intertwined and co-develop” (p. 109). Drijvers et al. suggest instrumental genesis as a way “[to make] tangible the interaction between the techniques involved in using the artifact and mathematical thinking” (p. 112). Drijvers and Gravemeijer (2005) claim: “The seemingly technical obstacles that students encounter while working in the computer algebra environment often have conceptual components, and the instrumental genesis approach helps to be conscious of this and to turn such obstacles into opportunities for learning” (p. 189).

Elements of the framework. Instrumental genesis is “seen as the combination of two processes” (Trouche, 2004, p. 289) that considers the “human-machine interaction” (p. 285). The framework denotes the difference between an *artifact* and an *instrument*. Vérillon and Rabardel (1995) discuss how the artifact, “a man-made material object” (p. 84), becomes an instrument, “a psychological construct” (p. 84), once the artifact is targeted by the subject for specific uses

particular to his or her individual needs—a process called *instrumentalization*. However, through instrumentalization, the constraints particular to the process also begin to shape how the user uses the technology for his or her needs—identified as *instrumentation*. These constraints include: the internal constraints exclusive to the hardware and software of the technology; the command constraints pertinent to use of the technology; and the organizational constraints related to the interface of the technology.

To provide further explication of the framework, Artigue (2002) identifies instrumental genesis as the “process . . . involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes” (p. 250). The identification of these schemes lies at the heart of the development of instrumental genesis as researchers attempt to identify ways and “*conditions of viability*” (Artigue, 2005, p. 251) for which students successfully and efficiently use technology in the mathematics classroom. Artigue (2002) points to the “complex world” (p. 249) that technology creates when one is studying the learning, teaching, and understanding of mathematics. She argues that technology offers a “great reduction in the cost of execution” (p. 249) of mathematical computations, which results in changes in the “pragmatic and epistemic values” (p. 249) of mathematical “techniques that are instrumented by computer technology” (p. 249). Artigue (2005) explains:

Any technique, if it wants to be more than a simple gesture mechanically learnt, must be accompanied by a more theoretical discourse. . . . For instrumented techniques, *it has to be built* and its elaboration raises specific difficulties as it necessarily brings both mathematical and technical knowledge into play. (p. 287)

Drijvers and Gravemeijer (2005) offer a similar argument in regard to “the close relationship between techniques and conceptual understanding within the instrumented action schemes” (p. 186). They explain: “Students can only understand the logic of a technical procedure from a conceptual background. Seemingly technical difficulties often have a

conceptual background, and the relation between technical and conceptual aspects makes the instrumental genesis a complex process” (p. 186). Drijvers and Gravemeijer claim that the formulation of the process occurs through “drawing up lists of key elements in instrumented action schemes” (p. 186). The lists, created from the observations of students interacting with technology while learning mathematics, “were found to be helpful for making explicit the relation between technical and conceptual components, and for indicating possible ‘end products’ of the instrumental genesis” (p. 186). Although they acknowledge the “prescriptive and rigid character” (p. 186) of the lists that often “ignore the process of individual instrumental genesis” (p. 186), they found them to be “helpful for designing student activities, for observing the interaction between students and the computer algebra environment, for interpreting it and for understanding what works out well and what does not” (p. 188).

Fruits of the framework. The results from the various research studies conducted in regard to instrumental genesis have not only contributed to the research on successful technological integration in the mathematics learning environment but also provided helpful descriptions about learners in such an environment. As discussed earlier, the research on technology use in mathematics education has identified various ways that technology can aid both the learner and teacher. However, the framework of instrumental genesis highlights the not-so-glamorous issues that technological use engenders in the learning environment. This research also points to the fact “that paper-and-pencil techniques and instrumented techniques [have] very different lives within the class” (Artigue, 2005, p. 286); it points to the need of “giving an *adequate status* to instrumented techniques and to *manage them institutionally*” (p. 286).

Attempting to explicate the individual and social aspects inherent in the development of “efficient instrumentation” (Artigue, 2005, p. 239), Trouche (2005b) discusses the necessity to

identify “the different forms that instrumental genesis takes by studying students’ behavior so as to establish a typology of work methods” (p. 197). The typologies consist of *local* aspects (the “privileged frame of work” (p. 202) that a student may enact) or, more broadly, *global* aspects (the “interactions” (p. 202) between various elements of the learning environment and their characteristics). The elements of the global typologies consist of: information sources, time of tool utilization, relationship of students to mathematics (specifically to different methods of proof), and metaknowledge. The basic tenets of instrumental genesis also consider “the central role of the subject’s *control* of his/her own activity” (Trouche, 2005b), p. 203), termed *command process*, and defined “as the ‘conscious attitude to consider, with sufficient objectivity, all the information immediately available not only from the calculator, but also from other sources, and to seek mathematical consistency between them’” (p. 203). The command process “takes place within a chart of essential knowledge, which is required in mathematical activity, in particular, when using symbolic calculators” (p. 203; see Figure 2) and distinguishes between two types of metaknowledge: first-level and second-level. First-level metaknowledge “makes it possible to seek information (*investigation*) from several sources” (p. 203), whereas second-level metaknowledge “makes it possible to process this information” (p. 203). Trouche cautions that the chart “does not completely describe a subject’s behavior” but is fruitful in how it can “provide us with a grid for analyzing” (p. 204) “the action of a given subject aiming at executing a given task in a given environment” (p. 204).

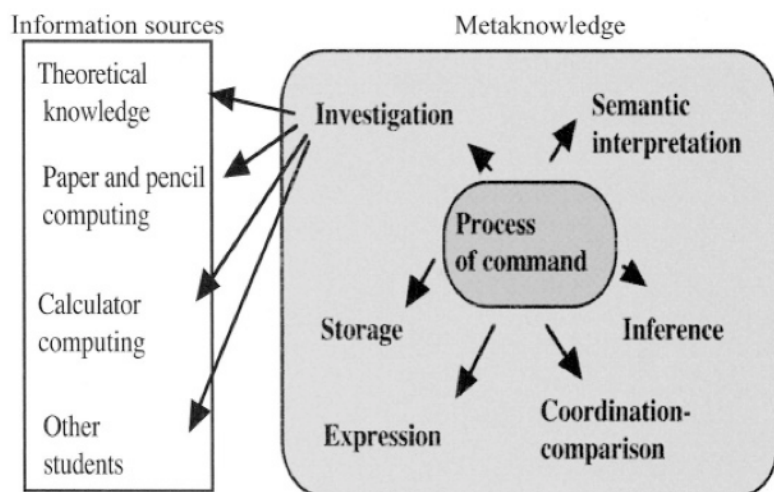


Figure 2. The chart of essential knowledge.

Note. From *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device Into a Mathematical Instrument* (p. 203), edited by D. Guin, K. Ruthven, and L. Trouche, 2005, New York, NY: Springer-Verlag. Copyright 2005 (Original French edition published by Pensees sauvages, 2002, 2005). Reprinted with kind permission from Springer Science+Business Media B.V.

Further, Trouche (2005b) also identified five working methods (Figure 3) for characterizing and explicating “extreme types of behavior” (p. 204) in a calculator (graphing and symbolic) environment. The student with a *theoretical* work method systematically calls upon mathematical references in order to accomplish tasks. The reasoning of such a student “is based essentially on analogy and over-excessive interpretation of facts with occasional use of calculator” (p. 204). The student possessing a *rational* work method primarily prefers “a traditional paper-and-pencil environment” with less regard to technology use. Possessing a “strong command,” for this type of student, “inferences [play] an important role in reasoning” (p. 204). The student with an *automatic* work method accomplishes tasks “by means of cut and paste observations” (p. 204) and displays similar conceptual difficulties regardless of the environment (i.e., paper-and-pencil versus calculator-based) with a “rather weak command” (p. 204). Further, this type of student uses “trial and error procedures with very limited reference to understanding of the tools used, and without strategies for verifying machine results” (p. 204).

The student with a *calculator-restricted-to* work method is observed to use “information sources more or less restricted to calculator investigations and simple manipulations” (p. 204). Also possessing a weak command, this particular type of student uses a reasoning “based on the accumulation of consistent machine results . . . with an avoidance of mathematical references” (p. 204). Finally, with “an average command” (p. 204), the student with a *resourceful* work method attempts to explore “all available information sources (calculator, but also paper-and-pencil work and some theoretical references)” (p. 204) when working through a task. This type of student evidences reasoning “based on the comparison and the confrontation” (p. 204) of all available information possible through the “investigation of a wide range of imaginative solution strategies: sometimes observations prevail, at other times theoretical results predominate” (p. 204).

Work method	Theoretical	Rational	Automatistic	Calculator restricted to	Resourceful
Privileged information source	Theoretical references	Paper and pencil	No single source	Calculator	No single source
Privileged metaknowledge	Interpretation	Inference	Investigation	Investigation	Comparison
Privileged proof method	Analogy	Demonstration	Copy and paste	Accumulation	Confrontation
Command process	Medium	Strong	Weak	Weak	Medium
Global time for calculator work	Medium	Short	Medium	Long	Medium
Time devoted to each instrumented gesture	Long	Short	Long	Short	Long

Figure 3. Typology of five work methods.

Note. From *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device Into a Mathematical Instrument* (p. 205), edited by D. Guin, K. Ruthven, and L. Trouche, 2005, New York, NY: Springer-Verlag. Copyright 2005 (Original French edition published by Pensee sauvage, 2002, 2005). Reprinted with kind permission from Springer Science+Business Media B.V.

Utilizing the Frameworks

Lapp's (1997) model for technological authority explicitly addresses issues of authority and suggests the role of the instructor as pivotal to how authority is perceived or where it is placed in the technology-rich learning environment. Grounded in sociocultural theory, instrumental genesis also includes the teacher as a crucial component of the technology-based mathematics-learning environment—although an explicit discussion of the construct of authority is virtually nonexistent in the literature on instrumental genesis. Artigue (2005) argues that, without “a *culture of instrumented mathematical work that had been progressively built up*” (p. 264) and “organized and coherently managed by the teacher” (p. 264), students come to view technology as a black box. Thus, in the two frameworks used in this study, the actions of the teacher (as the mathematics authority) are deemed as determining or influencing the placement of authority on other possible mathematics sources.

However, pivotal questions should be asked concerning the various mathematics sources students seek and use in various learning environments in order to gain a better understanding of issues of mathematical authority. What sources do students seek when the usual sources of mathematics are no longer available? For example, Lingefjård (2000) found that students in his study seemed “to abandon their trusted sources of authority” (p. 153) or even “introduced and defended contradictory ideas despite their records of satisfactory mathematical achievement” (p. 151) in a nontraditional environment where a course text was unavailable. What happens when the course instructor is removed as a mathematics source? Would students then rely upon their own accumulated mathematics knowledge and understanding from their formative schooling? What are the contributing factors for the mathematics decisions they ultimately make?

Decision-making happens at all levels of learning by all pivotal persons—with or without technology use. Teachers have to make a variety of instructional decisions in regard and response to their students' mathematics knowledge and conceptual understanding. And, although their decisions are not as obvious, students are forced to make decisions as well—decisions involved in the command process. The discussion of the command process implies that students try to match and make sense of the information available from different sources. Specifically in regard to technology, Hollebrands et al. (2007) claim that students “must consider whether what they are seeing is attributable to mathematics or whether what they are observing is a bias introduced by the tool” (p. 177). However, the factors involved for their decisions can vary and consist of a broader psychological basis than that considered by the typology offered by Trouche (2005b).

The underpinning for this study was not necessarily to create or enhance theoretical formulations but rather to explore various situations with plausible revealing factors for issues of mathematical authority. However, based upon the results of this study, I consider Lapp's (1997) framework and the typology of work methods discussed by Trouche (2005b) within the framework of instrumental genesis in regard to forming a plausible framework for conceptualizing mathematical authority with technology. Even though the typology of work methods was developed using graphing and symbolic calculators, I consider the relationship of the placement of mathematical authority in regard to a student's particular work method revealed by the data of the study to help ascertain plausible reasons for a student's behavior and his or her decisions.

CHAPTER THREE

METHOD

Conceptualizing mathematical authority with technology first necessitates the conceptualization of authority in various contexts. This step ensured that I was able to gain a sense of how the preservice teachers interacted with various sources of mathematics. Gaining such an understanding could help to contrast those interactions with the preservice teachers' interactions with technology and, ultimately, illuminate their respective placements of mathematical authority. The design of this study incorporated the use of the classroom-learning environment and individual interviews. The teaching and learning goals of the classroom instructor dictated activities and investigations that took place in the classroom environment. The structure of the individual interviews was dictated by what took place in the classroom environment.

Context

The study took place in an undergraduate geometry content course entitled Basic Concepts 2, designed for preservice teachers during the spring semester of 2010 at a university in the southeastern region of the United States. Two sections of the course were offered during this time and were taught by the same instructor. Basic Concepts 2 was designated as a Basic Studies⁴ course. Although the course focused on content relevant for teaching Grade K–8 students, prospective middle grades teachers (who would be certified to teach Grades 6–9) were

⁴ Basic Studies courses are taken before students may be formally admitted to the School of Education. Students at the university must meet basic requirements before actively pursuing their major degree.

also enrolled in the course. Additionally, the students in this course ranged from freshmen-level (in order to meet their Basic Studies requirements) to those who were licensure-only students (with bachelor's degrees in other fields of study).

The participants of interest in this study were those who would be certified to teach Grade K–6 mathematics—the elementary preservice teachers. The preservice elementary teachers at this university were required to complete a minimum of 6 credit hours of mathematics within their Basic Studies requirements. Three of those hours had to come from a mathematics course, whereas the other three could come from another course that was considered a mathematical science course. The mathematics course had to be taken from a list of seven possibilities.⁵ Basic Concepts 2 was not one of the seven possibilities. It was the second part of a series for which the first course, Basic Concepts 1, was focused on Numbers and Operations. However, Basic Concepts 1 was neither taught nor considered by the mathematics department as a prerequisite for Basic Concepts 2. Therefore students who took Basic Concepts 2 might not have taken Basic Concepts 1. The mathematics department did recommend, however, that the preservice elementary teachers take both courses to meet the 6-hour minimum requirement.

Additionally, the preservice elementary teachers at this university were required to select an academic concentration and take 18 semester hours of courses toward that concentration. The possible academic concentrations the students might choose from were: language arts, social studies, mathematics and technology, science and health, fine arts, English as a second language, international studies, and behavioral studies.

⁵ The students were required to choose one of the following courses: College Mathematics for the General Student 1, College Algebra, Trigonometry, Precalculus, Basic Concepts 1, Basic Calculus with Applications 1, and Calculus with Analytic Geometry 1.

Dr. Fikes,⁶ the course instructor for the two Basic Concepts 2 sections, joined the university faculty two years prior to data collection for this study. Her previous experience teaching Basic Concepts 2 consisted of two sections of the course, in a single semester, one year earlier. That was also the extent of her previous experience teaching Basic Concepts 1. Even though the mathematical content of the two courses was different, Dr. Fikes indicated that she used a similar pedagogy in the two courses. During the semester of data collection for this study, Dr. Fikes also supervised preservice secondary mathematics teachers as they completed their field practicum experiences.

The design of Basic Concepts 2 focused on geometrical concepts for the Grade K–8 learner but also incorporated topics from measurement and statistics and probability. Although the course was a content course (as opposed to a methods course), Dr. Fikes purposely structured the course to use pedagogy that engaged the students in mathematics discussion, helped them evaluate various mathematical strategies centered upon a particular mathematics concept, and helped them develop new mathematical ideas and understandings or deepen their current understandings. Additionally, Dr. Fikes wanted to capitalize on the process of the preservice teachers' changing identity from that of a student to that of a teacher.

Therefore, at times, Dr. Fikes incorporated examples of Grade K–8 students' work and videos of Grade K–8 students engaged in various tasks. The preservice teachers in the course were generally engaged with mathematical tasks designed to evoke a deeper understanding of the elementary concepts they might possibly teach their future students. A "class norm" was discussed and established on the first day of the course and strictly followed for the duration of the course. This discussion involved the conception of effective group norms that involved communication, cooperation, respect, and self-monitoring of one's contributions to small and

⁶ All participant names are pseudonyms.

whole group discussions. Specifically, Dr. Fikes called attention to the necessity of the students to listen to their peers' discussions in "trying to make sense of what they're saying" and emphasized such listening as an important skill to develop for their future careers.

Although the class sessions were not typical (in the sense that the investigations and ensuing discussions were quite varied), there was a general format used in the daily structure of the course. However, it should be noted that this format was not used exclusively; it changed based on varying factors. A typical class session would begin with attention to logistical aspects of the course that included addressing homework assignments, future class topics and sessions, and any other pertinent agenda item deemed necessary for discussion. Next, Dr. Fikes addressed the day's activity by detailing the materials available for the investigation as well as briefly noting the elementary mathematical concepts and ideas underlying the investigation in order to appropriately direct the students' mathematical focus. Then the students engaged in small group work as they explored and investigated the given task. The classroom setup consisted of tables at which the students would work in pairs (or joined with other pairs) to explore and investigate various concepts. While the students worked, Dr. Fikes moved around the classroom and addressed or posed various questions to different groups. Finally, a whole group discussion ensued regarding the outcomes of the small group investigations.

Dr. Fikes indicated she believed that students should have access to tools that help them solve problems and wanted to incorporate technology more into her pedagogy. However, the physical layout of the classroom was not conducive to technology use. There were a total of three computers in the room, one positioned in the front of the room and the other two positioned in a back corner of the room. Thus, Dr. Fikes's pedagogical uses of technology were limited to displaying videos of students engaged in mathematics tasks or the demonstration of mathematics

or teaching applets from the computer in the front of the room. Further, she did not object when students occasionally used their smartphones to access the Internet as a source or aid to finding information (e.g., definitions of mathematics terms).

I chose the context of this study, Basic Concepts 2, because I desired a classroom environment that would be less likely to evoke students' past behaviors stemming from (presumably) years of mathematics learning experiences in teacher-center (or lecture-styled) mathematics classes. Further, it was important that the instructor in such an environment have previous experience teaching with such pedagogy. Dr. Fikes's former experiences using her pedagogy in Basic Concepts 1 and Basic Concepts 2 assured me that she would be familiar with student reactions to her pedagogy for this study. Dr. Fikes indicated that her previous experiences teaching with such pedagogy revealed that she needed to be quite explicit and consistent regarding the class norms in order to help students learn from and cope with any unfamiliarity with such pedagogy.

Participant Selection

Data collection began on the first day of class for both sections of Basic Concepts 2. After Dr. Fikes explained the elements of the course syllabus and the course expectations, I introduced myself, described my research, and obtained written permission for student participation in the study. Data collection methods included videorecorded⁷ observations, a background information form (Appendix B), and a beliefs survey that the students completed. (A thorough discussion of the videorecording methods and the beliefs survey are provided in the next section.) The information form was developed by Dr. Fikes and asked for details about each preservice teacher's academic background and chosen academic concentration. I used the results

⁷ For the purposes of reporting, discussion of videorecording methods implies the use of simultaneous audiorecording as well.

of the information form and the observations of the first several days of the course to inform my participant selection.

Dr. Fikes structured the content investigation of Basic Concepts 2 so that concepts from statistics and probability were explored first. During that time, I collected data on both sections of Basic Concepts 2 in order to make a decision about which section and students I would focus on for the duration of the study. Although the two sections had similar class demographics (see Table 1), the dynamics of the two classes were markedly different.

Table 1

Basic Concepts 2 Cohort Responses to the Information Form

Category		Class section	
		One	Two
Gender	Female	24	22
	Male	2	4
Major	Elementary	15	20
	Middle	2	2
	Secondary	0	0
	Other	9	4
Concentration (elementary major only)	Behavioral Studies	4	4
	Language Arts	0	4
	Mathematics and Technology	2	5
	Science and Health	1	0
	Social Studies	1	1
	Unsure	5	4
	Other ^a	2	2
Rated mathematical ability	Excellent	0	1
	Good	7	10
	Average	15	12
	Below average	3	3
	Weak	1	0

^a*Other* includes English as a Second Language, Fine Arts, International Studies, and Early Childhood Concentration.

The students of the second section were more engaged, more vocal, and had a variety of backgrounds—features that were important for data collection purposes. Therefore, I decided that the majority of the data collection would continue only in the second section upon the completion of the statistics and probability unit.

I was interested in specific students who initially met two conditions. First, I chose participants who were vocal and active in the small and whole group discussions and who freely offered their mathematical thinking and knowledge. Having such participants would help reduce the amount of inferential analysis I had to do concerning their mathematical behaviors and understanding. Second, I focused only on elementary majors because they were the population of interest for my study. My final choice of participants meeting these two conditions included two with a mathematics concentration and two with a nonmathematics concentration. I hoped that the different concentrations would provide a contrast of results concerning placement of mathematical authority.

I used the results of the class observations and the information form to initially choose six potential participants from the second section. I continued to observe the students during the first three sessions devoted to geometry to make a well-formed decision of the study participants. I made the additional observations because of my concern that topics of statistics and probability might yield a different mathematical behavior from the participants than would be observed for the remainder of the semester as they explored concepts from geometry and measurement. Based on these observations, Garrett, Skyler, Janina, and Kayla were chosen to participate in the study. They readily agreed to commit their time outside of class to answer questions and engage in mathematics tasks relevant to their experiences in Basic Concepts 2. (See Table 2 for a detailed outline of the course and interview dates.)

Table 2

Basic Concepts 2 Section 2 Spring 2010 Meeting Schedule

Content	Date	Topic	Other information
	1/7	Intro/Syllabus	
Statistics/ Probability	1/12	Levels of Learning Statistics	
	1/14	Activities to Conceptualize Averages	
	1/19	More Activities to Conceptualize Averages and Other Statistics Terms	
	1/21	Quiz and String Length Activity	
	1/26	Average and Box Plots	
	1/28	Class Cancelled	
	2/2	Data Collection Activities and Box Plots	Skyler absent Kayla absent
	2/4	Warm Up and Woodpecker Sample Activity	Janina absent
	2/9	Exam 1*	
Geometry	2/11	Sorting Activity and Definition of Triangle	
	2/16	Quadrilateral Sorting Activity	
	2/18	Quiz and Symmetry	Kayla absent
	2/23	Warm Up, Properties of Rhombi and Finding Interior Angles of Polygons	Janina absent
	2/25	Finding Interior Angles and Sum of Angles of Polygons	Skyler absent
	3/2	Test Review & Tessellation Discussion	
	3/4	Exam 2*	
	3/9	Spring Break	
	3/11	Spring Break	
Measurement	3/16	Measurement Activities	Garrett Interview 1
	3/18	Crazy Cakes & Discussion of Linear versus Area Dimensions	Kayla absent Janina Interview 1
	3/23	Area of Different Figures	Skyler Interview 1
	3/25	Deriving & Justifying Formulas for Area	
	3/30	Deriving & Justifying Formulas for Area	Kayla absent
	4/1	No class – state holiday	
	4/6	Polyhedra	Skyler Interview 2 Garrett Interview 2
	4/8	Volume	Kayla Interview 1
	4/13	Volume	Skyler Interview 3 Janina Interview 2 Garrett Interview 3
	4/15	Exam 3*	Kayla Interview 2
Geometry	4/20	Transformational Geometry	Janina Interview 3
	4/22	Optional Quiz/HW/Final Exam Review*	Kayla Interview 3

Note. *Data were not collected for this class session.

Garrett

Garrett was classified as a junior and rated himself as “good” at mathematics on his information form. He began his studies as a middle school education major with a concentration in science and mathematics. At the beginning of his sophomore year, he changed his major to elementary education with a concentration in mathematics. Garrett had completed all of his education coursework and had two mathematics courses left to take “to finish up [his] concentration.”

In addition to his interest in education, Garrett also worked as a director at a local YMCA (a place of employment for him since he was 16 years old). Garrett’s schedule stayed quite full between his school and work obligations. He teetered back and forth between his future employment goals of continuing to work for the YMCA versus teaching elementary school. Although his scholarship obligations dictated that he would be teaching at least 4 years, he was not sure of his goals upon the completion of those obligations. At one point, he indicated that he loved children and “helping them grow” and that his “big plan [was] to be a youth services director under a CEO of a YMCA.” In other discussions, however, Garrett indicated that he was not sure what his future held and that he needed to “decide soon.”

There were a few reasons why I chose Garrett as a participant. I first considered him as a potential participant on the second day of class for which he helped organize the work of small groups on the class board for a whole group discussion. He exhibited a leadership quality that I thought might prove interesting for conceptualizing mathematical authority. Second, Garrett contributed daily to the different class discussions. And third, Garrett was the only male elementary major in the class who met the two conditions that I sought from study participants. Although the population of males was quite small in both sections, I thought having a male

representative in this study would provide a contrast of results concerning participants' placement of mathematical authority.

Skyler

Classified as a senior, Skyler had begun her studies as a nursing student. However, the experience of a patient's death motivated her to change her career choice to education—a field that she had desired to enter as a young child. Skyler was hesitant to teach high school level students but “wanted to do high school level math.” So she followed the advice of her academic advisor to major in elementary education but continue taking upper level mathematics courses. Therefore, at this point in her academic endeavors, Skyler had taken or was taking 18 semester hours of college-level mathematics courses (including Basic Concepts 1 with Dr. Fikes). She was also enrolled in Calculus 2 during the semester of this study.

Skyler was asked to participate in the study for a variety of reasons. First, she classified herself as “good” at mathematics on her information form and was a vocal participant who readily answered questions posed by Dr. Fikes. On the first day of class, I quickly noted Skyler as a potential participant because of her verbal contributions and her apparent high level of confidence. Second, since Skyler was an elementary major and had taken Basic Concepts 1 with Dr. Fikes, I thought she might possess an interesting perspective on the course that other participants may not have. Finally, Skyler's information form indicated that she had taken a higher-level mathematics classes that included a first semester of a basic calculus sequence and a sequence of calculus with analytic geometry. Thus, I felt that she could potentially offer a different perspective on placement of mathematical authority than the other participants.

Janina

A freshman with a concentration in language arts and a minor in history, Janina had initially decided to major in business because her mother's experiences as a middle-grades teacher made her anxious about a career in education. However, at the beginning of her second semester, Janina changed her major to elementary education because she "loved kids" and felt an obligation to provide "a positive influence" for and "to guide" children. Her academic commitments and her involvement in her sorority and campus organizations kept her schedule packed with activities. Janina generally enjoyed her academic endeavors although, at times, she acknowledged feeling "kind of overwhelmed." Janina had not completed much coursework because of her relatively short time at the university. She did, however, complete college algebra during her first semester at the university. After changing her major, she enrolled in Basic Concepts 2 to fulfill her mathematics requirements for her concentration and the basic studies program.

There were a number of reasons I asked Janina to participate in the study. First, Janina was not shy; she regularly contributed to the class discussions. She had an easy-going demeanor and was generally seen smiling or laughing with her group members. Janina was also vocally reflective; she would often provide discussion about why she believed others' mathematical strategies were correct in contrast to her own strategies. And, finally, Janina rated herself as "good" at mathematics on her information form but had chosen language arts as her concentration. Thus, I thought she might provide an interesting perspective in my attempt to conceptualize mathematical authority.

Kayla

Classified as a sophomore, Kayla had chosen social studies as her concentration because of her “love” for history. In an effort to “leave a mark,” Kayla felt “called to teach” because of her desire to “help the most people.” She explained her reason for choosing to teach children at the elementary school level: “They’re still innocent and you can encourage them to do their education better before they get to that peer pressure stage.”

Since coming to the university, Kayla had discovered, “You don’t really have to put forth that much work and you still get good grades.” Thus, as a “pretty athletic,” “out-doorsie,” and “super competitive” person, she spent most of her free time outside of school either on recreational athletic activities or visiting with her significant other (a mathematics major attending college in a different state). However, her personal and academic activities were dampened by the fact that she had been “sick a lot this semester,” which resulted in an abnormal “struggle” with her coursework.

Like the other participants, Kayla was chosen because of her class contributions. As early as the first day of class, Kayla often contributed to class discussions with mathematical sentiments that conveyed her beliefs about mathematical learning and understanding (as opposed to just answering a mathematics question). I was interested in exploring those beliefs in the study. Further, Kayla rated herself as “average” in mathematics and had taken Basic Concepts 1 with a different instructor. However, what really drew me to Kayla was the emotion in her response on her information form to the question: “Please list any additional mathematics courses that you plan to take after this semester.” Kayla’s response of “None!” indicated that she might harbor issues with mathematics; I was interested in exploring these issues concerning mathematical authority.

Data Collection

In addition to student responses to the information form and the pre- and post-beliefs surveys, the data I collected from the four participants were observations of class sessions and responses and observations gleaned from personal interview sessions. The collective data from these various sources were considered in the context of a multiple-case study design (Yin, 2003) concerning the four participants. The purpose of this design was to help me form an understanding of the participants' actions and behaviors observed during the class sessions and their individual interviews, to allow me to contrast those behaviors to each other, and to ascertain placement of mathematical authority in different contexts.

Beliefs Survey

As discussed earlier, data collection for both sections of the course occurred during the first unit of statistics and probability exploration. I distributed a beliefs survey to the students of both sections of Basic Concepts 2 on the first and last day of the course. The survey was not used to select the participants. Instead, the survey responses were used as a means to provide additional insight and evidence into their views of mathematics and mathematics learning and teaching. Further, the results of the pre- and post-beliefs surveys were used to help inform my inferences of placement of mathematical authority and possible changes in mathematics beliefs specific to the four participants and in contrast to the cohort of students.

The beliefs survey was developed by McCormick, Kapusuz, Al-Salouli, and The Preservice Beliefs Study Group (2004) in an effort to study the mathematics beliefs of preservice elementary teachers. The 75-item survey has nine categories of items that pertain to the beliefs, attitudes, and feelings the preservice teachers harbored in regard to mathematics and mathematics teaching and learning. (See Appendix C for sample items from each of the nine

categories.) McCormick et al. reported on a subset of the questions of the survey grouped into only four categories. For the purposes of this study, I categorized and grouped all but two of the remaining questions into the other five categories. I considered the wording of the two remaining questions too vague to reliably group either into a particular category.

The survey used a traditional five-point Likert-type scoring format. The questions were coded as either a positive or a negative statement. Each student's response for each question was scored accordingly and then summed with the other questions in its category. The sum for each category represents each student's placement along the scale for that category. The numerical range of the scale represents the spectrum on how positive or negative a student is in regard to the category. I conducted a statistical analysis in the form of a paired *t*-test to compare the means of the cohort of students' scores for the corresponding categories of the pre- and post-beliefs surveys. This statistical analysis also allowed me to gauge how the four participants changed over the course of the semester in contrast to the cohort of students in Basic Concepts 2.

Class Sessions

I video recorded all but four of the class sessions of the second Basic Concepts 2 section. The four class sessions not video recorded were the three inclass exams (because of the potential of distraction created from the equipment and myself) and the final day of the course. This last session was dedicated to making up or retaking quizzes and addressing questions about material on the final exam. The graduate teaching assistant directed this final session because Dr. Fikes was absent for the final 2 days of the course.⁸

My purpose for collecting the videorecorded data was multifold. First, this type of data provided a concrete timeline of the following: the mathematics content addressed throughout the

⁸ Dr. Fikes was absent for three class sessions. During these absences, the teaching assistant, a graduate student in the mathematics department who attended and observed every class session, substituted for Dr. Fikes.

course; the activities and investigations used in the course; the interactions between the students and between Dr. Fikes and the students; and the pivotal moments of teaching, learning, or discussions relevant to the study that transpired throughout the semester. Second, this data collection method allowed me to record different but simultaneous interactions in order to broaden my understanding of those pivotal moments. And finally, the video data were used to gain information about the four participants' mathematical thinking and understanding that could be elicited only through interactions with their peers as they struggled with various mathematics content and ideas.

I used two cameras, a main camera and a secondary camera, to capture the activities of the class sessions. Positioned in a back corner of the room, the main camera was used to capture as much of the class activity as possible as well as the whole group class discussions facilitated by Dr. Fikes. Because of the rectangular shape of the classroom and the limited scope of the camera, not all students were visible in the whole class discussions. The main camera's wireless microphone was attached to Dr. Fikes for each class session and thus captured her discussions with various students. I used the hand-held secondary camera to capture data from individual students during class sessions as I (as discreetly as possible) walked around the room. The data consisted of students' work or different mathematics conversations that were happening with (or in the absence of) Dr. Fikes. Once I had selected the four participants, a large percentage of the data captured with the secondary camera included at least one of the participants at any time. There were times, however, when both cameras would simultaneously capture Dr. Fikes engaged in discussion with various students, usually at least one of the four participants. These discussions included what, at the time, appeared to be potentially interesting conversations pertinent to the study.

The videos from the two cameras were streamed together (Figure 4) to create a *restored view* (Hall, 2000) of a “picture-in-picture” (p. 652) fabrication that allowed for a more holistic perspective of each class session rather than just one video angle. The restored view provides a better representation of actions and discussions that took place between the students and between the students and Dr. Fikes. For example, in several instances of data collection, the audio of Dr. Fikes’s wireless microphone provided evidence of her engaged in a particularly relevant or interesting conversation with a group of students. Meanwhile, with the other camera, I captured evidence of different students engaged in relevant or interesting discussions as well. However, without the video of either camera capturing Dr. Fikes’s location in the classroom, it would have been difficult to ascertain to whom she was speaking.

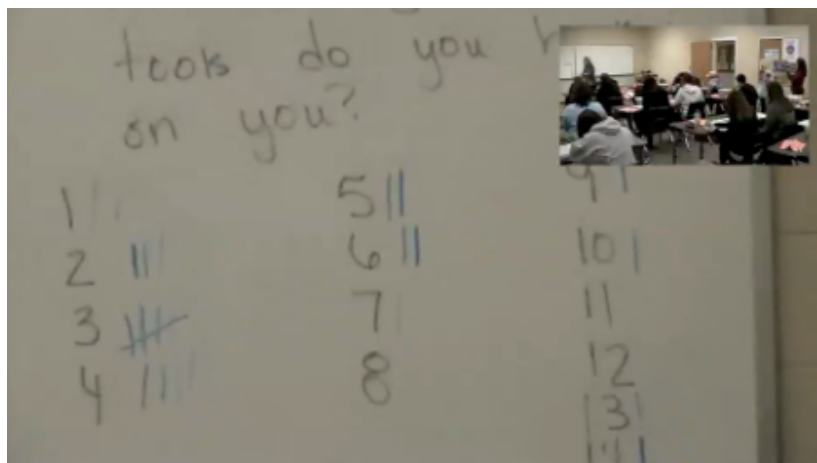


Figure 4. Restored view of two camera angles used for recording class sessions.

The classroom layout was such that each student would sit with at least one other student to investigate and explore problems. Prior to each class session, either Dr. Fikes or I would somewhat randomly distribute nametags to the desks as an indication of where the students were to sit. This method of seating helped Dr. Fikes maintain her attendance log and helped my efforts of data collection by ensuring that the four participants were sitting close to each other.

Interviews

I conducted three interviews, averaging approximately one and one-half hour each, with each participant over the course of the semester. Each interview comprised two components: a general interview portion and a clinical interview portion (Ginsburg, 1997) and was videorecorded with two cameras. One camera was used to capture the written work of the participant. The other camera captured the participant engaged in a mathematics task or captured the images on the computer screen as the participant worked on the mathematics task. Similar to the class sessions, the videos were streamed together to create a restored view to allow for a more holistic perspective of the interview.

With the questions from the general interview portion (Appendix D), I intended to collect data on each participant's beliefs about mathematics, mathematics learning and teaching, and technology. Most of the questions were developed by McCormick et al. (2004) to supplement or enhance the beliefs survey questions filled out by the Basic Studies 2 cohort on the first and last days of class. I incorporated mathematics tasks into the clinical portion of the interviews for each participant to explore and solve with the use of technology. The technologies the participants had available to them were the Geometer's Sketchpad [GSP] (Version 5.0) and the TI-83 Plus graphing calculator. I chose the mathematics tasks based upon particular mathematics tasks or activities that the participants had explored and discussed during class. Collectively, the characteristics of the different tasks and each participant's discussion of and engagement with those tasks provided me with his or her respective placement of mathematical authority.

In the Absence of the Instructor (Mathematics Task 1).

