ENDOTRIVIAL MODULES FOR CLASSICAL LIE SUPERALGEBRAS

by

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(Under the Direction of Daniel K. Nakano)

ABSTRACT

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over an algebraically closed field, k, of characteristic 0. An endotrivial $\mathfrak{g}$-module, M, is a $\mathfrak{g}$-supermodule such that $\mathrm{Hom}_k(M, M) \cong k \oplus P$ as $\mathfrak{g}$-supermodules, where k is the trivial module concentrated in degree $\bar{0}$ and P is a projective $\mathfrak{g}$-supermodule. Such modules form a group, denoted $T(\mathfrak{g})$, under the operation of the tensor product. We show that for an endotrivial module M, the syzygies $\Omega^n(M)$ are also endotrivial and for certain detecting Lie superalgebras of particular interest we show that $\Omega^1(k)$, along with the parity change functor, actually generate the group of endotrivials.

While it is not known in general whether the group of endotrivial modules for a given Lie superalgebra $\mathfrak{g}$ is finitely generated, the first classifications here support this result and another finiteness theorem maybe stated under under the additional assumption that a Lie superalgebra $\mathfrak{g}$ is classical and that $\mathfrak{g}_{\bar{0}}$ has finitely many simple modules of dimension $\leq n$ for some fixed $n \in \mathbb{N}$. In this case, we show that for the same fixed n, there are finitely many isomorphism classes of endotrivial modules of dimension n. While this result does not imply finite generation, it may be a useful tool in proving this result in the future.

The last result deals with relating the group of endotrivial modules for the Lie superalgebra $\mathfrak{gl}(n|n)$ to the group of endotrivial modules over a particular parabolic subalgebra $\mathfrak{p}$. The restriction map gives an embedding of the group $T(\mathfrak{g})$ into $T(\mathfrak{p})$. This result could

reduce the computation of the seemingly more complex $T(\mathfrak{g})$ to the understanding simpler case of $T(\mathfrak{p})$.

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CHAPTER 1

INTRODUCTION

The study of endotrivial modules began with Dade in 1978 when he defined endotrivial kG-modules for a finite group G in [16] and [17]. Endotrivial modules arose naturally in this context and play an important role in determining the simple modules for p-solvable groups. Dade showed that, for an abelian p-group G, endotrivial kG-modules have the form $\Omega^n(k) \oplus P$ for some projective module P, where $\Omega^n(k)$ is the nth syzygy of the trivial module k (defined in Section 3.2). In general, the endotrivial modules form an abelian group under the tensor product operation. It is known, via Puig in [22], that this group is finitely generated in the case of kG-modules and is completely classified for p-groups over a field of characteristic p by Carlson and Thévenaz in [13] and [14]. An important part in this classification is a technique where the modules in question are restricted to elementary abelian subgroups.

Carlson, Mazza, and Nakano have also computed the group of endotrivial modules for finite groups of Lie type (in the defining characteristic) in [9]. The same authors in [10] and Carlson, Hemmer, and Mazza in [8] give a classification of endotrivial modules for the case when G is either the symmetric or alternating group.

This class of modules has also been studied for modules over finite group schemes by Carlson and Nakano in [11]. It has been shown that all endotrivial modules for a unipotent abelian group scheme have the form $\Omega^n(k) \oplus P$ as well in this case. For certain group schemes of this type, a classification is also given in the same paper (see Section 4). The same authors proved, in an extension of this paper, that given an arbitrary finite group scheme, for a fixed n, the number of isomorphism classes of endotrivial modules of dimension n is finite (see [12]), but it is not known whether the endotrival group is finitely generated in this context.

We wish to extend the study of this class of modules to Lie superalgebra modules. First we must establish the correct notion of endotrivial module in this context. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over an algebraically closed field, k, of characteristic 0. A $\mathfrak{g}$-supermodule, M, is called endotrivial if there is a supermodule isomorphism $\operatorname{Hom}_k(M, M) \cong k \oplus P$ where k is the trivial supermodule concentrated in degree $\bar{0}$ and P is a projective $\mathfrak{g}$-supermodule.

There are certain subalgebras, denoted $\mathfrak{e}$ and $\mathfrak{f}$, of special kinds of classical Lie superalgebras which are of interest. These subalgebras "detect" the cohomology of the Lie superalgebra $\mathfrak{g}$. By this, we mean that the cohomology for $\mathfrak{g}$ embeds into particular subrings of the cohomology for $\mathfrak{e}$ and $\mathfrak{f}$. These detecting subalgebras can be considered analogous to the elementary abelian subgroups and are, therefore, of specific interest.

In this paper, we observe that the universal enveloping Lie superalgebra $U(\mathfrak{e})$ has a very similar structure to the group algebra kG when G is abelian, noncyclic of order 4 and $\operatorname{char} k = 2$ (although $U(\mathfrak{e})$ is not commutative). With this observation, we draw from the results of [7] to prove the base case in an inductive argument for the classification of the group of endotrivial $U(\mathfrak{e})$-supermodules. The inductive step uses techniques from [11] to complete the classification. As for the other detecting subalgebra $\mathfrak{f}$, even though $U(\mathfrak{f})$ is not isomorphic to $U(\mathfrak{e})$, reductions are made to reduce this case to the same proof.

The main result is that for the detecting subalgebras $\mathfrak{e}$ and $\mathfrak{f}$, denoted generically as $\mathfrak{a}$, the group of endotrivial supermodules, $T(\mathfrak{a})$, is isomorphic to $\mathbb{Z}_2$ when the rank of $\mathfrak{a}$ is one and $\mathbb{Z} \times \mathbb{Z}_2$ when the rank is greater than or equal to two.

We also show that for a classical Lie superalgebra $\mathfrak{g}$ such that there are finitely many simple $\mathfrak{g}_{\bar{0}}$-modules of dimension $\leq n$, there are only finitely many endotrivial $\mathfrak{g}$-supermodules of a fixed dimension n. This is done by considering the variety of all representations as introduced by Dade in [15]. In particular, this result holds for classical Lie superalgebras such that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie superalgebra.

The last chapter concerns the group of endotrivial modules for the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n|n)$. A particular parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is considered and it is shown that there

is a well defined map from $T(\mathfrak{g}) \to T(\mathfrak{p})$ by restricting a $\mathfrak{g}$-modules M to a $\mathfrak{p}$ module. The main goal of the section is to prove that this map is injective. This is done by showing that there is an induction functor which takes the trivial $\mathfrak{p}$-module to the trivial $\mathfrak{g}$-module, which means that this induction functor acts as a one sided inverse to restriction. This key result allows us to show that the kernel of the restriction map from $T(\mathfrak{g})$ to $T(\mathfrak{p})$ is just the trivial module, proving that the map is injective.

2.1 Superalgebras

We begin by defining the basic algebraic objects of interest in the sequel. Let R be a commutative ring.

Definition 2.1. An R-**graded vector space**, V, is a vector space with a family of subspaces $(V_r)_{r \in R}$ such that

$$V = \bigoplus_{r \in R} V_r.$$

Definition 2.2. A subspace $U \subseteq V$ is an R-**graded subspace** if

$$U = \bigoplus_{r \in R} (U \cap V_r).$$

That is to say that U contains all the homogeneous components of its elements. Note, there is a natural $\mathbb{Z}_2$ grading on any $\mathbb{Z}$ graded vector space given by taking

$$V_{\bar{0}} := \bigoplus_{i \in \mathbb{Z}} V_{2i} \text{ and } V_{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} V_{2i+1}.$$

Definition 2.3. Let W be another R-graded vector space. A linear mapping

$$f : V \to W$$

is said to be **homogeneous of degree** r, for some $r \in R$, if

$$f(V_s) \subseteq W_{r+s} \quad \forall s \in R.$$

A homomorphism of R-graded vector spaces is simply a homogeneous map of degree 0. An isomorphism of R-graded vector spaces is an invertible homomorphism and an automorphism of an R-graded vector space is an isomorphism of the vector space to itself.

Definition 2.4. Let V and W be two R-graded vector spaces. The tensor product $V \otimes W$ has a natural R-gradation given by

$$(V \otimes W)_r = \bigoplus_{s+t=r} (V_s \otimes W_t) \quad r, s, t \in R.$$

Definition 2.5. Let k be an algebraically closed field of characteristic 0. An algebra A over k is an R**-graded algebra over** k if the underlying vector space of A is R-graded and

$$A_i A_j \subseteq A_{i+j} \quad \text{for all } i, j \in R.$$

A homomorphism of R-graded algebras is a homomorphism of algebras which is also a degree 0 homomorphism of their underlying vector spaces.

A graded subalgebra of an R-graded algebra A is a subalgebra which is also an R-graded subspace of the underlying vector space of A. Similarly, a graded ideal of an R-graded algebra A is an ideal which is also an R-graded subspace of the underlying vector space of A.

Let R be the ring $\mathbb{Z}$ or $\mathbb{Z}_2$ and for a homogeneous element $r \in R_i$, define $|r| := i$. If we have two R-graded associative algebras, A and B, then we can define an algebra structure on the vector space $A \otimes B$.

Definition 2.6. Given A and B, R-graded algebras, the vector space $A \otimes B$ is an R-graded algebra under multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}(aa') \otimes (bb')$$

where a and b are homogeneous elements and we extend this to general elements by linearity. This is called the **graded tensor product of A and B** and is denoted by $A\overline{\otimes}B$.

Convention. From now on, we assume all elements to be homogeneous and all definitions are assumed to be extended to general elements by linearity, as is done above.

Now that we have defined the algebraic structures of interest, we consider modules over these algebras. In particular we are interested in modules that respect the grading structure.

Definition 2.7. Let A be an R-graded associative algebra and let V be an R-graded vector space that is also a left A-module. We say V is an R-**graded** A-**module** if

$$A_i V_j \subseteq V_{i+j} \quad \text{for all } i, j \in R.$$

An R-graded A-module homomorphism between V and W is an A-module homomorphism that is also a degree 0 homogeneous map of the R-graded vector spaces V and W.

Definition 2.8. A **superalgebra** is a $\mathbb{Z}_2$-graded algebra.

2.2   LIE SUPERALGEBRAS

Definition 2.9. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a superalgebra with the product denoted by $[-,-]$. If for all a, b, $c \in \mathfrak{g}$, we have

1. $[a, b] = -(-1)^{|a||b|}[b, a]$

2. $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$

then $\mathfrak{g}$ is called a **Lie superalgebra**.

This notation is intentionally suggestive of the (graded) commutator operation. In fact, if A is an associative superalgebra, then if we define

$$[a, b] := ab - (-1)^{|a||b|}ba \tag{2.1}$$

for all a, $b \in A$, then this operation gives A the structure of a Lie superalgebra and is denoted A_L.

We now consider a few observations about the structure of a Lie superalgebra, $\mathfrak{g}$. First, $\mathfrak{g}_{\bar{0}}$ is itself a Lie algebra since this is a closed subalgebra where graded anti-commutivity and graded Jacobi identity just become the standard versions. Second, we see that the bracket

operation gives us a map $\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$. So if we think of $\mathfrak{g}_{\bar{1}}$ as just being an abelian group, then we see that $\mathfrak{g}_{\bar{1}}$ becomes a $\mathfrak{g}_{\bar{0}}$-module. Furthermore, property 2 of definition 2.9 implies that $\mathfrak{g}_{\bar{1}}$ is actually a Lie algebra module. Finally, we note that (commutative) multiplication in $\mathfrak{g}_{\bar{1}}$ gives us a map into $\mathfrak{g}_{\bar{0}}$. Thus we have a symmetric bilinear map $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ which is $\mathfrak{g}_{\bar{0}}$ invariant by the graded Jacobi identity. Equivalently, we can think of this as a homomorphism of $\mathfrak{g}_{\bar{0}}$ modules $\phi : S^2 \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$. We also note that for x, y, and $z \in \mathfrak{g}_{\bar{1}}$, $\phi(x,y)z + \phi(y,z)x + \phi(z,x)y = 0$.

In fact, this information encapsulates all the information of a Lie superalgebra and can be used to define a Lie superalgebra. If we have a Lie algebra $\mathfrak{g}_{\bar{0}}$, with bracket $\langle , \rangle$, a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$, and a $\mathfrak{g}_{\bar{0}}$ homomorphism $\phi : S^2 \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ satisfying $\phi(x,y)z + \phi(y,z)x + \phi(z,x)y = 0$ for x, y, and $z \in \mathfrak{g}_{\bar{1}}$, then we can define

$$[a,b] := \langle a,b \rangle \quad \text{for } a,b \in \mathfrak{g}_{\bar{0}}$$
$$[a,x] := a \cdot x \quad \text{for } a \in \mathfrak{g}_{\bar{0}}, x \in \mathfrak{g}_{\bar{1}}$$
$$[x,a] := -a \cdot x \quad \text{for } a \in \mathfrak{g}_{\bar{0}}, x \in \mathfrak{g}_{\bar{1}}$$
$$[x,y] := \phi(x,y) \quad \text{for } x,y \in \mathfrak{g}_{\bar{1}}$$

to make the vector space $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ into a Lie superalgebra.

Example 2.10. Let V be an R-graded vector space. We can decompose the endomorphism ring $\mathrm{End}_k V$ as follows. If we define $(\mathrm{End}_k V)_r = \{\phi \in \mathrm{End}\, V | \phi(V_s) \subseteq V_{r+s}\}$, then

$$\mathrm{End}_k V = \bigoplus_{r \in R} (\mathrm{End}_k V)_r$$

which forms an associative R-graded algebra. In the case when $R = \mathbb{Z}_2$, $\mathrm{End}_k V$ is an associative superalgebra, and so using 2.1 to define a bracket on $\mathrm{End}_k V$, we obtain a Lie superalgebra. This is denoted $l(V) := (\mathrm{End}_k V)_L$.