Background. The first mathematics task was chosen for the purpose of introducing each participant to the software of GSP (Version 5.0) and to provide insight about the interactions

observed during the small group discussions that took place during the relevant class session. The task had been first presented to the class as a homework problem (Figure 5) and concerned the development and understanding of tessellations and, in particular, semiregular tessellations. Thus the students had engaged in a rudimentary exploration of semiregular tessellations through their homework efforts. The students then came together to continue that exploration in small group discussions as they tried to determine which of the two figures in Figure 5 met the conditions of the definition of semiregular tessellation. The mathematics investigation of the first interview continued that exploration through the use of GSP.

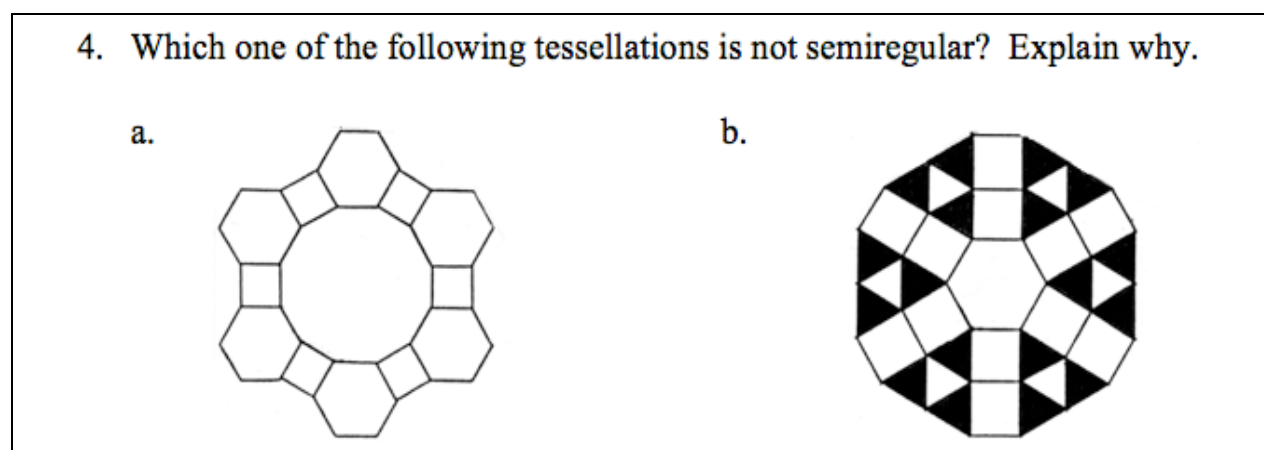


Figure 5. Homework assignment task.

I had several reasons for choosing this topic as the focus of the first mathematics task. First, tessellating the plane and understanding semiregular tessellations can easily be explored through and demonstrated with GSP. (At this point in the class, Dr. Fikes had used GSP only once during a whole group discussion for demonstration purposes of illustrating various polygons.) Thus, the task provided a nice context for the four participants to begin interacting with GSP. Second, Dr. Fikes did not directly address in class the topic of tessellating the plane.

The students were expected to explore the topic as part of a homework assignment and then draw on that understanding to answer the problem⁹ presented in Figure 5.

Up to that point in the course, the students' understanding of the topic was primarily developed through either their previous experiences with the topic (independent of the course) or (if they had no prior experiences with tessellations) their own means of learning the material for the homework assignment (independent of the instructor). Thus, a third reason for choosing this task for the first interview was that Dr. Fikes was not a direct factor in their ability to answer the question in Figure 5.

The students' discussions and interactions in regard to the task provided evidence of their attempts to flesh out an understanding of the definition of *tessellation* and *semiregular tessellation*. The different sources of mathematics information the students used included their previous experiences learning the topic, their peers in the course, the course textbook, and resources from the Internet. The tessellation exploration during the clinical interview included discussion of each participant's memories of his or her investigations during the class session so I might gain a better understanding of his or her relative placement of mathematical authority in the context of the task.

For the relevant small group discussion of tessellations, the four participants were sitting in close proximity. Garrett and Janina were coupled together, while Skyler worked with Julie¹⁰ at a table directly behind the two. Kayla and her partner sat at a table directly beside Skyler and Julie. Because of this close proximity, Garrett, Janina, Skyler, and Julie became engaged in a lengthy discussion about the review problem in Figure 5. Because three of the participants were

⁹ The problem presented in Figure 5 was part of a review sheet given to the students as an aid to study for their next exam. Prior to this class session, the students had been exploring how to determine interior angles and the sum of interior angles of various polygons as well as developing the pertinent formulas for each.

¹⁰ Julie was a nontraditional student majoring in special needs education. She regularly contributed to whole and small group discussions. She indicated she was "below average" in mathematics on her information form.

interacting with each other for a large portion of the class session, much of my time was spent within their vicinity. Therefore, I captured limited evidence of Kayla's discussion with her group members about the problem.

Interview components. After each participant responded to the general interview questions, he or she was then introduced to the technology of GSP. Each participant was informed that the purpose of the interviews was to gain a sense of how he or she thought about Dr. Fikes's pedagogy for Basic Concepts 2 as well as his or her feelings about the use of a specific type of technology (GSP) for learning mathematics. Specifically, in regard to the first interview, the participants were informed that they would be introduced to GSP using their knowledge of tessellations as formed and explored during the relevant class session.

Thus the clinical portion of the interview began with an introduction to the main features of GSP (e.g., menu items and creating basic shapes). Then, for the purposes of tessellating the plane and answering the question in Figure 5, each participant was provided with an explanation of how to create regular n -gons using GSP's premade scripts. Next, each participant was asked to recall his or her memories of the small group class discussions surrounding the review question in Figure 5. Although (with the exception of Kayla), the participants' small group discussions were captured through videorecording, it was important to ask them to recall their individual memories in order to gain a sense of their current understanding of the topic and how they remembered the events as they unfolded in class.¹¹

I then asked the participants to consider and discuss which of the regular polygons singularly tessellate the plane. I informed them that they could use GSP as necessary to aid in their discussion. A discussion of semiregular tessellations naturally followed their explorations

¹¹ A two-week period occurred (that included the second in-class exam and spring break) in the interval between the pertinent class session and the first participant interview. The topic of tessellations was not addressed on their exam.

of whether or not a regular octagon would singularly tessellate the plane. Finally, the participants explored the answer to the review question in Figure 5.

The Triangle Dissection Paradox (Mathematics Task 2).

Background. At the start of the second round of interviews, four class sessions had been devoted to the exploration and discussion of finding the area of various regular and irregular figures. In addition to finding the areas, the students were asked to develop and justify formulas for various standard geometric shapes and explain commonly used formulas for those shapes.¹² Based upon my observations of the participants' interactions and discussions during those four sessions, I decided to use a problem that could cause potential conflict with the mathematical knowledge they appeared to use in their class explorations.

The second mathematics task asked each participant to explore the Triangle Dissection Paradox (Weisstein, 2012), also known as Curry's Paradox or Curry's Triangle (Gardner, 1956). The "triangles" of the paradox are presented in Figure 6 as Triangle A and Triangle B. Once the pieces of the decomposed Triangle A are rearranged to form Triangle B, a mathematical anomaly appears: two congruent "triangles" with areas that differ by one square unit.

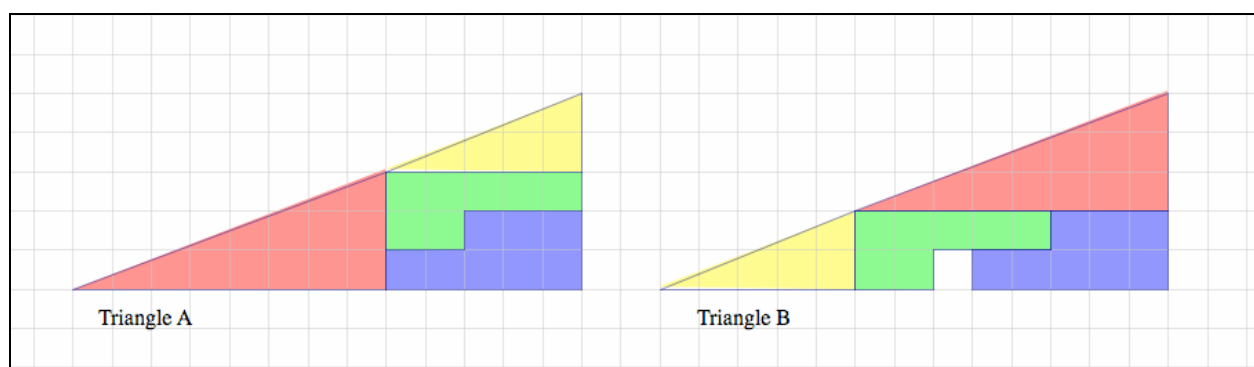


Figure 6. "Triangles" of the Triangle Dissection Paradox.

¹² For example, the commonly used formula for determining the area of a trapezoid is: $A = \frac{1}{2}(b_1 + b_2)h$.

One homework assignment and two consecutive class sessions played a pivotal role in my decision to use the paradox. For the homework assignment, the students were to think about the problems of a worksheet called Crazy Cakes¹³ (Appendix E). The worksheet was never addressed in a whole group discussion. Dr. Fikes explicitly addressed Crazy Cakes only in a short small group discussion devoted to how the students answered the questions of the worksheet. The problems of Crazy Cakes asked students to determine how to divide irregular shapes such that the line of division results in two sections that are of equal area but are not necessarily congruent. Later, in both small group and whole group discussions focused on the concept of area, the four participants periodically referred to the Crazy Cakes assignment. This reference provided evidence of their understanding of and ability to find the areas of figures using methods of decomposition and rearranging of shapes to justify their findings.

The first of the two class sessions was devoted to finding areas of different types of figures and, in particular, understanding the formula for finding the area of any triangle. Dr. Fikes engaged the students in a whole group discussion aimed at developing the formula for the area of a triangle using a virtual geoboard projected on the overhead screen. At the beginning of the first class session, Dr. Fikes informed the students of the necessity of their understanding of the development of basic mathematical ideas as future teachers. She asked the students to consider how they would answer the question, “What is area?” She said: “You need to know where that formula is coming from and why that formula works, as well as understanding the formula for the other shapes such as trapezoid and so forth.” At this point in the course, the students had engaged in exercises to determine areas of various shapes for different class investigations using their formative (and generally formulaic) knowledge of areas of different

¹³ The worksheet was developed as part of the Developing Mathematical Ideas materials found at <http://www.mathleadership.org/page.php?id=33>.

polygons learned in their prior mathematics experiences. However, for this class exploration, the *development* of the formula was the focus rather than the means.

Dr. Fikes then asked the students for the area of a one-unit square presented on the virtual geoboard. This question provided a conceptual foundation to help the students think about how to justify how they would find the area for other rectangles and right triangles projected on the geoboard. These discussions homed in on the idea that the area of a given rectangle can be found by counting the number of unit squares in the rectangle. When Dr. Fikes presented a right triangle to the class, a discussion ensued that every rectangle consists of two right triangles. Thus the students were able to use their knowledge of how to find the area of a rectangle to find the area of any right triangle. For example, the students used the rectangle to find the area of the right triangle whose shape was half of that rectangle.

Following the discussions of how to find the area of various right triangles projected on the virtual geoboard, Dr. Fikes then displayed an obtuse triangle (Figure 7a) and asked the students to consider how they would determine and justify the area for that figure. One student suggested creating “a mirror image” in order to form a parallelogram. As he described the method, Dr. Fikes created the image projected on the screen (Figure 7b). Dr. Fikes led the students through questions aimed at using their knowledge of transforming the parallelogram into a rectangle with the same area and then using the square units of the virtual geoboard to find the area of the parallelogram and then the area of the obtuse triangle.

The second class session was devoted to deriving and justifying the formulas for the areas of different quadrilaterals, specifically parallelograms. Here the students relied on their knowledge of the following: the area of a triangle (as developed in the previous class session); how to decompose and rearrange shapes of a given figure; how to use the areas of shapes that

compose a given figure to find its area; and how to relate the formulas for parallelograms and triangles.

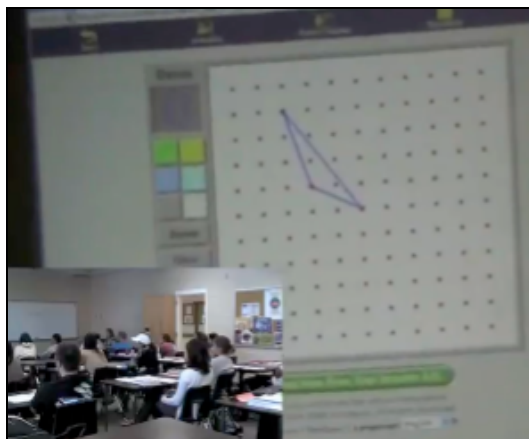


Figure 7a. Oblique triangle projected onto the virtual geoboard.

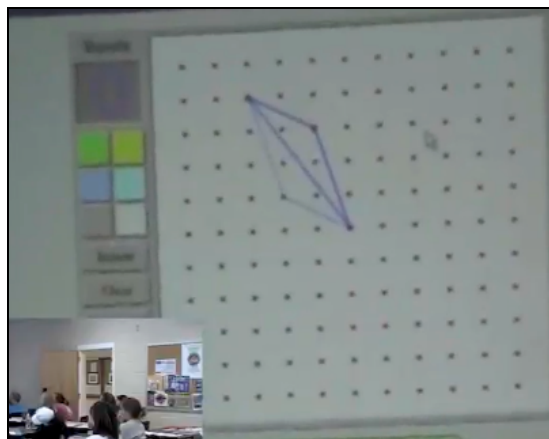


Figure 7b. Student's suggestion of how to create the corresponding parallelogram.

After their whole group discussion, Dr. Fikes had the students investigate problems of a handout (Appendix F) as a means for the students to use the mathematics they had just discussed. The first problem asked the students to develop a rule to find the area of any parallelogram and provide an explanation for how they know the rule works. The second problem asked the students to devise a general rule for finding the area of any trapezoid and explain why it would always work. For this problem, Dr. Fikes provided the commonly used formula for the area of a trapezoid on the board as a reminder to perhaps “help [the students] to think about how that formula is working.” The third problem presented the students with a picture of an obtuse triangle and rectangle sharing a base and informed the students that the two figures have the same height. The problem asked the students to determine whether the triangle’s area is equal to half of the rectangle’s area and to prove their argument.

Interview components. After each participant responded to the general interview questions, they began their individual explorations of the Triangle Dissection Paradox. Presented with the screenshot in Figure 8, the participants were asked to use the disjoint pieces below the

triangle to form another triangle with the same configuration and orientation as the triangle on top. After the rearrangement, each participant noticed a “hole” in the new triangle.

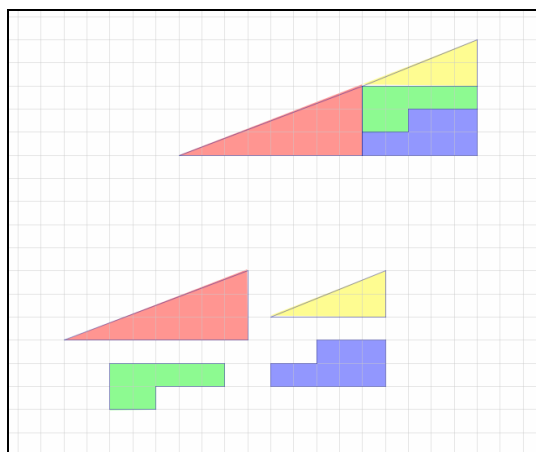


Figure 8. Screenshot of the second mathematics task.

I selected this task for the purpose of observing how each participant, using his or her mathematical knowledge and understanding as observed during the class sessions, reconciled the anomaly presented by the hole. I surmised that an issue of mathematical authority would arise as they tried to work through the mathematical conflict using technology as the medium for explorations. Based on each participant’s investigation and discussion of the task, I asked varying questions to get a better understanding of his or her placement of mathematical authority.

I did not provide the participants with any directions on how to complete the task. Further, while their investigations took place, none of the participants was told that he or she was explaining a paradox. Instead, I told them that the point of the interview was to continue learning about GSP through various explorations of mathematics tasks closely aligned with the mathematics that they learned and explored in Basic Concepts 2.

The Lie of GSP (Mathematics Task 3).

Background. The third mathematics task was the final and culminating task in regard to conceptualizing mathematical authority with technology. The task was designed with two purposes. The first purpose was to elicit the mathematical knowledge and understanding of the participant in regard to the topic of mathematical shearing in geometry without the aid of technology. The second was to determine each participant's placement of mathematical authority when the technology of GSP presented results different from what had been initially elicited.

At this point in the semester the students in Basic Concepts 2 had worked extensively with triangles by using them to find the areas of other figures through a process of decomposition. The third mathematics task (Driscoll, 2007), shown in Figure 9, concerned the understanding of mathematical shearing. The concept of mathematical shearing was explored in various formats throughout the semester, although Dr. Fikes never officially identified it as mathematical shearing. Not only did Dr. Fikes have the students explore the concept in small groups in different problem contexts, she also explicitly addressed the concept in a whole group discussion. In addition to addressing the concept of mathematical shearing, the third mathematics task also drew upon the four participants' knowledge and understanding of the necessary conditions for a polygon to be considered a triangle.

Two vertices of a triangle are located at $(0,6)$ and $(0,12)$. The triangle has area 12 square units. What are all possible positions for the third vertex? How do you know you have them all?

Figure 9. Third mathematics task taken from Driscoll (2007).

Class discussions relevant to the third mathematics task involved the concept of triangle and the preservation of area of a transformed figure with specific attention to the concept of mathematical shearing. The concept of a triangle was initially explored on the first day of the geometry portion of the course (see Table 1). Garrett proposed a definition for *triangle* during a

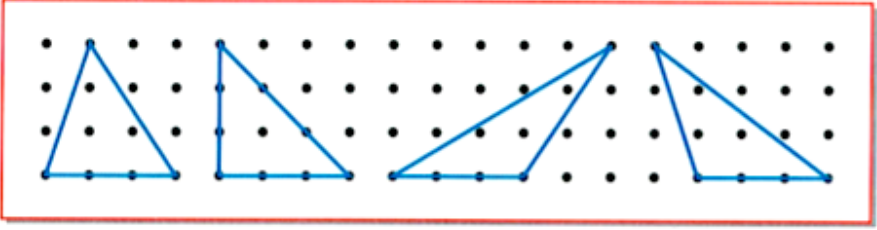
whole group discussion after he had developed it with his partner, Janina, during their small group discussion. By the time the participants engaged with the third mathematics task in the interview, the class had spent nearly 8 weeks in investigations and discussions of a variety of problems that used their initial exploration of the concept of a triangle. Further, the concept of preservation of area (as discussed in the results of the second mathematics task) had been explored in multiple instances prior to the third interview and used the concept of triangle as well. Throughout those instances, there were investigations specifically devoted to the exploration and development of the concept of mathematical shearing. The concept was treated as understanding the preservation of area through the transformation of a figure.

The class session in which Dr. Fikes explicitly addressed mathematical shearing (without actually using the term) was one of the two pivotal class sessions critical to the results of the second mathematics task. The main focus of that session was the development of the formula for the area of different polygons through a variety of means and justifications. Prior to the session, the students had been expected to complete a homework assignment whose purpose was to lay the groundwork for understanding mathematical shearing. For part of the assignment, the students were required to answer the problems shown in Figure 10 (as well as a similar problem that used parallelograms instead of triangles).

In a rare deviation from the typical class instruction of Basic Concepts 2, Dr. Fikes provided the students with the answers to the assignment (as opposed to initially allowing the students to discuss and check their results with their small group partners). Further, for the problems shown in Figure 10, Dr. Fikes also explicitly discussed the conjectures. This discussion helped to ensure that the students were aware that the areas of the figures were equal.

5. Use geoboard techniques to determine the areas of these triangles.

a. b. c. d.



Area: _____ Area: _____ Area: _____ Area: _____

Make a conjecture about the area of triangles with bases of equal length and common heights. Sketch three additional noncongruent triangles with bases of length 5 units and altitudes of length 4 units to test your conjecture. If your conjecture appears to be valid, record it in the provided space below.

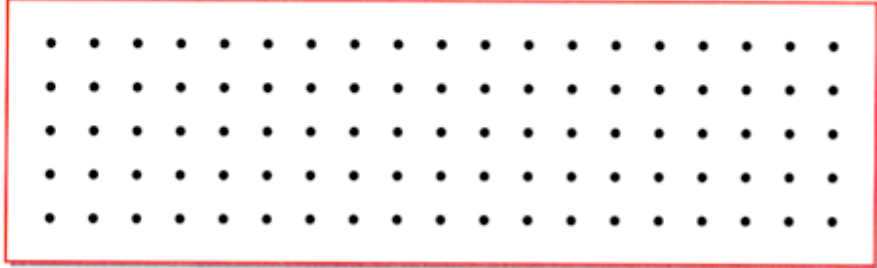


Figure 10. Mathematical shearing homework task from Bennett, Burton, and Nelson (2009).

Three of the four participants—Skyler, Kayla, and Garrett—were observed to understand the concept of mathematical shearing as they investigated different class exercises that used the concept. At some point during the class sessions, each one specifically made reference to the fact that triangles (or parallelograms) with equal heights and bases would have equal areas.

Unfortunately, data were not obtained on whether Janina understood and used the concept as she worked on the problems with her small group partner. However, because Dr. Fikes had explicitly addressed the required conjectures of the assignment and because mathematical shearing was the underlying concept in a number of problem contexts, it seemed plausible to assume that Janina had developed some understanding of the concept. Regardless, I attempted to elicit her understanding (as well as confirm the others' understanding) via the third mathematics task.

Interview components. I asked the participants to first attempt the task (Figure 9) on paper to ensure conceptualization without the interference of technology. As they began their exploration, I also asked them not to consider the formula for the area of a triangle and, instead, explore the task as they would in a typical Basic Concepts 2 class session. After their exploration with the task without the use of technology, my plan was to make available a GSP sketch (Figure 11) that I created prior to the interview. This sketch presented a triangle that appeared to meet the conditions of the problem in Figure 9. However, the location of the third vertex in the GSP sketch was different than what the participants found through their paper and pencil investigations.

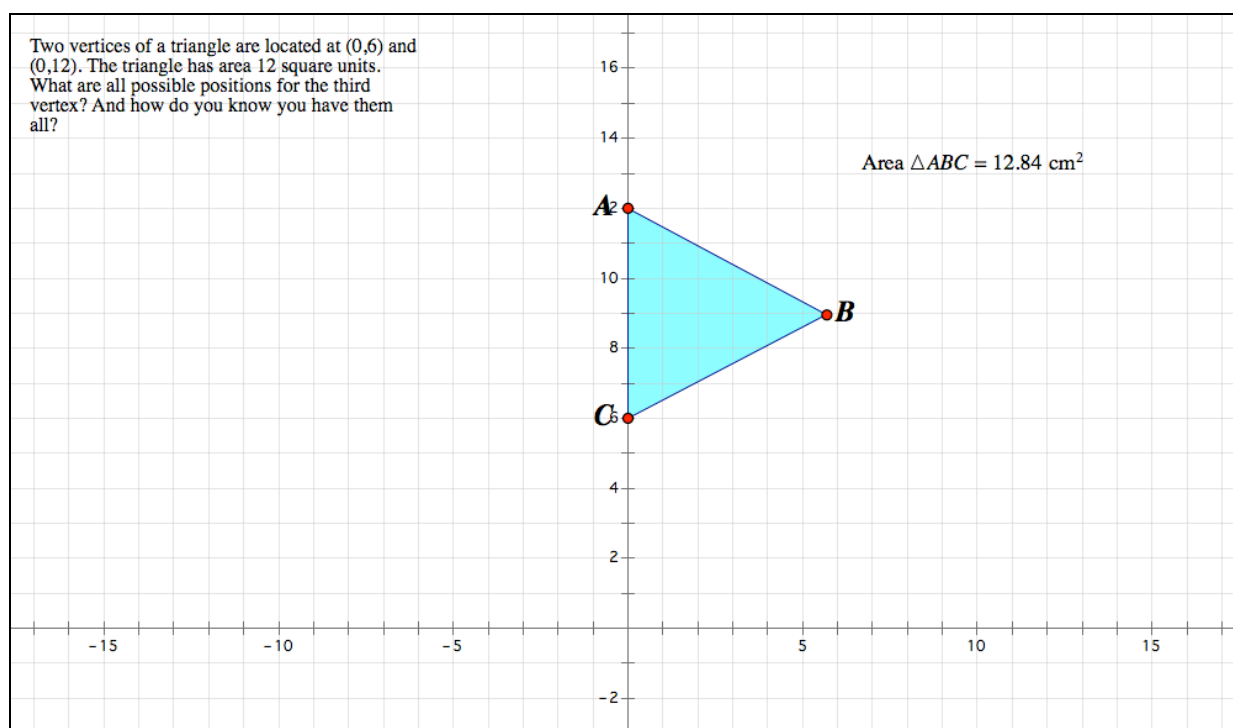


Figure 11. Screenshot of third task with certain elements visually available.

The *Display* menu of GSP allows for selected elements of the screen to be hidden while others remain visible to the user. Figure 11 shows the screen available to the participants, and Figure 12 shows the hidden and available elements of the task. In order to successfully alter what

was visually available on the screen, I created two dependent triangles shown in Figure 12. The area measurement of the smaller triangle in Figure 12 is displayed on the screen in Figure 11, but the larger triangle in Figure 12 is the one that is visually available in Figure 11. This larger triangle's area measurement is hidden from view in Figure 11 but can be seen highlighted in the bottom right of Figure 12. The segment at the bottom of the screenshot in Figure 12 is the scale (square root of 2) used to create the dependent triangles. The segment's length was chosen so that the ratio would not be easily recognized. After they were confronted with the conflicting results presented by GSP, I planned to discuss where each participant placed mathematical authority in regard to his or her interactions with the technology.

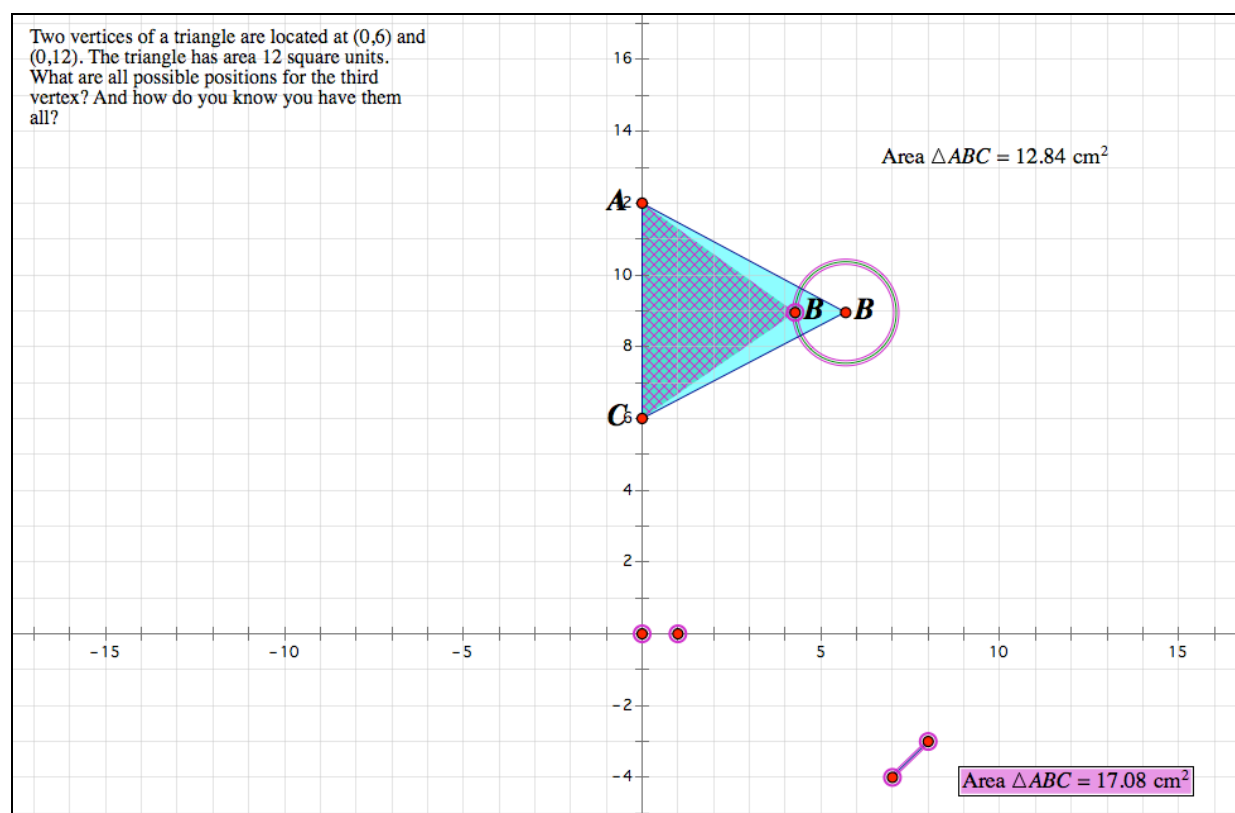


Figure 12. Screenshot of all elements of the third task.

Given the mathematics of the third clinical task, along with the class discussions devoted to the concept of mathematical shearing, I assumed that each participant would understand and be able to solve the task correctly and independent of technology. However, as the third

interview unfolded with Skyler, Garrett, and Janina, I could see that each understood the requirements of the task but had a limited understanding of mathematical shearing. Therefore I decided that an investigation with GSP that did not include a conflict presented by the technology might help them deepen their understanding of the concept. Thus, I altered the clinical interview to fit the outcomes of the participants' paper investigations. Only the interview session with Kayla unfolded as I had initially planned.

Data Analysis

Analysis is an ongoing and cyclic endeavor throughout any qualitative research study. Therefore, I conducted multiple passes at data analysis for the duration of the study on the three main types of data collected: the observations of class sessions; the results of personal interviews; and the results of the pre- and post-beliefs surveys. Collectively, I used the results of the various data analyses for each participant to identify his or her *work method* (Trouche, 2005b) and consider his or her assignment of mathematical authority.

Class Sessions

Lesson graphs. Given the copious amounts of data collected for this study, I found it imperative to implement strategies aimed at organizing and managing the data as soon as I began data collection. Thus, I made an initial pass at data analysis (that also served to organize the data collected from the class sessions) using the technique of lesson graphing (Izsák, 2008).

Lesson graphing provided me with an overview of the class activities as captured on videorecordings relevant to the purposes of the study. The lesson graph format consists of three columns: the time column (noting the time of a specific event); the description column (containing the description and, possibly, captured images of the event); and the comment column (consisting of any comments related to the event). (See Appendix G for a portion of a

sample lesson graph.) I would provide detail in the description column as I felt it necessary in regard to the research questions of the study. For example, if an event occurred that contained interesting dialogue between any of the participants, I essentially transcribed that dialogue as part of the description of that event. For other instances that did not yield data pertinent to the research questions, however, I provided a simple statement of the event. I used the comment column in one of three ways: to note questions or concerns I had in regard to the event; to provide a short account of my memories of the event as it unfolded during the pertinent class session (in support of or contrast to what the video showed); or to explicate my general thoughts in regard to the research questions of the study.

Coding lesson graphs. Once I had lesson-graphed the 23 videorecorded class sessions, I conducted another round of analysis of the class sessions in regard to the four participants. I extracted the portions of the lesson graphs that were relevant to each participant and organized them into a lesson graph synopsis. This document represented the interactions and behaviors of each participant as captured and noted from the class observations. I then coded the interactions and behaviors with the authority behaviors. (See Appendix H for a portion of a sample coded lesson graph synopsis.)

Authority behavior typology. I developed the authority behaviors to gain a better understanding of each participant's assignment of mathematical authority in regard to the purposes of this study. The authority behaviors fell into one of three categories: *giving authority*, *possesses authority*, and *authority neutral*. At times, the class discussions were not about mathematics but, instead, about pedagogy. Therefore, the collective set of authority behaviors included both pedagogical and mathematics authority actions.

Actions deemed to be giving authority included those for which a participant was observed placing the mathematical authority in someone (or something) other than himself or herself. The category of giving authority subdivided into two categories. The first category included codes for which the participant sought out or used a source (e.g., the course text, the Internet, or a generally accepted formula or rule) for mathematical investigation. The use of the source included a participant's acknowledgement or praise of another person, artifact, or technology. The second category included codes for which a participant sought clarification about solution methods or answers to mathematics tasks from another (i.e., his or her peers or Dr. Fikes). Behaviors in this category included instances when a participant asked for verification of the correctness of a mathematics solution or explanation and when a participant asked questions for purposes of making sense of the material at hand.

Actions deemed to be possesses authority included those for which a participant was observed either placing (or having others place) the mathematical authority in himself or herself. The category of possesses authority was subdivided into four categories. The first category included codes for which the participant was a mathematics source to his or her peers. Behaviors in this category included instances in which a participant provided clarification of the correctness of another's mathematical explanation or solution strategy and when a participant answered questions posed by a peer or Dr. Fikes targeted toward another classmate. The second category included codes for which a participant was sought or recognized by his or her peers as a mathematics source. Behaviors in this category included those instances in which a classmate asked a participant for a mathematical explanation or definition of a term and those instances in which a peer gave accolades to a participant regarding his or her mathematical abilities. The third category included codes for which a participant was observed to use his or her mathematics

knowledge and understanding in varying situations. Examples of a participant's personal use of mathematics knowledge and understanding included instances when the participant provided a mathematical justification or explanation of a solution strategy and when he or she provided mathematical examples to illustrate a concept. The final category included codes for which a participant acted as a pedagogical source. Opportunities to act as a pedagogical source happened only during whole group discussions with a focus on elementary students as they were observed engaged in mathematics tasks relevant to the course content.

Finally, actions that were coded authority neutral were behaviors that did not fit into either category but were noted as important in regard to participant behavior. For example, I thought it was important to note when a participant allowed a small group member to speak on behalf of the small group during a whole group discussion. But I did not feel confident in claiming that the participant was engaging in a giving authority behavior. Therefore, I would code this behavior as authority neutral.

Developing the typology. I began developing the typology of authority behaviors using techniques similar to the constant comparative method (Glaser, 1965). I started with the coding of one participant's lesson graph synopsis. This coding involved my initial reading through the lesson graph synopsis and generating codes based on the actions in that synopsis. Then I coded the second participant's lesson graph synopsis using the codes generated from the coding of the first participant's lesson graph synopsis. However, new codes were also generated unique to the second participant. I then reanalyzed the lesson graph synopsis of the first participant with the new codes and a fine-tuned meaning of the initially generated codes. This process occurred for each participant; it allowed me to clarify existing codes and to create, delete, or condense the number of authority behaviors coded as the process unfolded.

The completion of the coding process revealed a total of 11 giving authority behaviors, 19 possesses authority behaviors (4 of which were related to pedagogy), and 5 authority neutral behaviors. I designated the authority behaviors with a *W* (for whole group) or an *S* (for small group), indicating the form of class discussion for which I observed the action. Finally, I tallied and organized into tabular format the subcategories of the authority behaviors for the four participants upon completion of coding the lesson graph synopses.

Interviews

Transcribing. As a second type of initial data analysis, I transcribed verbatim each of the interviews conducted with the four participants. I then wrote *memos* (Maxwell, 2005) on pertinent ideas of each interview related to the class discussions and the study as the data from the transcriptions emerged in fuller detail (than my limited memories of observations and conversations would provide). These memos served to mark class events deemed important by the participant and contribute to themes formed from the analysis of transcripts (discussed next).

Coding transcripts. Upon completion of the memos, I coded the passages of the transcripts using a strategy similar to the process of coding the lesson graph synopses. The main difference between the two strategies is that I used a predetermined list of themes in the first pass of coding the transcripts (as opposed to first combing through the data to create an initial list of codes). I based the initial list of codes on the nine categories of the beliefs survey, but it grew as the analysis continued. Finally, I considered the results of the coding against the following seven distinct umbrella categories to form a case study description of each participant: background information; attitude toward mathematics; confidence in doing, learning, and teaching mathematics; major themes that arose from the interviews; impressions of Basic Concepts 2;

views on the use of technology for learning and teaching mathematics; and sources of mathematics learning and understanding.

Validity

Given the many concerns of validity issues that arise in qualitative research and interpretive analyses, I attempted to incorporate a variety of strategies to offset threats to the validity of results. First, I conducted a comparison of data obtained for each data type and for each participant to support resulting explanations. For example, I compared data obtained from the class sessions to data provided by the interviews as well as the results of the beliefs surveys. Then, I compared the data obtained from the interviews to the results of the beliefs surveys. Second, I used another form of validation technique that involved *searching for discrepant evidence* and *negative cases* (Maxwell, 2005) as I combed through the data upon the completion of connecting analyses. Finally, in an attempt to avoid being persuaded by preconceived notions I may have developed of any participant, I invited Dr. Fikes to read the results of the analyses to confirm that I portrayed the events and (in particular) the participants to the best of her memory.

Limitations

The present study had five types of limitations: the limitations inherent in the techniques of data collection; the constraints induced by the classroom environment; the constraints induced by the clinical tasks; the perceptions by the participants of me as a mathematical authority; and the limitations of the availability of each participant.

Limitations Inherent in the Methods of Data Collection

Even with videorecording technology, it was not possible to capture all interactions of the preservice teachers in the classroom environment. Even for the class sessions in which the four participants were sitting near each other, difficulties arose in trying to capture the conversations

on a topic. Thus, whereas using the two cameras provided more reliable data than the use of memory alone, this method of data collection was not fail-safe. Therefore it was imperative that I gleaned as much information from the participants as possible during their interviews to compare the results of the interviews and the videorecordings.

Further, collecting the data using videorecording inhibited my ability to write “rough observation notes” (Maxwell, 2005, p. 96). Thus, “media for recording, analyzing, and reporting primary data necessarily delete or reorganize aspects of the original phenomena, even as they add new dimensions to what we can learn” (Hall, 2000, p. 663). The recording media limited the data analysis to only the data captured on the recordings and, from the perspective of research design, *constrained* or *shaped* (Maxwell, 2005) the methods and results of the study. Hall (2000) discusses such an issue with data recording as “theory laden in the sense that theoretical interests focus researchers on which parts of ongoing interaction are relevant, reliable, or usable given existing methods of analysis” (p. 659). However, the affordances of such data collection methods greatly outweigh the cost. The affordances included an ability to revisit or adhere to marked or missed phenomena. As Hall notes: “Although they can always be questioned, the boundaries of phenomena in research on learning and teaching are largely fixed at the time primary data are constructed” (p. 659).

Constraints Induced by the Classroom Environment

Related to the first limitation, I had difficulty ascertaining mathematical knowledge and understanding of instances for which participants allowed their small group members to speak on their behalf (as well as the other small group members). Nearly all of the class sessions consisted of small group discussions with (time permitting) a follow-up whole group discussion about the explorations and discussions that took place during the small group discussions. Thus, it initially

seemed plausible to assume that the whole group contributions of any group member from a particular small group represented the mathematical ideas and understandings of the collective members of that group. However, as the observations ensued, I realized that the assumption was flawed. Therefore, the data that could reliably be used to make definitive statements about each participant's mathematical understanding and placement of mathematical authority were limited to statements or actions directly captured from the participant. This limitation posed a problem since, as noted above, even with two video cameras, I could not possibly capture all conversations that took place that involved all four participants.

Constraints Induced by the Clinical Tasks

The participants did not have knowledge of or experience with GSP prior to their investigations of the mathematics tasks during the clinical portion of the interviews. Therefore, they were not only learning about the software program and how to navigate its interface, but they were also investigating mathematics situations in what could be deemed as a deceptive context. Their lack of experience and knowledge of the program may have confounded the results of my attempts to ascertain their placement of mathematical authority regarding technology. Although this lack of knowledge about GSP is an undesirable limitation of the study, it was unavoidable given the constraint of the participants' schedules (discussed below). However, their lack of knowledge of GSP did ensure a common baseline from which to compare the participants' behaviors during the clinical interviews.

Perceptions of the Researcher as a Mathematical Authority

Regardless of the context (whether during the class sessions or in the interviews), I could not eliminate or account for my influence as an authority or source of mathematics. This influence may have played a role in how the four participants interacted with me in either

environment. I tried to lessen this limitation by ensuring that my interactions with them were limited during the class sessions. Restricting those interactions proved much more difficult during the interviews. I cannot be sure how much of the clinical interview was an accurate representation of how each participant would behave if he or she were to have explored the tasks independently or with a peer outside of class. Although being perceived as a mathematical authority can be an advantage when building a relationship with the participants, it was an undesirable aspect of the present study.

A final issue while conducting the interviews dealt with my views of GSP. My goal was to maintain a neutral discussion of the capabilities of the software in the hope of not providing an authoritative perspective of the technology or presenting the technology as a source of authority. My personal and professional experiences with GSP as a student, teacher, and researcher had generated a rather healthy respect for the affordances of the technology—a respect that may have been transparent during the clinical interviews.

Limitations of the Availability of Each Participant

Although having more data to corroborate my findings would certainly have enhanced the results of the study, there were time and logistical limitations to collecting more evidence. Because of the timeline of the course concepts as covered in class, the first interview with a participant did not occur until the middle of the semester. This limited timeframe worked against scheduling more interviews with the participants, who had many academic and nonacademic obligations. Therefore my discussions with them were condensed to the three interviews they could arrange given their busy lives.

CHAPTER FOUR

RESULTS

The focus of this study was to examine the allocation of mathematical authority of preservice elementary teachers when enrolled in a geometry content course focused on student mathematical learning and understanding and how the use of technology affected that allocation. I determined this allocation by observing the interactions and behaviors of four participants throughout their semester in Basic Concepts 2 and during three interviews that included explorations of mathematics concepts with the technology of the Geometer's Sketchpad [GSP].

The data obtained from the information form, the beliefs surveys, the observations of the class sessions, and the three interviews provided the context for developing a case study of each participant. The case studies begin to provide a mathematical portrait of the four participants throughout their semester in Basic Concepts 2. Given the complex nature of mathematics learning as it comes along with affective issues and preexisting beliefs about the nature of mathematics, it was not possible to predict with certainty where mathematical authority would lie for each participant. However, one can begin to understand reasons for various behaviors given a variety of evidence in the context of classroom learning.

In this chapter, I present the set of case studies imperative for gaining a sense of the personalities of the participants in an attempt to provide an explanation for their behavior during the clinical interviews. Specifically, I summarize the results from the different data sources for

each participant to conclude where he or she allocated mathematical authority and how technology affected that allocation.

Garrett

Attitude Toward Mathematics

Overall, Garrett had a positive attitude toward doing, learning, and teaching mathematics. He said that, as a young child, he “was like the mental math genius” and thought “[mathematics] was fun.” He explained, “I’ve always felt, like, I can relate to [mathematics]” and “I always liked math and science a lot more than I did the other classes.” When asked to rate his affinity toward mathematics on a scale of 1 to 10 (with 10 being the highest score), Garrett’s rating depended upon the type of mathematics. He explained:

If it’s like the geometry in middle school, like, geometry math, then it’s a 9 or 10—because I do enjoy that. When it gets up into, like, calculus and all that stuff, like, a 3 or 4. Kind of like, “Oh, let me just make it through the class.”

Unfortunately, however, Garrett’s middle school algebra class had left him with negative mathematics experiences and without sufficient knowledge of the “basics.” Thus Garrett struggled with algebraic concepts more than he did other mathematics topics. Despite those negative experiences, Garrett indicated that he “always did pretty good” in mathematics, although “a lot of the algebra classes were a little bit harder.” In particular, Garrett enjoyed the topic of geometry. He enjoyed his middle school geometry class so much that he took geometry again at the beginning of high school.

While he experienced success with and possessed an overall positive attitude toward mathematics, Garrett hoped that Basic Concepts 2 would be his last mathematics course. His plan was to attempt to “bypass” the last two mathematics classes he had left to take by substituting an art class for one of the math classes and a computer technology class for the

other. Garrett thought that the two mathematics courses left on his academic schedule (discrete mathematics and analytical geometry) were not particularly useful for his future career of teaching elementary school children.

Confidence in Doing, Learning, and Teaching Mathematics

Garrett contributed to every class session for both whole and small group discussions. He generally offered explanations of various concepts and, at times, would answer questions posed by classmates or Dr. Fikes. He explained his reason for contributing as, “I feel like, by engaging, that I’ve learned so much more.”

Garrett appeared to be quite confident in doing, learning, and explaining mathematics concepts during the class sessions and the interviews. He explained, “When it comes to geometry and like, child math, or like, elementary math, middle school, and most high school math, I feel really confident. . . . I can show you a way that I figured it out.” Garrett was also confident when working in small groups; he thought he could investigate the mathematics tasks regardless of his group members’ strengths and contributions. He elaborated:

I feel like, maybe ‘cause it’s that I feel like I’m stronger at math, that I can, like, I’m competent at it . . . it just helps you not to have to wait for—, if you’re as a kid raising your hands, waiting for someone to help you.

Although Garrett indicated possessing a high level of mathematics confidence, he stressed that his success was not effortless. He explained, “Like, I’ve always had to really work hard for all of my grades.” His desire to understand, as well as the fact that he enjoyed the subject, prompted him to ask questions for explanations that would help him when he began teaching the concepts to young children. During class sessions, Garrett would often ask for clarification of his classmates’ mathematical explanations and, in turn, would reword their

explanations as if he were internalizing their ideas for his own understanding. He elaborated on this technique for learning:

Um, a lot of times I'll ask questions like, I'm like, "I don't understand, could you explain that?" 'Cause it's—, I'm very hard-headed when it comes to a concept . . . either it makes sense and I'm like, "Yay, let me explain it to you." And if it doesn't, I'm like, "Please tell me how."

Major Themes That Arose From the Interviews

The two major themes that arose from the interviews with Garrett are the following: how external factors influenced his efforts in a course and his views of learning and teaching of mathematics. The first of the themes—how external factors influenced his efforts—contained two major external factors that greatly affected those efforts. The first factor was maintaining a high grade in the course. Although a typical motivating factor for most students, maintaining a high grade in the course was especially important for Garrett "to keep up [his] GPA for scholarships." Thus he would "spend more time" on an assignment if he knew it would be graded. The second external factor, which appeared to be more important than the first, was his perception of an instructor's level of "caring" about a course. This perception involved whether or not Garrett perceived that a course instructor deeply cared about his or her instruction, the course content, and the students' collective learning and understanding of the material. He explained that, without the caring aspect, "you won't get what you need to learn." He further clarified, "Like, if you have a teacher that acts like they really, really care, I'll do all the work in the world."

Garrett seemed particularly affected by his previous experiences in his early school years with teachers who he believed did not take seriously their charge of teaching students. He constantly referred to those early negative experiences and remarked:

I've had many teachers, they're like, "I hate math. I'm going to go ahead and tell you guys, I hate math, I'm not good at it, I'm terrible." . . . And they just kind of give you the problems, and you have to figure it out yourself.

In contrast, Garrett thought that Dr. Fikes "really cares about her field. She really enjoys math, and education, and it's easy to see that with the activities that we do." His perspective of her appeared to affect his drive to do well in the course. He elaborated:

Like, she would just spend all day long, and probably on the same topic, just going on, and making you think more and more about it. I'm pretty sure that if we had an 8-hour class, 5 days a week, that she would take care of covering those 8 hours, and it would fly by like it does the class now. . . . She wants to be, like she wants us to understand it, so we can be successful and hopefully not be like, "Oh, I hate math."

Garrett was passionate about the importance of the instructor for success in any class. He took the extra effort to seek out certain instructors for his classes at the university—instructors whom he believed would benefit his learning. He explained:

I've been able to pick classes with good professors that I've, like, known, before walking in, that this was going to be a good class. . . . Someone that I know is really good from either what everyone says is good or what the online ratings said.

The second major theme that arose out of Garrett's interviews was his view of how mathematics should be learned and taught. This view stemmed from his experiences of learning mathematics, his experiences tutoring and teaching mathematics, and his experiences in the education program at the university.

Initially, Garrett discussed mathematics as one-dimensional or "goal-oriented" and uttered statements that included remarks such as, "With [mathematics], you have a set of problems to do, and you can solve them and finish them." He seemed to champion memorization and believed that attaining mathematics skills required practice. For example, he believed that

multiplication facts “need to be learned until you memorize it” or that “you can’t logically think about something and try to reason with it if you don’t have any [computational] background.”

Later (in the same interview), and after a deeper probing into his thoughts and beliefs, Garrett stressed understanding connections between varying concepts and working from those existing understandings to gain more or deeper understandings with an emphasis on communication and explanation, particularly when teaching mathematics. He claimed it is important for children to be able to “explain [the mathematics] to you” and that “if they can explain [the mathematics] to you, they know what they’re doing. They don’t have to show you a thousand problems with a correct answer.” He further clarified, “I feel like, also, if you can explain it to someone else, then you really do know it. . . . And I feel like that can help anybody.” Garrett elaborated on his thoughts of the necessity “to associate” different ideas when helping students understand mathematics:

I think with all [mathematics], there’s logic to it, once you get the basics—or if you know something to associate it with—, I don’t feel like, you know, you have to learn every single thing. Like there is association with it. And, logically, you can be like, “I can’t do this, but I can do this.”

Garrett thought many of his former mathematics courses lacked a focus on “association” of concepts. Although the instructors of those courses emphasized “processes,” he felt ignorant of an understanding of the connections between those processes. For Garrett, understanding the “bigger picture” was most important for mathematical success. He explained, “I feel like you learn the whole concept, and you’re good to go. Like you can apply it to this problem, and it’s cool; it doesn’t change.” Garrett also recalled using copious amounts of memorization in his past mathematics courses to learn “all the processes,” but that technique failed him when taking the relevant exam. He said, “And that’s usually when I do bad, because I’m like, ‘Wait, which process? I can’t remember that one.’”

Garrett's extensive work with school-aged children through his employment at the YMCA played a crucial role in forming his views of teaching and learning mathematics. One pivotal example he provided was in trying to help a young boy learn operations on fractions. Garrett explained that, initially, he had thought the process of teaching the topic would simply "take one day" and "all you have to do is show [the rules]." He elaborated:

I thought I was being smart. And I went in there on my first day or my second day, and I was like, "Ok, fractions—this is how you add, this is how you subtract, this is how you multiply, this is how you divide." . . . And he regurgitated it back to me, and was like, "Ok, I got that." The next day, I came in there, and I was like, "Ok, show me." And he was like, "Show you what?" . . . And he took most of the semester just learning how to add and subtract and multiply and divide and simplify.

Additionally, Garrett provided various examples in which he had helped students to learn and think about mathematics. For example, he recalled helping a student make connections through learning strategies in regard to multiplication rules. He explained one such strategy: "So, if you don't know what six times eight is, you can do six times six. If you know that, you can do six times six is 36, then add six two more times to get six times eight."

Garrett's experiences in the education program at the university had also helped shape his views of mathematics teaching and learning. He provided examples of observing different levels of elementary and middle grades classes and discussed his observations of the "very redundant" nature of mathematics teaching and learning at those levels. To lessen such redundancy, he explained that time should be spent on the careful development of mathematics concepts with a focus on understanding. Therefore teachers can "take what you learn the year before and just add on to it." He further clarified, "Once you have the foundation built, then you can go deeper quicker."

Impressions of Basic Concepts 2

Garrett decided to take Basic Concepts 2 because he was “avoiding another teacher like the plague.” When walking by previous semester sessions of the course taught by Dr. Fikes, he had observed “[students] playing with stuff” and thought: “That must be a fun class, like, ‘cause I enjoy, like, hands-on learning.” And although the course was a mathematics content course, Garrett described it as being “an education math class” because of its “hands-on” nature.

Throughout the interviews, it was quite apparent that Garrett was greatly affected by Dr. Fikes’s pedagogical techniques. Garrett’s experiences in Basic Concepts 2 led him to change his perspectives from that of a student to that of a future teacher. After the second week of class, he began to change his goals from “learning to pass the test for myself” to “[knowing] how to do [the mathematics for] when I teach.” Garrett did not think that the course was a methods course in the sense that techniques were specifically taught for how to teach mathematics to young children. Instead, he saw the potential for each investigation as “a good activity to do with [his future] classes.” Garrett discussed that it was the responsibility of the preservice teachers enrolled in Basic Concepts 2 to think about how to incorporate those investigations as part of their future teaching. This perspective as a future teacher drove his desire to understand the material. Garrett did not want to just “get by” with the “least amount of work.” He remarked, “Like, I want to know how to do [the mathematics]. . . . I enjoy [the material], and I want to be able to teach it correctly.”

Another part of Dr. Fikes’s pedagogy that greatly affected Garrett’s views of teaching and learning mathematics was the constant requirement of explanation and justification. For Garrett, such pedagogy was tantamount to requiring an understanding of the material. This

pedagogy, in turn, created less need for large amounts of time devoted to studying and memorization. He explained:

I mean, it's all been kind of like, "You've learned it, and you've done enough with it that it makes sense to you." . . . I feel like by doing all that, it made it a better understanding than just telling, "This is the formula, and you need to memorize it." . . . By the time we get to the test, there's not as much studying that has to be done for it.

For Garrett, the pedagogy used in Basic Concepts 2 consisted of mainly "hands-on stuff," and he found Dr. Fikes's teaching style less "like a lecture" or "let me show you how to do problems." He remarked, "Like, she's told us maybe 10 problems exactly how to do." Thus, he found that the development of understanding was formed through ample opportunities to explore and discuss the various concepts with classmates with positive consequences. For example, Garrett stressed less about having his work "exactly right" because students are provided "time to talk about [the problems], get the work done, understand what you're doing, and then talk about it." He elaborated, "Because obviously we're all going to get different answers, you know, for some things. And you can tell, the harder the topic, the more we end up talking." He remarked:

I'm like, "Great. The homework will be turned in, I'll understand it by then, and that's the important thing." Like, that's one thing that as a teacher I would like to do is not have such an emphasis on turning it in to get a grade.

Another aspect of Dr. Fikes's pedagogy that particularly impressed Garrett was her questioning technique. On various occasions, he made frequent statements that conveyed the same sentiment as that illustrated below:

'Cause everything we do, she's like, "Well, how? Explain it to me." And then you're, like, starting to explain it, and you're like, "Oops, that doesn't make sense anymore." (laughs) . . . And every problem that we do, it's not a hundred problems of doing the same thing over and over. It's one problem, and then you have to write an explanation for it. . . . If you have a concept, and you can answer it, and you can explain it, then you're good to go.

Although Garrett thought this questioning technique was effective for his understanding, he did not believe that its use provided indication of whether or not his solutions were correct.

He explained:

I think [when Dr. Fikes asks] “Why?” [she means], “That’s interesting, elaborate more.” . . . And she’s never like, “Your answer is wrong, look at it again.” Nor if she says “Why?” it’s just automatically “Scratch that out and do it again.” It’s more like, “Hmm, explain that to me.”

Views on the Use of Technology for Learning and Teaching Mathematics

Garrett possessed positive views towards the use of technology for the learning and teaching of mathematics. However, he thought that technology should not be used when students were “learning a process” and provided the example of teaching concepts that use multiplication facts. He explained,

If the focus is something bigger than multiplication, then by all means. If it makes it easier, and you already know how to do the multiplication, then you’re not focused on getting that right versus getting the bigger picture.

For Garrett, students should learn concepts by first “[doing] it on paper and pencil, once or twice.” He thought that technology’s role was to act as “an aid,” not as a means for teaching; otherwise, some of the important aspects of different concepts “would be lost.” He explained the importance of independent investigation without technology using the lens of his own experiences: “Like, I feel like something that I struggled with, I feel like I’ll remember it more.”

However, once concepts were understood using paper-and-pencil techniques, Garrett thought that enhancing understanding through technology use would “be so rewarding” for students as they tried to work through time-intensive mathematical computations. For Garrett, technology’s ability to demonstrate mathematics concepts “quicker” enabled teachers to “get a whole lot more covered with a lot of new topics.” He cautioned, though, that using technology as

a learning tool was beneficial only “if the teacher knows what they’re doing.” Garrett also thought that learning with technology should be fully experienced by the student versus “if the teacher was just up there in front of the classroom doing it.” He explained, “I feel like you need a hands-on experience to realize [the concept].”

Sources of Mathematics Learning and Understanding

Although he indicated that he had sought the help of employees at the university tutoring center, Garrett identified his main source for learning mathematics as “definitely the teacher.” His use of the course text was limited to reviewing material or “to look up something specific.” He described himself as “not a big look-through-the-book type of person.” He explained: “I zone out when I try to read through a chapter and try to understand it that way.” However, the course text had once been a means of immediate feedback for Garrett. He observed,

Like, I killed the back of the book when I’m in math classes. Um, because I can check to see if I had it right. . . . I don’t have to turn [work] in and get it back wrong.

Garrett indicated that, for some of his past mathematics courses, he had purchased the teacher’s edition because it “had all the answers.” He was excited when, mid-semester, he found out that the course text for Basic Concepts 2 contained solutions to certain problems. However, given that mathematics understanding and connections between topics were stressed more in Basic Concepts 2 than correctly completed homework problems, Garrett developed a different perspective of the course text. He said, “I don’t feel like the back of the book is as important, because [Dr. Fikes is] not like, as you walk in, ‘You hand in your homework, and I’m going to grade it and give it back to you.’”

When discussing sources for mathematics learning in Basic Concepts 2, a course in which the instructor did not provide direct instruction for mathematics concepts, Garrett indicated that he would “try to use my peers.” However, he explained,

I feel like I probably use [Dr. Fikes] more than most people do. . . . I feel like, um, compared to everyone else, it’s probably a lot more, but not compared to, like, a normal classroom, I guess. . . . [Dr. Fikes] knows so much about [the concepts] that she can ask the right questions [even though] she never completely answers [the question]. . . . It’s just another question back. . . . Like you need somebody that questions you about [the concept].

When investigating mathematics concepts with his peers, Garrett explained how he reconciled situations in which group members found different solutions. “Um, usually we’ll go at it until someone either gets to my hard-headedness and gives a way that I can understand what they’re saying, or they get what I’m trying to say.” He gave an example of investigating with Skyler, another study participant who Garrett frequently worked with throughout the semester:

I feel like [Skyler’s] very knowledgeable in the math. And she’s very confident. And sometimes it doesn’t do us any good sitting next to each other, ‘cause I’m like “This is the way to do it.” And she’s like, “No, this is the way to do it.” And we have to have the third party come in and tell us that we’re both wrong, and this is how you really do it.

In regard to other typical sources of learning, Garrett used a cautionary tone when reconciling differences in mathematics results. For example, when addressing teachers, Garrett explained that they “know more than I do.” When discussing solutions provided by the course text, Garrett noted his potential to spend “an hour trying to figure out how I did [a problem] wrong.” In regard to technology, Garrett believed that discrepancies in answers were basically “a user problem” and elaborated, “Like the calculator always knows how to add three plus three whether I type it in correctly. Um, I find myself always checking anything I put in there twice.” Garrett indicated that his confidence with various technologies (e.g., particular types of

calculators or mathematics software programs such as MyMathLab¹⁴) would increase as he became more familiar with their individual interfaces.

Mathematical Authority in Different Contexts

In the Absence of the Instructor (Mathematics Task 1). As discussed in Chapter 3, the four study participants were sitting in close proximity as they worked in small groups to answer the review problem about semiregular tessellations in Figure 5. Garrett and Janina were coupled together, while Skyler and her partner, Julie, worked together at a table directly behind the two. Kayla and her partner sat at a table directly beside Skyler and Julie. During the time that the class worked on the review sheet, Garrett regularly turned around in his seat to talk with Skyler (presumably about questions on the review sheet), and Julie periodically asked the group members at Kayla's table for confirmation of answers on the review sheet.

Many of the small groups in the class began to discuss and explore the review problem at about the same time. Thus, there were various discussions captured through the audio of Dr. Fikes's microphone indicating students' attempts to conceptualize tessellating a plane. For example, Janina was heard stating to Garrett, "Like in a tessellation, you're like, they have to like fit. They have to like fit together." Also, Garrett was captured reading the definition of *semiregular tessellation* from the course text to make sense of it in regard to the problem. He said, ". . . tessellation is two or more types of regular polygons. . . . So this one has triangles, it has squares, it has hexagons. . . ."

Next, there was a lengthy discussion about tessellations and semiregular tessellations that began with just Julie and Skyler. Eventually, Garrett joined the conversation, followed by Janina.

¹⁴ Formerly known as Course Compass, information for this software can be found at: <http://pearsonmylabandmastering.com/?cc>.

The excerpts illustrate how the three participants tried to conceptualize tessellations and semiregular tessellations without Dr. Fikes as a mathematics source.

During the initial conversation, Julie was trying to remember how she had conceptualized tessellations based on her memories of her Internet research while attempting the problem for homework. It did not appear that Skyler used any type of external source, nor was there any evidence to indicate that she had experience exploring the topic in her prior mathematics classes to help answer the problem. This excerpt illustrates how Skyler began to change Julie's limited understanding and discussion of semiregular tessellations to help form her own understanding.

Julie: What's semiregular? I don't really get what a tessellation is. I know that if I'm looking at it, I know what it means. But I can't like, like give a definition. I know if you put them all together, you have the complete pattern or something.

Skyler: Hmm hmm (nods head in agreement).

Julie: Like triangles, you put them all together, and they make like a solid pattern.

Skyler: Right.

Julie: I would think this one would not be a tessellation. Or would not be a regular—would not be semiregular. I had look it up online.

Skyler: So semiregular is half regular?

Julie: I think *semiregular* means you'd have to put in another shape, I think.

Skyler: So which one is not? That one—I mean this would be semiregular, right? (Points to 4a in Figure 5) Because you have another shape you would put in with the trapezoid to close it off? And like, this one is already closed off (points to 4b in Figure 5). And this one might be a polygon because (inaudible) of dimensions.

Julie: Hmm hmm. And you could put another beside it.

Skyler: So that one would be semiregular? (Points to 4a in Figure 5.)

Julie: I don't know. You don't put it like side by side, do you? Or do you have a pattern? Like if you had another one of these (points to a hexagon in 4a in Figure 5).

Skyler: Right.

Julie: Could you lock this side into here? (Points to a different area in 4a in Figure 5.) Turn it around so, you could turn this around, and it'd fit in there? This fits in between two squares. It would fit in between these. Ok. See that makes sense. Because if you put these side by side it would—there would not be gaps in between.

Skyler: So this one would not be semiregular, right? (Points to 4b in Figure 5.)

Julie: Right.

Skyler: Because *semiregular* means you can add more.

Julie: Hmm hmm.

At the time of the next excerpt, Garrett had joined the conversation. He indicated that he had failed to make sense of the definition for *semiregular tessellation* provided in the course text. There was no evidence in the class observations to indicate whether he had any experience with tessellations. However, as the trio attempted to arrive at a conclusion about the problem, the conversation again morphed into the idea that semiregular tessellations lack completeness.

Garrett: Have you guys looked at number four yet?

Skyler: We're doing it right now. What does *semiregular* mean?

Garrett: We looked it up in the book but can't figure out anything that makes—it says it has to have two polygons that tessellate. One of the four chosen was like a triangle, a square, an octagon, and what's the other one?

Julie: Hmm hmm. Well, when I went online, it's like a pattern, like a tile pattern.

Garrett: Yeah.

Julie: If you put them all together you make like a solid. Like you put three triangles together, they um, they make like that solid line there.

- Skyler: She [Julie] said that a semiregular one was—.
- Julie: And it seemed, it seemed, I can't remember. It seemed like a *semiregular* meant that—.
- Skyler: It didn't close—(draws lines on 4a as shown in Figure 13), like if you closed it off.
- Julie: Like it doesn't meet completely, like you'd have to add another shape in there to make it, uh, make it a solid pattern.
- Skyler: Like that (shows her sketch in Figure 13).
- Garrett: So this one is not semiregular (points to 4b in Figure 5) because it's already done.
- Skyler: Right.
- Garrett: And this one (points to 4a in Figure 5) is semiregular because it needs to be—.
- Skyler: Right, if that's the definition.¹⁵

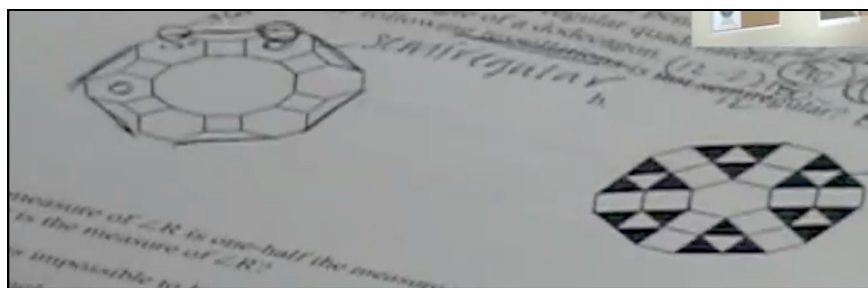


Figure 13. Skyler's written work illustrating how she would complete the figure.

In the following excerpt, Janina joined the conversation. As the idea of “completeness” continued to pervade their conceptualization of semiregular tessellations, Janina appeared to use an amalgam of the ideas to form the basis of how she understood tessellations.

¹⁵ At this point in their discussion, Julie, seeking affirmation of their answer, asks me, “Do you know what tessellation is?” When I indicate I do not plan on telling them, she asks whether they have the “right idea or no?” Since Julie was not a study participant, this exchange was not used as evidence to help answer the research questions of the study. However, I thought it was important to provide this footnote because two of the study participants were present to hear her question and my response.

Janina: Number four?

Garrett: Yeah.

Janina: I don't know.

Garrett: They're saying that [4]b is not semiregular, because it's—.

Julie: Jason¹⁶ knows.

Garrett: It's already a complete, um—.

Skyler: Figure.

Garrett: Figure. Like, it's a complete—.

Skyler: Like, see how a—.

Julie: Like, if you fit two of them together, it would, it would meet up, and you wouldn't—.

Skyler: You see how it has the little indention? (Points to 4a in Figure 5.) It means you couldn't count that as a decagon.

Janina: Right.

Skyler: You know what I mean?

Janina: Yeah.

Skyler: This one [4b] you can because it already has—.

Janina: It already has four sides—.

Skyler: Four sides. This one [4a] you would have to add those trapezoids or something to close it off.

Janina: Right.

Garrett: Right, does it have to be a regular decagon? Because all the sides aren't the same length.

Skyler: I don't know (laughs).

¹⁶ Jason was a middle school mathematics major who contributed almost daily to course discussions. He appeared very confident in his mathematics ability and rated himself as “excellent” in mathematics on the information form (Appendix D).

- Julie: I think, you'd have to add a different shape to it to make it a complete pattern.
- Skyler: I mean, this seems—.
- Janina: No, no, no. Because remember, a tessellation can't like—. Only, like, five figures can be regular—you know what I'm—you know what I'm trying to say?
- Skyler: Right.
- Janina: Like only, it's like a square. It doesn't go more than—. Like, a hexagon, I think, can be a regular tessellation. So a decagon, or like a dodecagon, isn't going to be a regular tessellation anyways.
- Julie: If you took this shape, and you added another shape up here, it would meet or make—.
- Garrett: So this one is a semiregular (points to 4a in Figure 5).
- Janina: Yeah.
- Garrett: And this one isn't (points to 4b in Figure 5).
- Janina: Well, see, semiregular—*semi* means half.
- Garrett: Yeah. This one's, like, halfway regular (points to 4b in Figure 5), but it looks, it's the same, but it doesn't have, these aren't the same maybe.
- Skyler: I don't know.
- Garrett: (Rubs his forehead and looks disturbed.)
- Skyler: I don't know what that means.
- Julie: 'Cause see, I'm looking at this (points to 4b in Figure 5) as like one whole piece. Not like, different pieces put together—I think that's the way it looks to me.
- Skyler: This one looks like it needs more pieces to be normal, or whatever (points to 4a in Figure 5).

As the discussion ensued, Skyler, Janina, and Julie collectively formed their reason for their choice of 4a as the answer to the problem. However, Garrett was still confused and sought clarity of their reasoning. The following excerpt displays Garrett's continued confusion and Skyler's authoritative stance on her choice of answer.

Garrett: You guys are saying [4]a is not semiregular?

Janina: No, [4]b.

Skyler: [4]a is semiregular 'cause it's halfway done.

Julie: Ok, it's not semiregular, that's what we're—.

Janina: We're saying—, we're saying—.

Skyler: [4]b is not semiregular.

Garrett: [4]b is regular?

Janina: No, [4]b is not [semiregular]—[4]b is the answer.

Julie: Right, [4]b would be regular.

Garrett: But why does it have to be regular? I thought that we were arguing that—.

Skyler: I mean, I don't know if *nonsemiregular* means it's regular. But if one of them is semiregular, then it's going to be [4]a.

Garrett did not seem willing to accept the conclusion of the other students. In the following excerpt, he was engaged in a conversation with Janina about how to write out their explanation of the conclusion. The idea of completeness still pervaded Janina's understanding of semiregular tessellations. Only then did it appear that Garrett finally agreed to their collective reasoning to support their chosen answer.

Garrett: Hold on, let me pull out the definition again. Ok. It says that a tessellation has two or more types of regular polygons arranged in the same order around—it has the same order. So they're arranged in the same order around every vertex point—it's called a semiregular. Every vertex point.¹⁷

Janina: Which would be right here (points to a vertex in 4b in Figure 5).

Garrett: Yeah.

Janina: Right here (points to a second vertex in 4b in Figure 5), right here (points to a third vertex in 4b in Figure 5).

Garrett: So that's—this one is arranged (points to 4b in Figure 5) this one isn't—doesn't have the same stuff at every vertex. So this vertex—oh, yeah, it does.

Janina: It does, yeah.

Garrett: There are only five regular polygons that can be used for a—.

Janina: Regular polygon.

Garrett: Three, six—.

Janina: I don't know—this is how I put it. I was just like, “[4]b is semiregular because it is already a completed figure; all sides form a dodecagon.” Like, on this one (points to 4b in Figure 5), we can't necessarily fit any shape, like, in that.

Garrett: Yeah.

Janina: Like she said (points to Skyler), like we can put like a trapezoid there. You know what I mean?

Garrett: Hmm hmm.

Janina: I mean, we could put like another like whatever (draws a hexagon as shown in Figure 14). On this one (points to 4b in Figure 5) it's already a complete—.

Garrett: It's already complete?

¹⁷ The definition of *semiregular tessellation* provided in the course text is operational only when the order of the regular polygons surrounding each vertex is considered the same regardless of analyzing the configuration in a clockwise or counterclockwise motion. The course text provided no examples to illustrate the definition provided. How the definition operates and the lack of examples may be reasons for Garrett's confusion as he tries to make sense of the definition in the context of the problem in Figure 5.

Janina: Yeah. And like it's so, like she was saying, it's like half-done. This one's like half-done (points to 4a in Figure 5) in a sense. You know?

Garrett: Hmm hmm.

Janina: (Erases marks.) Watch this all be completely wrong.

Garrett: It's not semiregular because it already completes . . . (continues to write down his thoughts).

Janina: Yeah.

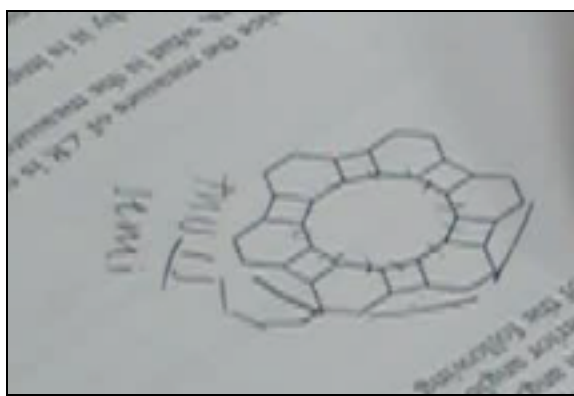


Figure 14. Janina's work indicating how she could fill in a hexagon.

Clinical interview results. Garrett correctly recalled most of the events as they unfolded in class during the discussion of the review problem. He did not think that, as a group, they had come to a consensus of the attributes for determining a semiregular tessellation. He correctly remembered, however, their collective reason for choosing Figure 4b in Figure 5 as the answer to the problem. Garrett did not indicate that he sought help outside of class for that particular review problem, although he mentioned that he had spoken with Dr. Fikes “for a second” in class that day¹⁸ about the problem.

Although he did not provide any indication of his knowledge during his small group discussion, Garrett recalled learning about tessellations prior to his experiences in Basic Concepts 2:

¹⁸ That discussion was not recorded, and Dr. Fikes had no recollection of it.