Definition 2.11. A **derivation of degree r** of a superalgebra A, is an endomorphism $D \in (\mathrm{End}_k A)_r$ satisfying

$$D(ab) = D(a)b + (-1)^{r|a|} a D(b).$$

The space of such derivations is denoted $(\text{der } A)_r \subseteq (\text{End}_k A)_r$ and we define $\text{der } A :=$ $(\text{der } A)_{\bar{0}} \oplus (\text{der } A)_{\bar{1}}$. We can further see that for two derivations, $D \in (\text{der } A)_r$ and $D' \in (\text{der } A)_s$, their bracket $[D, D']$ as defined in 2.1 is also a derivation by computing

$$[D, D'](ab) = (DD' - (-1)^{rs} D'D)(ab)$$

$$= D(D'(a)b + (-1)^{s|a|} a D'(b)) - (-1)^{rs} D'(D(a)b + (-1)^{r|a|} a D(b))$$

$$= D(D'(a))b - (-1)^{rs} D'(D(a))b$$

$$+ (-1)^{(s+r)|a|} a D(D'(b)) - (-1)^{rs+(r+s)|a|} a D'(D(b))$$

$$+ (-1)^{r(s+|a|)} D'(a)D(b) - (-1)^{rs+r|a|} D'(a)D(b)$$

$$+ (-1)^{s|a|} D(a)D'(b) - (-1)^{rs+(r+|a|)s} D(a)D'(b)$$

$$= (D(D'(a)) - (-1)^{rs} D'(D(a)))b$$

$$+ (-1)^{(r+s)|a|} (a D(D'(b)) - (-1)^{rs} a D'(D(b)))$$

$$+ (-1)^{r(s+|a|)} (D'(a)D(b) - D'(a)D(b))$$

$$+ (-1)^{s|a|} (D(a)D'(b) - (-1)^{2rs} D(a)D'(b))$$

$$= [D, D'](a)b + (-1)^{(r+s)|a|} a[D, D'](b).$$

We can see that $[D, D'] \in (\text{der } A)_{r+s}$ as is expected. Because of this computation, if we think of $\text{End}_k A$ as a Lie superalgebra, denoted as $l(A)$, by using the bracket defined in 2.1 and since $(\text{der } A)_r \subseteq (\text{End}_k A)_r$ (i.e. the grading of the subspace respects the grading of the original space), then $\text{der } A$ is a Lie subsuperalgebra of $l(A)$.

Example 2.12. If $\mathfrak{g}$ is a Lie superalgebra, then the graded Jacobi identity tells us that the map

$$\text{ad } a : \mathfrak{g} \to \mathfrak{g}$$

$$b \mapsto [a, b]$$

is a derivation of $\mathfrak{g}$. We refer to derivations of this kind as inner derivations. A similarly messy computation as the one above will yield that $[D, \text{ad } a] = \text{ad } D(a)$ which shows that the inner derivations form an ideal in $\text{der } \mathfrak{g}$.

CHAPTER 3

REPRESENTATIONS OF LIE SUPERALGEBRAS

3.1   THE UNIVERSAL ENVELOPING ALGEBRA

Just as in the case of standard Lie algebras, we can construct a Universal Enveloping Algebra.
Many of the ideas are very similar and the important results will be listed here for reference.

We construct the universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$ by the following
process. First, we take the tensor algebra $\mathcal{T}(\mathfrak{g})$ of the vector space $\mathfrak{g}$. We note that, by
definition 2.4, each of the tensor powers of $\mathfrak{g}$ inherits a $\mathbb{Z}_2$ grading, hence we can give the
entire algebra $\mathcal{T}(\mathfrak{g})$ a $\mathbb{Z}_2$ grading. It is clear that the canonical injection (into the first tensor
power) $\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ is an even homomorphism of supervector spaces. Furthermore, we can
endow $\mathcal{T}(\mathfrak{g})$ with the standard algebra (or in this case superalgebra) structure by taking the
tensor product to be the multiplication.

Now, we can take the two sided ideal, $\mathcal{I}$, generated by elements of the from

$$a \otimes b - (-1)^{|a||b|} b \otimes a - [a,b]. \tag{3.1}$$

This is the same construction as in the Lie algebra case but with the standard sign
grading added. By the definition of the product in $\mathcal{T}(\mathfrak{g})$ and of the bracket operation of $\mathfrak{g}$,
the element above is homogeneous of degree $|a| + |b|$. Thus, the quotient algebra inherits a
$\mathbb{Z}_2$ grading as well.

Definition 3.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. Consider the tensor algebra $\mathcal{T}(\mathfrak{g})$
and the two sided ideal generated by elements of the form found in equation 3.1. Then

$$U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g}) \Big/ \mathcal{I}$$

is defined to be the **Universal Enveloping Algebra** of the Lie superalgebra $\mathfrak{g}$.

Associated to $U(\mathfrak{g})$, we have a map obtained by composing the inclusion map $\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ with the projection map $\mathcal{T}(\mathfrak{g}) \twoheadrightarrow \mathcal{T}(\mathfrak{g})\big/\mathcal{I} = U(\mathfrak{g})$. We label this map by

$$\sigma : \mathfrak{g} \to U(\mathfrak{g}).$$

Since the operation in $U(\mathfrak{g})$ is assumed to be the tensor product, we take the convention of using juxtaposition to denote this multiplication. With this in mind, we make the comment that all elements of $U(\mathfrak{g})$ are linear combinations of products of the form $\sigma(g_1)\sigma(g_2)\ldots\sigma(g_n)$ for $g_i \in \mathfrak{g}$. Homogeneous elements of this form are of degree $|g_1| + |g_2| + \cdots + |g_n|$.

We refer to this algebra as being "universal" in the sense that it satisfies the following universal property. If A is a unital associative superalgebra such that there is a map $\phi : \mathfrak{g} \to A$ that satisfies $\phi([a,b]) = \phi(a)\phi(b) - (-1)^{|a||b|}\phi(b)\phi(a)$, then there exists a unique superalgebra homomorphism $\overline{\phi}$ such that $\overline{\phi}(1) = 1$ and the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \ \sigma\ \ } & U(\mathfrak{g}) \\ & {\phi}\searrow & \Big\downarrow{\overline{\phi}} \\ & & A \end{array} \qquad (3.2)$$

commutes.

Theorem 3.2 (Poincaré, Birkhoff, Witt)**.** *Let* $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ *be a Lie superalgebra where* $\mathfrak{g}_{\overline{0}}$ *has basis* $a_1, \ldots, a_s$ *and* $\mathfrak{g}_{\overline{1}}$ *has basis* $b_1, \ldots, b_t$*. Elements of the form*

$$\sigma(a_{i_1})^{k_{i_1}}\sigma(a_{i_2})^{k_{i_2}}\ldots\sigma(a_{i_m})^{k_{i_m}}\sigma(b_{j_1})\sigma(b_{j_2})\ldots\sigma(b_{j_n})$$

where $1 \le i_1 < \cdots < i_m \le s$ *and* $1 \le j_1 < \cdots < j_n \le t$ *and* $k_{i_r} > 0$ *give a basis for* $U(\mathfrak{g})$*. In the case of the empty product (i.e. $m = n = 0$), we define the product to be 1.*

It is sometimes valuable to note that we may in fact choose any ordering on the basis elements for $\mathfrak{g}$ in the above theorem rather than separating them according to the $\mathbb{Z}_2$ grading as given above.

Additionally, we observe that exponents are not necessary for the elements in $\mathfrak{g}_{\bar{1}}$ because for $b \in \mathfrak{g}_{\bar{1}}$, we have

$$\sigma(b)^2 = \sigma(b) \otimes \sigma(b) = \frac{2\sigma(b) \otimes \sigma(b)}{2} = \frac{\sigma(b) \otimes \sigma(b) + \sigma(b) \otimes \sigma(b)}{2} = \frac{[\sigma(b), \sigma(b)]}{2} = \frac{\sigma([b, b])}{2}.$$

Furthermore, since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}}$, it is clear that exponents are not necessary for elements in $\mathfrak{g}_{\bar{1}}$ as indicated in Theorem 3.2.

Corollary 3.3. *The canonical inclusion $\sigma : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Convention. Given the above corollary, we now take the convention that $\mathfrak{g}$ is identified with the graded subspace of $U(\mathfrak{g})$ via the mapping σ (unless the mapping is being emphasized).

3.2 Representations of Lie Superalgebras

Now that we have developed the analogue of enveloping algebras in the $\mathbb{Z}_2$ graded case, we can attempt to use these associative superalgebras to study the representation theory of Lie superalgebras. We begin by developing the definition of a representation of a Lie superalgebra.

Definition 3.4. A **Lie superalgebra homomorphism** between A and B is a homomorphism of the $\mathbb{Z}_2$-graded algebras (so it is homogeneous map of degree 0) that respects the bracket operation of the respective superalgebras. That is, if $\phi : A \to B$ is the homomorphism, then

$$\phi([a, a']_A) = [\phi(a), \phi(a')]_B.$$

Given the above definitions, we can now consider modules over Lie superalgebras.

Definition 3.5. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$ graded vector space. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, we define a **linear representation of a Lie superalgebra $\mathfrak{g}$ in V** to be a Lie superalgebra homomorphism $\phi : \mathfrak{g} \to l(V)$. It is common practice to refer to V as a $\mathfrak{g}$-supermodule and denote $g \in \mathfrak{g}$ as acting on V by $g(v)$ rather than $\phi(g)(v)$.

We note that, since ϕ is even, then the grading of elements mapping from $\mathfrak{g}$ to $\mathrm{End}_k V$ is preserved and so $\mathfrak{g}_i(V_j) \subseteq V_{i+j}$. Also, since we can explicitly write out the bracket operation in $l(V)$, we can observe that (in the supermodule notation)

$$[g, g'](v) = g(g'(v)) - (-1)^{|g||g'|} g'(g(v)).$$

We define a subsupermodule, U, of a $\mathfrak{g}$-supermodule V to be a $\mathbb{Z}_2$ graded subspace of V such that $\mathfrak{g}(U) \subseteq U$. A $\mathfrak{g}$-supermodule V is said to be irreducible if the only nontrivial subsupermodule of V is V itself.

If we are given a linear representation of $\mathfrak{g}$ in V, ϕ, we can use the universal property given in Equation 3.2 to obtain a map $\overline{\phi} : U(\mathfrak{g}) \to l(V)$. This is a unique homomorphism of associative superalgebras that satisfies $\overline{\phi}(g) = \phi(g)$ for all $g \in \mathfrak{g}$ and $\phi(1) = \mathrm{id}$. We then extend this to general products in $U(\mathfrak{g})$ to see that $\overline{\phi}(U(\mathfrak{g})_i)V_j \subseteq V_{i+j}$. Thus, we use ϕ to extend uniquely to $\overline{\phi}$ which is a representation of the associative superalgebra $U(\mathfrak{g})$ in the super vector space V.

Now, assume that we have ψ, a representation of the associative superalgebra $U(\mathfrak{g})$ in the super vector space V. We can restrict to $\mathfrak{g} \subseteq U(\mathfrak{g})$ to obtain a map, $\psi|_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}_k V$. Because of the relations imposed upon $U(\mathfrak{g})$ and the fact that ψ is a homomorphism of superalgebras, $\psi|_{\mathfrak{g}}$ is also a Lie superalgebra homomorphism and consequently, V is given the structure of a $\mathfrak{g}$-supermodule.

We can also see that $\phi = \overline{\phi}|_{\mathfrak{g}}$ and $\overline{\psi|_{\mathfrak{g}}} = \psi$ so these operations are inverses of each other and so these two notions of representations (or modules) are interchangeable. Thus, we can use either terminology in the following development of the representation theory, noting that we now have the advantage of using the associative property when we speak of $U(\mathfrak{g})$-supermodules.

Example 3.6 (Trivial Module). We can consider the vector space k as an associative super-algebra by taking $k_{\overline{0}} = k$ and $k_{\overline{1}} = \{0\}$. If $\mathfrak{g}$ is a Lie superalgebra, we can then give k the

structure of a $\mathfrak{g}$-supermodule by letting every element $g \in \mathfrak{g}$ act by 0, i.e.

$$g(c) = 0 \quad \text{for all } g \in \mathfrak{g} \text{ and } c \in k.$$

We can also see that, when we extend this module to $U(\mathfrak{g})$, all elements of $U(\mathfrak{g})$ act by 0 as well, except for the scalars in $U(\mathfrak{g})$, i.e. elements in the image of $T(\mathfrak{g})_0$ (note, we are using the $\mathbb{Z}$-grading here) under the projection map, which act via multiplication.

Example 3.7 (Adjoint Representation). Let $\mathfrak{g}$ be a Lie superalgebra. The map given in Example 2.12 allows us to define a representation of $\mathfrak{g}$ in itself by

$$\text{ad} : \mathfrak{g} \to \text{End}_k(\mathfrak{g})$$

$$g \mapsto \text{ad}\, g.$$

By (2) in Definition 2.9, we can check that this is a homomorphism of Lie superalgebras as well and thus, the ad map gives a representation of $\mathfrak{g}$ in itself.

Definition 3.8. Let $\mathfrak{g}$ be a Lie superalgebra. A **homomorphism of $\mathfrak{g}$-supermodules**, $\phi : V \to W$, is an even homomorphism of $\mathbb{Z}_2$ graded vector spaces such that for all $g \in \mathfrak{g}$ and $v \in V$

$$\phi(g(v)) = g(\phi(v))$$

or equivalently

$$\phi(u(v)) = u(\phi(v))$$

for all $u \in U(\mathfrak{g})$ and $v \in V$.