I did a lot of tessellations in school, like, as a kid. A tessellation to me is just an object that you can do over and over and over in rows and rows and rows—it all matches up together.

Further, Garrett had come to an understanding of the definition of *semiregular tessellation* provided in the course text, stating, “I guess I didn’t read that clearly—‘Arranged in the same order around every vertex point is called semiregular.’” He used the two figures in the problem, 4a and 4b in Figure 5, to illustrate the definition: “Um, so this one (points to 4b) was not semiregular because like, at this vertex point you had a square, triangle, square. And here you had a triangle, triangle, triangle.”

However, even after correctly illustrating the definition, Garrett still claimed confusion over how to identify a semiregular tessellation because of the discussion that took place during class. He recalled his memories of that “day of frustration”:

And [Skyler] was like, going off, like headstrong, into this thing. And I’m one of those people that you have to really explain it to me. I don’t just take “This is the answer.” I’m like, “Well, why?” I have to mentally connect something to it. Um, and she wasn’t doing that for me. She was kind of like, “This is it.” . . . So then we were asking the people in front of us, and they were like, “We don’t really know, that sounds right. Skyler, go for it.” And then the people behind us were like, “Go for it.”

After the discussion of his memories and understanding of tessellations and semiregular tessellations, Garrett used GSP’s premade scripts to explore which of the regular polygons singularly tessellates the plane. When his investigation turned to the octagon, he claimed that the octagon does not singularly tessellate the plane because a square would fit into the gaps. When I asked Garrett how he was sure whether those gaps were, indeed, squares, he immediately used GSP’s premade square script to fill in the space with a square of the appropriate size (Figure 15). He concluded that because a square fit into the space created by the four octagons, then the quadrilateral must be a square.

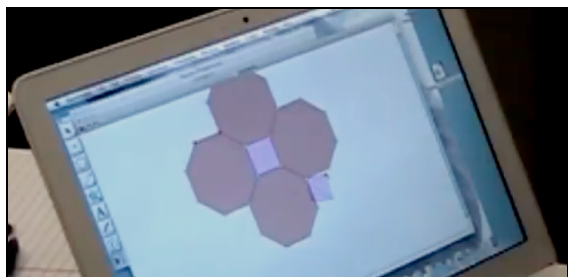


Figure 15. Garrett uses the square script to fill in the gaps.

Garrett's investigation of whether the regular octagon singularly tessellates the plane naturally led to the discussion of the definition of semiregular tessellation. Once I provided clarification of the definition, Garrett appeared to understand the conditions of what constitutes a semiregular tessellation and correctly explained why 4b in Figure 5 would not be a semiregular tessellation in contrast to how it was conceived during the small group investigation. However, he claimed difficulty in determining whether 4a in Figure 5 is a semiregular tessellation without continuing the pattern of shapes and expanding the area of the figure. Therefore, Garrett used GSP and its premade scripts for dodecagon, hexagon, and square to make his final conclusion. Garrett's exploration appeared to solidify his understanding of our discussion on semiregular tessellation. He said, "By looking at it with the way you said and the definition together, it makes more sense."

Discussion. Although Garrett did not explicitly discuss that he further pursued understanding the review problem, it was clear that he had made sense of it outside of class because no more class time was devoted to the topic. Additionally, although it seemed that he eventually agreed with his small group members' collective idea that a semiregular tessellation implies a lack of completeness, his interview discussion clearly indicated that it was not a sufficient reason for him. During the class observations, I observed Garrett on two occasions seeking out the definition provided in the course text to make sense of the concept of semiregular tessellation. Further, the data illustrate Garrett's attempts to make sense of the definition

provided in the course text in regard to how his group members were making sense of the concept using the idea of a lack of completeness.

The reasons that Garrett chose to use the definition of semiregular tessellation (as provided in the course text) as a source of mathematical explanation are not clear. However, one possible explanation rests with Dr. Fikes's pedagogy. On various occasions, Dr. Fikes directed students to use their chosen mathematics definitions for various terms to aid in their mathematics investigations and explanations of results.

The data on Garrett reveal a student who sought and contrasted information from multiple sources to aid and form his mathematics understanding until he was satisfied with his pursuits—that is, until the material made sense to him. Additionally, Garrett did not present himself as an authority on mathematics. He was the only participant who indicated having previous experience with tessellations, yet he did not share that information with his group members.

The Triangle Dissection Paradox (Mathematics Task 2). During a class session relevant to the second mathematics task, Garrett was partnered with Skyler as they worked on the problems of Appendix F. He clearly showed his understanding of the first problem as he drew different parallelograms (Figure 16a) and computed their areas by quickly counting the lengths of the base and height using the squares of the grids. Then, he discussed a different method with Skyler that one could “chop” parts of the parallelogram and move them to a different area to create a rectangle (Figure 16b). Skyler appeared to be in agreement with his two methods as he discussed them.

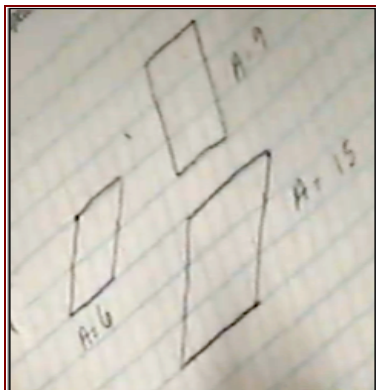


Figure 16a. Garrett's different parallelograms.

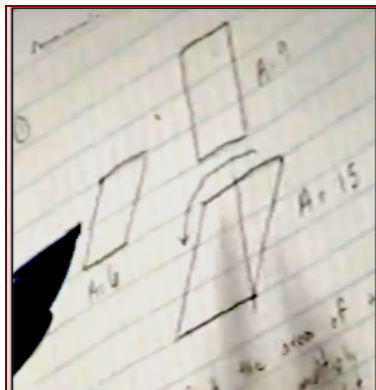


Figure 16b. Garrett's decomposition and rearrangement of parts of the parallelogram.

For the second problem in Appendix F, Garrett and Skyler discussed how they could devise a general rule for the area of a trapezoid. Garrett said he had difficulty “explaining formulas.” Skyler discussed her method, which involved the decomposition of each trapezoid into different parts (two or three parts, depending on the trapezoid). She then calculated and summed the area of those parts. Garrett questioned her method, stating, “That makes sense, but I was trying to think of like a rule—kind of like a triangle.” Skyler responded that he could use the formula of the area of a trapezoid to help find the rule. She explained, as she pointed to her work shown in Figure 17, “Basically it’s like area one, two, three. And basically you’re saying area of one, plus the area of two, plus the area of three, is going to equal one-half base one, plus base two, times height.” She informed Garrett that the second question used the mathematical reasoning of the first question. He agreed and stated, “If you chop up the trapezoid, you’ll get triangles and rectangles.” As a pair, however, they never actually answered the question as stated and thus did not devise a general rule for finding the formula for the area of any trapezoid.

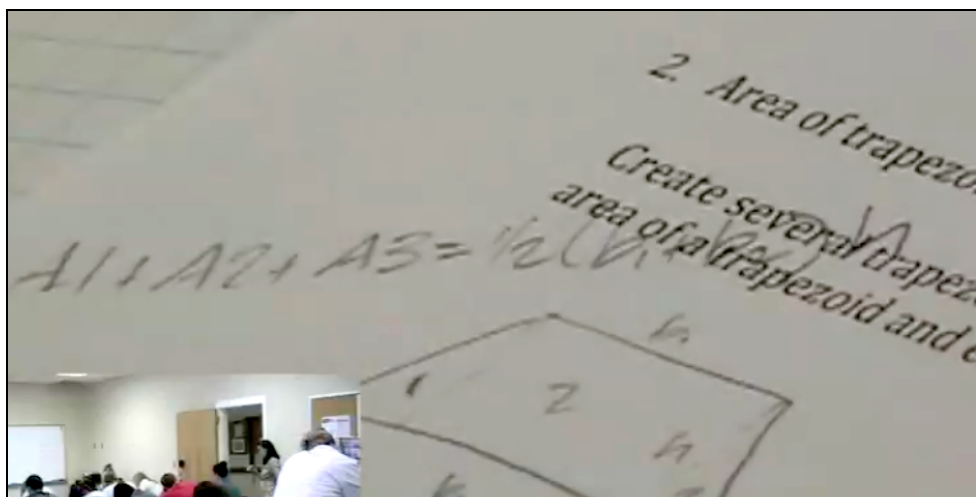


Figure 17. Skyler's work to generate a rule for a trapezoid.

Although Garrett initially questioned Skyler's explanation for devising a general rule, it appeared that he eventually agreed with her thinking even though it did not truly devise a rule as directed in the worksheet. Later, during the next class session and with a different group of table members, Garrett was presented with a method of how to devise the rule for finding the area of a trapezoid that, as he stated, "makes really good sense."

Clinical interview results. It took Garrett roughly 2 minutes to rearrange the parts of Triangle A in Figure 6 to form Triangle B; he was immediately perplexed by the presentation of the hole. In attempting to understand the mathematical reasons for the hole in the new triangle, Garrett's discussion and explorations provided evidence of a connected understanding of a variety of geometric concepts. On a basic level, he acknowledged that because the two triangles had the same base length and the same height, then they should have the same area. He tried to recall his memories of the effect on a shape's area because of changing its perimeter. He stated, "I guess it's kind of like that whole thing. That you can take a rectangle, and the perimeter can be the same, but the area can be different. Or vice versa, like you know how that goes." Because the perimeters of the two triangles appeared the same and the same four shapes were used to create

the new configuration, Garrett seemed sure that the areas of the triangles should be equal as well.

He pondered:

It is a good question though. How do they have the same shapes and same perimeter, per se, but they have different areas? . . . So, mathematically, you would think that the [triangles] were the same. . . . It would just make sense that there wouldn't be a gap there.

In addition to using his knowledge of triangles and preservation of area, Garrett used other strategies in regard to the technology of GSP. For example, he systematically counted the individual unit squares of both triangles by using GSP to mark dots on the unit squares (Figure 18). He also used GSP to confirm the measure of the areas of the two triangles.

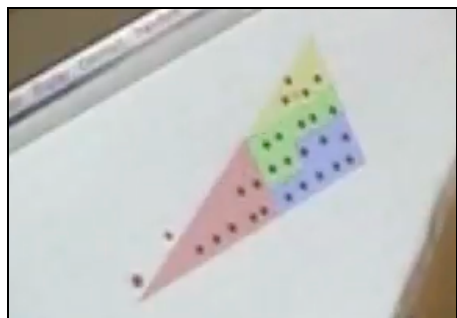


Figure 18. Screenshot of Garrett's exploration with GSP.

After approximately 21 minutes of exploration, Garrett could not think of any other explanation for the hole or generate any new ideas on how to explore the reason for its existence. He was, however, interested in the genesis of the anomaly and inquired, "How did you make that? How did you come up with these shapes? Were they given to you as like a problem? Or did you just think of them, like making them up?"

Once provided with the explanation for the paradox, Garrett said that he had initially been concerned in regard to how the hypotenuse of each triangle (Figure 6) did not appear to be a straight line. He said:

I was looking at that, like that's just—. I thought it was just user error for not drawing a triangle. But I guess it's not. . . . Um, 'cause I kept looking at this one (Triangle A), and I'm like, "It doesn't match up quite right." And I was, like, and

it kept bothering me, but I was like, “Oh, that’s just, you know, it’s just user problems.” Not the fact that it really isn’t a triangle.

At the conclusion of the second clinical interview, I asked Garrett if he thought the results of the task would have been the same had it been presented as just a paper investigation. He indicated that, although the task would have been the same, the paper investigation would have exaggerated the paradox. He said:

Um, I think so because it was taking the same shapes, and I think [doing it on GSP] is just a lot quicker. I think on paper, I would have been like, “Oh, it’s just the way I cut it out.” . . . They wouldn’t be exact so there would be not just, maybe, one square left over, but two, because of the way that it’s cut.

Discussion. Although Garrett could not explain the hole, he was unwilling to accept it using an explanation counter to his mathematics knowledge. He continued to place the mathematical authority in his own knowledge (instead of what was presented to him on the screen), and called on his mathematics background for his investigation. When I consider Garrett in the context of the class sessions, however, I would find it difficult to pinpoint where he placed mathematical authority in regard to his peers.

Garrett rarely presented himself as a mathematics source and at times appeared to concede that the mathematical reasoning presented by his group members was correct. His interviews, however, revealed a student who continued to reason mathematically about the concepts and investigations independent of the class sessions. Further, Garrett could be persuaded to reconsider his mathematical conclusions when presented with reasoning that aided or furthered his understanding and connected to his existing mathematics knowledge.

The Lie of GSP (Mathematics Task 3). The class session that provided evidence of Garrett’s understanding of mathematical shearing was the same one presented earlier in which Garrett and Skyler worked together on the problems of Appendix F. Both Garrett and Skyler

contributed during the whole group discussion led by Dr. Fikes that addressed the problems of Figure 10. Their contributions provided evidence of their understanding of mathematical shearing. As further evidence, the following conversation ensued between the two participants during their small group exploration of the third problem of Appendix F:

Garrett: (Writes assigned values to the base and height of the triangle and rectangle as shown in Figure 19a.) If this was 3, and this was 6, half of this would be, um, half of the rectangle would be 9. And then the area of the triangle is—6 times 3 is—18, and half of the triangle is 9. Right? Because half of 6 is 3, times 3 is 9. (Writes equation to the side of the problem as shown in Figure 19a.) So they would be the same. Half of this triangle—half of this rectangle is the triangle.

Skyler: Yeah, I thought the question was asking the entire—.

Garrett: The whole thing—.

Skyler: Rectangle. And the entire triangle. I was like there's no way it just fits in that little spot.

Garrett: That makes sense.

Skyler: Yea, yes.

Garrett: Even if you didn't want to do it with numbers—.

Skyler: Well, they have the same base times height.

Garrett: Yeah, so you could have like x and y (erases numbers and labels the base and height as shown in Figure 19b). The rectangle is equal to half x, y . And the triangle is equal to half x, y .

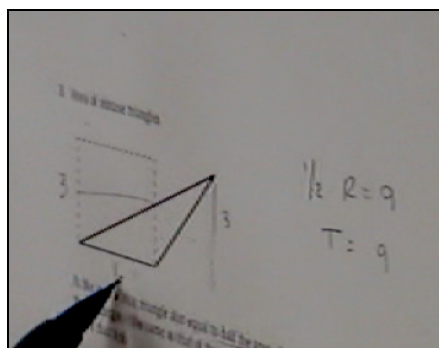


Figure 19a. Garrett assigns numbers to the base and height of the two figures.

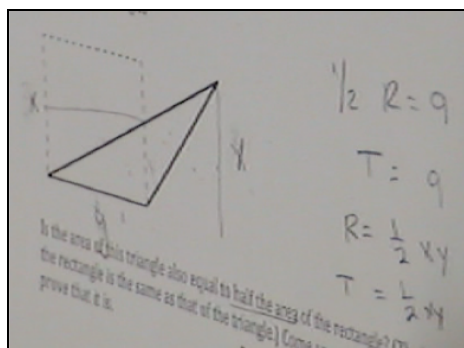


Figure 19b. Garrett changes the numbers to variables.

Clinical interview results. Garrett began the task (Figure 9) by counting unit squares to find possible locations of the third vertex. Quickly finding this method to be laborious, he discarded it in favor of using the formula for the area of a triangle to help structure his thinking. Garrett initially claimed that there would be only six triangles (four right triangles and two isosceles triangles) formed by the possible different locations of the third vertex. When prompted as to whether or not there could be more triangles that met the conditions of the task, Garrett acknowledged that the “possibility” existed but claimed that those triangles would have to be contained in the rectangular (8×4 square units) region shown in his work in Figure 20a. He explained, “Because if anything’s over the four—and they can’t, it’d be too big—like the area wouldn’t match twelve.” Upon further discussion, it became apparent that Garrett was indicating possible coordinates whose x -value was larger than 4 (on the right side of the y -axis). I decided to draw his attention to a possible point whose x -value was at 4 but whose y -value was larger than 6. Garrett drew a triangle that used the location of that point (shown in Figure 20a) and agreed that this triangle “could be one” that met the conditions of the task. He then motioned along the vertical line (that contains the right side of the rectangle in Figure 20a) and indicated that more triangles would exist if “you extended it up this way (motioned upward in the positive direction) and this way down here (motioned downward in the negative direction).” Garrett thought, however, that there was a limit to the number of triangles that would exist. He explained, “Um, I think that there’s a certain point that it has to stop. . . . It gets to be where it is so super small that it has no area.”

It was clear from his discussion that Garrett understood the mathematics of the task although it appeared that his concept of mathematical shearing seemed to block his conception of triangle. For example, Garrett explained, “I think [they have the same area] as long as you have

the same height and you have the same base. No matter how you draw, it is going to have the same area.” However, he questioned whether or not the process of mathematical shearing would alter the triangle, resulting in a figure other than a triangle. Garrett illustrated his thoughts with the drawing shown in Figure 20b:

Like there’s only a certain amount of times that you can—there’s only a certain amount of coordinates that you can go around it. . . . Um, it gets to a certain point where it’s not a triangle anymore. It just becomes a line because it gets so skinny. It was thick—wide before. Each time it gets a little skinnier to where here it’s a lot skinnier than it was before. It eventually turns into a line, to a—, the lines would be on top of each other.

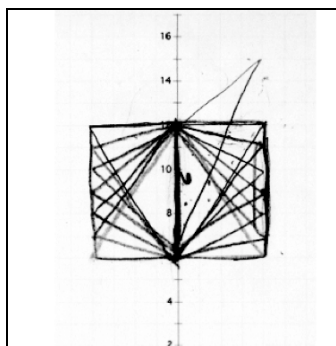


Figure 20a. Garrett’s investigation on paper.

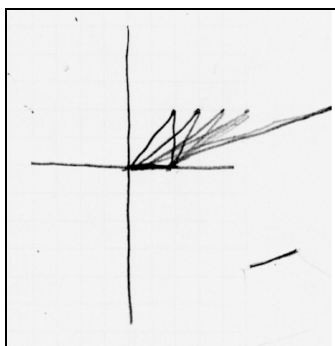


Figure 20b. Garrett’s work to illustrate his reasoning.

Because of this comment, I thought it was necessary to have Garrett explore the task with GSP to see whether the dynamic capability of the technology would quickly allow him to stretch the triangle and thus help him visualize that the shearing does not result in a transformation of the triangle to a different geometric shape.

Because Garrett’s main confusion was whether or not the placement of the third vertex would result in a triangle given its location, I asked him to create a GSP sketch to demonstrate the conditions of the task. As he created the sketch, he reiterated his goal to “see if there is a certain point” where the triangle was no longer a triangle. Garrett displayed his quick facility with GSP from his previous experience with the software during the first two interview sessions.

His use of the technology, however, was limited by what could be visually displayed when dragging the third vertex vertically down the screen. At one point, the two original vertices of the triangle were no longer visible. Based on what was visible, when all three vertices could be seen, Garrett said, “If you just did the outline, it would still be a triangle.” He further concluded that there are an infinite number of possible locations of the third vertex to complete the task, and “they’ll all be on the four” and “the negative four” of the x -axis. Further, Garrett acknowledged that the technology of GSP helped him to “quickly see” that the area and shape of the triangle were maintained. At this point in the clinical interview, I decided to ask Garrett to consider the GSP sketch designed to create conflict.

Upon his inspection of the premade GSP sketch, Garrett was immediately drawn to the fact that the area displayed in the new sketch was different from 12 when the third vertex was along the vertical line, $x = 4$. He initially proposed and explored two hypotheses to account for GSP’s area measurement. For each exploration, he used the same right triangle (shown in Figure 21) to form and test his hypotheses. The first proposal, which he later discounted, was that the scale of the axes was different than the initial sketch he made. For his second investigation, which did not provide satisfactory results, he counted the lengths of the sides of the triangle to be used in the area of a triangle formula. He then confirmed his count by using GSP to measure the lengths of the sides of the triangle. For both investigations, Garrett returned to the fact that the side-lengths of the right triangle displayed on the screen did not satisfy the formula for the area of a triangle. He explained:

So it’s lying, ‘cause the area is not 12. (laughs). . . . It should say, “This is four.” Like the four should match up. Because, even here it doesn’t make sense, because that said five point four. Five point four divided by two is two point seven. Two point seven times six is sixteen point two, not twelve. So it’s finding the area of something else besides the triangle I have.



Figure 21. Garrett's triangle used to determine his conclusions.

Although he was correct in his reasoning and statements, Garrett wanted to find an explanation for the discrepancy in results. Recalling his experiences with the Triangle Dissection Paradox from the previous interview, Garrett asked, “It doesn’t bend in, does it?” He then attempted to check that a bend was not a factor for a shortened or elongated length of any of the sides of the triangle by using the Pythagorean theorem. As the interview drew to a close, the following conversation ensued:

Me: Ok, so if you were to choose between yourself and the computer—this program, this sketch here, you—.

Garrett: I would like to say I was right, and the sketch was wrong.

Me: But you’re not one hundred percent sure?

Garrett: I know that it’s got to be right. ‘Cause it’s—, it’s user mistake. It’s not computer mistakes.

Me: So you just did another sketch, though, right?

Garrett: Hmm hmm.

Me: Where you did it, and you were able to calculate the area, and it was 12. Why do you think that sketch is different than this sketch?

Garrett: I don’t know—it’s what it decided to do. It’s—, I haven’t figured it out.

Me: It has a mind of its own?

Garrett: Yeah, I don't know what it was thinking. Um, if I knew how it got, why, if that was correct, and why—, then I could maybe figure out why.

Me: Oh, if you—, so you're saying, if this is correct, if something were to confirm that this was correct, then you'd be able to work backwards from it? That's what you were saying?

Garrett: Yeah, maybe. 'Cause I can't wrap my mind around why it's correct yet to be able to tell you why. To be able to be like, "Oh, it's 'correct cause of this.'" 'Cause I don't think it's correct.

Discussion. Initially I was surprised that Garrett did not hold a sufficient understanding to help him complete the third mathematics task on paper. One explanation for his initial solution of a fixed number of positions where the third vertex could lie was that he was only considering lattice points as possible locations for that third vertex. Considering only the lattice points is not unreasonable because relevant class explorations and the homework problem in Figure 10 used the geoboard or lattice points as the context to explore mathematical shearing. Garrett's initial exploration of the task with GSP allowed him to quickly realize the continuous aspect of the locations of the vertices.

Once he was able to move forward with the interview, the most remarkable feature of Garrett's explorations was his perseverance in trying to ascertain a plausible reason for the conflicting results presented by GSP—even though at some point he had concluded that the technology was "lying." His persevering behavior is not surprising given that Garrett had shown tenacity in trying to understand mathematical concepts that he initially found confusing. Calling on his mathematics background coupled with his quest for understanding, Garrett considered different options in order to make sense of the conflicting results. He claimed that he persevered because of the fact that he had to constantly check his work to ensure he was correct. Although he thought he had been "pranked" when shown the hidden features of the premade GSP sketch, Garrett also displayed an excitement about learning more about the program's ability to create

such an illusion. And realizing a potential aspect for the use of GSP with his future students, as well as learning another feature of the program, he inquired about the construction of the sketch.

Allocation of Mathematical Authority

Garrett was rarely a source of mathematics for his peers. In apparent attempts to make sense of the material at hand, Garrett often asked a variety of questions either of his peers or simply rhetorically. Further, he generally sought the help of typical mathematics sources such as the course instructor or the instructors in the university mathematics lab. In a class such as Basic Concepts 2, whose pedagogy was substantially different from that of his former mathematics courses, Garrett developed ways to seek mathematics help indirectly from Dr. Fikes.

The interviews revealed a mathematical confidence different from what Garrett displayed during the class sessions. In the interviews, his varied mathematics background came to the fore as he attempted a variety of strategies to explain what was presented by the technology. He did not readily accept a mathematical phenomenon (whether caused by technology or not), nor did he provide justification of the phenomenon simply because it existed.

Although Garrett's third interview did not unfold as I had initially planned, it was clear from other evidence that he would not easily submit to the authority of the technology. Further, Garrett had shown in multiple instances that he did not easily accept mathematical explanations from his peers. Garrett thought that mathematics explanations provided by peers needed to "make sense" in order for him to accept them. Mathematical sense for Garrett meant that the explanation also had to connect to his understanding of the concept at hand and needed to be as mathematically sound as anything he already understood about that same concept.

Garrett displayed many of the qualities of the geometric habits of mind (Driscoll, 2007) that are critical to forming a solid mathematical understanding of the topic and particularly

enjoyed the content of the course. Arguably, if algebra (the topic he struggled with the most) had been the content focus, Garrett's disposition might have mirrored Skyler's. However, given his desire to connect mathematical material, his experiences with algebra could have been quite similar to those observed in the present study.

The collective analysis of Garrett's data presents him as having a resourceful work method (Figure 3) and show that his placement of mathematical authority could be found in a variety of sources and not necessarily in a stable hierarchy. When necessary, Garrett evidenced proficient ability to utilize a variety of mathematics sources and held beliefs about the subject of mathematics that enabled him to persevere in the face of conflict. He did not appear to view technology as a possible mathematics source or mathematical authority but as a formidable aid to concept formulation and was observed to easily use the technology (sometimes in novel ways) in attempting to understand the anomalies encountered during the clinical interviews.

Skyler

Attitude Toward Mathematics

Skyler displayed a generally positive attitude toward doing, learning, and teaching mathematics and indicated it was her "favorite subject" and that she earned an A in all of her high school mathematics classes. On a scale of 1 to 10 (with 10 being the highest score), Skyler rated her affinity toward mathematics with a 9. Her reasons for the ranking and the fact that mathematics was her favorite subject included: her view that "there are so many different ways that you can do everything"; the challenges that she believed mathematics presented; and the rewards one could gain "when you figure out how to do something."

Skyler provided reasons for why she did not rate her affinity toward mathematics with a 10 that included a discussion of how she felt about geometry. Skyler's attitude toward

mathematics changed when it came to the topic of geometry. She described geometry as “always an issue for [her]” and characterized herself as “not strong” in the domain. She explained:

Um, geometry (laughs). I just don’t—. If you give me a formula, I can handle it. But when you start switching it, and changing it, and there’s shapes involved, and then you throw in extra steps and—. Then it just gets really, really clustered for me.

Skyler recalled geometry as the only mathematics topic in high school for which she did “homework problem after homework problem after homework problem” while gaining a limited understanding of the material. She discussed her inability to “see” geometrical concepts, such as those involving “rotations” or “twisting [shapes and angles] and rearranging them.” Skyler was not able to pinpoint where her issues with geometry originated but observed, “I hear the word *geometry*, I’m like, ‘Ehh.’ And it might be because of my high school class. But, you never know.”

Confidence in Doing, Learning, and Teaching Mathematics

Despite her issues with geometry, Skyler’s overall confidence in doing, learning, and teaching mathematics appeared to be consistently high as captured during the class sessions and in the interviews. She regularly contributed to whole and small group discussions and thought this contributing helped her understand the mathematics better. She explained, “I learn when I explain it to other people.” She clarified that talking aloud “stretches your vocabulary, it stretches your ways of thinking.” However, Skyler explained that she was not afraid to contribute “things that I don’t know [are] right.” She attributed this fearlessness to the structure of the class environment, “I feel like that [Basic Concepts 2] is a good opportunity to do that kind of stuff.”

Major Themes That Arose From the Interviews

The three major themes that arose out of the general interviews with Skyler consisted of the following: the importance of mathematical communication; the importance of understanding

the mathematics; and the appropriate methods for doing well in mathematics. Skyler generally intertwined her discussion of the first two themes. She believed that understanding mathematical concepts and possessing an ability to explain them were critical components for her future career and her future students' success. Skyler explained how her experiences in Basic Concepts 1 and Basic Concepts 2 affected those beliefs:

The main focus [of these courses] is understanding it—understanding the concepts, why you do this, why you do that. . . . I never had it where we focused so much on explanation, and we focused so much on why and how, . . . so I guess it was never really drilled in my head before that it was really important to understand why you do things.

In Skyler's other mathematics courses, sharing of answers and working with other people were frowned upon or prohibited altogether in favor of working "independently." She preferred the alternative in which students are able "to work with other people, and get their ideas, and feed off their ideas." But she also stressed the importance of "thinking on your own" as well as "[knowing] what you're doing first."

For Skyler, however, success in her mathematics courses was driven by one major factor that overshadowed the other important components: maintaining a good course grade. She believed that having a good grade could come at the cost of mathematical communication and understanding. She explained, "It should be more about you understanding the material, and sometimes the grades aren't a good, um, way of reflecting, you know, whether a student really understands the material or not." Although Skyler understood the necessity for regularly assigning students homework, given her busy schedule, she could focus only on graded homework assignments.

Aligned with her beliefs about the importance of mathematical understanding and communication, Skyler championed the pedagogical techniques used in Basic Concepts 2. For

her, those techniques included explorations and investigations through small group discussions of the mathematics concepts, as well as a focus on exploring multiple representations (also identified as “processes,” “different ways,” or “strategies”) of the mathematics of the investigations. She elaborated, “One thing about Dr. Fikes’s class is—that I will always have in my head until the day I die—is that there will always be more than one way of doing things.” However, for Skyler, simply knowing those multiple representations appeared to be more important for success in the class than for the connections between them. When discussing appropriate methods for studying mathematics, she explained, “I still think that practice is the number one thing you can do for math. Because you just have to get it drilled in your head to remember it and to remember different ways of looking at a problem.” Skyler likened the learning of mathematics to how one could study for other subjects:

[Spelling] is similar in the sense that you have to practice it religiously. . . . With history, it’s just memorization. You have to memorize in math. Um, with reading, spelling—it’s just repetition, seeing it over and over again. You have to do that with math. Um, science, it’s using strategies and theories and experimenting with stuff and, you know, you have to use that in math.

When I asked Skyler if one can be successful in mathematics without the use of memorization, she replied, “You just have to be really aware of the strategies that you use and just use them a lot.” She further clarified her ideas by explaining how she studied for Basic Concepts 2:

I mainly just go over the problems that we’ve done in homework and that we’ve done in class, um, videos that we’ve watched—just to remember different strategies . . . just remembering that train of thought you had when you were solving it.

Impressions of Basic Concepts 2

Skyler’s experiences in Basic Concepts 1 and Basic Concepts 2 greatly affected how she viewed her future mathematics teaching. She planned to emulate Dr. Fikes’s teaching style and

described it as demanding and, at first, both surprising and different. She recalled her first semester in Basic Concepts 1 with Dr. Fikes and how she had “almost dropped” the course. She explained, laughing:

Your brain needs to already be working before you walk in that door because she’s going to throw something at you, and you need to be prepared to answer it. . . . I would just come in there and be prepared to listen to her talk, and it was not like that.

Skyler had to adapt to this particular teaching style and explained, “I know that I should go ahead and start thinking about what’s coming. Because, you know, she’s going to ask you a follow-up question. You know she is.” Skyler came to view this teaching style as a critical factor for gaining proper mathematical understanding and began to emulate Dr. Fikes’s teaching style when she worked with her elementary-school-aged tutee. Through those efforts, Skyler noticed that her tutee engaged more in her learning when Skyler pressed her with questions in pursuit of mathematical understanding.

Although Skyler indicated that Dr. Fikes’s teaching style was different from that of virtually any other mathematics instructor she had previously had,¹⁹ her discussion of a typical Basic Concepts 2 session was not quite the same as how it actually unfolded. On different occasions, she provided similar discussions of how she viewed a typical class session in Basic Concepts 2. She provided the following explanation of how she would structure her own pedagogy based on her experiences in Basic Concepts 2:

I would explain the steps, explain why I was doing each step. Um, do like a practice problem or something. Let them go around, work in groups, work in pairs, do some problems, work on it by themselves. Or however they want to do it. . . . But I would want them to work in groups and then at the end, go back, go, have people share what they did, have people share their different methods

¹⁹ Skyler had had one class in high school in which the teaching style of the instructor was slightly similar to that used in Basic Concepts 2. She recalled her high school teacher “always calling on people in class . . . [to] come up and explain their answers . . . [and] to go over all different kinds of ways and strategies to solve different problems.”

But then have them explain certain things, and I guess just reiterate the fact of why we're doing this.

As discussed in Chapter 3, group work and the sharing of explanations were major pedagogical techniques used daily in Basic Concepts 2. However, I rarely observed Dr. Fikes providing a direct explanation of the concepts to be explored—whether initially or following concept exploration.

Views on the Use of Technology for Learning and Teaching Mathematics

Skyler provided conflicting thoughts about the use of technology for mathematics teaching and learning. Her first course in the education department had focused on “integrating technology in the classroom.” Her experiences in that class had solidified her decision to study elementary education. She claimed, “I loved it. I absolutely loved it.” She described technology as “hands-on” and discussed that its use, specifically the use of calculators, would not “hinder [students’ mathematics] skills as long as they’re not dependent on it.” She explained:

I guess technology, as a whole, is a good thing. I mean, we’re in the 21st century, you know? It’s going to happen, it’s going to be here. And especially like integrating technology in your classroom is going to be a big deal because students are so interested in that now. The more they’re interested in things, the more they’re going to pay attention, and the better they’re going to learn them.

However, Skyler’s later sentiments and actions revealed a reluctance to learn to use technology or even incorporate it into her future classroom. She explained that students may “take the easy way out” if the teacher introduces technology too soon. She traced her reluctance to experiences using technology as a student, which included the graphing calculator and software programs such as MyMathLab. She explained, “I mean, calculators are great . . . but, um, if you don’t type it in exactly how they want you to type it in, exactly like how they understand it, then you get the answer completely wrong.” She recalled her experiences with one software program that required mathematical “abbreviations” as part of its interface:

There's so many different words that you had to type in and just abbreviations, and you, if you didn't get the abbreviation exactly right, it would say "error." And it would just be frustrating and take too much time to type in the thing when you could just do it by hand, faster. Um, that's a lot of the reason why I don't use that stuff either. . . . But just for the simple fact of—I mean, you see how I am with this (laughs and points to the GSP sketch displayed on the computer screen)—it just takes a while to get used to one. You know? And, I don't know, I feel like you spend more time trying to figure out how to do things than actually using it.

Skyler stressed that if she could take a course that "really focused on one website, and then all through the class the teacher taught you how to do certain things or taught you the way," then she might be more amenable to using technology. She claimed: "It can be useful, though. As long as you know what you're doing."

Skyler's reservations about technology were especially apparent during the clinical interviews as she engaged with GSP. For example, she appeared apprehensive in instances when she did not use the mouse correctly to select a program feature. On one occasion, she anxiously remarked, "Oh, Lord," when asked to create a sketch in the program. And on another occasion, Skyler called the program "a cheat" when it displayed the area of a selected figure. She explained, "It's just, you don't know that—the computer's just telling you that, you know?"

Sources of Mathematics Learning and Understanding

Skyler indicated her main sources for learning mathematics as "what I learn in class." She used the "lecture notes or worksheets or assignments that are given in class" to help her understanding. Alternatively, Skyler said that she would seek the help of a classmate or referred to the course text if she either missed class or needed help on concepts that she was "not sure about." However, given her busy schedule outside of class, she could only seek help from classmates during class time. And although Skyler indicated possibly seeking help from classmates, she indicated a preference for autonomy when doing mathematics, especially when working problems she found particularly difficult. She explained:

I'm the kind of person who it makes me feel so much better when I figure it out for myself. Um, so, I particularly would go look it up on the Internet, or in the book, um, or in my notes or something like that. . . . I like the challenge in figuring it out. 'Cause when I figure out the answer, I feel, I don't know, I feel really good about myself.

Often, Skyler's desire for autonomy gave the impression that she viewed herself as the source for her mathematics learning. She explained, "And I actually get excited about [mathematics] when I can figure something out that I didn't know before, you know?" The data collected from the class observations provide a picture of Skyler as a source of mathematics for other students. She was rarely observed asking for help from others, or even from Dr. Fikes, on concepts and problems being explored. During small group discussions, she readily explained (sometimes incorrectly) a variety of mathematical concepts and ideas. On occasion, she convinced (sometimes incorrectly) other students to change their minds about their mathematical conclusions. During whole group discussions Skyler appeared confident in her mathematical explanations and contributions and would interpret or elaborate on fellow students' mathematical explanations for others' clarification, with and without solicitation.

When confronted with solutions different from her own, however, Skyler indicated using caution in examining those solutions of classmates who she thought were particularly smart in mathematics. She gave an example of one classmate, Jason, with whom she worked on several occasions. Skyler described Jason as "very smart," someone who "understands the material and everything"—someone who she went to mathematical "battle" with "all the time." As she recalled their conversation on the first day of class, discussing their similar mathematics backgrounds, Skyler elaborated on her mathematical interactions with Jason:

But there's certain things, there's certain ideas and concepts that we have, we feel differently about, you know? Most of the time we both can get the right answer, but we use two different ways of thinking about it. And so it takes a while for us to see, "Ok, wait, your way works too." But it's still the same, you know? So

yeah, when he talks out loud, and he starts explaining stuff, and I'm like, "Where's he going with this?" But then he gets the right answer, you know? It's like, wait, did his method work? It makes you question it. But, you're hesitant to say something to him. You're more resistant to say something to him than you are somebody who doesn't really speak up in class, um, and you don't already have this picture in your head that, you know, they're really good at what they're doing.

When considering other typical mathematics sources (e.g., the course instructor or the course text), Skyler also discussed using caution in examining mathematics results different from her own. This approach required carefully questioning a course instructor for an explanation of the conflict in results or reworking her own methods to ascertain the reason for discrepancy with solutions of the course text. Skyler elaborated on the resulting frustrations that could occur from her efforts: "The back of the book doesn't show their process. They just show the answer, [and] I think the process is more important." In regard to technology use, Skyler indicated that any error in results presented by technology was more likely a fault of the user.

Mathematical Authority in Different Contexts

In the Absence of the Instructor (Mathematics Task 1). Earlier, when I presented data from the results of Garrett's first clinical interview, I also presented a conversation that included Garrett, Skyler, and Janina to provide evidence of how the three participants attempted to conceptualize tessellations and semiregular tessellations. This small group discussion provides an entry point for identifying the various mathematics sources that the participants sought to help form their understanding of tessellations and semiregular tessellations.

During this conversation, Skyler did not appear to use any type of external source, nor was there any evidence to indicate that she had experience exploring tessellations in her prior mathematics classes to help answer the problem. Instead, the excerpt illustrated how Skyler used (and subsequently changed) the knowledge from Julie's limited understanding and discussion of

semiregular tessellations to help form her own understanding. This change resulted in a conceptualization of semiregular tessellations that did not incorporate the definition as presented in the course text or said out loud by Garrett. Further, as the conversation ensued, Skyler seemed to become more confident in her conceptualization of semiregular tessellations as tessellations that lack completeness.

Clinical interview results. Skyler correctly recalled most of the events as they unfolded in class during the discussion of the review problem. She did not think that her small group members had come to a consensus of the attributes for determining a semiregular tessellation, although she correctly remembered their collective reason for choosing 4b in Figure 5 as the answer to the problem. Further, Skyler did not indicate that she had sought help outside of class for that particular review problem.

Although Skyler did not recall previous experiences learning tessellations, she had formed some conceptual understanding of the topic. Skyler said, “Um, to me, I guess this isn’t in mathematical terms, but it’s when the shapes are all connected to one another and they fit perfectly together.” In regard to conceptualizing a semiregular tessellation, Skyler returned to the small group discussion that only certain polygons tessellate and provided reasons based on how she visualized the shapes on the plane. Skyler indicated that it became increasingly difficult for her to visualize whether a polygon singularly tessellates the plane as the number of sides of the polygon increases. She explained, “Just because it’s hard, I can’t see figures without actually looking at them.”

After her discussion of her memories and understanding of tessellations and semiregular tessellations, I asked Skyler to use GSP’s premade scripts to investigate which of the regular polygons singularly tessellates the plane. When her investigation turned to the octagon, she

claimed that the octagon does not singularly tessellate the plane because a square would fit into the gaps. Skyler demonstrated her knowledge by pointing to the figures on the screen and performed calculations on paper, with a method she learned in class, to conclude that the octagon tessellates with a square. She drew a rough sketch of an octagon on paper and created eight congruent triangles formed from the center of the octagon (Figure 22). Skyler then used her knowledge of congruent base angles and the exterior angle of a figure to conclude that one of the angles of the quadrilateral in question measures 90 degrees. Further, she argued that the sides of the quadrilateral consist of sides of congruent octagons, so the quadrilateral must be a square.

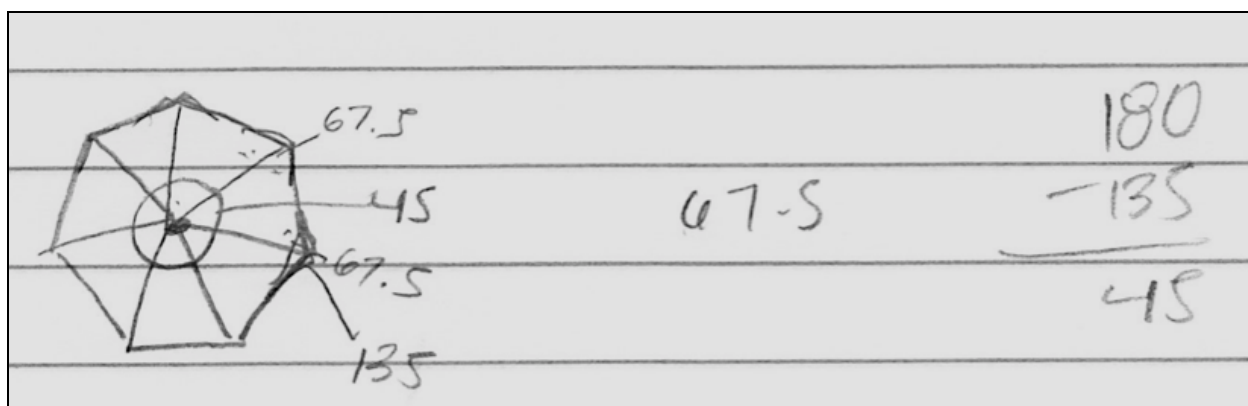


Figure 22. Skyler's scratch work on tessellating with the octagon.

Skyler's investigation of whether the regular octagon singularly tessellates the plane naturally led to the definition of semiregular tessellation. After I provided clarification of the definition, she appeared to understand the conditions of what constitutes a semiregular tessellation. She then correctly explained why 4b in Figure 5 would not be a semiregular tessellation in contrast to how it was conceived during the small group investigation. However, Skyler thought it would be difficult to visually know if 4a in Figure 5 would be a semiregular tessellation without physically expanding its area with more shapes to see if the pattern continued. Therefore, she used GSP and its premade scripts for dodecagon, hexagon, and square to draw her conclusion.

Skyler's investigation appeared to solidify her understanding of the discussions that took place earlier in the clinical interview. She repeated the clarification of why 4a in Figure 5 was a semiregular tessellation and why 4b was not. However, at the next class session, Skyler indicated that she had thought about the problem more and "was going in circles" in an attempt to figure out the reason for the answer. At the beginning of the second interview, Skyler remarked that, in the end, she "just figured it out" and again correctly repeated the definitions.

Discussion. During the small group discussion, Skyler's understanding of semiregular tessellation was based on how she interpreted information presented by Julie, who lacked confidence in her own understanding. Skyler used that information to form an authoritative stance on the requirements of a semiregular tessellation, determine the answer to the review problem, and solidify Janina's interpretation of the concept.

Additionally, because of the exploration she had done in her clinical interview, I was surprised by Skyler's quick comment to me during a later class session (while I was videorecording near her table) that she was still confused about semiregular tessellations. During the interview, Skyler had seemed rather confident in her understanding of the concept and even correctly stated the reason 4a in Figure 5 was semiregular.

The data on Skyler revealed a complex individual who, as a possible mathematical authority for her peers, was not necessarily able to use mathematical connections or other typical sources of mathematics (such as the definition provided in the course text) to help form or solidify her own understanding. Unchallenged, Skyler was able to heighten the characteristics of what seemed to initially mark her as an authority.

The Triangle Dissection Paradox (Mathematics Task 2). Skyler presented evidence of her understanding of the underlying mathematical idea of the second mathematics task in an

excerpt I presented earlier for data on Garrett. This data came from the relevant class session for his second clinical interview. During this session, Skyler and Garrett were paired to work on the problems of Appendix F and engaged in discussion of how they would generate a rule for a trapezoid. As a pair, they did not devise a general rule for a trapezoid as instructed in the handout. Although it appeared that Skyler did not understand the instructions of that handout, she demonstrated how to use decomposition to find the area of a figure. Further, Skyler later discussed how she answered the second question of Appendix F with the partners sitting at the table in front of her and Garrett. After one of the partners stated she did not understand the formula for the area of a trapezoid, a conversation ensued that provides additional data of Skyler's understanding of how to use decomposition to find area. This discussion, however, also provides evidence of how Skyler's confidence appeared to elevate her as a source of mathematical authority to her peers:

Skyler: A way that might be easy for you to understand is, ok, basically you have these four items in this formula. Ok, how do you find the area of a triangle?

Partner: Just one-half base times height.

Skyler: (Marks on the formula she has written in Figure 17 for the area of a trapezoid.) Ok, so you have the one half (underlines the one half). You have the base and then you have the height, right? (Underlines the first base and the height.) Well, then, how do you find the area of a rectangle?

Partner: Base times height.

Skyler: So then you have the base (underlines the second base in the formula). These two will always share the same height because they're in the trapezoid (draws vertical lines on the trapezoid, indicating she's discussing the three areas of the trapezoid).

Partner: Hmm hmm.

Skyler: So basically this one's just being used twice (circles the letter h in the formula) because they're sharing it. Does that make sense?

Partner: That makes a lot more sense.

Skyler: That's basically if you want to break it down to where the terms come from. But, just, in order to get that formula and then in order to explain like how it works, would be setting those two equal to each other (underlines her work shown in Figure 17). I mean, if they're not equal, then it wouldn't work.

Partner: Ok, yeah, that makes a lot more sense.

Clinical interview results. As Skyler began her investigation of the second mathematics task, she did not believe that it could be accomplished. Thus, although it seemed that Skyler understood what I asked of her in regard to the task, she found it difficult to form Triangle B in Figure 6. For each of her multiple attempts, her efforts always resulted in the configuration shown in Figure 23.

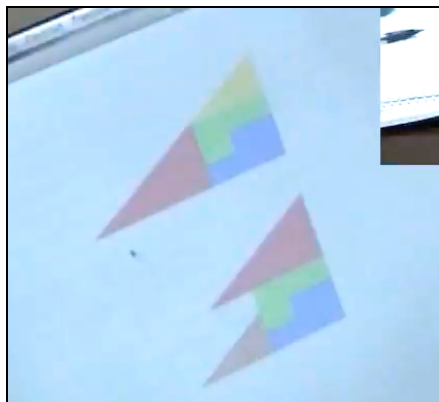


Figure 23. Screenshot of Skyler's attempts to rearrange the pieces of the triangle.

Once I presented her with the correct configuration, she exclaimed, "Ohhh! Clever." She explained her initial doubts: "I didn't think—because the pieces didn't match, um, up together—I didn't think it would recreate that shape. Um, I mean obviously, you can recreate other shapes." However, she was immediately drawn to the hole and initially explained its existence with the statement: "But I guess it depends on how you put [the four shapes] together."

Skyler used a variety of strategies, sometimes repeatedly, to form a plausible explanation for the hole. In addition to using her knowledge of how to calculate the area of a triangle with the

formula $A = \frac{1}{2}bh$, Skyler recounted the lengths of the base and height for both triangles multiple times. She used GSP's *Measure* feature to find the area of the colored region of both triangles to confirm that the areas of the two triangles were off by one square unit. She also used the feature to calculate the area of each of the four corresponding pieces of both triangles to confirm that they were the same area. And once she had confirmed that those areas were the same, Skyler then used GSP's *Calculate* feature to sum the areas to ensure that the summations of the colored areas of each triangle were equal. Additionally, Skyler checked that the pieces forming the two triangles were of equal area by laying each piece on top of its corresponding piece in the other triangle. Throughout these different investigative attempts, and on multiple occasions, Skyler indicated that she would “normally” compute and sum the areas of the four different shapes to determine the area of the entire figure. She explained:

But I think if you were to add, still, all the pieces together, it would be the same. (She again counts the length of the base of the new triangle.) I don't know—that's a tough question. Because it doesn't, obviously you're missing that piece right there (points to the hole). . . . There definitely is [an issue here]. I mean, you would have to take that 32.5 and then minus one—which would be 31.5. So that would be a completely different area.

Although she relentlessly pursued the task, Skyler did, however, question whether I knew the reason for the hole after roughly 13 minutes of exploration—but that did not stop her attempts to ascertain an understanding. When left with no other strategy to explain the hole, Skyler revisited her initial discussion of how the pieces were placed in each triangular configuration:

It's something about how it's stretched out or something. . . . But, you know how I was talking about how you could take these two figures (points to Figure 24a) and make them into a rectangle like that (points to Figure 24b). . . . Well this isn't a rectangle anymore because it's spread out. There's not a—, it doesn't fit together perfectly. . . . But for some reason, since it's not that rectangle, and it's stretched out like that, it's making the area more stretched out.



Figure 24a. Skyler points to the two pieces that create the hole.

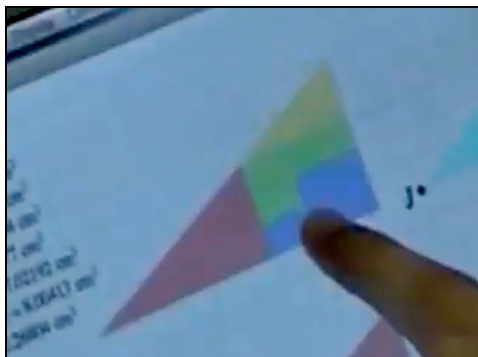


Figure 24b. Skyler indicates the pieces in their original configuration.

During the entire clinical interview, Skyler's frustration was shown only through her verbal statements. At one point, Skyler exclaimed, "Whew, oh, I really don't know. It's kind of frustrating." And on a separate occasion, she said, "It just stresses me out." Similar to how she acted during the class sessions, Skyler possessed a calm demeanor and appeared confident during her discussions and explanations of her mathematics knowledge as well as in her attempt to explain the hole using the idea of the placement of the four different shapes.

At the conclusion of the second clinical interview and after I had explained the paradox to Skyler, I asked her if she thought her exploration of the task would have been different had she explored it on paper only. Skyler indicated that she thought the task would have been "more confusing" and that the technology provided affordances that investigating the task on paper could not. She explained:

Just because, um, I mean with this program, like you can, you can see that it fits. Now, if you draw it on a sheet of paper, it's a little bit harder to make it absolutely perfect, and I mean, just drawing it on the board, your squares aren't going to be perfect. You know? Um, so you would see that and you would be like, "Oh well, there's just an error in the way I drew it."

Discussion. Skyler's mathematics knowledge pointed to an issue for the existence of the hole. She used a variety of strategies to confirm that the two triangles had different areas. Eventually (although reluctantly), she provided an explanation in the face of the failures of her

mathematics techniques during her investigation. Skyler's admission of her apprehension and frustration with the problem was a huge contrast to how she was observed during the class sessions (confident, mathematically capable, and authoritative). But, similar to the Skyler of the class sessions, her calm demeanor did not belie her emotions. Though she experienced anxiety, doubt, and frustrations, the most remarkable feature of Skyler's clinical interview was her perseverance in attempting to make mathematical sense of the hole.

The Lie of GSP (Mathematics Task 3). Skyler presented evidence of her understanding of mathematical shearing during whole and small group discussions of a class session in which Dr. Fikes explicitly addressed the concept and Skyler was partnered with Garrett to work on the problems of Appendix F. These data were presented earlier when I discussed the results of Garrett's third interview.

Clinical interview results. Like Garrett, Skyler also used the method of counting unit squares to determine possible locations of the third vertex and quickly identified 10 vertices meeting the conditions mandated by the task. She said that she knew the triangles formed by those vertices would have the requisite area of 12 square units by discussing the task in terms of using "silly putty." She explained, "You know you still have the same amount, but then you can stretch the figure." As Skyler explained that any vertex she found on the right side of the y -axis would have a mirror image on the left side, she drew vertical lines to indicate the placement of those vertices. However, Skyler did not discuss the distance of those lines from the y -axis. Her discussion implied that she could not identify a value for that distance, only that she knew that mirror images of the vertices in question existed.

When I asked Skyler if she thought there were more possibilities for the location of the third vertex, she immediately drew a vertex and completed the triangle from the other two

vertices (shown in Figure 25a). As she drew, she observed, “Which I’m sure, I mean you, if you wanted to keep doing it, you could go like, you know, do something like that.” She then drew the triangle’s mirror image on the other side of the y -axis:

Me: Would that third vertex down here be on this line? (pointing to the vertical lines she indicated earlier).

Skyler: No.

Me: Ok?

Skyler: ‘Cause that would be too much area. I mean if it was all the way down here. Um, I mean maybe you could keep stretching it out to like here. You know? And then that would equal, but eventually, I would think that the further you go down, the closer you would have to go to the y -axis to get that same area. Because, if you, um let me see, if you took it all the way down here (indicates below the x -axis), if you go all the way, it would have to stop somewhere.

When I asked Skyler at what point she believed the vertices would start getting closer to the y -axis, she laughed as she said, “I knew you were going to ask me that. I have no idea. Um. (pauses) I don’t know. Maybe the x -axis.” Skyler continued to discuss how, as the triangle became more oblique, the area of the triangle would get larger. However, she then returned to her original discussion that involved the idea of silly putty.

Skyler: I don’t know it just all goes back to the whole, the putty. I guess you can stretch it and stretch it and stretch it, and you can stretch it as long as you want, but it’s going to get skinnier.

Me: Ok. But the area increases, as it gets skinnier?

Skyler: No.

Me: So, how is that different than this? (Pointing to Figure 25a.) When you stretch it?

Skyler: Well, because if you stretch it far out, then the area, I mean, you have this and this as the vertice [*sic*] (points to the two mandated vertices). . . .

(On a new sheet of paper Skyler drew the triangles shown in Figure 25b.)

Ok. I guess if you did this. Like, I just don't think that this right here (points to the two triangles on the left in Figure 25b) is the same area.

Because of her uncertainty of whether the triangles would have the same area, I decided Skyler might benefit from GSP's dynamic capability to calculate the area of a figure as it changes shape. I hoped that her investigation with the technology would draw upon her knowledge of mathematical shearing as I had observed during the relevant class session. Thus, I asked her to create a GSP sketch that met the conditions of the problem to provide her with a dynamic representation of what she found difficult to visualize on paper only.

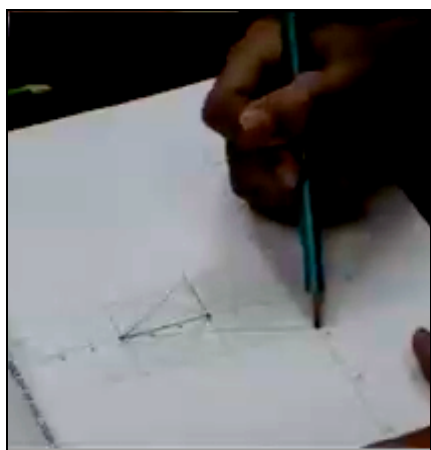


Figure 25a. Skyler indicates that the areas would not be the same.

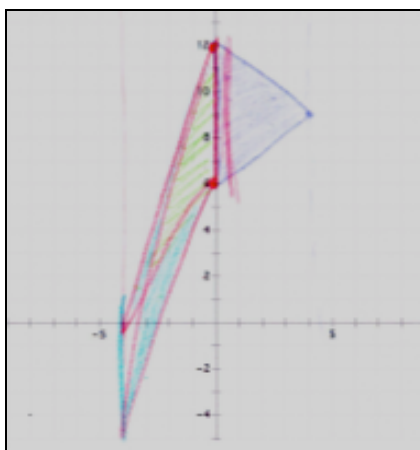


Figure 25b. Skyler's work to illustrate her thinking.

Skyler displayed her apprehension of working with GSP as she replied to my request with "Oh, Lord. . . . You could just give me the sketch." However, as she attempted to create the appropriate GSP sketch, she recalled working with the measurement feature of GSP from the previous interviews. She inquired, "Is there a way—what was that thing we did the other day? That it would show the area as we were moving it, and it changed or whatever?" Once she created the sketch, Skyler confirmed that the area stayed fixed as she stretched the triangle using the third vertex placed along the vertical lines at $x = 4$ and $x = -4$. Although it appeared that she

did not have a mathematical basis for why the vertices could be found along those lines, Skyler indicated that GSP was convincing her of the fact in a way her investigation on paper could not.

She elaborated:

How I drew it or however—, maybe this area right here doesn't look equivalent to that area to me (points to the two triangles on the left side of the y -axis in Figure 25b). That's why I said that [I don't know if they have the same area]. I mean [GSP's] telling me that the area is [the same] so it's making me believe it. . . . Um, but I mean, see, like that—, if I—, if you were to tell me to look at that (points again to the two triangles). And that's so close that it's like, "Uh, it could be [the same area]." But it might not be the same. See, how would I really know? You know what I mean? I guess the closer it gets to being similar, it's harder for me to understand.

In response to this observation, I attempted to elicit Skyler's knowledge of mathematical shearing as it had been captured in class and inquired why she thought all of the vertices lay along one of the two vertical lines. Skyler returned to her discussion of silly putty and discussed critical features of the concept of mathematical shearing:

Um, I don't know. I know I'm talking about silly putty a lot. But, if you do this, and you stretch it, you know, like, as long as you have that same, I guess, width maybe? Or, no you could call that—, you could call that the height. No, you couldn't. I don't know.

Skyler's comments revealed that her mathematical issues were more with orienting the triangles in ways that could help her visualize their respective heights and bases. After orienting her paper investigation so that the vertices at $(0, 6)$ and $(0, 12)$ were horizontal, Skyler was finally able to identify the respective heights and bases of the triangles. She elaborated on her initial issues and mathematical conclusions:

I'm one of those people who looks at a triangle—, like today in class, looks at a triangle, and the bottom is the base, and the height is the height. You know, that kind of thing. But, in this formula, as long as you get how tall it is and how wide it is, um, it doesn't matter what order it is. . . . So, I guess they all have the same base and the same height. Which makes them all have the same area. That would be my conclusion.

Although it seemed that Skyler had finally arrived at a conclusion with a mathematical basis for her initial findings on paper (and then confirmed with her exploration on GSP), her conception of triangle was not formed fully enough to allow her to finalize her conclusions for the task at hand. Further, the limitations of GSP's ability to visually display the mathematical shearing from the original two vertices hindered her efforts:

Me: So how many triangles do you think there are?

Skyler: I don't know. I don't know. Can you move this down? (Indicating the third vertex on the bottom of the screen.) How do you move this down? (Skyler starts to move the vertex.) Wow, this guy is stretching. . . . I don't know, 'cause I don't know what, if it would count. You know, like, obviously as you're stretching further down, this is getting narrower and narrower. So when you do this, it's going to get really narrow to where you can't even see the space in between it. I don't know if that counts. Do you, do you see what I'm saying?

Me: So you don't think [the area] would be twelve? You're not sure?

Skyler: No, I think [the area] will be twelve.

Me: Ok.

Skyler: But—.

Me: Oh, you don't know if you have a triangle?

Skyler: You know? 'Cause eventually it's going to be so narrow. Do you see how narrow it is?

Me: So you think it will be, uh—.

Skyler: Like that I would never—.

Me: You think it would be something so close to looking like—.

Skyler: A line. The area is so stretched out that it has to be so narrow that—.

Me: Would it be a triangle anymore?

Skyler: I don't know where that, where you could draw that stopping point.

Me: Ok. So you don't have a number then?

Skyler: No.

Me: But you think it's a finite number.

Skyler: Hmm hmm.

Given Skyler's conclusion of the task and the limitations of GSP's visual display, I decided that I would not pursue the clinical interview as initially planned. At that moment, I was unsure of the strength of Skyler's understanding of mathematical shearing and did not want to compound any existing confusion or misunderstandings by asking her to consider another sketch that presented incorrect results.

Discussion. Although I was initially surprised that Skyler did not have a sufficient understanding of mathematical shearing to be able to successfully complete the task on just paper, the results of the third clinical interview were not necessarily unexpected when considered against the other data gathered from the study. First, Skyler continued to provide evidence of her apprehension about working with technology. Second, she provided more evidence of her inability "to see" geometric movement. And third, Skyler had difficulty, at least initially, in connecting her knowledge of mathematical shearing to the task at hand. Ultimately, however, Skyler was able to call upon what knowledge she did have to understand GSP's dynamic presentation—although her weak conception of triangle held her back from fully grasping the concept of mathematical shearing.

Allocation of Mathematical Authority

Observed as possessing a high level of mathematics self-confidence during class sessions, Skyler revealed a less confident mathematical disposition during the personal interviews when it came to the topic of geometry. In the personal interviews, Skyler periodically discussed her

inability to “see” the geometry. This belief may have led to a psychological block toward the topic. Although this inability could certainly have contributed to her feelings about geometry, a larger factor was Skyler’s lack of the geometric habits of mind (Driscoll, 2007)—habits that would result in a deeper and more connected geometric understanding.

Despite her reservations about geometry, Skyler continued to appear highly confident in her mathematical explanations and conclusions of investigations during the class sessions. Her successes in the various mathematics courses she had taken over the years, as well as her tutoring experiences with elementary-aged children, helped form her overall mathematics disposition. The data revealed a complexity in her mathematical confidence and placement of mathematical authority given her varied and fairly extensive mathematics background.

During the clinical interviews, Skyler appeared quite reluctant to use GSP, although she had praised the use of technology for the learning and teaching of mathematics. Unfortunately, because Skyler did not participate in the third clinical interview as I had initially planned, it is unclear whether she would submit to the authority of technology when confronted with results different from her own. Arguably, her apprehension in using technology, compounded with her issues with geometry, could potentially confound any results of an investigation that tried to conceptualize mathematical authority with technology in a geometric learning context. Given her confidence, as observed in class, the results of Skyler’s personal interview data were quite surprising—almost contradictory to her behavior as she interacted with her peers.

The collective analysis of Skyler’s data indicate that she had an automatic work method (Figure 3) and that she primarily placed mathematical authority in her own mathematics knowledge while ignoring more useful mathematics sources. She did not indicate that she had used any typical mathematics source (i.e., classroom teacher or course text), and she pointed to

her experiences in class or her personal class notes as means for her to form her mathematical understanding. At times, the use of these sources led to piecemeal behavior that meshed together a variety of concepts and ideas in order to fit the mathematics investigation at hand, yielding unpredictable results in regard to mathematical correctness. Further, given her varied mathematics background, Skyler appeared unable to efficiently or sufficiently call upon the knowledge accumulated from that mathematics. It is difficult to decide whether this inability was due to her views of mathematics and mathematics learning or to the geometrical content of the course.

Janina

Attitude Toward Mathematics

Janina held a moderately negative attitude toward mathematics. During the course of the interviews, she periodically labeled herself as “not a big math person.” She occasionally made statements such as: “I never really enjoyed math,” and “Math isn’t my forte.” However, Janina said that her decision to select language arts as her concentration was based on “personal interest” and that she “was not against math or anything.” Further, she was using her experiences in Basic Concepts 2 as “a little trial run” to be “one hundred percent sure” that she had correctly chosen language arts as a concentration. Otherwise, Basic Concepts 2 would be the last mathematics class she would take at the university.

Although she had been successful in all of her previous mathematics classes, Janina thought that the subject, in contrast to language arts, was restrictive in many aspects of learning. She described mathematics as “very strategic” and “takes a lot of practice.” She described herself as “a very visual learner” and “always the one that’s drawing up the diagrams and, like, writing out the equation on top of my paper.” For Janina, mathematics exercises result in only “one right

answer” with little room for “interpretation” of various results. She explained: “And I feel like that’s one reason why I don’t really enjoy math that much, is because it is sitting there and very technical, and you’re just like, ‘Ok, like, it doesn’t interest me at all.’” In contrast, Janina felt that language arts allowed her to “go with my own thoughts, do my own thing” since “it does give me the power to be able to explain myself—which I’ve never done in math.”

When discussing the conditions to be successful in mathematics, Janina appeared to equate success with getting high grades in the subject and thought that a mathematically capable person easily understood the subject without much, if any, required practice of requisite skills. Further, she thought that one must be “a logical thinker” and be born with a special talent to do well in mathematics and, for such people, mathematics “just comes to them easier.” She remarked that she did not have such a talent, but that success in mathematics could exist without such a talent because good performance “just comes from, like, practicing and studying.” For Janina, studying for a mathematics class was different than studying for other subjects— which require “a lot of memorization.” She elaborated:

In other subjects I can take flashcards, big flashcards, and memorize it and do fine on the test. In math, I can’t do that at all. I have to repeatedly, like, practice the problems and make sure I’m understanding what I’m doing.

Confidence in Doing, Learning, and Teaching Mathematics

Despite her negative views of the subject, Janina felt confident to teach mathematics at the elementary school level. Her confidence stemmed from her belief of how mathematics teaching and learning looked at that level. She explained:

But I guess in elementary education [teaching is] more of a “Why?” It answers that question. It’s more getting [the students] to explain [the concept] in different ways, to actually be able to explain: “This is why I do this; this is how I got this.”

Janina's experiences in Basic Concepts 2 helped to build her confidence to teach mathematics. She thought she was being prepared "to look at different views" of how students engage with mathematics with the emphasis on "how we got our answer."

Janina regularly contributed to the whole group discussions and actively engaged in the small group activities—smiling as she spoke with her classmates. She generally appeared mathematically confident during class discussions. At times, however, she would seek help from her peers in explaining solutions or expressing her thought processes for various mathematics discussions. She explained, "I've never had a problem talking to people or answering questions in class or anything. I mean, even if I don't know the right answer, I still don't have a problem saying anything out loud."

Major Themes That Arose From the Interviews

Only one major theme arose over the course of the three interviews conducted with Janina: the contrast between her impressions of the pedagogy used in Basic Concepts 2 and how she would teach her future students. Janina constantly described Basic Concepts 2 as "really different than any other class I've taken before." (A thorough discussion of her impressions is provided in the next section.) She described Dr. Fikes's pedagogical techniques as aligned with why she had chosen language arts as her concentration and how liberated that subject made her feel. On several occasions, Janina discussed how she would have benefitted from such experiences as a student learning mathematics. However, although Janina provided many accolades of the pedagogy used in Basic Concepts 2, she indicated that she would not use such techniques to teach her future students—a most unexpected result of her interviews.

Janina had already decided that she wanted to teach second grade and thought that students at this age needed "structure" and someone to help them make "that connection" and

“be able to get them to explain the problem of why they got [their answers]—to be able to help them explain [their thoughts] eventually.” Although Janina thought that Dr. Fikes’s pedagogy was a necessary factor in her preparation for her future career, she indicated that she would not use the questioning technique employed by Dr. Fikes. Janina believed that second graders needed to first learn “basic” and “computational skills” before struggling with mathematics concepts so that they might have “a basis on what they’re doing.” She added, “You can’t just throw them in like—, throw them in the middle in there and be like—, and ask them like—, have them solve it, and they’ve never even, like, seen anything like it before.”

Impressions of Basic Concepts 2

Janina had enrolled in Basic Concepts 2 because of a slight “push” from her academic advisor and the fact that her roommate (who had previously completed Basic Concepts 1 with Dr. Fikes) was also enrolled in the course. Her experiences in the course changed Janina’s views of mathematics and her affinity toward the subject. On a scale of 1 to 10 (with 10 being the highest score), Janina indicated that she would have rated her affinity toward mathematics with a 4 before her experiences in Basic Concepts 2. However, at the end of the third interview, Janina rated her affinity as a 7. Janina discussed the teaching style used in Basic Concepts 2 as “[opening] my eyes to a whole different side of math,” altering the “biased judgment” she previously held against it. She described the course as “completely different” from any other mathematics course she had taken. She explained, “This is the first math class I’ve ever sat in, especially in math, and had to explain how I got this answer, and why I feel that this is the right answer, why I’m doing it this way.”

Although Janina thought that Basic Concepts 2 was not a methods course, she described the activities used as “really good” and suitable for teaching her future students. Rather than a

course goal of “teaching us how to do the math problems,” Janina thought that Dr. Fikes’s pedagogical goals were to provide the students “techniques on learning how to explain math problems.” These goals provided the means to anticipate the types of mathematical thinking the preservice teachers could potentially encounter with future students or, at the very least, the knowledge that such diverse thinking actually exists. Janina elaborated:

[Basic Concepts 2] prepares us to think about: that you’re going to have 25 different students who all think differently, and this is the first time that they’re going to be in a situation where they actually have to explain how they got this answer.

Janina also appreciated the “interaction” required in the course. And, because her roommate was also in the class, she was able to continue the interaction outside of the classroom to “collaborate together” on their homework. Janina indicated it would have been “beneficial” to have a similar learning environment to that of Basic Concepts 2 in her previous mathematics courses. She explained, “I mean, just because your teacher thinks it’s this way doesn’t mean that every student in the class is going to—it’s not going to be understandable by everyone that way.”

Janina described herself as “a big over-achiever” and “a perfectionist” with the need to “satisfy” herself in the quality of her work. However, her greatest motivation for working hard and doing well in mathematics was the course grade. Although she viewed the pedagogy used in Basic Concepts 2 as helpful to her as a future teacher, there were various instances in which she expressed being “frustrated” with the questioning technique used by Dr. Fikes. At times, she indicated the desire for “a direct answer” to her questions and explanations. However, Janina recognized the benefit of such pedagogical techniques and indicated that the lack of a direct response was important because “it forces us to actually be confident in what we think.” She explained:

I guess a lot of the students expect the teacher just to [give the answer] because, like I said, that's how most math classes are. [Teachers are like], "No, this is wrong." Like, "This is how you do it, you didn't do it right." And [Dr. Fikes is] not like that at all. I don't think she's ever actually told me or really answered any problem that I've ever done in that class. And so, yeah, it's kind of frustrating sometimes. But I guess it's good because, I mean, it helps me understand it better.

Views on the Use of Technology for Learning and Teaching Mathematics

Janina appeared amenable to the use of technology for mathematics teaching and learning. In class, she periodically used her smartphone to search the Internet for definitions of various terminologies. She thought that learning mathematics with the aid of technology was acceptable as long as students "understand how to do it on their own first." She clarified, "But if they know how to do it—they can tell you how to do it—and then just use the calculator to get the answer." For Janina, technology was an acceptable means to "check our answer," but she was adamant that the learner, and especially those who are "teaching other people," should know "how to do [the mathematics] first."

Janina's own experiences using technology for mathematics learning were limited to the graphing calculator; she had no experience with applets or GSP prior to Basic Concepts 2. Janina did not appear apprehensive when using GSP during the clinical interviews and, at times, described the software's capabilities as "awesome." However, even with the capabilities of GSP, Janina thought that students would best be served by first exploring mathematics concepts by hand. For example, in regard to tessellations, she claimed that the "visual learner" would benefit more from first exploring the concept with pattern blocks. She elaborated: "So, I like the whole, like getting out the [pattern blocks]. Like getting them out, and playing around, and letting them figure it out on their own, and playing with it so that they would visually see that." Further, Janina thought that older learners would benefit more from learning with technology than the younger elementary-school youth that she intended to teach.

Sources of Mathematics Learning and Understanding

Janina indicated that in previous courses, the instructor had been her sole source of mathematics learning. However, in Basic Concepts 2, she generally relied on her class notes. And, as mentioned earlier, she periodically used the Internet in attempting to understand concepts or to retrieve definitions of various terminologies. I also observed Janina seeking out help or confirmation of her answers from her peers. Specifically, she often sought the help of Jason (sometimes from across the room), whom she identified as “a math whiz.”

Although Janina often sought the help of various sources for her mathematics explorations, I never saw her blindly accept the mathematics results of others without first attempting to make sense of the explanations offered—especially if those results differed from her own. Given the requirements of the Basic Concepts 2 course, explanation was the primary means that Janina used to discern the correct mathematical conclusion when different results arose. Janina described herself as “stubborn” with the need to make “sense” of another’s explanation in order to accept his or her mathematical conclusions—even if the other person was deemed a math whiz.

In regard to other mathematics sources, Janina indicated that she had never encountered an occasion when a course text (specifically the solutions provided in the back of the text) was wrong. And in regard to technology (specifically the calculator), Janina said that any issue she had with its use was a direct result of an error she had made—such as “not plugging in what I really needed to be plugging into the calculator.”