Alternatively, if we have a $\mathfrak{g}$-supermodule V and an $\mathfrak{h}$-supermodule W, we can consider the space of k-linear mappings $\text{Hom}_k(V, W)$. This space has a natural $\mathbb{Z}_2$ grading given by

$$\text{Hom}_k(V, W)_i = \left\{ \phi \in \text{Hom}_k(V, W) \mid \phi(V_j) \subseteq W_{i+j} \right\}.$$

We can then give this space the structure of a $\mathfrak{g} \times \mathfrak{h}$-supermodule where the action on some $\phi \in \text{Hom}_k(V, W)_r$ is given by

$$((g, h)\phi)(v) = h(\phi(v)) - (-1)^{r|g|}\phi(g(v)).$$

In the special case when $\mathfrak{g} = \mathfrak{h}$, we can use the diagonal homomorphism of $\mathfrak{g}$ into $\mathfrak{g} \times \mathfrak{g}$ to turn $\text{Hom}_k(V, W)$ into a $\mathfrak{g}$-supermodule with action given by

$$(g\phi)(v) = g(\phi(v)) - (-1)^{r|g|}\phi(g(v)).$$

We can also consider maps from one $\mathfrak{g}$-supermodule to another which respect the action of $\mathfrak{g}$. Let V and W be two $\mathfrak{g}$-supermodules. An element $\phi \in \text{Hom}_k(V, W)_r$ is $\mathfrak{g}$-invariant if and only if

$$g(\phi(v)) = (-1)^{r|g|}\phi(g(v)) \tag{3.3}$$

for all $g \in \mathfrak{g}$ and $v \in V$. We note that this is exactly the subset of $\text{Hom}_k(V, W)$ on which $\mathfrak{g}$ acts trivially. This set is denoted $\text{Hom}_{\mathfrak{g}}(V, W)$ or $\text{Hom}_{U(\mathfrak{g})}(V, W)$, depending on which language is appropriate. We make the observation that $\text{Hom}_{\mathfrak{g}}(V, W)_{\bar{0}}$ is the set of homomorphisms of V into W.

With this in mind, we can now construct and consider two important modules of this form: $\text{Hom}_k(V, k)$ and $\text{Hom}_k(V, V)$.

Example 3.9 (Dual Module)**.** First, we examine $\text{Hom}_k(V, k)$, denoted V^*, the dual module. Recall that k is a supermodule (see Example 3.6) with the odd part being $\{0\}$. We can see that V^* has its $\mathbb{Z}_2$ gradation given by

$$(V^*)_i = \left\{ \phi \in V^* \mid \phi(V_{i+\bar{1}}) = \{0\} \right\}.$$

Given the above action, we can simplify this to be

$$(g\phi)(v) = -(-1)^{r|g|}\phi(g(v)).$$

Example 3.10 (Endomorphism Module)**.** Given a $\mathfrak{g}$-supermodule M, we can consider the $\mathfrak{g}$-supermodule $\text{Hom}_k(M, M) = \text{End}_k(M)$ of endomorphisms of M. This kind of module will be of primary interest for the remainder of this paper.

We also have another tool for producing new modules from ones we already know about.

Definition 3.11. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. We can define

$$\Pi : \mathrm{mod}(\mathfrak{g}) \to \mathrm{mod}(\mathfrak{g})$$

by $\Pi(V) = \Pi(V)_{\bar{0}} \oplus \Pi(V)_{\bar{1}}$ where $\Pi(V)_{\bar{0}} = V_{\bar{1}}$ and $\Pi(V)_{\bar{1}} = V_{\bar{0}}$. This operation is known as the **parity change functor**.

Note that, since the vector spaces are the same, the endomorphisms of $\Pi(V)$ are the same as V and the grading of the endomorphisms is preserved. Thus we can see that if V is a $\mathfrak{g}$-supermodule, then the same map turns $\Pi(\mathfrak{g})$ into a $\mathfrak{g}$-supermodule as well, however, this module does not necessarily have to be isomorphic to the original module.

At this point, we now wish to specialize to the categories of interest for the remainder of this paper. We say that a module M is *finitely semisimple* if it is isomorphic to a direct sum of finite dimensional simple subsupermodules. Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{t}$ a subsuper-algebra of $\mathfrak{g}$. Let $\mathcal{C} = \mathcal{C}_{(\mathfrak{g},\mathfrak{t})}$ be the full subcategory of the category of all $\mathfrak{g}$-supermodules such that the objects are finitely semisimple as $\mathfrak{t}$-modules. The projective objects in this category are $(\mathfrak{g}, \mathfrak{t})$-projective objects which are defined as follows as in [21, Appendix D].

If $\mathfrak{g}$ is a Lie superalgebra and $\mathfrak{t} \subseteq \mathfrak{g}$ a subalgebra, a sequence of $\mathfrak{g}$-supermodules

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

where each f_i is even is called $(\mathfrak{g}, \mathfrak{t})$-exact if it is exact as a sequence as $\mathfrak{g}$-supermodules and when the sequence is considered as $\mathfrak{t}$-supermodules, $\ker f_i$ is a direct summand of $M_i|_{\mathfrak{t}}$ for all i. A $\mathfrak{g}$-supermodule is called $(\mathfrak{g}, \mathfrak{t})$-projective if for any $(\mathfrak{g}, \mathfrak{t})$-exact sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

and $\mathfrak{g}$-supermodule map $h : P \to M_3$ there is a $\mathfrak{g}$-supermodule map $\tilde{h} : P \to M_2$ such that $g \circ \tilde{h} = h$. When it is clear from context which subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is being considered, $(\mathfrak{g}, \mathfrak{t})$-projective modules will be referred to as relatively projective modules, i.e., projective relative to the subalgebra $\mathfrak{t}$. Note that any projective $\mathfrak{g}$-module is necessarily $(\mathfrak{g}, \mathfrak{t})$-projective. Relatively injective modules are defined in a dual way.

Next, we define $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{t})}$ to be the full subcategory of $\mathcal{C}$ where the objects are finite dimensional modules. In this work, the particular case of interest will be $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ and $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projective modules. We note that this category has enough projectives as detailed in [2] because if M is any $U(\mathfrak{g})$-supermodule, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M$ is a $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projective module (as shown in [21]) which surjects onto M. Dually, any such M also has an injective module I into which M injects given by $\mathrm{Hom}_{U(\mathfrak{g}_{\bar{0}})}(U(\mathfrak{g}), M)$.

Now, we can introduce a module which will be of particular interest for the remainder of this paper. We may use this definition in finite dimensional, self injective module categories with enough projectives.

Definition 3.12. Let $\mathfrak{g}$ be a Lie superalgebra and let M be a $\mathfrak{g}$-supermodule. Let P be a minimal projective which surjects on to M (called a projective cover), with the map

$$\psi : P \twoheadrightarrow M.$$

Then we define the **1st syzygy of** M to be $\ker \psi$ and denote it by $\Omega(M)$ or $\Omega^1(M)$. This is also referred to as a Heller shift in some literature. We can extend this to all positive integers by inductively defining $\Omega^{n+1} := \Omega(\Omega^n)$.

Similarly, given M, let I be the injective hull of M with the inclusion

$$\iota : I \hookrightarrow M$$

then we define $\Omega^{-1}(M) := \mathrm{coker}\,\iota$. This is extended to negative integers by defining $\Omega^{-n} := \Omega^{-1}(\Omega^{-n+1})$.

Finally, define $\Omega^0(M)$ to be the compliment of the largest projective direct summand of M. In other words, we can write $M = \Omega^0(M) \oplus Q$ where Q is projective and maximal as a projective summand.

Thus, we now have defined the n-**th syzygy of** M for any integer n.

ENDOTRIVIAL MODULES

4.1 DEFINITION AND PRELIMINARY RESULTS

Note. We are working in the category $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$

Definition 4.1. Given a category of modules, $\mathcal{A}$, consider the category with the same objects as the original category and an equivalence relation on the morphisms given by $f \sim g$ if $f - g$ factors through a projective module in $\mathcal{A}$. This is called the **stable module category** of $\mathcal{A}$ and is denoted by $\text{Stmod}(\mathcal{A})$.

Definition 4.2. Let $\mathfrak{g}$ be a Lie superalgebra and M be a $\mathfrak{g}$-supermodule. We say that M is an **endotrivial module** if $\text{End}_k(M) \cong k \oplus P$ where k is the trivial module discussed in Example 3.6 and P is a projective supermodule.

Since we have the supermodule isomorphism $\text{Hom}_k(V, W) \cong W \otimes V^*$ for two $\mathfrak{g}$-supermodules V and W, we will often times rephrase the condition for a module M being endotrivial as

$$M \otimes M^* \cong k \oplus P.$$

Lemma 4.3 (Schanuel)**.** *Let* $0 \to M_1 \to P_1 \to M \to 0$ *and* $0 \to M_2 \to P_2 \to M \to 0$ *be short exact sequences of modules where* P_1 *and* P_2 *are projective, then* $M_1 \oplus P_2 \cong M_2 \oplus P_1$.

The proof is simple and can be found in [1]. This lemma is useful because it tells us that, in the stable category, we do not have to necessarily use the projective cover to obtain the syzygy of a module. Indeed, any projective module will suffice because the kernels of the projection maps will only differ by projective summands.

Proposition 4.4. *The category $\mathcal{F}$ is self injective. That is, a $\mathfrak{g}$-module M is projective if and only if it is injective.*

The proof can be found in Proposition 2.2.2 in [3].

Lemma 4.5. *For a $\mathfrak{g}$-module M, we have the following*

$$\Omega^0(M) \cong \Omega^{-1}(\Omega^1(M)) \cong \Omega^1(\Omega^{-1}(M)). \tag{4.1}$$

Proof. First, we recall that $M \cong \Omega^0(M) \oplus P'$ for some projective module P'. We then take the projective cover of M, $P \oplus P'$, where P is the projective cover of $\Omega^0(M)$ and have the short exact sequence

$$0 \longrightarrow \Omega^1(M) \longrightarrow P \oplus P' \longrightarrow \Omega^0(M) \oplus P' \longrightarrow 0.$$

If we want to compute $\Omega^{-1}(\Omega^1(M))$, we need to find an injective hull for $\Omega^1(M)$ and take the cokernel of the map. This cokernel will be $\Omega^{-1}(\Omega^1(M))$. However, we can see that, since P is projective, by Proposition 4.4, P is also injective. Moreover, it is the injective hull of $\Omega^1(M)$ since P was a projective cover of $\Omega^0(M)$. So the cokernel of the injection is just $\Omega^0(M)$ and we conclude that $\Omega^{-1}(\Omega^1(M)) \cong \Omega^0(M)$

Similarly, we have a short exact sequence of the form

$$0 \longrightarrow \Omega^0(M) \oplus I' \longrightarrow I \oplus I' \longrightarrow \Omega^{-1}(M) \longrightarrow 0$$

relating M and $\Omega^{-1}(M)$ via the injective hull of M. Since I is injective (and minimal), it is also projective (and minimal). Thus, this gives a minimal projective cover of $\Omega^{-1}(M)$ whose kernel is $\Omega^0(M)$. This is the definition of $\Omega^1(\Omega^{-1}(M))$ and so we have $\Omega^1(\Omega^{-1}(M)) \cong \Omega^0(M)$. $\square$

Corollary 4.6. *For a $\mathfrak{g}$-module M, $\Omega^n(\Omega^m(M)) \cong \Omega^{n+m}(M)$ for any $n, m \in \mathbb{Z}$.*

Proof. We note that if n and m are of the same sign then this is simply the definition of the syzygy. When they are of different signs, this follows from a simple induction argument using Lemma 4.5. $\square$

Proposition 4.7. *For a $\mathfrak{g}$-module M, $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$ for any $n \in \mathbb{Z}$.*

Proof. When $n = 0$, the claim is clear since we are in a self injective category. We proceed by induction on n. First we consider the definition of $\Omega^1(M)$. We take a projective cover of M and take the kernel to be $\Omega^1(M)$ as follows

$$0 \longrightarrow \Omega^1(M) \longrightarrow P \longrightarrow M \longrightarrow 0.$$

We now dualize this short exact sequence (i.e. apply $\mathrm{Hom}_k(-, k)$ to the sequence) to get a new short exact sequence,

$$0 \longrightarrow M^* \longrightarrow P^* \longrightarrow (\Omega^1(M))^* \longrightarrow 0.$$

Because of our assumptions on P, we see that P^* is the injective hull of M^*. So by definition, the cokernel of the injective map is then $\Omega^{-1}(M^*)$. However, we can also see that we have completed the base case since $(\Omega^1(M))^* \cong \Omega^{-1}(M^*)$.

Now, we assume that $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$ for some positive $n \in \mathbb{Z}$ and we will show that the same is true for $n+1$. We do this just by recalling that the definition of $\Omega^{n+1}(M) = \Omega^1(\Omega^n(M))$. So

$$(\Omega^{n+1}(M))^* = (\Omega^1(\Omega^n(M)))^* \cong \Omega^{-1}((\Omega^n(M))^*) \cong (\Omega^{-1}(\Omega^{-n}(M^*))) = \Omega^{-(n+1)}(M^*)$$

This completes the inductive step and proves the claim for all positive integers.

If n is negative, then $-n$ is positive so

$$(\Omega^{-n}(M^*))^* \cong \Omega^n(M^{**}) \cong \Omega^n(M).$$

Now, apply the dual once more to get

$$(\Omega^{-n}(M^*))^{**} \cong \Omega^{-n}(M^*) \cong (\Omega^n(M))^*$$

so $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$ for negative n as well. $\qquad\square$

Lemma 4.8. *Let P be a projective $\mathfrak{g}$-module. Then $P \otimes_k N$ is also projective for any $\mathfrak{g}$-module N.*

Proof. Let M be a $\mathfrak{g}$-module and P and N as above. By the tensor identity found in (see Lemma 2.3.1 in [2]), we have

$$\operatorname{Ext}_{\mathcal{F}}^n(M, P \otimes N) \cong \operatorname{Ext}_{\mathcal{F}}^n(M \otimes N^*, P) = 0$$

for $n > 0$ since P is projective. Hence, $P \otimes N$ is also projective. $\qquad \square$

Lemma 4.9. *Let M and P be $\mathfrak{g}$-modules, and let P be projective. Then $\Omega^n(M \oplus P) \cong \Omega^n(M)$.*

Proof. When $n = 0$, note that $M \cong \Omega^0(M) \oplus P'$ where P' is a maximal. Then $P' \oplus P$ is a maximal projective summand of $M \oplus P$ by construction and so we conclude that $\Omega^0(M \oplus P) \cong \Omega^0(M)$.

The rest of the proof follows directly from Corollary 4.6 since

$$\Omega^n(M \oplus P) \cong \Omega^n(\Omega^0(M \oplus P)) \cong \Omega^n(\Omega^0(M)) \cong \Omega^n(M)$$

holds for any $n \in \mathbb{Z}$. $\qquad \square$

Proposition 4.10. *Let M and N be $\mathfrak{g}$-modules. Then for any $n \in \mathbb{Z}$, $\Omega^n(M) \otimes N \cong \Omega^n(M \otimes N) \oplus P$ for some projective module P.*

Proof. First we consider the case $n = 0$. First we compute $\Omega^0(M \otimes N)$. Since $M \cong \Omega^0(M) \oplus P$ and $N \cong \Omega^0(N) \oplus P'$, when we take $M \otimes N \cong (\Omega^0(M) \oplus P) \otimes (\Omega^0(N) \oplus P')$ and distribute the tensor product, (by recalling Lemma 4.8) we see that the only non-projective term is $\Omega^0(M) \otimes \Omega^0(N)$. Thus, $\Omega^0(M \otimes N) \subseteq \Omega^0(M) \otimes \Omega^0(N)$. Now, we compute that

$$\Omega^0(M) \otimes N \cong \Omega^0(M) \otimes (\Omega^0(N) \oplus P') \cong (\Omega^0(M) \otimes \Omega^0(N)) \oplus (\Omega^0(M) \otimes P')$$

$$\cong (\Omega^0(M) \otimes \Omega^0(N)) \oplus P'' \cong \Omega^0(M \otimes N) \oplus P'''.$$

For positive n, we again use induction. We begin the base case by considering the sequence defining $\Omega^1(M)$,

$$0 \longrightarrow \Omega^1(M) \longrightarrow P \longrightarrow M \longrightarrow 0.$$

We can now take this sequence and tensor it with N. Since we are tensoring over k, this is an exact operation. So we obtain a new sequence

$$0 \longrightarrow \Omega^1(M) \otimes N \longrightarrow P \otimes N \longrightarrow M \otimes N \longrightarrow 0.$$

Since $P \otimes N$ is a projective module, then by Lemma 4.3 $\Omega^1(M) \otimes N \cong \Omega^1(M \otimes N) \oplus P'$ for some projective module P'.

For the inductive step, we assume the claim for n. Then,

$$\Omega^{n+1}(M) \otimes N \cong \Omega^1(\Omega^n(M)) \otimes N \cong \Omega^1(\Omega^n(M) \otimes N) \oplus P'$$

$$\cong \Omega^1(\Omega^n(M \otimes N) \oplus P'') \oplus P' \cong \Omega^{n+1}(M \otimes N) \oplus P'$$

where the last isomorphism is given by Lemma 4.9.

We can use Proposition 4.7 to handle the case where $n < 0$. In this case, we already have that

$$\Omega^{-n}(M) \otimes N \cong \Omega^{-n}(M \otimes N) \oplus P'.$$

We can take duals on both sides and replace M and N by their duals as well without changing anything. So now we have

$$(\Omega^{-n}(M^*) \otimes N^*)^* \cong (\Omega^{-n}(M^* \otimes N^*) \oplus P')^*$$

$$\Omega^{-n}(M^*)^* \otimes N^{**} \cong \Omega^{-n}(M^* \otimes N^*)^* \oplus P'^*$$

$$\Omega^n(M^{**}) \otimes N \cong \Omega^n(M^{**} \otimes N^{**}) \oplus P'^*$$

$$\Omega^n(M) \otimes N \cong \Omega^n(M \otimes N) \oplus P'^*.$$

$\square$

Corollary 4.11. *Let M and N be $\mathfrak{g}$-modules and m, $n \in \mathbb{Z}$. Then $\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{m+n}(M \otimes N) \oplus P$ for some projective module P.*

Proof. We us two applications of Proposition 4.10.

$$\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^m(M \otimes \Omega^n(N)) \oplus P$$

$$\cong \Omega^m(\Omega^n(M \otimes N) \oplus P') \oplus P$$

$$\cong \Omega^m(\Omega^n(M \otimes N)) \oplus P$$

$$\cong \Omega^{m+n}(M \otimes N) \oplus P$$

where the third isomorphism is given by Lemma 4.9. □

Before the main theorem of this section, we prove one last proposition.

Proposition 4.12. *Let M and N be $\mathfrak{g}$-modules. Then $\Omega^n(M) \oplus \Omega^n(N) \cong \Omega^n(M \oplus N)$ for any $n \in \mathbb{Z}$.*

Proof. We will just prove the basic case and the rest of the proof will be clear from the approaches used previously in the section.

By the definition of Ω^1, we have two short exact sequences

$$0 \longrightarrow \Omega^1(M) \longrightarrow P \longrightarrow M \longrightarrow 0$$

associated to M and

$$0 \longrightarrow \Omega^1(N) \longrightarrow P' \longrightarrow N \longrightarrow 0$$

associated to N. We can take the direct sum of the two to get

$$0 \longrightarrow \Omega^1(M) \oplus \Omega^1(N) \longrightarrow P \oplus P' \longrightarrow M \oplus N \longrightarrow 0.$$

By construction, $P \oplus P'$ is the projective cover of $M \oplus N$, and so by definition $\Omega^1(M) \oplus \Omega^1(N) \cong \Omega^1(M \oplus N)$. □

We now have enough tools to prove our first theorm.

Theorem 4.13. *If a $\mathfrak{g}$-module, M, is endotrivial, then so is $\Omega^n(M)$ for any $n \in \mathbb{Z}$.*

Proof. By assumption, we know that $M \otimes M^* \cong k \oplus P$ for some projective module P. We can apply Proposition 4.7 and Proposition 4.10 to see that

$$
\begin{aligned}
\Omega^n(M) \otimes (\Omega^n(M))^* &\cong \Omega^n(M) \otimes \Omega^{-n}(M^*) \\
&\cong \Omega^0(M \otimes M^*) \oplus P' \\
&\cong \Omega^0(k \oplus P) \oplus P' \\
&\cong k \oplus P'.
\end{aligned}
$$

$\square$

4.2 THE GROUP $T(\mathfrak{g})$

As we have noted before, given a fixed Lie superalgebra, $\mathfrak{g}$, we can consider which $\mathfrak{g}$-modules are endotrivial. So define the set

$$
T(\mathfrak{g}) := \left\{ [M] \in \mathrm{Stmod}(\mathfrak{g}) \ \middle| \ M \otimes M^* \cong k \oplus P_M \text{ where } P_M \text{ is projective} \right\}.
$$

Proposition 4.14. *For* $\mathfrak{g}$*, a Lie superalgebra,* $T(\mathfrak{g})$ *forms an Abelian group under the operation* $[M] + [N] = [M \otimes N]$.

Proof. First we note that, if $[M]$, $[N] \in T(\mathfrak{g})$, then

$$
\begin{aligned}
(M \otimes N) \otimes (M \otimes N)^* &= (M \otimes N) \otimes (M^* \otimes N^*) \\
&= M \otimes M^* \otimes N \otimes N^* \\
&= (k \oplus P_M) \otimes (k \oplus P_N) \\
&= (k \otimes k) \oplus (k \otimes P_N) \oplus (P_M \otimes k) \oplus (P_M \otimes P_N) \\
&= k \oplus P_{M \otimes N}.
\end{aligned}
$$

Since tensoring with a projective module yields another projective module, we see that $[M \otimes N] \in T(\mathfrak{g})$ as well and so the set is closed under the operation $+$. We note that this operation is associative by the associativity of the tensor product.

Next, we observe that, since $k^* \cong k$, we have $k \otimes k^* \cong k \otimes k \cong k$ and so $[k] \in T(\mathfrak{g})$. Since we tensor over k itself, we also see that for any $[M] \in T(\mathfrak{g})$, (indeed any $\mathfrak{g}$-module),

$$[M] + [k] = [M \otimes k] \cong [M] \cong [k \otimes M] = [k] + [M]$$

and so $[k]$ is the identity in $T(\mathfrak{g})$.

Finally, since M is finite dimensional, we have that $(M^*)^* \cong M$ and so if $[M] \in T(\mathfrak{g})$,

$$M^* \otimes (M^*)^* \cong M^* \otimes M \cong M \otimes M^* \cong k \oplus P_M$$

and so $[M^*] \in T(\mathfrak{g})$ is as well. Since we are working in the stable category,

$$[M] + [M^*] = [M \otimes M^*] = [k \oplus P_M] = [k]$$

so for each $[M]$, $T(\mathfrak{g})$ contains the inverse $[M^*]$.

Lastly, we make the (by now) obvious comment that, by the property of the tensor product, for any $[M], [N] \in T(\mathfrak{g})$, we have $[M] + [N] = [M \otimes N] = [N \otimes M] = [N] + [M]$. Thus, we have shown that $T(\mathfrak{g})$ satisfies the properties of an Abelian group. $\square$

The following lemma simplifies computations involving both syzygies and the parity change functor and will be useful throughout this work.

Lemma 4.15. *Let k be either the trivial supermodule, k_{ev}, or $\Pi(k_{ev}) = k_{od}$ in $\mathcal{F}$, then*

$$\Pi(\Omega^n(k)) = \Omega^n(\Pi(k))$$

for all $n \in \mathbb{Z}$.

Proof. In the case where $n = 0$, the claim is trivial.

The parity change functor, Π, can be realized by the following. Let M be a $\mathfrak{g}$-supermodule, then

$$\Pi(M) \cong M \otimes k_{od}$$

and if N is another $\mathfrak{g}$-supermodule and $\phi : M \to N$ is a $\mathfrak{g}$-invariant map, then

$$\Pi(\phi) : M \otimes k_{od} \to N \otimes k_{od}$$

$$m \otimes c \mapsto \phi(m) \otimes c$$

defines the functor Π. Let

$$0 \longrightarrow \Omega^1(k) \longrightarrow P \longrightarrow k \longrightarrow 0$$

be the exact sequence defining $\Omega^1(k)$. Then $\Pi(P)$ is the projective cover of $\Pi(k)$ and since the tensor product is over k, the following sequence

$$0 \longrightarrow \Pi(\Omega^1(k)) \longrightarrow \Pi(P) \longrightarrow \Pi(k) \longrightarrow 0$$

is exact. Thus, $\Pi(\Omega^1(k)) = \Omega^1(\Pi(k))$ as desired. We can easily dualize this argument to see that $\Pi(\Omega^{-1}(k)) = \Omega^{-1}(\Pi(k))$ and the proof is completed by an induction argument. $\qquad \square$

COMPUTING $T(\mathfrak{g})$ FOR RANK 1 DETECTING SUBALGEBRAS

Determining $T(\mathfrak{g})$ for different Lie superalgebras will be the main goal for the next three chapters. In particular, we will be focusing on the case where $\mathfrak{g}$ is a detecting subalgebra, whose definition depends on the Lie superalgebras $q(1)$ and $\mathfrak{sl}(1|1)$.

5.1 DETECTING SUBALGEBRAS

Recall the definitions of $q(n) \subseteq \mathfrak{sl}(n|n) \subseteq \mathfrak{gl}(n|n)$. The Lie superalgebra $q(n)$ consists of $2n \times 2n$ matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right)$$

where A and B are $n \times n$ matrices over k. The Lie superalgebra $q(1)$ is of primary interest and has a basis of

$$ t = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \qquad e = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). $$

Note that t spans $q(1)_{\bar{0}}$ and e spans $q(1)_{\bar{1}}$. The brackets are easily computed using Equation 2.1,

$$ [t,t] = tt - tt = 0, \quad [t,e] = te - et = 0, \quad [e,e] = ee + ee = 2t. $$

The Lie superalgebra $\mathfrak{sl}(m|n) \subseteq \mathfrak{gl}(m|n)$ consists of $(m+n) \times (m+n)$ matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

where A and D are $m \times m$ and $n \times n$ matrices respectively, which satisfy the condition $\text{tr}(A) - \text{tr}(D) = 0$.

For this work, $\mathfrak{sl}(1|1)$ is of particular importance, so note that $\mathfrak{sl}(1|1)$ consists of 2×2 matrices and has a basis of

$$t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A direct computation shows that $[x, y] = xy + yx = t$ and that all other brackets in $\mathfrak{sl}(1|1)$ are 0.

We may now define the detecting subalgebras, $\mathfrak{e}$ and $\mathfrak{f}$ as introduced in [2, Section 4]. Let $\mathfrak{e}_m := q(1) \times q(1) \times \cdots \times q(1)$ with m products of $q(1)$, and $\mathfrak{f}_n := \mathfrak{sl}(1|1) \times \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1)$ with n products of $\mathfrak{sl}(1|1)$.

Let $\mathfrak{a}$ denote an arbitrary detecting subalgebra (either $\mathfrak{e}$ or $\mathfrak{f}$). We define the rank of a detecting subalgebra $\mathfrak{a}$ to be $\dim(\mathfrak{a}_{\bar{1}})$, the dimension of the odd degree. The rank of $\mathfrak{e}_m$ is m and the rank of $\mathfrak{f}_n$ is $2n$ and, in general, a detecting subalgebra of rank r is denoted $\mathfrak{a}_r$. We start by considering rank 1 subalgebras.

With these definitions established, we now turn to the question of classifying the endotrivial modules for rank 1 detecting subalgebras.

5.2 ENDOTRIVIAL MODULES FOR THE $\mathfrak{e}_1$ DETECTING SUBALGEBRA

It is possible to classify $T(\mathfrak{e}_1)$ by considering the classification of all indecomposable $q(1)$-modules found in [2, Section 5.2].