Mathematical Authority in Different Contexts

In the Absence of the Instructor (Mathematics Task 1). Earlier, I presented data for Garrett that included a conversation of how he, along with Skyler and Janina, attempted to flesh

out an understanding of semiregular tessellation. This small group discussion began to illuminate how the three participants sought and used mathematics sources to form their understanding.

Initially, Janina used the course text to form an understanding of tessellation. She then called upon her knowledge of the meaning of *semi* to form an understanding of semiregular tessellation. However, her understanding of semiregular tessellation did not include the definition provided in the course text. Although Garrett had read the text definition for semiregular tessellation with Janina on two different occasions in an attempt to answer the question in Figure 5, it appeared that Skyler's support of Janina's reasoning was sufficient for her to conceptualize semiregular tessellation as whether or not a pattern possesses completeness.

Clinical interview results. Like Garrett and Skyler, Janina correctly recalled most of the events as they unfolded in class during the discussion of the review problem. She did not think that the group had come to a consensus on the attributes for determining a semiregular tessellation but did correctly remember her group members' collective reason for choosing 4b in Figure 5 as the answer to the problem. Further, Janina did not indicate that she had sought help outside of class for the problem.

Although Janina did not recall previous experiences learning tessellations, she had formed some conceptual understanding of the topic. She explained, "I guess if you have a tiled floor, and you're trying to do a tessellation, how do you fit these different shapes inside this tile floor and there would be no white space?" Further, Janina recalled that she used the course text to read the definition of *tessellation*. She elaborated, "And so I had to go back and refresh my memory on it. And I, I mean, I kind of taught myself, retaught myself how to do it. I do not remember ever doing [tessellations]." Concerning how she conceptualized a semiregular tessellation, Janina continued to use her reasoning that "*semi* means, like, halfway." She also

indicated that she recalled using the book to read that only regular polygons “up to a certain number of sides” will tessellate.

After discussing her memory and understanding of tessellations and semiregular tessellations, Janina used GSP’s premade scripts to explore which of the regular polygons singularly tessellates the plane. Like Garrett and Skyler, when her investigation turned to the octagon, she claimed that the octagon does not singularly tessellate the plane because a square would fit into the gaps. To prove her claim, Janina, used scratch paper and calculated the measure of an interior angle of a regular octagon with the formula, $\frac{180(n-2)}{n}$, where $n = 8$. This allowed her to find the measure of an exterior angle of the octagon. She then doubled that angle’s measure to conclude that the measure of an angle of the quadrilateral in question is 90 degrees. Like Skyler, Janina argued that the sides must be equal in length because they are formed from sides of congruent and regular octagons.

Janina’s investigation of whether the regular octagon singularly tessellates the plane led to a natural discussion of the definition of semiregular tessellation. I provided clarification of the definition, and Janina appeared to understand the conditions of what constitutes a semiregular tessellation. She was able to correctly explain why 4b in Figure 5 was not a semiregular tessellation in contrast to how it was conceived during the small group investigation. However, Janina thought that it would be difficult to identify whether 4a in Figure 5 was a semiregular tessellation without expanding the area of the figure to continue the pattern. Thus, she explored the pattern with GSP’s premade scripts for dodecagon, hexagon, and square to draw her conclusion. Her investigation appeared to solidify her understanding of the requisite features of a semiregular tessellation. When I asked her for her thoughts on the investigation, Janina said, “I

didn't know about what you said about the vertices [having the same configuration of shapes at each one].”

Discussion. Janina did not seek out the definition of semiregular tessellation but, instead, tried to use her knowledge of the English language and her limited knowledge of how many regular polygons can singularly tessellate the plane to make sense of the reason to choose 4b in Figure 5 as the answer. It seemed that Skyler's contribution of the need for completeness further solidified Janina's reasoning that *semi* means “half” when considering tessellations.

I found it interesting that Janina claimed ignorance about the requirement of the pattern of the shapes around each vertex for a semiregular tessellation. In fact, it was rather surprising given that she was coupled with Garrett, who, on two occasions, tried to make sense of the definition of semiregular tessellation presented in the course text as he worked with her.

The data on Janina suggest that she did not systematically seek out typical sources of mathematics (e.g., the text definition) to attempt to understand the material. For example, Janina did use the course text to try to understand regular tessellations and, specifically, which regular polygons tessellate a plane. However, she did not realize how to use that information or whether such knowledge would even aid in her efforts to solve the problem at hand.

The Triangle Dissection Paradox (Mathematics Task 2). Janina was coupled with Jason for the relevant class discussion in regard to the second clinical interview. As the pair worked on the first problem in the handout in Appendix F, they indicated to Dr. Fikes that they were confused about the directions:

Dr. Fikes: Yeah, how would you justify that [base times height formula] to me? Why does that work?

Janina: Because that's how, when it's a parallelogram, that's how you find the total area, or the total surface, like, that it's covering.

- Dr. Fikes: Ok. I'm not convinced.
- Jason: Well, a parallelogram has equal sides. So you can always turn it into a rectangle. Then you can just count the squares.
- Dr. Fikes: Can you always turn it into a rectangle?
- Jason: I mean—.
- Dr. Fikes: You sure?
- Jason: No. I mean you can make that kind of—.
- Janina: You can kind of make a rectangle, like—. (laughs)
- Jason: It just is, I mean—.
- Janina: It's just something you have to know. It's just a rule.
- Dr. Fikes: Can you take a couple of parallelograms and show me that it turns into a rectangle? I might be convinced if you can show me that it works.
- Janina: Like this one. We can move this, move this around, bring it over here. Oh, that's not right (laughs).
- Dr. Fikes: I'm not seeing it. . . . You just told me that a parallelogram can always be made into a rectangle.
- Janina: He said that (points to Jason), not me (laughs).
- Jason: You're dogging me out!
- Dr. Fikes: Well, do you believe him?
- Janina: No.
- Jason: You're gonna dog me out?
- Dr. Fikes: No? Ok. So your buddy here doesn't believe you. You're going to have to come up with some sort of justification.
- Jason: Can I Google it please?
- Janina: (laughs) I've Googled so many things in this class.

Jason: I mean it's just, I don't know—.

Janina: It's just a rule, like you learn—.

Jason: Yeah, I learned from kindergarten that if I—.

Janina: No one explained why. They're just, "Here's the rule. This is all you need to know."

The above discussion provides evidence that, although Janina was not blindly handing the mathematical authority over to her partner—a person she claimed to be “a math whiz”—she appeared to believe that there were some concepts in mathematics whose understanding was beyond her as a learner. However, she could have been questioning his mathematics knowledge in the presence of Dr. Fikes, who she might have believed to be a higher mathematical authority than Jason. Her reluctance to believe him may have been due to the fact that he indicated a lack of confidence in his method when questioned by Dr. Fikes. Later in the class session, Dr. Fikes followed up with Janina and Jason regarding the task:

Janina: I take back what I said.

Jason: I was right.

Janina: Jason was right.

Dr. Fikes: You're convinced?

Janina: Convinced. Hmm hmm.

Jason: It just took someone else to explain it to her.

Janina: No, [Kayla and her partner] actually just explained it. Jason just didn't explain it.

Clinical interview results. After working on the Triangle Paradox task for only 15 seconds, Janina used the shapes in Triangle A in Figure 6 to form Triangle B; she immediately acknowledged the hole in the new arrangement. In a direct contradiction to her mathematical

knowledge, Janina transformed her mathematical understanding in an attempt to explain the anomaly. However, if the following episode had not taken place, there would have been relatively little evidence (other than a short observation of her discussion of Crazy Cakes during one class session) of her mathematical understanding of the paradox.

- Janina: Oh well, missed one though, is that ok? Is there allowed to be a little white space?
- Me: So that's the question. Do you think, I mean, can you tell me—, do you think there should be a white space there?
- Janina: No.
- Me: Why's that?
- Janina: 'Cause I don't think—, it's the same shape. Hold on, I can tell you (quietly counts the blocks). Well, I mean, I guess they have the same base. It's thirteen units. So, I guess it, I mean that's just not a complete triangle though.
- Me: So, if you were comparing this triangle (points to Triangle A in Figure 6) to this one (points to Triangle B in Figure 6), their bases are the same. Are their heights the same?
- Janina: Hmm hmm. Yeah, five.
- Me: Ok, so you would expect their areas to be the same.
- Janina: Right.
- Me: But yet, we're missing a hole.
- Janina: A piece, right.
- Me: And you would not, you would not expect that?
- Janina: No.

When I asked Janina to consider why the hole existed, her explanation homed in on the idea that the reason was due to the placement of the four pieces. She initially said, “I mean, it's just the way that the shapes are, like, laid out.” As justification for her reasoning, Janina referred

to a recent class discussion on a task that had involved an exploration of volume (Figure 26). Janina and her partner had arrived at different answers on their work for this problem. She concluded that it would take 108 cubes to fill the prism, while her partner, deciding that the cubes could not be decomposed, found that only 96 cubes would fill the prism. As she presented the conflict of answers during a whole group discussion, it was quite clear that she understood the reason for the different results and concluded that her partner's answer was correct. Although Janina acknowledged that the task at hand for the clinical interview involved area (as opposed to volume), her recollection of the class discussion seemed pivotal for her justification of why the hole existed. She explained, "Well, yeah, because I mean we see two different, I mean, orientations of the shapes and so I mean here, they're obviously two different areas by just the way the shapes are placed."

5. How many plastic 2-unit-by-2-unit-by-2-unit cubes can be stacked neatly in an 8-unit-by-9-unit-by-12-unit box? Explain.

Figure 26. Problem investigation Janina referred to in order to explain the paradox.

Later, Janina tentatively agreed that the area of the two regions should be the same when taking into account that the triangles were congruent. In an attempt to confirm this claim, she counted the squares of each of the four pieces of both triangles. However, throughout the remainder of the discussion, Janina maintained her argument that the two areas would be different based on the placement of the four shapes in the two triangles. Other than counting the squares, Janina appeared content with her mathematical conclusion and did not make any attempt to understand the paradox. Therefore, I tried to provide a counterargument using her experiences with Crazy Cakes. Janina recalled her work on Crazy Cakes and indicated that the area of the figure would remain the same once she had decomposed the figure and moved pieces of it around. The following episode ensued after her recollection (which took place roughly 23

minutes into the clinical interview) and concluded with her asking me if I knew the answer and if I planned to tell her what it was.

Me: But, I want to make sure I heard you correctly when you said [that] when you moved the shapes around (pointing to the computer screen), you could change the area, right?

Janina: Yes.

Me: So if you move these shapes (pointing to Crazy Cakes), around you don't change the area?

Janina: No.

Me: And that's ok?

Janina: (laughs) Ok for, like—.

Me: For those purposes for this problem? Or—? I mean—.

Janina: Yeah.

Me: Do you see what I'm saying here? How you are telling me that—.

Janina: No, I do, I do. I feel like you're tricking me (laughs).

Janina never appeared fully confident of her mathematics discussion during the clinical interview. Unlike her usual lighthearted and spirited demeanor, she spoke in a low tone of voice and somewhat sheepishly as she consistently maintained that the hole was a result of the placement of the shapes. And although it seemed that the hole was initially problematic for her, she arrived at a conclusion that appeared sufficient for her, even as it contradicted what she knew mathematically.

After I explained the paradox to Janina, I asked her if she thought that the task would have been the same had she done it on paper only. She indicated that she thought the results of the task would have been the same but her ability to understand why would have been different because "it's way faster on the computer." She explained:

Well, I mean, [doing the task on paper] could have been really frustrating. I would have never looked at [the hypotenuse], and it would have probably taken me hours and hours and hours to examine this to figure out why it's like that.

Discussion. The collective results of the second clinical interview began to reveal the fragility of Janina's mathematics knowledge in the face of a mathematical anomaly. It was clear that the hole was problematic for her. She used only one strategy, however, in attempting to ascertain the reason for its existence and felt compelled to explain its existence in direct contrast to what she knew was true through her work on Crazy Cakes. Further, the explanation she used for the existence of the hole seemed based on her recent class investigation of Figure 26. Her use of this reasoning in the context of volume provides more evidence of Janina's piecemeal method to make sense of the mathematics at hand.

Janina's demeanor changed to one of a sheepish nature throughout her investigation, suggesting that she was experiencing a feeling of conflict because of the hole. It is difficult to say whether her general mathematics beliefs, backgrounds, and experiences necessitated an explanation for the hole's existence in the face of mathematics conflict—especially when presented by technology. Her behavior during the interview was in contrast to her behavior observed in class. While she never presented herself as confident in the mathematics to the level of Garrett or Skyler, she was not observed to blindly accept the mathematics of others.

The Lie of GSP (Mathematics Task 3). As explained in Chapter 3, data were not obtained on whether Janina understood and used the concept of mathematical shearing as she worked on the problems of Appendix F with her small group partner, Jason, during the relevant class session. However, Dr. Fikes had explicitly addressed the required conjectures of the assignment and mathematical shearing was the underlying concept in a number of problem

contexts. Thus, I deemed it plausible that Janina had developed some understanding of the concept.

Clinical interview results. To solve the task in Figure 9, Janina first counted unit squares to find a triangle with an area of 12 square units and determine a location of the third vertex. The triangle she found is bolded in her work shown in Figure 27. She then attempted to use the Pythagorean theorem to help find the location of other possible vertices. However, when her efforts were unsuccessful, she relied on the formula for the area of a triangle to confirm that the location of the single vertex she had initially found yielded the correct area.

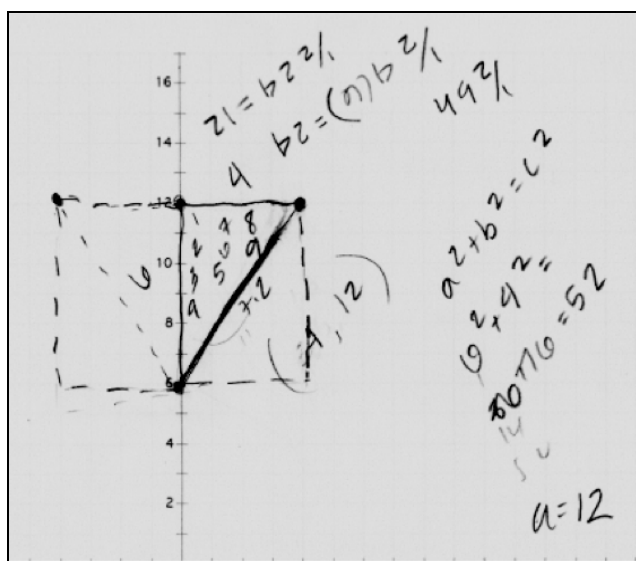


Figure 27. Janina's investigation on paper.

Next, Janina identified the mirror image of the single vertex reflected across the y -axis to form a second triangle. She said that those two triangles were the only ones that used the coordinates mandated by the task and that had an area of 12 square units. The explanation she provided for her reasoning was, "I just know." When I asked if she believed there were other triangles that met the conditions of the task, she outlined with her pencil two additional right triangles in her drawing. She said that although the two new triangles both had an area of 12 square units, they did not use the coordinates mandated by the task. Because of this exchange, I

decided to have Janina create a GSP sketch to dynamically demonstrate the task to help her conceptualize mathematical shearing.

After Janina created a GSP sketch that included the use of the *Measure Area* feature to measure the area of the dynamic triangle, I asked her to move the third vertex around and “convince me that those two [triangles] are the only two triangles that have 12 for their area.” She laughed as she replied, “Ok, I don’t even know what I’m thinking, but there’s definitely more.” Janina went on to find two more triangles to conclude that a total of four triangles met the mandated conditions. After I asked her a second time to move the vertex, she then found 12 possible locations at which the third vertex could be placed. When I asked Janina if she thought 12 was the final number, she responded, “Now I feel like I don’t even know if I should say yes now, since I started with two, and now I’m with twelve, so— (laughs).”

I probed further, as Janina interacted with GSP, to ascertain whether or not she understood the concept of mathematical shearing. She said, “So, if we look at it, we can make [the triangle] smaller, and, I mean, you can still have 12. It’s just going to be a skinnier triangle.” To further assess her understanding, I probed to find out if Janina could form a mathematical basis for what the measurement feature of GSP presented to her. She observed,

Well, one of our bases is four. Not base—yeah, I guess base. . . . And if you had to find the area of twelve, it’s one half base times height. So, the other base is going to have to be four. . . . You have the height already. And that has to stay the same.

Janina indicated that, although she thought there were a lot more possible locations for the third vertex, the total would be a finite quantity. She elaborated:

Just because, I mean, I guess it’s going to be a large number, I guess. But we still have to stay within a certain rectangle or certain—. Yeah, a certain rectangle that all these triangles are going to fit into. Like, in our heads, it’s going to be really big, but I don’t think we can keep letting this go on forever. I could be wrong, but I don’t think so.

Janina used the mouse to pull the third vertex down to see if she was correct in her thinking and noted, “I mean, I feel like that would eventually (pauses), it wouldn’t really have anywhere else to go.” I asked Janina whether her conclusion was because she did not think the figure would be a triangle, she confidently responded, “It will always be a triangle because there’s three vertices there.” In that moment, it seemed the mathematics issue at hand for Janina was whether or not the area of the triangle would be maintained at 12 square units. However, as she dragged the third vertex down the screen and after a considerable amount of time had passed, the following conversation ensued:

Me: Why do you think the area would not be preserved?

Janina: (Long pause.) You’re going to, like, run out of triangle to stretch, I guess, in a way. Does that make sense?

Me: So you’re saying that it will be so thin that it won’t be a triangle?

Janina: Yeah.

Me: Even though you have three vertices?

Janina: Well, I mean it would just be, like I mean, we’re almost (laughs)—I feel like I can just keep going and going. (Continues dragging the third vertex.) I mean I guess we could, I feel like there’s only, like, so much space, like, I feel like this is almost going to be a segment, like—.

Me: Ok. So the condition of the three vertices would go out the window when it starts to—.

Janina: Right.

Me: Get to some point.

Janina: At some point, yeah. . . . I just don’t know how long it can go on (laughs).

Me: So your conjecture is that it’s still—you think there’s a finite number?

Janina: I guess not. . . . I just feel like this is going to come together eventually. It’s just, like, not going to work.

Me: But as you go down you see that it does work?

Janina: Yeah, it's still working.

To gain further access to her mathematical understanding, I asked Janina if she thought there was a mathematical explanation for what was presented by the technology. She returned to her original claim that the height and the base were fixed and indicated that the locations of the different vertices were all on the “vertical line” with an x -value of four. Because of this observation, Janina concluded that there were an infinite number of vertices that met the conditions of the task. After making her conclusion, however, she asked whether she was correct in that conclusion. I confirmed her answer and decided it was time to present her with the GSP sketch I had originally created to present a conflict of mathematical results.

Upon her inspection of the premade GSP sketch, Janina was immediately drawn to the fact that the area displayed in the new sketch was different from 12 when the third vertex was along the vertical line, $x = 4$. Janina was quite reserved in her quest to find a plausible reason for the difference in results presented by the GSP sketch and did not actively pursue different strategies. She said:

I don't know, because I feel like the area, the formula that we used to find the area of a triangle wouldn't work for this. I mean if you have like six and five, it's going to give us an area of like fifteen.

Because she was “confused,” Janina discussed the first thing she would do to find an explanation for the discrepancy: “Ask my teacher.” Further, the only explanation Janina considered was that “the units aren't as big.” And, after only 5 minutes of simply moving around the third vertex, Janina inquired, “What's wrong with it? . . . Are you going to show me?” I asked Janina if perhaps the computer was wrong. She hesitantly responded, “Maybe something

like (paused) not like set right. . . . I mean, it is a computer so it can like (laughs), I mean, it's the information that we put into it."

The most remarkable aspect of Janina's interview was her failure to pursue a feasible reason for the incorrect measurement display. However, she did not appear to lack confidence in her initial findings—findings that were confirmed by her investigation with GSP as well as me. Although she had two mathematics sources to help boost that confidence, her final reason for not necessarily acquiescing to the results of the premade GSP sketch rested upon the fact that "the formula doesn't work." Although it was clear that she was no longer interested in investigating the discrepancy between the results, Janina indicated in the wrap-up conversation of the third interview that she would have used methods other than her initially chosen ones to investigate the differences. She elaborated:

I wouldn't go back to the whole infinite thing, because—, like, how I got the infinite thing was using [counting] the squares and the formula. And here [counting] the squares and the formula didn't even work. So I couldn't use that to justify why it was like that.

Discussion. Similar to Skyler and Garrett, the results of Janina's third clinical interview were not necessarily unexpected when considered against the other data gathered for the study. Because I had limited data from the class sessions on her understanding of the concept of mathematical shearing, I did not know whether she would have sufficient understanding of the concept to be able to successfully complete the task on paper only. However, similar to Garrett's paper investigation of the third mathematics task, Janina also provided an initial explanation of a fixed number of positions where the third vertex could lie. And, similar to Garrett, her reasoning may also be explained by the consideration of only lattice points as possible locations for that third vertex.

Allocation of Mathematical Authority

Not necessarily possessing the level of mathematics confidence shown by Skyler during the class sessions or by Garrett during his interviews, Janina was periodically observed standing her mathematical ground against conflicting explanations offered by her peers. At times she appeared confident in her mathematical solutions but not nearly as confident in her mathematical explanations—especially when providing them to her peers during whole group discussions. Janina was frustrated with Dr. Fikes’s questioning technique and failure to provide confirmation of her mathematical correctness. To effectively handle the lack of mathematical confirmation from Dr. Fikes, Janina appeared to utilize peer-provided explanations that “make sense” as well as to morph her existing mathematical understandings extracted from various experiences to accommodate conceptual discrepancies that she encountered.

It is difficult to predict how Janina would have behaved in the third clinical interview had it unfolded as I initially planned. She was clearly alarmed by the unexpected results during both the second and third clinical interviews, and her discussions and actions during those two interviews were quite sheepish (vastly different from her general happy demeanor) when working with the technology. Although she provided an explanation for the mathematical anomaly in the second clinical interview, she clearly was not comfortable with it. Further, she was uncomfortable with what was presented by the premade GSP sketch during the third clinical interview. However, she did not display any desire to figure out the underlying reason for the conflicts, as Garrett or Kayla had during their interviews. Regardless, Janina’s reactions during those two interviews suggest that she viewed technology as not only a source of mathematics but a mathematical authority as well. She was willing to provide explanations for the technology’s results when they were contrary to her own mathematics understanding.

Like Skyler, the data collected in regard to Janina point to an automatistic work method (Figure 3) although she did not appear to have a stable source of mathematical authority given the pedagogy and design of Basic Concepts 2. In the course context, at times Janina sought answers or clarification from peers or would use her own knowledge in a piecemeal fashion to form a mathematical understanding of the concept at hand. Arguably (as evidenced by Janina's frustration with the absence of direct confirmation from Dr. Fikes), had Dr. Fikes made herself available as a mathematics source during the class sessions, Janina would have seen Dr. Fikes as the absolute mathematical authority over her peers' explanations or her own mathematics knowledge.

Kayla

Attitude Toward Mathematics

Kayla harbored a particularly negative attitude toward mathematics, describing it as not her “strong point.” Her attitude stemmed from her former learning experiences—experiences that included courses as recent as Basic Concepts 1, the only other mathematics course she had taken at the university just one year earlier. She recalled her earliest experiences in elementary school with teachers who expressed sentiments such as, “It’s this way or no way.” She elaborated:

And it didn’t matter if I came up with an answer differently. They wouldn’t count it if it wasn’t their way. But my way was just as right. . . . I mean, you had to go by what they were saying. But if you didn’t understand [their way] as well, then you were more likely to do something wrong with that part. So I don’t think that was fair.

Her later learning experiences did not provide much improvement in regard to her feelings about the subject. Although Kayla consistently earned A’s and B’s in her high school mathematics classes, she described those experiences as “awful” and said she had lacked “good math teachers.” She explained, “Honestly, I don’t remember what I did in high school. I took, I mean,

math every year—twice some years. I mean I took up to calculus. But I really don't remember it." A major reason for Kayla's frustrations was the nature of her mathematics classes, which had given her limited opportunity to "play with the math." She explained:

Because in high school, I hated math. . . . It was harder for me than anything else. . . . I didn't always understand it, so I'd get frustrated. And there was other stuff I'd rather be doing. I mean, playing sports and stuff like that. We would have 30 problems a night. So it was an overload. And, I mean, I understand it's important and stuff, but you'd get through the first 10, and the other 20 were just like them. . . . It was too time-consuming.

Confidence in Doing, Learning, and Teaching Mathematics

Kayla periodically indicated lacking confidence in her mathematical ability and, in particular, felt she lacked a natural ability to do well in mathematics. She explained:

It's just harder for me to be good at it. Some people can make A's on the test and not have to do their homework. I have to do my homework in order to make a B on the test—and I study, and that kind of thing—which some people don't have to do.

However, Kalya was confident in her ability to teach mathematics to elementary school students. She indicated that she "liked elementary math" and thought she would have a variety of mathematics resources available to her once she started teaching. She explained: "And [for the mathematics that] I don't understand, I mean, in a school you have plenty of other people you can ask, . . . [and] a lot of good books and stuff are out there." Kayla thought that the necessity for her to "work harder at math than at anything else" would ensure that she would "be fine with [teaching it]."

Although she described herself as someone who has "always been kind of shy," Kayla regularly contributed to both the whole and small group discussions by asking thoughtful questions about the material as well as answering questions posed by Dr. Fikes or her peers. She observed that she had "learned to talk to people a lot more" in various environments, such as

when employed as a waitress. Further, in class, Kayla never appeared as though she lacked confidence with the material.

Kayla possessed a strong desire to understand the material. She noted how important it was “to ask questions that other kids would ask, like if they’re in elementary” and clarified that her questions might help her classmates to “see how someone else would think of [the concepts]” as well as help her “think about and play with [the concepts].” She elaborated:

Like sometimes, well, I just like to argue a point that’s not necessarily, maybe the most—maybe that’s not what everybody else is saying. Because I know Dr. Fikes doesn’t always say whether you’re right or wrong. Um, so I just like to think about other ways, um, maybe the same way, but just a different way of thinking.

Although Kayla indicated having a lack of overall confidence in her mathematical ability, I periodically observed her as having a rather intuitive mathematical sense during the class sessions and discussions. For example, in one mathematics investigation of justifying the angle measures of various figures without the use of standard measuring tools (e.g., a protractor), Kayla instinctively wrote a mathematical equation in an attempt to represent her mathematical discussion—although she eventually abandoned the representation in favor of other more concrete methods. This behavior was not solicited from her peers or Dr. Fikes; instead, it seemed to come to her naturally as she sought a way to facilitate her mathematical explanation and justification.

Major Themes That Arose From the Interviews

There were two major themes that arose out of the interviews with Kayla. First, given Kayla’s cumulative years of negative experiences learning mathematics, she appeared to possess a view of learning that was in direct contrast to those experiences. Second, Kayla was particularly affected by the pedagogy of the learning environment she experienced in Basic

Concepts 2. Her experiences in the course seemed to align with her views of how learning mathematics should be manifested in an academic setting.

As discussed earlier, Kayla held a particularly negative view of mathematics, attributed to her prior learning experiences. Those experiences, as well as how she thought elementary students should learn mathematics, appeared to form the foundation for her views of teaching. Kayla believed that every class consists of “students that think about math differently” and that teachers need to be “open-minded” and to “look for multiple ways to solve [a] problem.” She stressed, “Kids don’t know that formula. So you have to think of it differently, you have to be like, ‘Well, if I don’t use the formula, how do I get the answer?’” She described students as “brilliant” versus adults who tend to “complicate things when we think about math.” Kayla believed that students should be given a voice in their learning and that teachers should “let the students interact” and “listen” to them as they solicit “their [students’] feedback and what they think about certain problems.” Kayla thought students deserved “to think freely”; otherwise, “they don’t really know the math—they just memorize it.” Further, she emphasized the importance of understanding the development of concepts: “If you don’t understand where it comes from, then you don’t really understand the math.”

Throughout her interviews, Kayla frequently said that memorizing or practicing problems in any subject, and specifically in mathematics, should not be considered the means to learning the content; instead, the emphasis should be placed on understanding. She explained, “In fact, I know you’re better [at mathematics] if you don’t memorize it. Because then you understand it, and you don’t just know it for that problem—you know it for all problems that are like that problem.” Kayla further clarified:

I mean, because in math—well, in every subject—it all relates somehow. And if you find how it can relate, then you understand the relationships and how things

work better. Like, memorization is really important with multiplication tables. But you need to understand the multiplication and how it works before you memorize stuff. So, like, basically math should be taught in a way that you're taught how to understand stuff and how it relates, and relationships, and how to think about that problem.

Further, as Kayla stressed the importance of understanding, she observed that teachers who “explain everything” are really “just telling the students, ‘It is the way it is.’” Thus, students are stripped of the ability to think for themselves about the mathematics. She explained:

No child, or person, can think while the teacher is telling you how to do it. . . . [Students] will understand it a lot better if they can get the answer [for themselves]—if you let them play with the problem.

Kayla substantiated her ideas by discussing her experiences in an internship in an elementary school where she had observed the students working on “simple word problems” but having “no clue what to do with them.” She explained that the students “didn’t know how to apply what, like, multiplication or addition or subtraction or division to a certain problem.”

Impressions of Basic Concepts 2

The pedagogy used in Basic Concepts 2 and her resulting experiences in the course seemed to positively affect Kayla’s views of mathematics. She thought, “Most math classes should be taught like [it is in Basic Concepts 2].” On a scale of 1 to 10 (with 10 being the highest score), Kayla would have rated her affinity for mathematics as either a 3 or 4 based on her mathematics experiences in previous courses. However, after her experiences in Basic Concepts 2, Kayla rated her affinity toward mathematics as either a 7 or 8, explaining that it would just “depend on the day.”

Kayla indicated that she enjoyed the subject “a lot better now than I actually ever have” because she “[understood] it better.” She explained, “I just feel like I actually enjoy this class so I’ll work harder on it.” Although she had not yet found mathematics to be a “passion” or her

“love,” she thought her experiences in Basic Concepts 2 helped her in her pursuit to “be a good teacher.” Because of Dr. Fikes’s pedagogy, Kayla found that she now liked “to think about [mathematics]” and was now “willing to put the work in to try and figure out the best way to teach students.” For Kayla, having this perspective for her future students was “completely different than doing math.”

Kayla provided a range of examples of how she thought Basic Concepts 2 affected her beliefs about the subject of mathematics and doing mathematics. She described the “need to talk” to her classmates to get a sense of how certain topics worked and explained: “When you’re doing math, it’s as much about how you think about it, um, and how you go about problem solving as it actually is doing the problem.” She explained:

So, it’s been pretty neat because I could think about something one way, and someone else could think about something else a different way. And then you have what you believe, and then they affect it. So, they change it, and you end up with a—, maybe a more right answer than what you had to start with.

Kayla recalled her former experiences in mathematics, even as recently as in Basic Concepts 1, as not providing students with the opportunity to “play with the math.” And, as a “very hands-on” learner, Kayla believed she needed to “see it and do it myself” to be able to “remember it.” She explained that she “got through” her previous mathematics courses by “memorizing stuff” that she “kind of understood—but not really.” Further, Kayla explained that those former instructors only explained the mathematical reasoning as: “Well, it is the way it is because math said so.” To illustrate the contrasting experiences, Kayla provided an example of one experience in Basic Concepts 2 that left her feeling “much better and confident”:

Well, [Dr. Fikes] doesn’t just tell me why—why it is the way it is. She asks, I think, the whole class. They come up with why they think it is. Then we talk about it. Um, the most, the best, example I can give you is when we were classifying shapes. And we had rhombi and parallelograms and squares and rectangles, I mean all those—[and] kites. Um, we had to come up with our own

conclusions about what they were. And that stuck with me. ‘Cause it wasn’t something that I just had to memorize. It was something I understood after we talked about it.

And although Kayla described Dr. Fikes’s questioning technique as initially “frustrating” and “irritating,” she believed that it was the sole reason for her newfound views about mathematics. She elaborated, “But I, I realized that it’s helping me think about [the mathematics]. And I’ve come to the conclusion about math that I have now because of that.” Additionally, Kayla found Dr. Fikes’s pedagogical techniques appealing in ways that the pedagogies of her former course instructors were not. For example, the assignments were “not an overload” of “30 questions a night.” Kayla described the assignments as generally consisting of “three problems” that students had “to understand.” Further, Kayla thought the mathematics content as “applicable to what I’m going to be doing.” She explained, “As long as I can see how it relates instead of just being told, ‘Oh, you’re just going to need to know this.’ Then it works a lot better.” Thus, Kayla found that her old mathematics behaviors were no longer necessary for Basic Concepts 2. She explained:

Um, I don’t memorize anything in this class. Like I said, I have a new approach to math, and I’m trying to understand it instead of just memorizing it, because I’m going to have to turn around and teach it. And if I memorize it, then I can’t explain it.

Kayla’s views of Dr. Fikes’s assignments seemed to align with her general academic demeanor in regard to her drive and determination for her academic pursuits. Although she wanted to do well in her studies, Kayla admitted, “I like to say that I don’t do the least amount of work, but I don’t do the most amount of work either.” She clarified that she sought “a happy medium” in order to “try and balance things” and that she tried “not to study too much because you over-think things.” Through Kayla’s studies, she found that if she spent more than “10 minutes staring at the problem, then I’m not going to get the answer.” Thus, for her academic

endeavors, she generally followed the rule to “not to spend more than 3 hours doing homework” because, she believed, “it shouldn’t take you longer than that [to complete it].” She explained, “At a certain point it doesn’t matter how hard you look at [a problem], you’re not going to figure it out.”

Views on the Use of Technology for Learning and Teaching Mathematics

Kayla’s experiences with using technology for learning mathematics were limited to the graphing calculator, although other types of technology had been available to her during her previous schooling. She said, “How we interact with the computers has changed significantly since I was, like, in school. . . . We didn’t use computers in math when I was growing up. . . . [We used them for], like, word processing.” And although her experiences with learning with technology were fairly limited, she possessed a fairly neutral view of using technology for the learning and teaching of mathematics. During the clinical portion of the interviews, Kayla never seemed anxious about engaging with GSP. In fact, she generally interacted with the software with gusto and interest while investigating the program and called herself “a little kid” when exploring its various attributes.

Kayla thought it was important for students to do computations “like adding” in their head and that she, personally, did not “like carrying a calculator around” and so did “more stuff on paper now.” She claimed that, although technology offers the student the advantage of making computations “a lot quicker,” it does not “tell you anything that you couldn’t have figured out on paper.” She elaborated:

Because I know, for me, I do a lot better if I’m sitting there playing with the box or, um, just like the geoboards or just something like that. Um, because it’s hands-on. It keeps my attention because I can play with it, and I’m still learning something from it. . . . I like technology, and I think it’s a good tool. But I think sometimes it can be used too much. And I think you just have to be careful with that.

Sources of Mathematics Learning and Understanding

For Kayla, the sole source for learning mathematics was “definitely the teacher.” She said that, while in class, she often sought the help of Dr. Fikes—even though she thought Dr. Fikes would not provide a direct verbal indication of mathematical correctness. Kayla explained that she would “look at [Dr. Fikes’s] body language” in order to “judge whether I’m just like completely off-base.” Kayla indicated that, at times, she relied on the course text to aid her learning and understanding. However, she turned to the text only if it contained examples that were explained “step by step by step” without “skipping anything.” Although Kayla thought highly of the course text used in Basic Concepts 2 because it did not contain “a lot of text,” she also thought it lacked enough examples to make sense of the material.

I rarely observed Kayla seeking help from her peers. Her interactions in class were generally of a collaborative nature, with the goal of understanding the mathematics rather than seeking solutions from other students. When there was a discrepancy of mathematics results between her and her group members, however, Kayla stressed that analyzing the explanation offered was the single most important factor in reconciling the differences. Regardless of how Kayla viewed a person’s mathematical ability, she placed a greater emphasis on their explanation of their process of arriving at a particular solution. She thought “the process” of finding a mathematics solution was much more important than the actual answer in regard to her understanding. She said, “I mean, ‘cause the answer could be right or wrong. But, if you’re close in the process then, I mean, you understand it better, or you probably know what you’re doing.”