We know that $k \times \mathbb{Z}_2$ parameterizes the simple $q(1)$-supermodules and the set $\{L(\lambda), \Pi(L(\lambda)) \mid \lambda \in k\}$ is a complete set of simple supermodules. For $\lambda \neq 0$, $L(\lambda)$ and $\Pi(L(\lambda))$ are two dimensional and projective. Thus, the only simple modules that are not projective are $L(0)$, the trivial module, and $\Pi(L(0))$. The projective cover of $L(0)$ is obtained as

$$P(0) = U(q(1)) \otimes_{U(q(1)_{\bar{0}})} L(0)|_{q(1)_{\bar{0}}}.$$

Since $U(q(1))$ has a basis (according to the PBW theorem) of

$$\left\{ e^r t^s \mid r \in \{0, 1\}, \ s \in \mathbb{Z}_{\geq 0} \right\},$$

we see that $P(0)$ has a basis of $\{1 \otimes 1, e \otimes 1\}$ and there is a 1 dimensional submodule spanned by $e \otimes 1$ which is isomorphic to k_{od}, the trivial module under the parity change functor. The space spanned by $1 \otimes 1$ is not closed under the action of $U(q(1))$ since $e(1 \otimes 1) = e \otimes 1$. It is now clear that the structure of $P(0)$ is

$$
\begin{array}{c}
k_{\text{ev}} \\
\big| \, e \\
k_{\text{od}}
\end{array}
$$

and the reader may check that $P(\Pi(L(0))) \cong \Pi(P(0))$. Thus, directly computing all inde-composable modules shows that the only indecomposables which are not projective are k_{ev} and k_{od} which are clearly endotrivial modules.

The group $T(\mathfrak{e}_1)$ can be computed in terms of the syzygies. The kernel of the projection map from $P(0)$ to $L(0)$ is $\Omega^1(k_{ev}) = k_{od}$. In order to compute $\Omega^2(k_{ev})$, consider the projective cover of k_{od}, which is $\Pi(P(0))$. The kernel of the projection map is again k_{ev}, the trivial module. The situation is the same for $\Pi(L(0))$ only with the parity change functor applied.

Now we have the following complete list of indecomposable endotrivial modules,

$$
\Omega^n(k_{ev}) = \begin{cases} k_{\text{ev}} & \text{if } n \text{ is even} \\ k_{\text{od}} & \text{if } n \text{ is odd} \end{cases} \qquad \Omega^n(k_{od}) = \begin{cases} k_{\text{od}} & \text{if } n \text{ is even} \\ k_{\text{ev}} & \text{if } n \text{ is odd} \end{cases}
$$

and an application of Corollary 4.11 proves the following proposition.

Proposition 5.1. *Let $\mathfrak{e}_1$ be the rank one detecting subalgebra of type $\mathfrak{e}$. Then $T(\mathfrak{e}_1) \cong \mathbb{Z}_2$.*

5.3 Endotrivial Modules for the $\mathfrak{f}_1$ Detecting Subalgebra

Now we consider the Lie superalgebra $\mathfrak{sl}(1|1)$. By the PBW theorem, a basis of $U(\mathfrak{sl}(1|1))$ is given by

$$
\left\{ x^{r_1} y^{r_2} t^s \mid r_i \in \{0, 1\} \text{ and } s \in \mathbb{Z}_{\geq 0} \right\}. \tag{5.1}
$$

Not all endotrivial $U(\mathfrak{sl}(1|1))$-supermodules will be classified yet since this is a rank 2 detecting subalgebra. First consider modules over a the Lie superalgebra generated by one element of $\mathfrak{sl}(1|1)_{\bar{1}}$.

Note that, since $[x, x] = [y, y] = 0$, an $\langle x \rangle$-supermodule or a $\langle y \rangle$-supermodule will also fall under the classification given in [2, Section 5.2]. For modules of this type, there are only four isomorphism classes of indecomposable modules, k_{ev}, k_{od}, $U(\langle x \rangle)$, and $\Pi(U(\langle x \rangle))$. It can be seen by direct computation that $U(\langle x \rangle)$ is the projective cover of k_{ev} (and the kernel of the projection map is k_{od}) and $\Pi(U(\langle x \rangle))$ is the projective cover of k_{od} (and the kernel of the projection map is k_{ev}).

Alternatively, let $z = ax + by$ where $a, b \in k \setminus \{0\}$. then $U(\langle z \rangle) \cong U(q(1))$. Thus, we have the same result and proof as in Proposition 5.1.

Proposition 5.2. *Let $\mathfrak{f}_1|_{\langle z \rangle}$ be a rank 1 subalgebra of $\mathfrak{sl}(1|1)$ generated by z, an element of $\mathfrak{sl}(1|1)_{\bar{1}}$. Then $T(\mathfrak{f}_1|_{\langle z \rangle}) \cong \mathbb{Z}_2$.*

COMPUTING $T(\mathfrak{g})$ FOR RANK 2 DETECTING SUBALGEBRAS

The main result of this section is the classification of $T(\mathfrak{a}_2)$, stated in Theorem 6.11. Given this goal, the Lie superalgebras of interest in this section are $q(1) \times q(1)$, denoted $\mathfrak{e}_2$, and $\mathfrak{sl}(1|1)$, denoted $\mathfrak{f}_2$. The classification of $T(\mathfrak{a}_2)$ is more complex than the rank one case and will require some general information about representations of the detecting subalgebras to prove the main theorem of this section.

6.1   RANK r DETECTING SUBALGEBRAS

Since the this section requires considering arbitrary detecting subalgebras, consider the following. Rank r detecting subalgebras are defined to be subalgebras isomorphic to either $\mathfrak{e}_r \cong q(1) \times \cdots \times q(1) \subseteq \mathfrak{gl}(n|n)$ or $\mathfrak{f}_r \cong \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1) \subseteq \mathfrak{gl}(n|n)$ where there are r copies of $q(1)$ and $\mathfrak{sl}(1|1)$ respectively.

Recall, $\mathfrak{e}_r$ has a basis of

$$\{e_1, \cdots, e_r, t_1, \cdots, t_r\}$$

and there are matrix representations of t_i and e_i which are $2n \times 2n$ matrices with blocks of size $n \times n$. Let d_i be the $n \times n$ matrix with a 1 in the ith diagonal entry and 0 in all other entries. Then

$$t_i = \left(\begin{array}{c|c} d_i & 0 \\ \hline 0 & d_i \end{array} \right) \quad e_i = \left(\begin{array}{c|c} 0 & d_i \\ \hline d_i & 0 \end{array} \right)$$

is a representation of $\mathfrak{e}_r$. The only nontrivial bracket operations on $\mathfrak{e}_r$ are $[e_i, e_i] = 2t_i$, thus all generating elements in $U(\mathfrak{e}_r)$ commute except for e_i and e_j anti-commute when $i \neq j$ and

by the PBW theorem,

$$\left\{ e_1^{k_1} \cdots e_r^{k_r} t_1^{l_1} \cdots t_r^{l_r} \;\middle|\; k_i \in \{0,1\} \text{ and } l_i \in \mathbb{Z}_{\geq 0} \right\}$$

is a basis for $U(\mathfrak{e}_r)$.

For $\mathfrak{f}_r$, the set

$$\{x_1, \cdots, x_r, y_1, \cdots, y_r, t_1, \cdots, t_r\}$$

forms a basis. The matrix representation of each element are also $2n \times 2n$ matrices with blocks of size $n \times n$. If d_i is as above then

$$
t_i = \left(\begin{array}{c|c} d_i & 0 \\ \hline 0 & d_i \end{array} \right)
\qquad
x_i = \left(\begin{array}{c|c} 0 & d_i \\ \hline 0 & 0 \end{array} \right)
\qquad
y_i = \left(\begin{array}{c|c} 0 & 0 \\ \hline d_i & 0 \end{array} \right)
$$

gives a realization of $\mathfrak{f}_r$. The only nontrivial brackets are $[x_i, y_i] = t_i$. So, in $U(\mathfrak{f}_r)$, $x_i \otimes y_j = -y_j \otimes x_i$ when $i \neq j$ and t_i commutes with x_j and y_k for any i, j, and k. Observe that $x_i \otimes y_i = -y_i \otimes x_i + t_i$ for each i. Finally, the PBW theorem shows that

$$\left\{ x_1^{k_1} \cdots x_r^{k_r} y_1^{l_1} \cdots y_r^{l_r} t_1^{m_1} \cdots t_r^{m_r} \;\middle|\; k_i, l_i \in \{0,1\} \text{ and } m_i \in \mathbb{Z}_{\geq 0} \right\}.$$

is a basis for $U(\mathfrak{f}_r)$.

Note that if $\mathfrak{g}$ and $\mathfrak{g}'$ are two Lie superalgebras, and $\sigma : \mathfrak{g} \to U(\mathfrak{g})$ and $\sigma' : \mathfrak{g}' \to U(\mathfrak{g}')$ are the canonical inclusions, then [23] gives an isomorphism between $U(\mathfrak{g} \times \mathfrak{g}')$ and the graded tensor product $U(\mathfrak{g})\overline{\otimes}U(\mathfrak{g}')$ where the mapping

$$\tau : \mathfrak{g} \times \mathfrak{g}' \to U(\mathfrak{g})\overline{\otimes}U(\mathfrak{g}')$$

$$\tau(g, g') = \sigma(g) \otimes 1 + 1 \otimes \sigma'(g')$$

corresponds to the canonical inclusion of $\mathfrak{g} \times \mathfrak{g}'$ into $U(\mathfrak{g} \times \mathfrak{g}')$. The corresponding construction for modules is the outer tensor product. If M is a $U(\mathfrak{g})$-module and N is a $U(\mathfrak{g}')$-module, then the outer tensor product of M and N is denoted $M \boxtimes N$. This is a $U(\mathfrak{g})\overline{\otimes}U(\mathfrak{g}')$-module where the action is given by

$$(x \otimes y)(v \boxtimes w) = (-1)^{|y||v|}x(v) \boxtimes y(w)$$

for $x \otimes y \in U(\mathfrak{g})\overline{\otimes}U(\mathfrak{g}')$.

This correspondence is relevant to the work here because it implies that $U(\mathfrak{e}_r) \cong U(\mathfrak{e}_1)\overline{\otimes}\ldots\overline{\otimes}U(\mathfrak{e}_1)$ and $U(\mathfrak{f}_r) \cong U(\mathfrak{sl}(1|1))\overline{\otimes}\ldots\overline{\otimes}U(\mathfrak{sl}(1|1))$ with r graded tensor products respectively. It will be useful to think of the universal enveloping algebra and corresponding representations in both contexts.

6.2 Representations of Detecting Subalgebras

Because $q(1)_{\overline{0}}$ and $\mathfrak{sl}(1|1)_{\overline{0}}$ consist only of toral elements, for an arbitrary detecting subalgebra $\mathfrak{a}$, the even component $\mathfrak{a}_{\overline{0}}$ consists of only toral elements as well. Thus, any $\mathfrak{a}$-module will decompose into a direct sum of weight spaces over the torus $\mathfrak{a}_{\overline{0}}$. Furthermore, this torus commutes with all of $\mathfrak{a}$, as can be observed by the bracket computations given above, and so the weight space decomposition actually yields a decomposition as $\mathfrak{a}$-modules.

This decomposition is consistent with the standard notion of a block decomposition for modules and a case of particular importance is that of the principal block, i.e., the block which contains the trivial module. All elements of $\mathfrak{a}$ act by zero on the trivial $\mathfrak{a}$-module, and so in particular, all toral elements do as well. This defines the weight associated with this block to be the zero weight and $\mathfrak{a}_{\overline{0}}$ acts by zero on any module in the principal block.

This structure has implications for the projective and simple modules in each block. As noted, $\mathfrak{a}_{\overline{0}}$ is toral, and hence the only simple modules are one dimensional with weight λ. Then for a simple $\mathfrak{a}$-module S of weight $\lambda = (\lambda_1, \ldots, \lambda_r)$, the restriction $S|_{\mathfrak{a}_{\overline{0}}} \cong \oplus T_i$ where each T_i is a simple, hence one dimensional, $\mathfrak{a}_{\overline{0}}$-module of weight λ. Thus, $T_i \cong k_\lambda$ where k_λ denotes a one dimensional module, concentrated in either even or odd degree, with basis $\{v\}$ and the action of $\mathfrak{a}_{\overline{0}}$ is given by $t_i.v = \lambda_i v$. Then by Frobenius reciprocity,

$$\mathrm{Hom}_{U(\mathfrak{a})}(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}_{\overline{0}})} k_\lambda, S) \cong \mathrm{Hom}_{U(\mathfrak{a}_{\overline{0}})}(k_\lambda, S|_{\mathfrak{a}_{\overline{0}}}) \neq 0$$

and since S is simple, the nonzero homomorphism is also surjective. Thus, the projective cover of any simple $\mathfrak{a}$-module can be found as a direct summand of an induced one dimensional module.

When $\mathfrak{a} \cong \mathfrak{sl}(1|1)$, these modules are small enough to be explicitly described and are important for the next proposition. By the considering the basis in (5.1), these induced modules will be 4 dimensional. Furthermore, when $\lambda = 0$ and concentrated in the even degree, the induced module, denoted $P(0)$, is indecomposable with simple socle and simple head, and hence is the projective cover of the trivial module k_{ev} and $\Pi(P(0))$ is the projective cover of $\Pi(k_{ev}) = k_{od}$.

When $\lambda \neq 0$, a direct computation shows that the induced module splits as a direct sum of two simple $\mathfrak{sl}(1|1)$-modules each of which are two dimensional, with basis $\{v_1, v_2\}$ and action

$$x.v_1 = v_2 \quad \text{and} \quad x.v_1 = 0$$
$$y.v_1 = 0 \qquad\qquad y.v_2 = v_1.$$

For one of the summands v_1 is even and v_2 is odd, and for the other v_2 is even and v_1 is odd. Thus the simple $\mathfrak{sl}(1|1)$-modules outside of the principal block are two dimensional and projective. Moreover, in the terminology of [5], this computation shows that these modules are absolutely irreducible. This will be relevant in the following proposition.

Another useful property of this block decomposition is that it yields information about the dimensions of the modules in certain blocks.

Proposition 6.1. *Let $\mathfrak{a}_r$ be a rank r detecting subalgebra. Then any simple module outside the principal block has even dimension.*

Proof. Since $q(1)_{\overline{0}} = \mathfrak{sl}(1|1)_{\overline{0}}$, let $\{t_1, \ldots, t_r\}$ be the basis for $(\mathfrak{a}_r)_{\overline{0}}$ such that t_i is a basis for the even part of the ith component of either $q(1)$ or $\mathfrak{sl}(1|1)$. Since the module is outside of the principal block, the associated weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ must have some $\lambda_i \neq 0$.

According to a theorem of Brundan from [4, Section 4], the simple $\mathfrak{e}_r = q(1) \times \cdots \times q(1)$ (r products) modules, denoted $\mathfrak{u}(\lambda)$ for $\lambda \in \mathbb{Z}^n$, have characters given by

$$\mathrm{ch}\, \mathfrak{u}(\lambda) = 2^{\lfloor h(\lambda)+1/2 \rfloor} x^{\lambda}$$

where $h(\lambda)$ denotes the number of t_i which do not act by 0. Thus all modules are even dimensional except in the case when $h(\lambda) = 0$, i.e. all simple modules outside of the principal block are even dimensional.

For the case of $\mathfrak{f}_r$, by [23, Section 2.1, Proposition 2], $U(\mathfrak{f}_r) \cong U(\mathfrak{sl}(1|1))\overline{\otimes}\ldots\overline{\otimes}U(\mathfrak{sl}(1|1))$. By [5, Lemma 2.9], we can construct any irreducible $\mathfrak{f}_r$-module as outer tensor products of irreducible $\mathfrak{sl}(1|1)$-modules. Since all simple $\mathfrak{sl}(1|1)$-modules are absolutely irreducible, the outer tensor product of such modules is also absolutely irreducible. We saw that a simple $\mathfrak{sl}(1|1)$-module has dimension one if the weight is 0, and dimension two otherwise. Thus the dimension of a simple $\mathfrak{f}_r$ is $2^{h(\lambda)}$, and so all simple modules outside of the principal block are even dimensional. $\qquad\square$

Now that something is known about the dimensions of the simple $\mathfrak{a}$-modules, we consider the dimensions of the projective $\mathfrak{a}$-modules. By the previous proposition, any projective module outside of the simple block will be even dimensional as well. Furthermore, in the previous sections we have shown that the only simple modules in the principal block are k_{ev} and k_{od}, and for $\mathfrak{e}_1$ and $\mathfrak{f}_1$, the direct computations have shown the projective covers of these are indecomposable modules of even dimension.

By the rank variety theory of [2, Section 6], restriction of any projective $\mathfrak{e}_r$ or $\mathfrak{f}_r$ module, must be projective when restricted to $\mathfrak{e}_1$ or $\mathfrak{f}_1$, respectively. Thus, any projective $\mathfrak{a}$-module is a direct sum of modules which are each even dimensional, and thus even dimensional as well.

Given these results, the following lemma will greatly restrict our search for endotrivial modules.

Lemma 6.2. *Let M be an indecomposable $\mathfrak{e}_2$-supermodule or an $\mathfrak{sl}(1|1)$-supermodule. If M is an endotrivial supermodule, then M must be in the principal block, i.e. all of the even elements must act on M by 0.*

Proof. Since M is endotrivial, $M \otimes M^* \cong k_{ev} \oplus P$ for some projective module P. Since $\dim P = 2m$ for $m \in \mathbb{N}$ by the previous observations, $\dim M \otimes M^* = \dim M^2 \equiv 1 \pmod 2$.

Since all modules outside of the principal block are even dimensional, M must be in the principal block. $\square$

This simplifies the search for endotrivial modules and we also can conclude that the only simple endotrivial modules are k_{ev} and k_{od}. Now we wish to show that the only endotrivials are $\{\Omega^n(k_{ev}), \Pi(\Omega^n(k_{ev}))|n \in \mathbb{Z}\}$.

Note. Since endotrivial $\mathfrak{a}$-modules are restricted to the principal block, the even elements act via the zero map on any module. With this in mind, it is convenient to think of endotrivial $\mathfrak{a}$-supermodules in a different way. Since t_i acts trivially for all i, considering $\mathfrak{a}$-modules as a $\mathfrak{a}_{\bar{1}}$-modules with trivial bracket yields an equivalent representation. The representations of these superalgebras are equivalent and the notation for this simplification is $V(\mathfrak{a}) := \Lambda(\mathfrak{a}_{\bar{1}})$.

With these reductions, in general, endotrivial $\mathfrak{a}_r$-modules are simply endotrivial modules for an abelian Lie superalgebra of dimension r concentrated in degree $\bar{1}$. For simplicity, denote a basis for $(\mathfrak{a}_r)_{\bar{1}}$ by $\{a_1, \ldots, a_r\}$. Then it is clear that $V(\mathfrak{a}_r) = \langle 1, a_1, \ldots, a_r \rangle$ generated as an algebra.

6.3   Computing $T(\mathfrak{a}_2)$

Given the above simplification, a basis for $V(\mathfrak{e}_2)$ is given by

$$\left\{ e_1^{r_1} e_2^{r_2} \mid r_i \in \{0, 1\} \right\}.$$

If we consider the left regular representation of $V(\mathfrak{e}_2)$ in itself with this basis, we have a 4 dimensional module with the structure
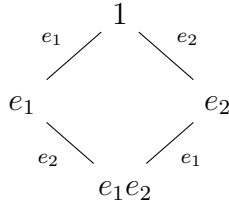
which is isomorphic to the projective cover of k thought of as an $\mathfrak{e}_2$-module.

In the $\mathfrak{sl}(1|1)$ case, now that the search has been restricted to the principal block, $V(\mathfrak{sl}(1|1))$ has the same structure. $V(\mathfrak{sl}(1|1))$ has a basis given by
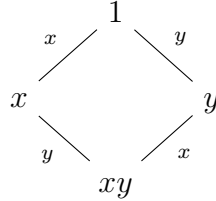
$$\left\{ x^{r_1} y^{r_2} \mid r_i \in \{0, 1\} \right\}$$

and $V(\mathfrak{sl}(1|1))$ has the structure

which is isomorphic to that of $V(\mathfrak{e}_2)$ and is the projective cover of k as an $\mathfrak{sl}(1|1)$-module.

For the rank 2 case, the algebra $V(\mathfrak{a}_2)$ is similar to the group algebra $k(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and endotrivials in the superalgebra case will be classified using a similar approach to Carlson in [7]. First we give analogous definitions and constructions to those in Carlson's paper.

Let M be a $\mathfrak{g}$-supermodule. The rank of M, denoted $\mathrm{Rk}(M)$, is defined by $\mathrm{Rk}(M) = \dim_k(M/\mathrm{Rad}(M))$. Any element of M which is not in $\mathrm{Rad}(M)$ will be referred to as generators of M. The socle of M has the standard definition (largest semi-simple submodule) and can be identified in the case of the principal block by $\mathrm{Soc}\, M = \{m \in M | u.m = 0 \text{ for all } u \in \mathrm{Rad}(V(\mathfrak{a}_r))\}$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{a}_r$ with a basis of $\mathfrak{h}_{\bar{1}}$ given by $\{h_1, \ldots, h_s\}$, then $\tilde{\mathfrak{h}} := \bigotimes_{i=1}^{s} h_i$ is a useful element of $V(\mathfrak{h})$. This is because in $M|_{V(\mathfrak{h})}$, $\tilde{\mathfrak{h}}.M \subseteq \mathrm{Soc}\, M|_{Vh}$.

Now we prove a lemma in the same way as [7].

Lemma 6.3. *Let M be an endotrivial $\mathfrak{a}_r$-supermodule for any $r \in \mathbb{N}$. Then*

$$\dim \mathrm{Ext}^1_{V(\mathfrak{a}_r)}(M, \Omega^1(M)) = 1$$

and M is the direct sum of an indecomposable endotrivial module and a projective module.

Proof. By definition, $\mathrm{Hom}_k(M, M) \cong k_{ev} \oplus P$ for some projective module P. It is clear from the definitions, that $\mathrm{Hom}_{V(\mathfrak{a}_r)}(M, M) = \mathrm{Soc}(\mathrm{Hom}_k(M, M))$. We have observed that $\tilde{\mathfrak{a}}.M \subseteq \mathrm{Soc}(M)$ and in the case of a projective module, equality holds. So then

$$\tilde{\mathfrak{a}}.\mathrm{Hom}_k(M, M) = \tilde{\mathfrak{a}}.(k \oplus P) = \mathrm{Soc}(P).$$

Since $\operatorname{Soc}(\operatorname{Hom}_k(M, M)) = k \oplus \operatorname{Soc}(P)$, we can see that $\tilde{\mathfrak{a}}. \operatorname{Hom}_k(M, M)$ is a submodule of $\operatorname{Hom}_{V(\mathfrak{a}_r)}(M, M)$ of codimension one.

Let P' be the projective cover of M. Apply $\operatorname{Hom}_k(M, -)$ and the long exact sequence in cohomology to the short exact sequence defining $\Omega^1(M)$ to get the following commutative diagram

$$0 \longrightarrow \operatorname{Hom}_k(M, \Omega^1(M)) \longrightarrow \operatorname{Hom}_k(M, P') \longrightarrow \operatorname{Hom}_k(M, M) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Hom}_{V(\mathfrak{a}_r)}(M, \Omega^1(M)) \longrightarrow \operatorname{Hom}_{V(\mathfrak{a}_r)}(M, P') \xrightarrow{\psi^*} \operatorname{Hom}_{V(\mathfrak{a}_r)}(M, M)$$
$$\longrightarrow \operatorname{Ext}^1_{V(\mathfrak{a}_r)}(M, \Omega^1(M)) \longrightarrow 0$$

where the vertical maps are multiplication by $\tilde{\mathfrak{a}}$. Since the diagram commutes and the map into $\operatorname{Hom}_k(M, M)$ is surjective, the image of ψ^* contains $\tilde{\mathfrak{a}}. \operatorname{Hom}_k(M, M)$, and we conclude that the dimension of $\operatorname{Ext}^1_{V(\mathfrak{a}_r)}(M, \Omega^1(M))$ is at most 1. The dimension is nonzero since the extension between a non-projective module and the first syzygy does not split. Since $\operatorname{Ext}^1_{V(\mathfrak{a}_r)}$ splits over direct sums, the claim is established. $\qquad\square$

Lemma 6.4. *Let M be a $V(\mathfrak{a}_r)$-module and let $\mathfrak{b}$ be a subalgebra of $\mathfrak{a}_r$. Then*

$$\Omega^n_{\mathfrak{a}_r}(M)\big|_{\mathfrak{b}} \cong \Omega^n_{\mathfrak{b}}(M|_{\mathfrak{b}}) \oplus P$$

for all $n \in \mathbb{Z}$, where P is a projective $V(\mathfrak{b})$-module.

Proof. The case when $n = 0$ is proven by considering the rank varieties of $\mathfrak{a}_r$ and $\mathfrak{b}$. Let M be as above and

$$0 \longrightarrow \Omega^1_{\mathfrak{a}_r}(M) \longrightarrow P \longrightarrow M \longrightarrow 0$$

be the short exact sequence of $V(\mathfrak{a}_r)$-supermodules defining $\Omega^1(M)$. Then by the rank variety theory of [2, Section 6], the module $P|_{\mathfrak{b}}$ is a projective $V(\mathfrak{b})$-module (although perhaps not the projective cover of $M|_{\mathfrak{b}}$) and

$$0 \longrightarrow \Omega^1_{\mathfrak{a}_r}(M)|_{\mathfrak{b}} \longrightarrow P|_{\mathfrak{b}} \longrightarrow M|_{\mathfrak{b}} \longrightarrow 0$$

is still exact. Then by definition, $\Omega^1_{\mathfrak{a}_r}(M)|_{\mathfrak{b}} \cong \Omega^1_{\mathfrak{b}}(M|_{\mathfrak{b}}) \oplus P$. This argument applies to $\Omega^{-1}(M)$ as well and so by induction, $\Omega^n_{\mathfrak{a}_r}(M)|_{\mathfrak{b}} \cong \Omega^n_{\mathfrak{b}}(M|_{\mathfrak{b}}) \oplus P$ for all $n \in \mathbb{Z}$ $\qquad\square$

The previous lemma indicates that the syzygies of a module commute with restriction up to a projective module (and truly commute in the stable module category). With this in mind, the subscripts on the syzygies will be suppressed as it will be clear from context which syzygy to consider.

In proving Theorem 6.11, we work with 2 conditions on a $V(\mathfrak{a}_2)$-supermodule, M. For notation, recall that $V(\mathfrak{a}_2) = \langle 1, a_1, a_2 \rangle$. The conditions are

(1) $\text{Rk}(M) > \text{Rk}(\Omega^{-1}(M))$;

(2) for any nonzero $a = ca_1 + da_2 \in (\mathfrak{a}_2)_{\bar{1}}$, $M|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus P$ where $t = 0$ or 1, P is a projective $\langle a \rangle$-supermodule, and $c, d \in k$.

Note that in condition (2), $U(\langle a \rangle) \cong V(\mathfrak{a}_1)$ and so the structure of such modules is detailed in Sections 5.2 and 5.3.

The technique is to show that some endotrivial modules have these properties and use them to classify all endotrivial modules.

Let M be an endotrivial $V(\mathfrak{a}_2)$ module. Because the complexity of an M is nonzero (see [3, Corollary 2.71]), $\Omega^n(M)$ satisfies (1) for some $n \in \mathbb{Z}_{\geq 0}$, so we can simply replace M with $\Omega^n(M)$ initially and proceed. Since the endotrivials for rank 1 supermodules have been classified, and $M|_{\langle a \rangle}$ is isomorphic to a $V(\mathfrak{a}_1)$-supermodule, M satisfies (2) as well.

The classification of $T(\mathfrak{a}_2)$ is approached in the same way as in [7], but the techniques are altered to suit the rank 2 detecting subalgebra case. We begin by establishing several supplementary results.