Kayla’s interview responses revealed other mathematics sources that would not have been detected during classroom observations. On several occasions, Kayla mentioned that she often sought the help of others when she found herself at a roadblock of mathematical

understanding. For those occasions, she either called upon a close in-town friend (who majored in mathematics) or her significant other (also a mathematics major). The discussion of Kayla's significant other took a prominent role in many of the mathematics discussions throughout the interviews. She explained:

[My significant other] is really good at math. And he's really good. He tutors people, too. So he's really good at explaining stuff. So I know if I go to him, even over the phone, he can explain it to me better than most of my friends here can He's—, he's helped me with math like through high school and now. . . . Usually, if nobody's around, then I'll just call [him] 'cause I know how to tell him over the phone (laughs).

When discussing how she handled a discrepancy of mathematics conclusions between herself and typical sources of mathematics (e.g., the course instructor or course text), Kayla's explanations further illuminated her general lack of confidence in her mathematical abilities—although her interactions with her peers did not betray such a characteristic. For example, when discussing how she reacted to answers provided in the back of the course text that were different from what she had found, she explained, “No, the back of the book is probably not wrong (laughs). It's probably me.” Kayla remarked that she often used the solutions presented in texts to “work backwards to figure out where I went wrong or how to do the problem.” She extended this same sentiment to results presented by technology, saying “No, the calculator is always right. I don't argue with that.” She explained, “But, a lot of the times, like the calculator will give you a different answer than maybe you were expecting or got in the book, and you left out a parenthesis. So, it's user error, not calculator error.” Kayla elaborated on her reasoning for her deference:

You ought to be able to look at a math problem and figure out if you're close to being right. And I'm not the best at that because I think I'm right. And then I'm just like, “Oh, well, this has to be right 'cause I don't know what else it could be.” So, I just assume that it is. But you should be able to look at the problem and

guess whether you're close or not. And then, um, you will, you'll have an idea of whether to try again or stay with what you have.

Mathematical Authority in Different Contexts

In the Absence of the Instructor (Mathematics Task 1). As discussed in Chapter 3, for the relevant class session for the first mathematics task, Kayla was sitting in close proximity to the other three study participants. However, Kayla did not interact with them as they tried to conceptualize tessellations and semiregular tessellations in an attempt to answer the question in Figure 5. The videos revealed that the group members at Kayla's table appeared to be in rather limited discussion with each other. Instead, they were diligently, and mainly independently, working on the review sheet—only periodically speaking with each other.

Clinical interview results. Kayla, the only participant who did not participate in the small group discussion about the review problem, had a brief first interview because she had less time available to her than the other three participants. Therefore, there was little time to devote to the clinical portion of the interview. However, even if time had been available, Kayla did not require much of it.

Kayla was very quick to engage with GSP and provided a concise, and correct, discussion of her reason for choosing 4a in Figure 5 as semiregular. She stated, "Um, yes, I said that that one (pointed to 4a) was a semiregular tessellation. Um, because the patterns continued and they were arranged in the same order." Kayla recalled that, initially, she did not remember the definition of a tessellation. She explained, "I looked up what a tessellation was, and then I tried to look at the figures and see what happened. . . . I haven't seen this stuff since geometry." Even though she had the correct understanding of why 4a was a semiregular tessellation, she still explored GSP through the concept of tessellating. And when it came time to explore whether or not the octagon singularly tessellates the plane, Kayla quickly explained, "It doesn't because [the

angles] don't add up to 360 degrees." Given that she was the only participant who appeared to have this mathematical understanding, I asked her how she knew that was the reason. She simply replied, "Because I studied for my test."

Discussion. Kayla, the participant whose interactions during the class sessions were the least captured of the four, held a solid understanding of why 4a in Figure 5 was a semiregular tessellation and did not need GSP to investigate the review problem. Further, Kayla understood the reason that one or more figures would be able to tessellate a plane. However, given the short amount of time I had with her, it was not clear how she studied the material to make sense of it, although there are multiple possible reasons for her mathematical understanding.

First, another question on the review sheet asked students to "Explain why it is impossible to have a tessellation made only from regular pentagons." Thus, Kayla may have extended her understanding of the mathematics for that question to the clinical interview investigation. Second, on a number of occasions, Kayla indicated that she sought the help of her significant other or a good friend of hers—both mathematics majors—when investigating problematic exercises. Therefore, their help may have formed the basis of her confidence for this problem as she sought to understand it. Additionally, she may have been able to get help from one of her partners in her small group. But, as noted earlier, the video data did not reveal that her group members were engaged in a conversation about any problem for any length of time. And third, Dr. Fikes confirmed that Kayla did seek her help outside class on the topic of tessellations. However, Dr. Fikes was unsure of when (before or after the in-class exam).

The Triangle Dissection Paradox (Mathematics Task 2). Although Kayla had a partner during the relevant class session for the second mathematics task, she primarily worked independently on the problems of Appendix F. She stopped working only to address questions

from Dr. Fikes. The following interactions between Kayla and Dr. Fikes during that class session illustrate Kayla's mathematical knowledge and understanding in regard to the mathematics of the second clinical interview task.

For the first problem, Kayla drew a variety of parallelograms on grid paper and computed their areas using two approaches. First, she decomposed each parallelogram (that was not already a rectangle) into two parts and rearranged the parts to form a rectangle. Then she used the formula, base times height, to find the area of the newly formed rectangles. Additionally, she counted the unit squares of the grid paper of the parallelograms and the rectangles to confirm her answer. Her detailed work is shown in Figure 28.

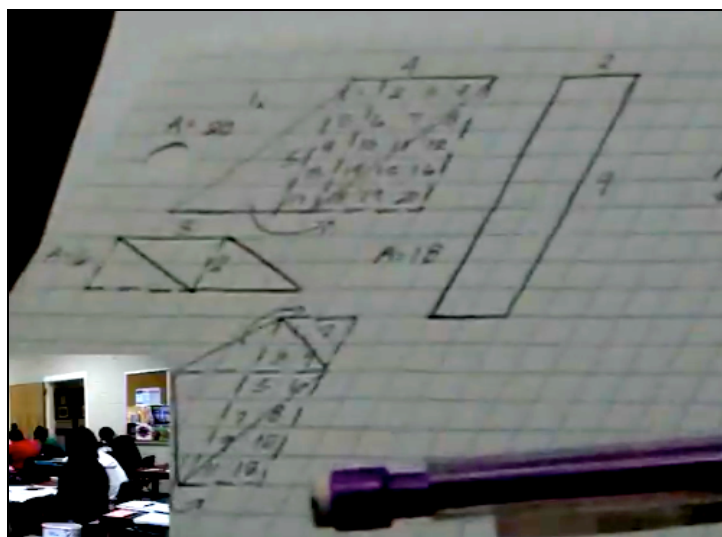


Figure 28. Kayla's work to show how to decompose and rearrange a parallelogram.

When Dr. Fikes questioned Kayla about this method, Kayla said, "You can shift them all into rectangles." She provided an example of how she would find the area of a different parallelogram. The following episode illustrates Kayla's confidence in her ability to transform any parallelogram into a rectangle. More importantly, Kayla used two justifications to show that she had correctly computed the area of the parallelogram.

Kayla: (Draws the parallelogram in Figure 29a.) So this is your square that goes straight down the middle. And that's gonna give two right-angle triangles.

So you can move this triangle over here and have three spaces over and one, two, three, four, four spaces up. And then all this in here (shading rectangular area shown in Figure 29b) is what you're going to count for your area.

Dr. Fikes: How do you know those triangles are the same size?

Kayla: Just because you—. The height and the length are the same. The hypotenuses are going to be the same.

Dr. Fikes: Ok. And you can do that with any parallelogram?

Kayla: Hmm hmm.



Figure 29a. Kayla's parallelogram.

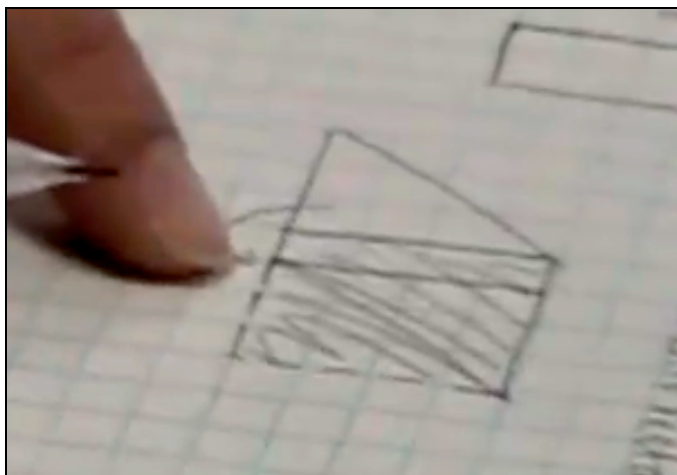


Figure 29b. Shaded area to count as the actual area of the parallelogram.

Clinical interview results. It did not take Kayla long (a little over a minute) to find the arrangement of the pieces of Triangle A in Figure 6 to form Triangle B and satisfy the requirements of the third mathematics task. Her efforts to understand the reason for the hole were equally short. Kayla did not appear alarmed that a hole existed in the new triangle. She casually explained the hole's existence as: "Just the way it's put together." When probed for more explanation, Kayla said: "I just figured when you asked me to put them together that it had to fit without any white space left over. That was just my first thought. But, I mean, it's there, and it's not doing anything wrong." This last statement represents the tone of the rest of the clinical

interview with Kayla—which mainly consisted of my probes to ascertain whether her mathematics knowledge could create any issue of conflict regarding the existence of the hole.

In an effort to create conflict, I asked Kayla to recall her memories of class explorations that involved the decomposition of figures and the rearranging of the parts of those figures.

Kayla understood the point of the question and explained:

Yeah, I know exactly what you're saying. It's a good question. . . . I mean I don't, I don't know why there's white space and that the area is the same with the white space. 'Cause you would think it wouldn't be. I thought it wouldn't be.

However, even with her acknowledgement that the hole's existence seemed contradictory to what she expected, Kayla seemed willing to just accept its existence. This acceptance was the most prominent feature of the clinical interview with Kayla. Like Skyler and Janina, Kayla again pointed to the placement of the four shapes as the reason for the hole. She elaborated:

I don't know, I mean, I know [their areas are not the same]. But, and I'm guessing, I think it's because you—, otherwise you can't combine and have the same space on the bottom. You have to have 13 across to match this figure (points to Triangle A in Figure 6). And no other combination besides this and that (points to the bottom two shapes of Triangle A in Figure 6) will give you the 13 across. So you have to have a space. . . . I just can't figure out why they don't match. Like why there's a white space. Like I know that there is, and that that's ok. But I don't know why it's there.

As I questioned Kayla as to how she would explore the reason for the hole using GSP, she asked, “Can you ask [GSP] for an explanation? . . . I'm sure there would be [something I could try] but I don't know a process to get to the answer.” Kayla seemed reluctant to use GSP for any exploration as she pointed out that any single feature of the program “isn't going to tell you anything that you can't figure out on the graph.” Perhaps because of my questioning, however, Kayla eventually used GSP to measure the total area of both triangles. Upon getting confirmation that the areas were off by one square unit, she remarked, “Well, that makes sense. It's telling you something that you already knew.” Kayla once again confirmed that she expected

the areas of the two triangles to be the same, but with the same level of nonchalance, she explained, “But they’re not. So, you know, because the formulas in math tell you they’re not.”

Once I realized that Kayla was no longer going to investigate the task, I explained the paradox to her. I asked if she thought the task would be the same if she had explored it on paper only. She thought the task would result in the same outcome and explained, “If you’re given the same shapes, then yeah. . . . Um, just because it’s on the computer, it’s not going to change anything.” However, Kayla indicated that doing the task on the computer offered some affordances that doing it on paper would not, such as the ability to quickly compute the area of the different shapes.

Kayla explained her apparent lack of initiative for pursuing the reason for the hole: “Because once you get to a certain point, it doesn’t matter how hard you look at it, you’re not going to figure it out. It doesn’t matter.” In a follow-up conversation during the third interview, Kayla indicated that she pursued mathematics problems “until I feel like I really am not going to get anywhere.” She elaborated:

So, just however long that takes. Like this triangle problem last time, I was done with it. I’m not going to lie. I was done. But, if I thought I was go—, if I had any chance of making any sort of headway, then, I’d been fine. I’d [have] kept going.

Kayla clarified that if the task to explain the hole had been given as a class assignment, she would have sought out the help of others and (as she had in the previous interviews) mentioned her significant other. She explained:

Well, I’d look at it a while, and then I’d think about it. And then I’d get irritated. And then I’d call [my significant other] and be like, “I don’t know.” And then he would be like, “This is why.” And then I would be like, “So what does this mean?” And then I’d figure it out.

Discussion. Kayla’s lack of interest in and nonchalant attitude toward the existence of the hole was a truly remarkable feature of her second clinical interview. Because she had expressed a

new outlook on mathematics with a focus on understanding, it seemed odd that she so readily accepted the hole without a pursuit of understanding even though its existence was contrary to what she expected.

Further, providing a nebulous reason for what was presented by GSP explained through “the formulas in math,” Kayla’s reasoning suggested that the existence of the hole was acceptable in the face of her mathematics knowledge. Perhaps with the knowledge that I was there to explain the hole or that she could rely on others to help explain its existence, she did not feel inclined to pursue the task any further. Alternatively, perhaps the mathematical anomaly did not pique her interest enough to justify further investigation of the hole.

The Lie of GSP (Mathematics Task 3). Kayla held a firm understanding of the concept of mathematical shearing as was evidenced by the data from the relevant class session that I presented for her second clinical interview. These data can be found in the conversation Kayla had with Dr. Fikes that displayed Kayla’s confidence in, and ability to, transform any parallelogram into a rectangle and maintain the parallelogram’s area.

Clinical interview results. Kayla was the only participant who, rather quickly and correctly, answered the task in Figure 9 as she claimed that the locations of the third vertex of the different triangles are “all at four.” Understanding that the distance between the y -coordinates provided the triangles with a base of 6, she justified her reasoning in a variety of different instances claiming that “as long as the height’s four, you’re going to have [an area of] twelve.” She elaborated, “Yeah, you can go out as far as you want to because you’re losing, as you go out, you’re losing area this way. So it makes up for going further out.” Based on her discussion and the fact that Kayla clearly understood the concept of mathematical shearing, I decided to move

on with the interview as I initially planned. I asked her to consider the GSP sketch that I had created with the intention of presenting conflicting results.

As soon as Kayla moved the third vertex to the vertical line at $x = 4$, she asked, “Is that the area? (She indicated the measurement shown on the screen.) Hmm. I was wrong. I hate it when that happens.” The following conversation ensued:

Me: So, what are you thinking about?

Kayla: I don’t know why. Because I think it’s the same. It’s just out further than I said it was.

Me: But when, what you said was, you know the height was four, and the base was six, so you used the area formula, and you counted. And you still confirmed that it was twelve.

Kayla: Well, it’s possible that I miscounted. It’s hard to count things that aren’t whole.

Me: But what about that formula? That worked for you, right?

Kayla: Yeah, that should have been right. I don’t know. It’s just different. I wasn’t expecting that.

Me: (Pauses.) Do you have any thoughts about why it might be different?

Kayla: Um (pauses), well, I thought originally that it could be because (pauses) it was an equilateral triangle. But then I moved it, and it wasn’t because of that.

Me: So you moved it to—?

Kayla: To a scalene triangle. Or a right—, I mean, it doesn’t matter. Right triangle, scalene triangle, isosceles, it wouldn’t matter (continuing to move around the third vertex).

Given her behavior in the previous interview session, I was surprised by Kayla’s determination to understand the reason for the difference between the result presented by GSP and what she expected. She continued to move the third vertex around the screen and remarked that “the computer should be right” and that the question was “probably another trick question”

that involved “bent lines” (in reference to the second mathematics task). Her confidence in her initial exploration on paper only was rapidly diminishing:

Me: But, um, had I not shown you this program, this sketch, you would have been fine with your answer here?

Kayla: I would have thought I was right.

Me: Ok. Why would you have thought you were right?

Kayla: Just cause one half base times height says—, I mean, maybe I’m not doing the formula right. But your base is six, and your height’s four. And half of that is twelve.

Me: Ok. Why would you think you’re not doing the formula right?

Kayla: Well, because I’m obviously wrong somewhere. I just don’t know where.

As she continued to move around the third vertex, Kayla indicated that she had no other mathematical means or methods to ascertain the reason for the different result provided by GSP. She laughed as she discussed that at this point she “would give up again” and “just be done and ask somebody else.”

The most remarkable aspect of Kayla’s interview was how quickly she appeared to defer to the technology of GSP in regard to the conflict of results. Kayla confidently used two methods to explore and justify her results for the third mathematics task on paper. However, when prompted for a plausible reason for the results of the premade GSP sketch, Kayla blamed herself for incorrectly counting the squares as well as incorrectly using the formula for the area of the triangle. When asked why she so easily dismissed her own methods in favor of GSP’s results, Kayla explained: “You’d have to have confidence in your answer, which I don’t have, because I know there’s a good chance that I’m wrong. . . . I mean, I just know how shoddy my math skills can be sometimes.”

During the wrap-up portion of the third interview, Kayla said that she was unaware of GSP's ability to hide elements of the screen—a feature she did not witness me use but the other three participants had. Additionally, Kayla continued to stand by her notion that technology is “always right,” as she thought that I manipulated the program in order to “[make] math say what you want it to say.”

Discussion. In contrast to the other three participants and considering Kayla's stated issues with mathematics, her mathematical understanding and knowledge regarding the concept of mathematical shearing initially surprised me. However, she clearly demonstrated her understanding of the concept during the relevant class session. And, although her quick deference to the results of GSP was unexpected in view of her mathematics knowledge, her submission to the authority of GSP was not necessarily surprising when considered against her stated beliefs about the infallibility of technology.

Allocation of Mathematical Authority

During the class sessions, Kayla was rarely a source of mathematics for either herself or her peers. But she appeared to have a strong mathematical instinct during her investigations and was quite confident in many of her explanations during both small and whole group discussions. However, her interviews revealed that Kayla actually lacked confidence in her mathematical abilities and her prior mathematical learning experiences seemed to be a driving factor in the amount of effort she expended in various mathematics explorations.

Kayla utilized a variety of sources when she did not understand a concept. She often said that when she thought she could no longer pursue a mathematics investigation, she would seek out the help of someone who she deemed more mathematically capable. Her significant other appeared to be a large factor in her mathematics learning and understanding beginning as early

as her high school experiences. And, like Garrett, she also discussed seeking the help of Dr. Fikes by developing a particular questioning method to help in her mathematics investigations. Kayla definitely viewed technology as a source of mathematics and believed that technology is “never wrong,” although she stated technological results provided no more mathematical clarity than what she could deduce on her own on paper.

The results of the collective data for Kayla show her as having a rational work method (Figure 3) with an established hierarchy for placement of mathematical authority depending on the context. For example, in regard to technology, if Kayla could not find a viable explanation for an unexpected result presented by the technology, she would defer to the authority of the machine—even if it countered what she knew. If other sources (e.g., her significant other) were available, however, Kayla would consult them to explain any conflict of results. Thus, it is arguable that she also presented a resourceful work method (Figure 3)—using only those sources that she viewed as possibly contributing useful information.

Cross-Case Comparison

Authority Behaviors

Table 3 displays the tallies of the authority behaviors of the participants in whole and small group discussions, respectively. I organized the behaviors using the subcategories of giving authority and possesses authority (with the exception of possesses pedagogical authority) in tabular form to help provide a picture of the form of the participants’ interactions during the class sessions. The data are limited in the sense that I could not record all small group discussions, especially when the participants were not sitting together. Data collection was also limited for any class session that a participant was absent (Table 2).

Table 3

Authority Behaviors Whole and Small Group Setting

Category		Setting	Participants			
			Garrett	Skyler	Janina	Kayla
Giving Authority	uses a source for investigation	whole	1	2	0	1
		small	12	3	7	1
	seeks clarification or answers from peers	whole	10	2	2	2
		small	14	1	3	3
Possesses Authority	acts as source to peers	whole	9	22	3	4
		small	1	5	2	5
	peers use as source	whole	0	0	0	0
		small	1	8	4	1
	self-knowledge used as source	whole	31	24	17	18
		small	16	22	11	9

Although it may be useful to contrast the authority behaviors in terms of the two participants with a mathematics concentration versus those of the two without a mathematics concentration, the results from the analysis of authority behaviors do not lend themselves to such a simple distinction. For example, I expected that the two participants with nonmathematics concentrations to present more giving authority behaviors than the two participants with mathematics concentrations. This lack of expected presentation of specific behavior types is not necessarily surprising when one considers the context of and pedagogy used in the course as well as the data obtained in forming case studies of the four individuals.

Garrett presented the most giving authority behaviors. This was especially evident as he sought clarification or answers from his peers in either whole or small group situations. However, as discussed earlier, Garrett was often observed asking questions as if attempting to make sense of the material at hand. Further, he constantly considered various mathematics sources to form or aid his mathematical understanding. Skyler presented the most possesses authority behaviors and generally did so during the whole group discussions. Her high count of

possesses authority behaviors during the whole group discussions presents an interesting contrast between Skyler and Garrett, considering he had nearly as many mathematics courses under his belt as Skyler, possessed a similar disposition to the subject of mathematics, and seemed equally confident in his mathematics abilities (although he did not portray such confidence during the class sessions).

As noted earlier, I expected the two participants with nonmathematics concentrations, Janina and Kayla, to present high counts of giving authority behaviors. However, each had considerably low counts of giving authority behaviors that may be explained by the two participants' collective data of the study. Even though Janina did not choose mathematics as her concentration, she did not lack mathematical confidence or easily submit to the mathematical authority of her peers. I was surprised, however, by Kayla's low counts of giving authority behaviors considering that she often conveyed little confidence in her mathematical abilities. One explanation for this result is that Kayla was changing in regard to her beliefs about mathematics.

The data from the counts of authority behaviors do reveal, however, a distinction between the two types of participants (i.e., with or without a mathematics concentration) regarding the category of how each participant used his or her self-knowledge as a source when presenting possesses authority behaviors. The two participants with a mathematics concentration, Garrett and Skyler, each had nearly double the total count of the presentation of behaviors using his or her self-knowledge as a source than the respective total counts of Kayla and Janina. One possible reason for the difference in presentation of this particular possesses behavior category between the two types of participants could be attributed to the level of the participants' confidence in

their mathematics abilities. Garrett and Skyler were more confident in their overall mathematical abilities than Janina and Kayla.

Beliefs Survey Results

Basic Concepts 2 Cohort. The preservice teachers in the two sections of Basic Concepts 2 represented a range of academic concentrations and beliefs about their mathematical abilities (Table 1). The beliefs survey, distributed to both sections on the first and last day of the course, further detailed this range in regard to the preservice teachers' beliefs about mathematics and mathematics teaching and learning. As discussed in Chapter 3, a student's score for each category of the survey represents the location of the point on the spectrum for that category. Table 4 presents the results of the paired t -tests that I conducted to compare the means of the scores of the cohort for the corresponding categories of the pre- and post-beliefs surveys.

Three of the nine categories had a statistically significant difference in the means of the pre- and post-beliefs surveys. The two categories with a statistically significant increase in the means involved how the students characterized mathematics and how they viewed the importance of communicating when learning and doing mathematics. Both of these categories were main focal points of the design of Basic Concepts 2. The category with a statistically significant decrease in the means involved the students' attitudes toward mathematics. This decrease also may be due to the pedagogy used in the course. Such a learning environment may have been new to many of the students, creating an increase in frustration because of their expectations of learning mathematics. Further, Dr. Fikes's emphasis on understanding mathematics formulas and basic geometric ideas may have challenged students who had not been asked to consider mathematics in such a way before.

Participants. The participants' changes in regard to their responses to the pre- and post-beliefs surveys are presented in Table 5. The Basic Concepts 2 Cohort's mean percent change for each category is also presented to aid in determining whether changes the participants had were unique or reasonable in contrast to the collective changes in beliefs of the cohort.

Kayla had the most striking change in her responses for her beliefs about mathematics and mathematics teaching and learning. She had an average 25% change in her beliefs and moved toward the positive end of the spectrum of all categories except how she characterized mathematics (which had no change). Kayla's greatest change occurred in her beliefs about how personal effort affects one's mathematical success; her second biggest change occurred in her beliefs about personal authority in learning. Kayla, who (among the cohort of students in Basic Concepts 2) scored as having one of the most negative attitudes toward the subject of mathematics, was greatly affected by her experiences in the course. Further, her views of learning appeared to align with the pedagogy of and expectations set forth by Dr. Fikes. This match of beliefs about learning seemed to account for the positive change in Kayla's attitude about mathematics and how she felt about personal authority.

Skyler had the next largest change in beliefs with an average (negative) 7% change. She moved toward the negative end of the spectrum for all but two of the categories. Her changes, however, were not as significant as those shown by Kayla. Further, Skyler experienced no change in her attitude toward mathematics. She held great confidence in her abilities with the subject matter but felt inadequate when it came to the topic of geometry. Thus, her experiences with the topic over the course of the semester may have affected her overall beliefs about the subject—but not her attitude toward it.

The other two participants did not experience much change in their beliefs about mathematics and mathematics teaching and learning. Garrett had an average 5% change in his beliefs and moved toward the positive end of the spectrum for six of the categories. Janina had an average 1% change in her beliefs and moved toward the positive end of the spectrum for four of the categories. However, they both experienced the largest change in their beliefs in the category of personal authority in learning.

Collectively, the category of personal authority in learning had the largest overall change in beliefs for the four participants. This change may appear surprising when compared to the student cohort of Basic Concepts 2 who, on average, did not change in this category. However, a closer analysis of the percent changes for each individual student in the cohort revealed that a substantial number of students did change in their beliefs about personal authority in learning. Out of the 49 students, 18 changed toward the positive end of the spectrum, whereas 22 changed toward the negative end of the spectrum. Nine students experienced no change.

Table 4

Basic Concepts 2 Cohort Beliefs Pre- and Post-Survey Means by Category

Category	Spectrum		Pre-Survey		Post-Survey		<i>t</i> (96)	<i>p</i>	95% CI	
	NE	PE	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>			LL	UL
Characteristics	15	75	48.61	3.58	50.12	4.78	2.11	.04*	0.07	2.95
Authority	4	20	13.31	1.84	13.27	2.33	-0.15	.88	-0.59	0.51
Attitude	6	30	20.55	3.71	19.55	4.42	-2.57	.01*	-1.78	-0.22
Communication	5	25	18.04	2.78	19.27	2.14	3.25	.002*	0.47	1.98
Effort	7	35	28.18	3.56	28.06	4.10	-0.19	.85	-1.41	1.16
Outside Influences	5	25	18.92	1.97	18.57	2.14	-1.31	.20	-0.88	0.19
Confidence	14	70	49.04	8.38	47.90	9.73	-1.39	.17	-2.80	0.52
Understanding	7	35	29.04	2.30	28.61	3.14	-1.06	.30	-1.24	0.38
Usefulness	9	45	36.33	3.85	35.29	4.99	-1.85	.07	-2.18	0.09

Note. NE = negative endpoint; PE = positive endpoint; CI = confidence interval; LL = lower limit; UL = upper limit.
 **p* < .05.

Table 5

Participants' Beliefs Pre- and Post-Survey Results by Category

Category	Participant													
	Spectrum		Cohort		Garrett		Skylar		Janina		Kayla			
	NE	PE	Change	Pre	Post	Change	Pre	Post	Change	Pre	Post	Change	Pre	Post
Characteristics	15	75	3%	49	55	12%	56	53	-5%	46	48	4%	53	53
Authority	4	20	0%	15	17	13%	17	15	-12%	15	17	13%	10	14
Attitude	6	30	-5%	24	25	4%	26	26	0%	20	20	0%	15	18
Communication	5	25	9%	24	25	4%	22	23	5%	20	21	5%	17	22
Effort	7	35	1%	30	33	10%	34	29	-15%	31	28	-10%	16	29
Outside	5	25	-1%	20	19	-5%	22	18	-18%	20	20	0%	17	21
Confidence	14	70	-2%	58	58	0%	59	55	-7%	51	52	2%	42	48
Understanding	7	35	-1%	31	32	3%	32	31	-3%	29	29	0%	29	31
Useful	9	45	-3%	41	41	0%	44	40	-9%	37	36	-3%	39	43

Note. NE = negative endpoint; PE = positive endpoint.

CHAPTER FIVE

DISCUSSION

The general focus of this study centered upon conceptualizing mathematical authority with technology. In order to explore this focus, I first considered the construct of authority and authority issues within technology-based mathematics learning. Issues of mathematical authority in the literature are generally manifested in the results of studies and point to the “passive role” (Erlwanger, 1973, p. 15) that students assume in their mathematics learning while forming or deepening undesirable perceptions of and beliefs about the subject. Further, research suggests that students may adjust their behavior to accommodate to the authority (Artigue, 2005; Erlwanger, 1973; Hativa, 1988; Lingefjärd, 2000). Not necessarily evident in a teacher-centered environment (in which the voice of the learner is virtually absent), existing issues of mathematical authority can be further compounded by the use of technology. Noss (1988) explains, “In technological cultures, practical activities have become increasingly complex and the sciences have become deeply interwoven with everyday life, and paradoxically, increasingly invisible” (p. 255). Studies that tout the usefulness of technology for aiding successful mathematics learning do not necessarily take into account the authority of the technology—a psychological construct that cannot be ignored (Cooney, Shealy, & Arvold, 1998). Thus issues of mathematical authority in certain technology-based classrooms may be even more pronounced and elusive than those in other learning environments.

Understanding the placement of mathematical authority may provide educators with crucial knowledge of what information sources students seek, why they seek out those sources, and how they use the information available from various sources. In ascertaining such knowledge, educators can begin to tailor their learning objectives and pedagogical techniques to help “free the pupil to think for himself” (Erlwanger, 1973, p. 22). This freedom is especially important in a technology-based learning environment, where “there is a great risk that technology will create a new sort of authority” (Lingefjård, 2000, p. 160) and in regard to “how easy it seems for students to ‘get lost’ and trust the technology far too much” (p. 161).

Summary

In this study, I sought to find where four preservice elementary teachers allocated mathematical authority and how the use of technology affected that allocation. To examine their placements of mathematical authority, I needed to understand other important aspects of their mathematical being, which included: their beliefs about mathematics and mathematics teaching and learning, their confidence in their mathematical abilities, and how issues of mathematical authority were manifested for them in various nontraditional learning situations. Observations of the interactions and behaviors of the preservice teacher participants with others in the geometry course they were taking, coupled with their discussions during individual interviews, offered evidence as to where they placed mathematical authority.

The study took place in a geometry course for preservice elementary mathematics focused on student mathematical learning, understanding, and reasoning. The course instructor used a pedagogy consisting of small and whole group discussions as students explored a variety of mathematics investigations to gain a deeper understanding of geometry concepts. This context afforded me a variety of avenues to pursue the two research questions of the study:

1. Where do preservice elementary teachers allocate mathematical authority in the context of a geometry content course using pedagogy consistent with student-centered learning?
2. How does the use of technology, while engaged with concepts of the geometry course, affect these preservice elementary teachers' allocation of mathematical authority?

First, to conceptualize mathematical authority with technology, I had to conceptualize student placement of mathematical authority in general and in different learning situations. I needed a learning context in which students were less likely to behave with actions that mirrored years of mathematics experiences in teacher-centered or lecture-style classrooms—behaviors that may be deemed more like actions from muscle memory—so as to “honor an individual’s ingenuity in transforming and applying knowledge to specific dynamic situations” (Cooney et al., 1998, p. 308). Second, the pedagogy of the geometry course, using various modes of investigations through mathematics explanation and justification, provided evidence of the participants’ mathematics understanding and issues of mathematical authority that I later explored during interviews. Third, the content of the geometry course incorporated concepts (i.e., content geared toward teaching the Grade K–8 learner) that the participant may have felt familiarity with. Such concepts are an integral aspect of studying mathematical authority because attempts to learn material that students perceive as new or difficult can confound the discernment of factors influencing the placement of mathematical authority. Fourth, the participants of the study had a common goal of understanding the mathematics of the course because it was deemed important for their future careers as elementary teachers. This goal was important from the

aspect of their motivation—another psychological construct that could possibly confound the discernment of the placement of mathematical authority.

I followed the four participants for the duration of their semester in the geometry course and interviewed them on separate occasions. They had different backgrounds, interests, and were at different points in their academic pursuits, providing varying perspectives on mathematics learning and teaching that yielded contrasting results. The participants had relatively similar high school mathematics content backgrounds, which provided a baseline for contrasting their mathematics knowledge. Further, given their previous mathematics experiences, the preservice teachers' data provided an entry point from which to explore and understand existing issues of mathematical authority rather than issues arising as students enter formal schooling.

Data collected for the study came mainly from videorecordings of class sessions and interviews. An information form and pre- and post-beliefs surveys provided additional data. Collectively, the data obtained were used to form a case study analysis of each participant in light of the theories of instrumental genesis (Artigue, 2002; Trouche, 2005b; Vérillon & Rabardel, 1995) and student perception of technological authority (Lapp, 1997).

The data from the class sessions revealed the participants' individual interactions and behaviors with peers and the course instructor that could not have been manifested during interviews. The interviews were divided into two segments: a general interview portion and a clinical interview portion. The responses to the general interview questions provided detailed information about the participants' beliefs and attitudes about mathematics, mathematics teaching and learning, and technology. Further data on the participants' beliefs about technology and placements of mathematical authority came from the clinical portion of the personal interviews as they interacted with a dynamic geometry software program to explore a

mathematics task. The first task asked the participants to continue a class investigation to analyze whether two figures satisfied a definition. The second task asked each participant to resolve a paradox in which the polygons of a triangle could be rearranged to form a seemingly congruent triangle with a different area. In the third task, I asked the participants to first solve a problem on paper and then resolve a conflict that arose as they considered a different mathematical solution of the problem presented by a specially altered dynamic geometry software sketch.

Conclusions

This study demonstrated that placement of mathematical authority for a single individual is not consistent between varying situations and does not necessarily exist in a hierarchal fashion. The lack of consistency of where an individual may allocate mathematical authority is dependent on the context of the problem situation and learning environment, the mathematical content that the individual is engaged with, and the mathematics sources available. Placement of mathematical authority also greatly depends upon the individual's personal views of and beliefs about the nature of mathematics, how he or she interacts with various mathematics sources, and his or her past mathematics learning experiences. These views, beliefs, and interactions are considerably affected by deep-rooted psychological constructs pertaining to issues of mathematical confidence and related to the general work methods (Trouche, 2005b) that concern how the individual privileges and uses various mathematics sources. Further, technology, seen as a source of mathematics, can become an absolute authority in the face of mathematical conflict regardless of one's personal mathematical conviction or understanding.

Allocation of Mathematical Authority

Research Question 1: In the context of the course. Within the context of the geometry course of this study, the four participants allocated mathematical authority in different ways, and

that allocation was not dependent on whether a participant had a mathematics concentration. One participant with a mathematics concentration primarily allocated mathematical authority in her mathematics knowledge as she ignored more useful mathematics sources. The second participant with a mathematics concentration placed mathematical authority in a variety of sources, and his placement of authority in different mathematics sources did not take place in a hierarchal fashion. The third participant with a nonmathematics concentration did not have a stable source of mathematical authority within the context of the course of the study. She either sought out clarification from peers she deemed mathematically capable or pieced together (not necessarily connected) mathematical ideas to form an understanding of a particular concept. The fourth participant with a nonmathematics concentration had an established hierarchy for placement of mathematical authority depending on the context.

Research Question 2: The effect of technology. The effect of technology on the allocation of a participant's mathematical authority in a mathematics source was dependent upon how that participant viewed or trusted technology and his or her placement of mathematical authority in general as discussed above. Similar to the placement of mathematical authority within the context of the course environment, the effect of technology on that placement of authority was not dependent on whether a participant had a mathematics concentration.

One participant with a mathematics concentration was apprehensive about the use of technology for the learning and teaching of mathematics. Her apprehension limited her ability to view the technology as a source, and therefore the technology did not affect how she allocated mathematical authority in general. The second participant with a mathematics concentration indicated that he trusted particular technologies the more he became familiar with them. He used technology as an aid for mathematical understanding but did not use it as a mathematics source.