Lemma 6.5. *A $V(\mathfrak{a}_2)$-supermodule M satisfies (2) if and only if $\Omega^n(M)$ satisfies (2) for all $n \in \mathbb{Z}$.*

Proof. Since $M|_{\langle a \rangle}$ is endotrivial, then $\Omega^n(M)|_{\langle a \rangle}$ is as well by Lemma 6.4, and so by the classification of the $V(\mathfrak{a}_1)$ endotrivials, $\Omega^n(M)$ satisfies (2) for all n if and only if M satisfies (2). □

Lemma 6.6. *Let M be a $V(\mathfrak{a}_2)$-supermodule which has no projective submodules and which satisfies condition (2) and let b be some nonzero element of $(\mathfrak{a}_2)_{\bar{1}}$ not in the span of a. Let v be a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of some decomposition of $M|_{\langle a \rangle}$, then M satisfies condition (1) if and only if every such element v satisfies $b.v \neq 0$.*

Proof. First, note that since M has no projective summands, $a.M \subseteq \mathrm{Ann}_b(M)$ and $b.M \subseteq \mathrm{Ann}_a(M)$, i.e. that radical series of M has length 1. Otherwise if $(a \otimes b).m \neq 0$ for some $m \in M$, then this would generate the socle of a projective submodule, and hence summand since $V(\mathfrak{a}_2)$ is self injective. Recall also that in the principal block $\mathrm{Ann}_x(M) = \mathrm{Soc}(M|_{\langle x \rangle})$.

Let M be as above and assume that $b.v \neq 0$. Since M is endotrivial, it must have odd dimension, and by the decomposition given in (2) and the knowledge of projective $U(\langle a \rangle)$-modules, if $\dim M = 2n + 1$, then $\dim \mathrm{Soc}(M|_{\langle a \rangle}) = n + 1$.

Since M satisfies (2) and the choice of a was arbitrary, we also know that $M|_{\langle b \rangle} \cong \Omega^t(k|_{\langle b \rangle}) \oplus P$ and since $b.v \neq 0$, v is not in the socle of $M|_{\langle b \rangle}$, which also has dimension $n + 1$. Since, as noted before, $a.M \subseteq \mathrm{Ann}_b(M)$ and $b.M \subseteq \mathrm{Ann}_a(M)$, but $b.v \neq 0$, then $\mathrm{Soc}(M) = \mathrm{Ann}_a(M) \cap \mathrm{Ann}_b(M) = a.M = b.M$ which has dimension n and $\mathrm{Rk}(M) = n + 1$. Thus, if $0 \to M \to I \to \Omega^{-1}(M) \to 0$ is the exact sequence defining $\Omega^{-1}(M)$, then since M has no projective submodules, the following holds

$$\mathrm{Rk}(\Omega^{-1}(M)) = \mathrm{Rk}(I) = \dim(\mathrm{Soc}(M)), \tag{6.1}$$

and so

$$\mathrm{Rk}(M) = n + 1 > n = \mathrm{Rk}(\Omega^{-1}(M))$$

as was desired.

Now, assume that $\mathrm{Rk}(M) > \mathrm{Rk}(\Omega^{-1}(M))$ and let a and b be as above. Since $M|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus P$ and $M|_{\langle b \rangle} \cong \Omega^t(k|_{\langle b \rangle}) \oplus P'$ which both have socles of dimension $n + 1$ and intersect in at least n dimensions since $a.M \subseteq \mathrm{Ann}_b(M)$ and $b.M \subseteq \mathrm{Ann}_a(M)$ and at most in $n+1$. However, by Equation 6.1, they cannot intersect in $n+1$ or else condition (1) would

be violated. Thus, if v is a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of $M|_{\langle a \rangle}$, then it is not in the socle of $M|_{\langle b \rangle}$, and so $b.v \neq 0$. $\qquad \square$

A new condition is introduced to encapsulate the previous lemma.

(3) Let M satisfy condition (2). If v is a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of $M|_{\langle a \rangle}$, then v is a generator for M. If m is any generator for M then either $a.m \neq 0$ or $b.m \neq 0$ where b is some nonzero element of $(\mathfrak{a}_2)_{\overline{1}}$ not in the span of a.

Lemma 6.7. *Let M be a $V(\mathfrak{a}_2)$-supermodule which has no projective submodules and which satisfies condition (2), then M satisfies (1) if and only if M satisfies (3).*

Proof. First, assume M satisfies (3). Since M satisfies (2), let v be some generator for the $\Omega^t(k|_{\langle a \rangle})$ component in some decomposition of $M|_{\langle a \rangle}$. Since $a.v = 0$, $b.v \neq 0$ by (3) and by Lemma 6.6, M satisfies (1).

Next, assume that M satisfies (1). By Lemma 6.6, $b.v \neq 0$. Assume that v is not a generator for M. Then, by definition, v is in $\mathrm{Rad}(M)$ and so we can write $v = a.m_1 + b.m_2$ for some $m_1, m_2 \in M$. By assumption

$$b.v = b.(a.m_1 + b.m_2) = c\widetilde{\mathfrak{a}_2}.m_1 \neq 0$$

for some $c \in k$. However, then m' generates a projective submodule which is a contradiction, so v is also a generator of M.

The only thing left to show is that if m is any generator for M, either $a.m \neq 0$ or $b.m \neq 0$. If m is any such generator, then m must also be a generator for $M|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus P$. The case when m generates $\Omega^t(k|_{\langle a \rangle})$ has been handled, so assume that m is a generator for P. Since P is a projective $\langle a \rangle$-module, and m is a generator, then $a.m \neq 0$. $\qquad \square$

Proposition 6.8. *Let M be a $V(\mathfrak{a}_2)$-supermodule which has no projective submodules satisfying (1), (2), and (3), then $\Omega^n(M)$ does as well for all $n \geq 0$.*

Proof. By Lemmas 6.5 and 6.7, it is sufficient to show that (3) holds for $\Omega^n(M)$ for all $n \geq 0$. This proposition it trivial when $n = 0$ since $\Omega^0(M) = M$ by assumption.

Let m be a generator for $\Omega^1(M)$, then m is also a generator for $\Omega^1(M)|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus P$ in some decomposition of $\Omega^1(M)|_{\langle a \rangle}$. Assume that m is a generator for P. Since P is a projective $\langle a \rangle$-module, $a.m \neq 0$.

All that remains to be shown is that if w is a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of $\Omega^1(M)|_{\langle a \rangle}$, then w is a generator for $\Omega^1(M)$ and $b.w \neq 0$.

Let $0 \to \Omega^1(M) \to P \xrightarrow{\psi} M \to 0$ be the exact sequence defining $\Omega^1(M)$. Let v be a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of $M|_{\langle a \rangle}$. This generator can be chosen so that $v = \psi(p)$ where p is a generator of P. We claim that $a.p$ is a generator for $\Omega^1(M) \subseteq P$. If not, then write

$$a.p = a.l + b.m$$

for some elements $l, m \in \Omega^1(M)$. Then

$$ba.p = ba.l.$$

Define $\omega := b.p - b.l$. Since $a.\omega = b.\omega = 0$, by definition, $\omega \in \mathrm{Soc}(P) = \tilde{\mathfrak{a}}.P \subseteq \Omega^1(M)$. So then $b.p = \omega + b.l \in \Omega^1(M) = \ker \psi$ (as in Definition 3.12). Now

$$\psi(\omega + b.l) = \psi(b.p) = b.\psi(p) = b.v \neq 0$$

by assumption since M satisfies (3). This is a contradiction, thus, $a.p$ is a generator for $\Omega^1(M)$.

Note that in $\Omega^1(M)|_{\langle a \rangle}$, $a.p$ generates a trivial $\langle a \rangle$-module. Thus, any generator for the $\Omega^1(k|_{\langle a \rangle})$ component of $\Omega^1(M)|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus Q$ is equivalent to $a.p$ modulo $\mathrm{Soc}(Q) = a.Q$. So if w is any such generator, then it is possible to write $w = ca.p + a.\nu$ for some $0 \neq c \in k$ and $\nu \in \mathrm{Soc}(Q)$. Then

$$b.w = b.(ca.p + a.\nu) = c'\tilde{\mathfrak{a}}.p \neq 0.$$

These computations show that condition (3) holds for $\Omega^1(M)$, hence condition (1) does as well. An inductive argument completes the proof of the proposition. $\qquad\square$

Proposition 6.9. *Let M be a $V(\mathfrak{a}_2)$-supermodule which has no projective submodules satisfying (1), (2), and (3), then either $\Omega^{-1}(M)$ satisfies all three conditions or $\Omega^{-1}(M)$ has a summand which is isomorphic to k (either even or odd).*

Proof. Let $0 \to M \to I \xrightarrow{\psi} \Omega^{-1}(M) \to 0$ be the exact sequence of $V(\mathfrak{a}_2)$-supermodules defining $\Omega^{-1}(M)$. By Lemma 6.5, $\Omega^{-1}(M)$ already satisfies (2). Recall that because we are working in a self-injective category, the module I is also projective and we can take advantage of the previous description of these modules.

Let $v \in M$ be a generator for the $\Omega^t(k|_{\langle a \rangle})$ component of some decomposition of $M|_{\langle a \rangle}$. Our previous work shows that we may choose v so that $v = a.p$ for some $p \in I$. We also know that for some nonzero $c \in k$, $c\tilde{\mathfrak{a}}.p = b.v \neq 0$. Then p is a generator for I and $\psi(p)$ is a generator for $\Omega^{-1}(M)$. There are two cases to consider.

First, if $b.\psi(p) = 0$, then $\psi(p) \in \text{Soc}(\Omega^{-1}(M))$, since $a.\psi(p) = \psi(a.p) = \psi(v) = 0$ because $v \in M$. If this happens, then $k \cdot \psi(p) \cong k$ is a direct summand of $\Omega^{-1}(M)$.

For the rest of the proof, assume that for any such element p, $b.p \notin M$ and we will show that $\Omega^{-1}(M)$ satisfies all three conditions. Note that, by Lemma 6.7, we only need to establish (3). Indeed, it has been observed that $\psi(p)$ is a generator for $\Omega^{-1}(M)$ and by the assumption, $b.p \notin M$ yields that $b.\psi(p) \neq 0$ in $\Omega^{-1}(M)$.

Now let m be a generator for $\Omega^{-1}(M)$ such that $a.m = 0$. By (2), $\Omega^{-1}(M)|_{\langle a \rangle} \cong \Omega^t(k|_{\langle a \rangle}) \oplus P$ where P is a projective $\langle a \rangle$-module. Since $a.\psi(p) = 0$, any generator of the $\Omega^t(k|_{\langle a \rangle})$ component will be equivalent to $\psi(p)$ modulo $\text{Soc}(P)$. This case has been covered since we assumed $b.p \notin M$ so assume that m does not generate the $\Omega^t(k|_{\langle a \rangle})$ component and, thus, must be a generator for the projective summand. However, since $a.m = 0$, $m \in \text{Soc}(P)$ and we conclude that m cannot be a generator for P, a contradiction. So if $m \not\equiv \psi(p) \mod \text{Soc}(P)$ is any generator, then $a.m \neq 0$. Thus, $\Omega^{-1}(M)$ satisfies (3) and thus (1) in this case, and the proof is complete. $\square$

Theorem 6.10. *Let M be an endotrivial $V(\mathfrak{a}_2)$-supermodule in $\mathcal{F}$. Then $M \cong \Omega^n(k) \oplus P$ for some $n \in \mathbb{Z}$ and where k is either the trivial module k_{ev} or $\Pi(k_{ev}) = k_{od}$ and P is a projective module in $\mathcal{F}$.*

Proof. It has been observed that if M is endotrivial, then M satisfies condition (2). Additionally, for some $r \geq 0$, $\Omega^r(M)$ satisfies (1). So by Lemma 6.7, $\Omega^r(M)$ satisfies (3) as well.

By (1), we can see that $\Omega^r(M)$ has no summand isomorphic to k. Assume that $\Omega^{-s}(\Omega^r(M))$ has no such summand for for all $s > 0$. By Proposition 6.9,

$$\text{Rk}(M) > \text{Rk}(\Omega^{-1}(M)) > \text{Rk}(\Omega^{-2}(M)) > \cdots$$

which is clearly impossible since $\text{Rk}(M)$ is finite for any module in $\mathcal{F}$. Thus, $\Omega^{-n-r}(\Omega^r(M)) \cong k \oplus Q$ for some $n \in \mathbb{Z}$. Since $\Omega^{-n-r}(\Omega^r(M))$ satisfies (2), $Q|_{\langle a \rangle}$ is a projective $\langle a \rangle$-module and by considering the rank variety of $V(\mathfrak{a}_2)$, Q is a projective $V(\mathfrak{a}_2)$-module. The k summand may either be contained in $\Omega^{-n}(M)_{\bar{0}}$ or $\Omega^{-n}(M)_{\bar{1}}$ and since $\Omega^{-n}(M)$ contains no projective submodules, $\Omega^{-n}(M) \cong k$ and $\Omega^0(M) \cong \Omega^n(k)$. By Lemma 6.3,

$$M \cong \Omega^n(k) \oplus P$$

where P is a projective $V(\mathfrak{a}_2)$-supermodule and k is either k_{ev} or k_{od}. $\qquad\square$

Given this theorem, it is now possible to identify the group $T(\mathfrak{a}_2)$.