Thus technology did not affect his allocation of mathematical authority. The third and fourth participants with a nonmathematics concentration viewed technology as a source and provided explanations for results presented by the technology counter to what they knew mathematically. Their willingness to explain the results of the technology indicated that they viewed technology as an authority. Technology did not affect the third participant's placement of authority in the context of the geometry course, since her allocation of authority was not stable. However, technology did affect the fourth participant's allocation of mathematical authority that existed in a hierarchal fashion as the authority of the technology superseded her mathematical knowledge and understanding.

Plausible Reasons for Placement of Mathematical Authority

Reasons that the participants gave for their mathematical decisions and conclusions varied and depended on the following: the types of work methods (Trouche, 2005b) the participants used throughout the study; their respective beliefs about the nature of mathematics and mathematics learning; and their trust (Lingefj rd, 2000) in the information presented by various mathematics sources.

Two participants presented an automatistic work method, as was demonstrated through their piecemeal mathematical reasoning used in order to make sense of varying mathematical ideas. Neither was observed to blindly accept mathematical reasoning from any particular source. However, one primarily sought information from different sources and used various mathematical aspects of the information she found to form her mathematical conclusions. In contrast, the other participant generally relied on her own mathematics knowledge and understanding and was particularly distrustful of the results presented by technology. Additionally, neither possessed geometric habits of mind (Driscoll, 2007) and did not hold a

connected view of mathematics learning as the other two study participants discussed in their personal interviews.

The third participant presented a resourceful work method evident from his productive use of his mathematics knowledge as well as how he contrasted and compared the mathematics information he gleaned from the various mathematics sources he sought. He was unique in his quest for mathematics understanding in contrast to the other three participants. This quest was especially evident as he pursued mathematics knowledge outside of class explorations and investigations and explored many mathematical avenues attempting to understand the unexpected results presented by the technology in the second and third interviews. Additionally, he displayed a facility and ingenuity with the technology to help him explore the clinical tasks. However, he did not submit to the authority of the technology when it presented results that did not match the mathematics knowledge he had acquired. His trust was not placed in one single source. Instead, he considered how the items of information from various sources compared to each other in order to make a viable mathematical decision—a hallmark of the student with a resourceful work method.

In contrast, the fourth participant presented a rational work method evident in her lack of use of technology. Her indication that she periodically sought assistance from other mathematics sources (particularly those persons she viewed as more mathematically capable than she was), also suggested a resourceful work method. The sources she sought, however, were those that she viewed as contributing useful or new information—a view she did not hold about technology. Given this perspective, it is not surprising that both the third and the fourth participants indicated that they sought mathematics help from the instructor using covert methods of inquiry. Regardless, her trust in the technology led her to accept it as a mathematical authority in the face

of conflict with her own understanding or when she had no viable explanation for the results presented by the technology. She presented an interesting aspect of conceptualizing mathematical authority in regard to technology. Her belief that technology did not add new information that one can obtain using paper alone suggested that she viewed her mathematics knowledge as a higher authority. But, in the absence of user error, she viewed the results of technology as unfailing and thus conceded to the incorrect results presented by the technology in the third interview.

The Complexity of Mathematical Authority

The study of mathematical authority revealed the complexity of the construct as the participants made mathematical decisions in the learning environment. Typical markers (e.g., an individual's level of mathematical confidence, mathematics background, or mathematical knowledge and understanding) that may be used to predict an individual's propensity to allocate mathematical authority in a particular source were not always evident for the participants in this study. For example, this study demonstrated there is not a simple connection between an individual's level of mathematical confidence and his or her ultimate placement of mathematical authority in a given context. One participant, who exhibited great confidence during the class sessions, lacked confidence in her geometrical ability that was revealed only in the interviews. However, this participant continued to predominantly use her mathematics knowledge and understanding as a mathematical authority. Another participant, who was quite confident in the results of various mathematics tasks that she found on paper, quickly submitted to the authority of technology when it presented a conflict of results. Her mathematical knowledge and understanding no longer served as a mathematical authority for her against the authority of the

technology as evident from her discussion for why she believed the technology was correct over her mathematically valid and viable reasons.

There were a number of factors that affected a participant's decision to seek out a mathematics source and use that source as an authority. Some of the factors (e.g., the mathematics of the investigations or an individual's mathematics knowledge) can be readily identified through observations of the classroom environment. There are also unknown factors, however, that may be present and outside the scope of the classroom environment that may influence an individual's decision to allocate mathematical authority in certain sources. For example, one participant frequently indicated how she sought mathematics help from her significant other who was an undergraduate student majoring in mathematics. Another participant indicated that he occasionally sought mathematics help from the employees of the university mathematics lab. These outside factors may contribute to the mathematical confidence an individual may hold regarding a particular concept. An individual may allocate mathematical authority to a new source if the mathematical understanding gleaned from these outside factors that serve as authorities match the information presented by the new mathematics source.

Typical mathematics sources available to students include the classroom teacher, the course text, and peers. The removal of one mathematics source may not necessarily affect how an individual allocates mathematical authority if he or she did not generally seek out that source as an authority. In this study, the course instructor used pedagogy consistent with student-centered learning through the use of mathematics investigations and explorations from which the students were expected to generate understanding in a collaborative effort. Although she removed herself as a typical mathematics source and authority, she pushed and pursued the students' mathematical understanding by questioning their mathematical knowledge and

justifications. Two participants in this study continued to use the instructor by studying her body language or using her questioning techniques as a type of affirmation. This adjustment to their behaviors as they accommodated the instructor's self-removal as a mathematics source and authority indicates adaptability to how they sought out sources to which they allocated mathematical authority.

As the instructor removed herself as a mathematical authority, the authority of other students became more salient in the classroom. The removal of the classroom teacher as a regular mathematics source necessitated a complicated process for contrasting and choosing between information provided by other available mathematics sources. A participant's ability to effectively or efficiently contrast and choose information from various sources and his or her ultimate decision to allocate authority in a particular source were dependent on that participant's views of and confidence in the source and how he or she made sense of the mathematics within the constraints of his or her mathematical knowledge and understanding. For example, one of the participants of the study contrasted information provided by the course text with explanations from his peers in order to make sense of a mathematics problem. The mismatch in the information presented by the two available sources caused him to search out additional information outside class in order to make sense of the mathematical concept inherent in the problem. In this example, the participant used both his peers and the course text as a mathematical source but did not use them as a mathematical authority because he could not make sense of the information that either presented in regard to the problem.

Oppression and Freedom of Authority

This study shows that the construct of authority is directly related to the notion of power as either an oppressive or liberating construct. In certain contexts, the placement of mathematical

authority can oppress constructive and meaningful learning. This complex aspect of mathematical authority was especially evident with the two participants who did not choose mathematics as a concentration. In their interviews, both pointed to the freedom of learning that their respective concentrations provided. One indicated that she longed for the power of expression that she found in her concentration. The second expressed a desire “to play” with and explore mathematics concepts and form her own understanding of underlying concepts—rather than the understanding mandated by her former mathematics teachers.

In a similar vein, both participants with a mathematics concentration indicated having a personal power and freedom from the joy in “figuring out” the mathematics. During their investigation of the second task, both displayed remarkable perseverance in resolving the paradox. It is difficult to provide demarcation of the line that separates their views of mathematical autonomy from those of the other two participants’ views. Perhaps because of some aspect of their personalities, the two participants with a mathematics concentration had been able to work within the structure of the mathematics tasks mandated by their former experiences, whereas the other two participants found such structure disagreeable.

Change Is Possible

Change has been deemed difficult to achieve and does not necessarily happen in one semester (Cooney et al., 1998; Lingefjärd, 2000). However, the results of this study indicate that it is possible to change an individual’s beliefs about the nature of mathematics and mathematics learning and teaching—however small that change may be. Change in beliefs was evident in the two participants (one with a mathematics concentration and the other with a nonmathematics concentration) who held a connected view of learning—believing that all mathematics concepts are connected in meaningful ways. Their previous learning experiences did not support their

views of learning. However, the pedagogy used in the geometry course did align with their views of learning. The participant with a nonmathematics concentration changed in her beliefs about the subject of mathematics. The participant with a mathematics concentration formed new goals from learning to pass the test for his own means to learning for understanding in regard to his future students.

The Complexity of Technology

As with beliefs formed about mathematics, various factors contribute to beliefs about technology. Such factors include: the many years of technological use for everyday and academic purposes, advances in modern technology resulting in powerful machinery, the effects of the collective social psyche about technology, and how technology is presented and used in the learning environment. For example, the participants all possessed a view of technology as being fault-free—evident from their individual discussions that incorrect results provided by technology are rooted in user error.

In this study, technology was definitely viewed as a possible source of mathematics and, through its use, only offered certain types of information available in particular formats. For example, unlike their view of other manipulatives, the two participants with a nonmathematics concentration did not discuss and did not appear to view the technology as an aid to be used to further their mathematics understanding. Instead, they indicated their need for “hands-on” learning and preferred the use of pattern blocks, rather than the technology, to help visualize the mathematics.

This study also suggests that the trust in technology is not necessarily proportional to the amount of technological use. All four participants were familiar with the graphing calculator from previous mathematics experiences. However, the two participants with mathematics

concentrations were the only ones to have extensive experience with other forms of technology—although their views of technology were markedly different from each other. One participant was quite apprehensive in regard to technological use. In contrast, the other participant indicated that he trusted particular technologies the more he used them and gained familiarity with their interfaces. This participant interacted with the technology in a manner quite different than that of the other three participants through his nonconventional use of the features of the dynamic geometry program to help him investigate the mathematics tasks.

Technology, touted as providing mathematical independence and giving students opportunities to develop mathematical understanding as well as engage in meaningful mathematical activities (Heid, 1997), has also been shown to be perceived by students as a black box (Artigue, 2005; Drijvers & Gravemeijer, 2005) from which mathematics information can be uncritically obtained. Artigue (2005) argues that, with the proper use of technology carefully built up and “*tightly piloted*” (Artigue, 2005, p. 265) by the teacher, the technology can serve “as a *revelator of the fragility* of the students’ mathematical knowledge” (p. 246). However, the results of this study indicate that technology can also reveal issues in regard to placement of mathematical authority. This revelation was particularly evident in one participant’s submission to the authority of the technology for the third mathematics task, where it was quite clear that she did not lack an understanding of the mathematics inherent in the task but yet thought the incorrect results presented by the technology were correct.

Revisiting Theory

The theory of student conceptualization of technological authority (Lapp, 1997) can aid in understanding the reasons that students seek various mathematics sources when the course instructor guides the use of mathematics tools (e.g., technology) or other mathematics sources

(e.g., peers and textbooks). However, the theory does not offer a way to consider how students seek or place mathematical authority when the course instructor is either removed as a mathematics source or does not offer guidance on interacting with other sources of mathematics. I used the tenets of instrumental genesis and, in particular, the work methods (Trouche, 2005b) of the theory to provide insight into the participants' placement of mathematical authority. Considered together, and in the context of conceptualizing mathematical authority with technology, I propose a meld of the two theories that incorporates affective issues as well.

The results of this study suggest that the construct of mathematical authority is related to the mathematics background of a student and his or her beliefs about the various mathematics sources available. The mathematics background of the individual includes his or her mathematics knowledge and mathematics experiences. Together, the student's mathematics knowledge and experiences help to contribute to how the student functions mathematically and his or her beliefs and perceptions about mathematics. Further, the student's knowledge and experiences contribute to form the confidence the student holds in his or her mathematics ability as well as the trust (Lingefjärd, 2000) he or she places in other mathematics sources. When the instructor is available and is viewed as the single source of mathematics, the placement of mathematical authority may be explained through Lapp's (1997) model. However, when the instructor is removed as a source of mathematics, the individual is left to consider other available sources and assign mathematical authority according to his or her work methods as described by Trouche (2005b). This placement of authority may be in a hierarchal fashion or connected through a web.

Implications

Lingefjärd (2000) argues that future teachers need a different type of educational experience that (rather than consisting of more mathematics courses) engages "them in reasoning

and in constructing mathematical modes, in assessing the extent to which a mathematical argument is valid, and in developing, comparing, and evaluating alternative solution processes” (p. 152). The complexity of mathematical authority complicates the notion that students, particularly preservice teachers, need more mathematics classes. The two participants with mathematics concentrations were also upper-level college students. They exuded more mathematical confidence and had significantly more mathematics experiences under their belts than the two with nonmathematics concentrations. In particular, during the class sessions, one of the participants presented a high level of self-confidence and as a source of mathematics.

The experiences provided in the geometry course align with Lingefjård’s pedagogical suggestion and are an example of successful use of such pedagogy, especially in regard to the change of one participant’s beliefs about mathematics. This participant held negative beliefs about mathematics and indicated having a low self-confidence concerning her mathematical abilities. She exhibited, however, the most positive change of the participants in regard to her beliefs about mathematics and mathematics teaching and learning. One must consider the accessibility of the mathematics of the geometry course and how such accessibility contributed to her change and successes. The experiences offered in this course included weeks of classroom discourse and mathematics investigations on a few main topics. But, in the end, her personal beliefs on the authority of technology negated her belief in the results of her own mathematics understanding as she quickly deferred to the results presented by the technology in the third clinical interview. This complexity provides an area for future research to further consider the role of the construct of authority concerning mathematics teaching and learning as well as defining successful learning.

Anderson (2000) posits that mathematics educators need to consider “a new definition of success, one that depends on authorship and ownership of mathematics” (p. 183). Noss (1988) claimed that the use of technology for mathematics learning aids in such ownership as students come to view the computer screen “as theirs” (p. 257). However, the participant, who clearly understood the mathematics of the third mathematics task and solved it using two distinct and mathematically viable explanations, immediately gave up ownership of the mathematics and her own authority when confronted with conflicting results provided by the technology. Under normal circumstances, this participant would be deemed to have successfully learned the concept at the heart of the third mathematics task. However, her submission to the authority of the technology presented her as a student who lacked confidence in her own mathematics understanding. Further, the results of her clinical interviews may have been confounded with her lack of familiarity with the technology. Thus future research concerning mathematical authority could incorporate a closer examination of the concept of success in mathematics learning and understanding. Such research should also consider how technology, when used as a familiar tool for mathematics investigation, affects what is generally deemed as successful learning in a nontechnological learning environment.

Technology is pivotal in understanding certain mathematics concepts and provides affordances in particular areas (e.g., discrete mathematics and applied logic) that were previously inaccessible with paper-and-pencil techniques (Heid, 1997; Trouche, 2005a). This sentiment is not lost when considering mathematics germane to the area of computer science—a crucial need in today’s society. However, technology infusion in mathematics learning also leads to the concern of the sacrifice of meaningful learning (Heid, 1997) or personal authority (Lingefjård, 2000). Trying to alleviate these concerns by eliminating the use of technology for the

introduction of new mathematics concepts would hinder efforts to mathematically prepare students in the twenty-first century. This dichotomy of concerns points to the need for research that considers the construct of mathematical authority in order to ascertain where a balance exists that creates successful learning without sacrificing personal authority.

Further, there is a need to understand and explicate the mathematics learning environments that not only create the necessary mathematical understanding but also promote and maintain personal mathematical authority. Such research could begin to tease out how a teacher could foster students' sense of mathematical autonomy to help students feel inclined to push forward in their quest for mathematics understanding. This research could also illuminate critical characteristics that reveal a level of mathematical authority sufficient for independent learning. Identifying such characteristics is especially important when considering the transitions students encounter as they move between various learning environments. For example, what would happen to the mathematical views of the two participants who (because of the pedagogy used in the geometry course) experienced a change in their beliefs about mathematics and mathematics teaching and learning if they were, in a subsequent semester, to return to learning experiences similar to those they had prior to their enrollment in the course?

The theory of instrumental genesis provides a context to begin to further study the type of environment that would promote mathematical understanding as well as promote and maintain mathematical authority. However, at present, the theory lacks direct explication of the construct of mathematical authority. Future research could begin to look at the schemes inherent in mathematics learning, understanding mathematical authority in light of the conceptual framework I provide in Figure 30, and in regard to existing schemes identified in instrumental genesis (particularly those inherent in the command process (Trouche, 2005b)). Further, my

proposal for a conceptual framework discussed above was designed in light of the results of this study—a context in which the instructor was not necessarily an available mathematics source and in which there were only four participants. However, future research could reconsider the framework in regard to the authority of the technology versus the authority of the teacher. Finally, just as the theory of instrumental genesis considers how the interface and design of various technologies necessitate a different type of interaction and engagement from the individual, future research could also investigate how the designs of various technologies also affect the placement of mathematical authority.

Concluding Remarks

The past several decades have witnessed a surge of technological advances to aid the everyday lives of those in developed countries. The children of Generation X both experienced and helped create this explosion of available technologies. This resulting boom gave way to the varied forms through which technology has become an integral part of nearly every aspect of our lives. Arguably, this technological expansion created the technological dependency of the children of Generation Y (also known as the Millennials). Technology is no longer available only to a small percent of society's privileged individuals. Instead, technological use has become a common everyday habit for many individuals; for the younger set, technology is rooted deep in the social psyche. Affordances of technology include the vast amount of information quickly gleaned off the Internet with various technological mediums (e.g., smartphones and tablets) that can easily fit in one's back pocket as well as instant social gratification of communication with a large network of people through various forms of social media. Further, certain technologies in the mathematics classroom (e.g., the scientific or graphing calculator) are deemed as necessary as having a pencil and paper. And, by the time students enter a high school mathematics

classroom, they have already had a substantial amount of time interacting with technology as a tool or means for learning in a variety of subjects.

Having been through a slew of advancements and improvements, the technology of today is rarely viewed as faulty or lacking. Instead, it is touted as a way to streamline ideas and communicate with others both effectively and efficiently. Given the wide array of technologies available in our everyday lives, one has to wonder about all the effects of technology, including the unseen psychological consequences that might affect successful mathematics learning and understanding.

Of course, it would be quite difficult to ascertain the psychological consequences of using technology while learning without conducting a long-term study utilizing a particularly fine-grained analysis of students engaging with technology. That study would begin with students' introduction to the technology and would include their continued use of it through various aspects of their lives. That could not be the case in the present study because the preservice elementary teachers had already used technology (typically scientific and graphing calculators) in much of their previous mathematics learning. Further, the use of the mathematics tasks aimed at presenting mathematics conflict can only go so far in any study of placement of mathematical authority. This was especially evident in the present study when all four participants began to speculate about the "trick" of the paradox of the second mathematics task as they worked on the third mathematics task.

As a final comment, I would be remiss not to acknowledge the bravery of the four participants to participate in this study and offer their mathematics knowledge, beliefs, frustrations, and anxieties. Although I set out to conceptualize mathematical authority with technology, these four individuals taught me a fair amount about other aspects of mathematics

teaching and learning as well. Further, my interactions with the participants reaffirmed my belief of the importance to listen to and be attuned with students' mathematical and psychological needs. The two participants with a nonmathematics concentration sought an academic freedom that reminded me of how the institutionalization of mathematics teaching can hinder great potential. The participant with a mathematics concentration who, during the class sessions, exuded great mathematical confidence and acted as a mathematics source for her peers did not really have such confidence in her geometric abilities. She made me realize that students not only have mathematical decisions to make and act upon, but also a certain expectation of proper mathematical fulfillment. And finally, the other participant with a mathematics concentration, through his persistent quest for understanding in honor of his future students' mathematics learning, provides me with hope for the future of mathematics education.

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APPENDIX A

Habits of Mind

Driscoll (1997) discuss four geometric habits of mind that mathematics teachers should aim to understand, develop, or enhance in their students' experiences with geometric concepts and ideas. They are discussed briefly below.

Reasoning with relationships

"This is actively looking for relationships (e.g., congruence, similarity, parallelism) within and between geometric figures in one, two, and three dimensions, and thinking about how the relationships can help your understanding or problem solving" (p. 12).

Generalizing geometric ideas

"This is wanting to understand and describe the 'always' and the 'every' related to geometric phenomena" (p. 12).

Investigating invariants

"An *invariant* is something about a situation that stays the same, even as parts of the situation vary. This [geometric habit of mind] shows up, for example, in analyzing which attributes of a figure remain the same and which change when the figure is transformed in some way (e.g., through translations, reflections, rotations, dilations, dissections, combinations, or controlled distortions)" (p. 13).

Balancing exploration and reflection

"This is trying various ways to approach a problem and regularly stepping back to take stock. This balance of 'What if . . .' with 'What did I learn from trying that?' is representative of this habit of mind" (p. 14).

Cuoco, Goldenberg, and Mark (1996, pp. 389–392) provide the following habits of minds that geometers use in their approach to mathematics:

- Geometers use proportional reasoning
- Geometers use several languages at once
- Geometers use one language for everything
- Geometers love systems
- Geometers worry about things that change
- Geometers worry about things that do not change
- Geometers love shapes

APPENDIX B

Information Form**Basic Concepts 2 Spring 2010
Background Questionnaire**Background:

Please fill in the following information before answering the survey questions.

Gender: Female / Male

Age: under 23 / 23 to 30 / over 30

How good are you at mathematics? Excellent / good / average / below average / weak

Please put a check by the mathematics courses that you took in middle or high school.

- ☐ first-year algebra (or integrated math I)
- ☐ geometry (or integrated math II)
- ☐ second-year algebra (or integrated math III)
- ☐ trigonometry or pre-calculus
- ☐ statistics
- ☐ calculus
- ☐ other – please list _____

Please list all college mathematics courses that you have taken or are taking.

Please list any additional mathematics courses that you plan to take after this semester.

What is your area of concentration in the elementary education program? (If not sure, please indicate that).

APPENDIX C

Beliefs Categories with Sample Beliefs Survey Items

Category	Sample Item
Characteristics of Mathematics and the Learning and Doing of Mathematics	<i>Mathematics is a set of inter-related concepts.</i> <i>Computation is the single most important skill in mathematics.</i>
Authority in Learning Mathematics	<i>A teacher should always tell a student who asks if she or he has done a problem correctly.</i> <i>Students should be responsible for figuring out if they have solved a problem correctly.</i>
Attitude Towards the Subject of Mathematic	<i>I do not want to take more mathematics.</i> <i>I like mathematics.</i> <i>I am only taking mathematics because of a requirement.</i>
The Importance of Communication and Group Work While Learning Mathematics	<i>Students should discuss their solutions strategies with the whole class.</i> <i>Students should be expected to solve math problems with a partner or in a small group.</i>
The Importance of Understanding Mathematics	<i>Getting a right answer in math is more important than understanding why the answer works.</i> <i>Time used to investigate why a solution to a math problem works is time well spent.</i> <i>A person who doesn't understand why an answer to a math problem is correct hasn't really solved the problem.</i>
The Importance of Personal Effort for Success in Learning Mathematics	<i>By trying hard, one can become smarter in math.</i> <i>Ability in math increases when one studies hard.</i>
The Importance of Outside Influences for Success in Mathematics	<i>I care about whether my peers think I am a good student in mathematics.</i> <i>Grades are an important motivator for me.</i>
The Usefulness of Mathematics	<i>People who don't know much math can get along just fine in today's world.</i> <i>Studying mathematics is a waste of time.</i> <i>Mathematics is useful for solving everyday problems.</i>
Confidence in the Ability to Learn and Teach Mathematics Successfully	<i>If I can't solve a math problem quickly, I quit trying.</i> <i>I understand most of what goes on in math class.</i> <i>I think I can be a very effective teacher of math in elementary school.</i>

APPENDIX D

General Interview Questions

Interview Questions Set 1

Initial Instructions to the Student

I am interested in knowing your thoughts and feelings about math and math teaching. I'll be asking you questions and I hope that you will be at ease and give honest responses.

There are no right or wrong answers for any questions, I would just like to know how you personally think and feel. If there is a question that you don't want to answer, you don't have to. Just say that you don't want to answer it. Your responses are completely confidential. They will only be reported in a way that doesn't reveal your identity. For example, a report may say "Ellen, a junior in elementary education, indicated that math was one of her favorite subjects in high school but she indicated that she did not like doing group work in T104." [pseudonyms will be used for all names] Additionally, Dr. Fikes will not know any of your responses until after the class is over. The earliest she may know of them would be in the summer or the fall.

General Background and Feelings about Mathematics

1. Why do you want to be a teacher?
2. How do you feel about "doing mathematics" and being a "mathematics teacher" to elementary grades children? Are you confident that you can do a good job? (*Follow up with A through D if these are not mentioned.*)
 - A. Is it important for an elementary teacher to have strong math skills? Do upper elementary grades teachers need more math skills than primary grades students?
 - B. Are your math skills strong enough to be a good teacher?
 - C. How much knowledge of how children think about math and learn math should an elementary teacher have?
 - D. How much knowledge of specific techniques for teaching mathematics should an elementary teacher have?
3. How did you do in mathematics before you came to this university? Did your performance change over time? (Probe with respect to elementary, middle, and high school. Make sure you record the high school math courses taken and note how well the student did in

those courses.)

4. Tell me about the mathematics courses you have taken in college. What were they and how well did you do? Are you planning to take more math courses (if yes, which ones and why?)
5. Why are you taking Basic Concepts 2?
 - A. Did you deliberately sign up for Dr. Fikes section?
 - B. How does this class differ than other mathematics class you've had? How is it the same?
 - C. What do you think about the techniques used in this class for teaching/learning math? Would you use any of them when you teach?
6. On a scale of 1 to 10, with 10 being the highest, how much do you like mathematics? (Explain.) Are there specific things you like and dislike about math?

Mathematics Study Habits

7. How do you study for a test or quiz in math (not math methods)?
8. Has anyone taught you study skills for math? Is studying for math different than studying for other subjects?

Mathematics Learning and Teaching

9. How important memorization is in mathematics? Can someone be good in math without being good at memorizing?
10. Do you feel that you memorize a lot, somewhat, or a little for Basic Concepts 2? Why?
11. Some elementary teachers feel that children must be taught basic skills and that they will be frustrated by hard problems until they know these skills. Other elementary teachers think that children can use their intuition to figure out problems without having to memorize computational skills first. What is your feeling about this issue?
12. Do you think working with others in math is a good idea? (Probe for when it is appropriate and when it isn't.)
13. Is it appropriate to use calculators or computers in math class or on math homework? What would you use them for? Is it ever "cheating" to use a calculator in math class?

Questions to begin the GSP exploration:

14. What do you remember about the tessellations activity you did 2 weeks ago?

15. Did you ever come to a consensus of what a regular and a semiregular tessellation are?
16. (After exploring with GSP) What do you think about this program for learning tessellations (compared to manipulatives)?

Interview Questions Set 2

Effort

1. Most students have a balance between studying and social/non-academic activities. What is that balance like for you? Do you attend all of your classes? How much time do you spend studying each day/each week?
2. What causes you to work hard in **math** (not math methods)?
Allow the student to answer and then follow-up by asking how important each of the following are in terms of effort in a math course:
 - a. grades
 - b. believing that knowing math is important for teachers
 - c. believing that knowing math is important for anyone
 - d. doing math is enjoyable
 - e. other issues (allow student to specify what these might be)
3. How important is it to you to be viewed as one of the top students in a math class? Is it important to you to look like a good math student or poor math student to your friends and teacher? *(The point of this question is to look for evidence of an ego-orientation [feeling that the only way to succeed is to make yourself look better than others])*
4. How often do you do the least amount of work you can to get by? *(The point of this question is to look for a work-avoidant orientation)*
5. How do you like math in comparison to other subjects? Are the factors that make you work hard in other subjects different from the ones that make you work hard in math?
6. What do your friends think about math? Do they like it? Do they see it as useful? Do they work hard in math?

Mathematics Learning and Teaching

7. When someone makes mistakes in mathematics, does it mean that person is a weak math student? (explain)

8. Are there times when the answer you arrive is different than what someone else arrives at? *(follow up with these questions and be sure to differentiate between mathematics and other areas of their studies)*

- If it is someone you are working with?
- If it is someone that you think is good at mathematics?
 - How do you know if someone is good at mathematics?
 - How do you know if someone is good at mathematics in Basic Concepts 2?
- If it is a teacher?
- If it is the back of the book?
- If it is technology?

9. Which of the following statements do you agree with most? Why?

"Elementary students need guidance in math and thus a good math teacher explains everything."

or

"Elementary students remember math best when they figure it out for themselves and thus a good math teacher may let them struggle on a challenging problem."

Interview Questions Set 3

Nature of Mathematics

1. Some people argue elementary school math should be based on problem solving and logical reasoning while others think the only real focus should be computational skills. What is your feeling?
2. If someone asked you what a mathematician does, what would you tell that person? *(As a follow-up, you might ask the student if she/he has any idea what faculty in the math department do when they are doing "research.")*

Mathematics Learning and Teaching

3. Do you think it takes special talent to do well in mathematics? Do you have such talent? Can people do OK in math even without special talent?
4. When learning mathematics, what would you say is your main source for learning the material? *(Differentiate between Basic Concepts 2 and their other previous mathematics classes.)*
 - Teacher?
 - Book?
 - Notes?

- Friends/Peers?
- Classmates?
- People you perceive to be good at mathematics?
- Math lab?
- Internet/technology?

5. If you find that a mathematics problem is particularly difficult for you, how long do you spend on a mathematics problem before you seek assistance on it?

- In Basic Concepts 2 vs. other previous mathematics courses
- On homework for Basic Concepts 2 vs. other previous mathematics courses

6. When seeking mathematics assistance, what type do you normally seek?

- Teacher?
- Book?
- Friends/Peers?
- Classmates?
- People you perceive to be good at mathematics?
- Internet/technology?

7. What would you keep/change about the structure of Basic Concepts 2?

APPENDIX E

Crazy Cakes²⁰

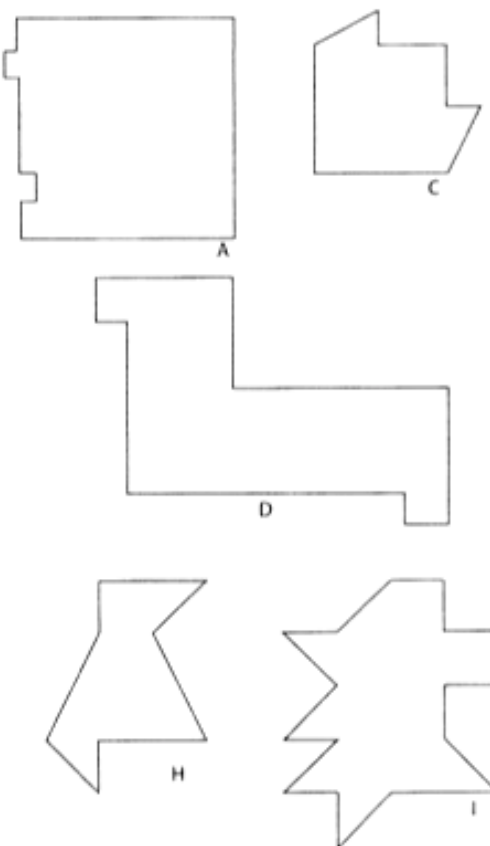
PAGE 2 OF 2

SESSION 1

MEASURING SPACE IN ONE, TWO, AND THREE DIMENSIONS

Math assignment: Crazy Cakes

Divide each of the "strange cakes" below into two parts with equal area. The two parts need not be congruent.



A C

D

H I

Note: Lettering is discontinuous because these are just five of the nine figures (A through I) on the "Crazy Cakes for Two" student sheet in *Different Shapes, Equal Pieces*.

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²⁰ From *Developing Mathematical Ideas (DMI): Measuring Space in One, Two and Three Dimensions* (p. 39), by D. Schifter, V. Bastable, and S. J. Russell, 2001, Lebanon, IN: Dale Seymour. Copyright 2001 by Education Development Center.

APPENDIX F

Exploring Area Handout

MAT 142
Exploring Area Formulas

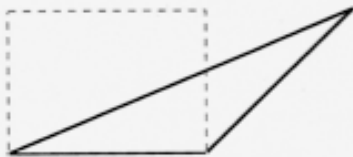
1. Area of parallelograms

Draw three different parallelograms below. Determine a way to find the area of each. Devise a general rule for the area of a parallelogram and explain how you know it will always work.

2. Area of trapezoids

Create several trapezoids and find the area of each. Devise a general rule for the area of a trapezoid and explain how you know it will always work.

3. Area of obtuse triangles



Is the area of this triangle also equal to half the area of the rectangle? (The height of the rectangle is the same as that of the triangle.) Come up with an argument to prove that it is.

APPENDIX G

Portion of a Sample Lesson Graph

Date: 02/18/10

Class Description: Quiz and Symmetry

Note:

- T denotes course instructor
- No Time Stamp – must use time on QuickTime play
- None of the participants are at a table together – Kayla is absent from class.

Time	Description	Comments
00:01:00	T gives students quizzes to begin and also instructs them to read the pages 282-283 in the book when they complete their quiz.	
00:19:15	T brings class back to whole group discussion about symmetry as read in the text and their prior knowledge. Discussion around: *What does symmetry mean? (Garrett states, “Same on both sides.”) * What does it mean to be the same on both sides? *Drawing a line through a figure (a line of symmetry) you can “see both sides and they would be the same if you laid them on top of each other.” *Reflection symmetry *Rotational symmetry	
00:20:35	T instructs class to write the word “MOM” on their paper and to draw a dotted line to represent the line of symmetry and discusses the results with the whole group.	
00:22:35	T instructs class to use the mirror to confirm whether their line of symmetry is correct – “When you look in the mirror do you see the exact same thing on the other side?” T poses to whole group, “Are there any other lines of symmetry?” T asks class to consider Melanie’s comment about seeing a line of symmetry through an M. There is small group and whole group discussion about using the first or second M of the word depending on the mirror and the reflection that can be seen. There is a discussion that the mirror creates a “problem” because “you can’t actually see through them”; T introduces the Mira (as opposed to	Interesting use of a tool – could be a basis of authority?

	the mirror) to the class.	
00:27:50	T discusses that they will be discussing rotational symmetry. T shows the class the National Library Virtual Manipulatives (NLVM) on the overhead projector and demonstrates the pattern block applet as students try by hand what she demonstrates. T asks, “How many lines of symmetry does a square have?” Answers include: horizontal, vertical and diagonals T asks, “Does it have rotational symmetry?” Skyler responds, “Since it has all sides equal it should.” T discusses “Since it has rotational symmetry, it should match up before you rotate it 360 degrees.” T instructs the class to find the number of rotational symmetries for a square. Students use their square pattern block to determine a number.	
00:35:26	T discusses how to check for rotational symmetry with the applet – stating “Do we get the same image before the black dot gets back to the same spot?” T discusses: how much did the square rotate for each image that matches – “What’s the angle measures for each one of those symmetries?” Discussion includes how to determine the angle measure of each rotation, a diamond in relation to a square, how to determine the movement to get to a specific angle measure rotation (for instance, what would the image look like with a 45 degrees rotation), does the shape change to a different shape once a rotation occurs?	Skyler discusses the measure through the diagonal of the square that goes through the vertex point of reference for the rotation of the square. T confirms that she is correct.

APPENDIX H

Portion of a Sample Coded Lesson Graph Synopsis

Kayla Lesson Graph Notes Coded

Note: Highlighted codes represent instances for which a participant is speaking directly to the course instructor (T).

Date	Page	Content	
01/07/10	3	Kayla discusses her thoughts on why a “pretend” student did a computation the way she did.	WCCE
01/12/10	4	Kayla’s group is incorrectly making a bar graph – the graph looks like a continuum with peaks and lows (more like a line graph). Kayla is with Michael’s group – she questions T about whether or not her graph is correct and even answers (correctly) what T asks in regard to interpreting the graph. (Should revisit this on page 5).	SAA SMJ/E
	18	Kayla offers up her reasoning for thinking about calculating mode is an interpretation of the results.	WMJ/E
01/14/10	2	Kayla discusses how students may have difficulty in finding conclusions (or interpretations) of the data they analyzed. Her discussion is echoed by another student.	WPJ WME/C
	5	Kayla discusses the analyzing that is occurring (she correctly states/uses the word).	WPJ WCCE
	8	Kayla discusses a video clip and her philosophy on what it means to “do math.”	WTP WPJ
	11	Kayla and Michael’s group work on a linking/block statistical activity but do appear confused as to what they are supposed to be doing.	
	15	Kayla’s group is using their linking blocks to illustrate the final answer – but not the process of finding the answer (in other words, the mean). (Although, later, on page 17, Michael clearly shows that he has an understanding of mean – but perhaps not in the context of the tower activity.)	
	18	Kayla clearly understands the activity now.	WMJ/E
	19	Kayla and Michael are discussing the next task of using mean to find the answer – but they are confused on the process or question (? – not sure which).	SED