Theorem 6.11. *Let $\mathfrak{a}_2$ be a rank 2 detecting subalgebra of $\mathfrak{g}$, then $T(\mathfrak{a}_2) \cong \mathbb{Z} \times \mathbb{Z}_2$ and is generated by $\Omega^1(k_{ev})$ and k_{od}.*

Proof. Let M be an endotrivial $V(\mathfrak{a}_2)$-supermodule. By Theorem 6.10, in the stable module category, $M \cong \Omega^n(k_{ev})$ or $M \cong \Omega^n(\Pi(k_{ev}))$. By Lemma 4.15, this can be rewritten as $M \cong \Omega^n(k_{ev})$ or $M \cong \Pi(\Omega^n(k_{ev}))$. Since the group operation in $T(\mathfrak{a}_2)$ is tensoring over k and by Corollary 4.11,

$$M \cong \begin{cases} \Omega^1(k_{ev})^{\otimes n} \otimes_k (k_{od})^{\otimes t} & \text{if } n > 0 \\ \Omega^{-1}(k_{ev})^{\otimes n} \otimes_k (k_{od})^{\otimes t} & \text{if } n < 0 \\ \Omega^1(k_{ev}) \otimes_k \Omega^{-1}(k_{ev}) \otimes_k (k_{od})^{\otimes t} & \text{if } n = 0 \end{cases}$$

where $t \in \{1, 2\}$. Thus, there is an isomorphism, ϕ, from $T(\mathfrak{a}_2)$ to $\mathbb{Z} \times \mathbb{Z}_2$ given by

$$\phi(M) := \begin{cases} (n, t) & \text{if } M \cong \Omega^1(k_{ev})^{\otimes n} \otimes_k (k_{od})^{\otimes t} \text{ for } n > 0 \\ (n, t) & \text{if } M \cong \Omega^{-1}(k_{ev})^{\otimes n} \otimes_k (k_{od})^{\otimes t} \text{ for } n < 0 \\ (0, t) & \text{if } M \cong k_{ev} \otimes_k (k_{od})^{\otimes t} \end{cases}$$

and it is now clear that $T(\mathfrak{a}_2)$ is generated by $\Omega^1(k_{ev})$ (and its inverse) and k_{od}. $\qquad \square$

COMPUTING $T(\mathfrak{g})$ FOR ALL RANKS INDUCTIVELY

Now we wish to proceed by induction to classify endotrivial modules for the general case $\mathfrak{a}_r$ where $r > 2$. The structure of $\mathfrak{e}_n \cong q(1) \times \cdots \times q(1) \subseteq \mathfrak{gl}(n|n)$ and $\mathfrak{f}_n \cong \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1) \subseteq \mathfrak{gl}(n|n)$ where there are r copies of $q(1)$ and $\mathfrak{sl}(1|1)$ respectively is given in Section 6.1.

By Lemma 6.2, any endotrivial module for a detecting subalgebra is in the principal block, we can consider (equivalently) endotrivial representations of $V(\mathfrak{e}_n) = \Lambda((\mathfrak{e}_n)_{\bar{1}})$ and $V(\mathfrak{f}_n) = \Lambda((\mathfrak{f}_n)_{\bar{1}})$. The support variety theory of [2, Section 6] will be used as well. For an endotrivial module M, since $M \otimes M^* \cong k_{ev} \oplus P$, we have $\mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(M) = \mathcal{V}_{(\mathfrak{a},\mathfrak{a}_{\bar{0}})}(k_{ev}) \cong \mathbb{A}^r$ (which is also equivalent to the rank variety of M).

Our first step in the classification comes by following [11, Theorem 4.4]. Recall that $\{a_1, \ldots, a_r\}$ denotes a basis for $(\mathfrak{a}_r)_{\bar{1}}$ and that $V(\mathfrak{a}_r) = \langle 1, a_1, \ldots, a_r \rangle$ when generated as an algebra.

Theorem 7.1. *Let M be an endotrivial $V(\mathfrak{a}_r)$-supermodule, where $\mathfrak{a}_r$ is a rank r detecting subalgebra. Let $v = c_1 a_1 + \cdots + c_r a_r \in (\mathfrak{a}_r)_{\bar{1}}$ with $c_i \neq 0$ for some $i < r$ and let $A = \langle v, a_r \rangle$ be the subsuperalgebra of $V(\mathfrak{a}_r)$ of dimension 4 generated by v and a_r. Then, for some s independent of the choice of v, $M|_A \cong \Omega^s(k|_A) \oplus P$ for some $\mathfrak{a}_r$-projective module P where $k|_A$ is either the trivial module k_{ev} or $\Pi(k_{ev}) = k_{od}$.*

Proof. First, note that since $\mathfrak{a}_r$ is a purely odd, abelian Lie superalgebra, $v \otimes v = \frac{[v,v]}{2} = 0$ and $a_r \otimes a_r = 0$ but $v \otimes a_r = -a_r \otimes v \neq 0$ and so $A \cong V(\mathfrak{a}_2)$. Also note that if $v' = c_1 a_1 + \cdots c_{r-1} a_{r-1}$, since $c_i \neq 0$ for some $i < r$, then $\langle v, a_r \rangle \cong \langle v', a_r \rangle$ by a change of basis. So without loss of generality, redefine $v = c_1 a_1 + \cdots c_{r-1} a_{r-1}$ and $A = \langle v, a_r \rangle$ for the new v and identify all such v with the points in $\mathbb{A}^{r-1} \setminus \{0\}$.

By the previous classification, $\Omega^0(M|_A) \cong \Omega^{m_v}(k)$ where k is either even or odd. We now show that m_v is independent of the choice of v.

Since $\dim \Omega^m(k|_A) = \dim \Omega^{-m}(k|_A) > \dim M$ for large enough m, then there exist b, $B \in \mathbb{Z}$ such that $b \leq m_v \leq B$ for any $v \in \mathbb{A}^{r-1} \setminus \{0\}$. Moreover, we can choose b and B such that equality holds for some v and v'. Now replace M by $\Omega^{-b}(M)$. Once this is done, we assume $b = 0$, and for all $v \in \mathbb{A}^{r-1} \setminus \{0\}$, $0 \leq m_v \leq B$ where the bounds are actually attained.

Let $C \in \mathbb{Z}$ be such that $0 \leq C < B$ and let

$$S_C = \{v \in \mathbb{A}^{r-1} \setminus \{0\} \mid m_v > C\}$$

We claim that S_C is closed in the Zariski topology of $\mathbb{A}^{r-1} \setminus \{0\}$.

Recall that, since we are working with $V(\mathfrak{a}_2)$-modules, i.e., in the principal block, there are, up to the parity change functor, a unique simple module, k, and indecomposable projective module, which is isomorphic to the left regular representation of $V(\mathfrak{a}_2)$ (see Sections 6.2 and 6.3). Thus, any projective module P has dimension $4n$ for some $n \in \mathbb{N}$.

Since $m_v = 0$ for some v, it follows that $\dim M \equiv 1 \pmod 4$. This implies that m_v is even for all v since $\dim \Omega^n(k|_A) = 1 + 2|n|$. Thus, for any v, $\dim \Omega^{2s}(k|_A) = 1 + 4s$ for $s \geq 0$. With this in mind, define

$$t = (\dim M - \dim \Omega^{2c}(k|_A))/4$$

where $c = C/2$ if C is even and $c = (C-1)/2$ if C is odd. In either case, $2c \leq C < 2c + 2$. This construction is done to ensure that for any v, the statement that $m_v \leq C$ means that the dimension of the projective part of $M|_A$ is

$$\dim M - \dim \Omega^{m_v}(k|_A) \geq 4t.$$

In other words, if $m_v \leq C$, then $M|_A$ has an A-projective summand of rank at least t so the rank of the matrix of the element $\omega_v = v \otimes a_r$ (which generates the socle of A) acting on M is at least t. Otherwise, if $m_v > C$, then $M|_A$ has no A-projective summand of rank t. Consequently, the rank of the matrix of ω_v is strictly less than t.

Let $d = \dim M$ and let $\mathcal{S}$ be the set of all subsets of $\mathfrak{N} = \{1, \ldots, d\}$ having exactly t elements. For any $S, T \in \mathcal{S}$ define $f_{S,T} : \mathbb{A}^{r-1} \setminus \{0\} \to k$ by

$$f_{S,T}(v) = \mathrm{Det}(M_{S,T}(\omega_v))$$

where $M_{S,T}$ is the $t \times t$ submatrix of the matrix of ω_v acting on M having rows indexed by S and columns indexed by T. The functions $f_{S,T}$ are polynomial maps and so their common set of zeros $\mathcal{V}(\{f_{S,T}\}_{S,T \in \mathcal{S}})$ is a closed set of $\mathbb{A}^{r-1} \setminus \{0\}$. If $M|_A$ has no A-projective summand of rank t, then each determinant must always be 0, hence in the vanishing locus, and otherwise, at least one of the $f_{S,T}(v)$ will be nonzero. Thus we have constructed a set of polynomials such that $f_{S,T}(v) = 0$ on each polynomial $f_{S,T}$ if and only if $v \in S_C$. We conclude that S_C is closed in $\mathbb{A}^{r-1} \setminus \{0\}$.

It is also true that for any C, S_C is open in $\mathbb{A}^{r-1} \setminus \{0\}$. First, replace M with M^* (which is also endotrivial). Since $(\Omega^n(M^*))^* \cong \Omega^{-n}(M)$, for M^*, the bounds are $-B \leq m_v \leq 0$. Replacing M^* with $\Omega^B(M^*)$ again yields $0 \leq m_v \leq B$. However, now we have that for any v,

$$M|_A \cong \Omega^{m_v}(k|_A) \oplus P,$$

and by the above computation, we also have

$$(\Omega^B(M^*))|_A \cong \Omega^{B-m_v}(k|_A) \oplus P.$$

Thus, $S_C = (S_{B-C})^c$ and so S_C is open. Since S_C is both open and closed and $\mathbb{A}^{r-1} \setminus \{0\}$ is connected, we conclude that S_C is either the empty set or all of $\mathbb{A}^{r-1} \setminus \{0\}$. By assumption, there is a v such that $m_v = 0$, so S_0 is nonempty. Thus, $S_0 = \mathbb{A}^{r-1} \setminus \{0\}$ and $B = 0$ as well (since the bounds are attained). Thus, the number m_v is constant over all $v \in \mathbb{A}^{r-1} \setminus \{0\}$ and

$$M|_A \cong \Omega^s(k|_A) \oplus P$$

for any subsuperalgebra $A \cong V(\mathfrak{a}_2)$ where k is either k_{ev} or k_{od}, by the classification of $T(\mathfrak{a}_2)$.

We also claim that for any such A, the parity of $k|_A$ is constant as well. This can be seen by assuming that there are A and A' such that $M|_A \cong \Omega^s(k_{ev}) \oplus P$ and $M|_{A'} \cong \Omega^s(k_{od}) \oplus P'$.

Now consider the dimensions of $M_{\bar{0}}$ and $M_{\bar{1}}$. Since $\dim \Omega^s(k_{ev}) = \dim \Omega^s(k_{od})$, it follows that $\dim P = \dim P'$. Note that $\dim P_{\bar{0}} = \dim P_{\bar{1}}$ and consequently, $\dim P_{\bar{0}} = \dim P'_{\bar{0}}$ and $\dim P_{\bar{1}} = \dim P'_{\bar{1}}$. Finally, recall that $\dim \Omega^s(k_{ev})_{\bar{0}} \neq \Omega^s(k_{ev})_{\bar{1}}$. Without loss of generality, assume that $\dim \Omega^s(k_{ev})_{\bar{0}} > \dim \Omega^s(k_{ev})_{\bar{1}}$, i.e. s is an even integer. Then

$$\dim \Omega^s(k_{od})_{\bar{0}} = \dim \Omega^s(\Pi(k_{ev}))_{\bar{0}} = \dim \Pi(\Omega^s(k_{ev})_{\bar{0}}) = \dim \Omega^s(k_{ev})_{\bar{1}}$$

and similarly,

$$\dim \Omega^s(k_{od})_{\bar{1}} = \dim \Omega^s(\Pi(k_{ev}))_{\bar{1}} = \dim \Pi(\Omega^s(k_{ev})_{\bar{1}}) = \dim \Omega^s(k_{ev})_{\bar{0}}.$$

This implies that $\dim \Omega^s(k_{od})_{\bar{0}} < \dim \Omega^s(k_{od})_{\bar{1}}$. These different decompositions combine to yield that $\dim M_{\bar{0}} > \dim M_{\bar{1}}$ by considering $M|_A$ and $\dim M_{\bar{0}} < \dim M_{\bar{1}}$ by considering $M|_{A'}$. This is a contradiction and so the parity of the k in the decomposition of $M|_A$ is constant for any choice of A as well. $\qquad\square$

Theorem 7.2. *Let M be an endotrivial $V(\mathfrak{a}_r)$-supermodule, then $M \cong \Omega^n(k) \oplus P$ for some $n \in \mathbb{Z}$ where k is either the trivial module k_{ev} or $\Pi(k_{ev}) = k_{od}$ and P is a projective module in $\mathcal{F}$.*

Proof. Let M be an endotrivial $V(\mathfrak{a}_r)$-supermodule and let $A = \langle v, a_r \rangle$. By Theorem 7.1, $M|_A \cong \Omega^m(k|_A) \oplus P$ where $v = c_1 a_1 + \cdots + c_{r-1} a_{r-1}$ for some $(c_1, \ldots, c_{r-1}) \in \mathbb{A}^{r-1} \setminus \{0\}$ and m is independent of the choice of v. The goal is to prove that $M \cong \Omega^m(k) \oplus Q'$ or, equivalently, $\Omega^{-m}(M) \cong k \oplus Q$. For simplicity, replace M by $\Omega^{-m}(M)$ and assume that $M|_A \cong k|_A \oplus P$.

The first step is to show that the module $\hat{M} = a_r.M$ is a projective $\hat{V} = V(\mathfrak{a}_r)/(a_r)$ module. We do this by considering the rank variety $\mathcal{V}_{\mathfrak{a}}^{\mathrm{rank}}(\hat{M}|_{\hat{V}})$ (see [2, Section 6.3]).

As in the previous proof, we are working in the principal block and so, there is, up to the parity change functor, a unique indecomposable projective $V(\mathfrak{a}_r)$-module, which is isomorphic to the left regular representation of $V(\mathfrak{a}_r)$ in itself (see Section 6.2). Note that the dimension of these projective indecomposable modules is 2^r.

Recall that $A = \langle v, a_r \rangle \cong V(\mathfrak{a}_2) = \langle a_1, a_2 \rangle$ and we assume that $M|_A \cong k|_A \oplus P_A$ where P_A is a projective A-module. Then $a_r.M|_A \cong a_r.k|_A \oplus a_r.P_A \cong a_r.P_A$. The action of a_r on these modules is trivial, so think of them now as v-modules. We also know that $a_r.P_A$ is still projective as a v-module, since $a_2.V(\mathfrak{a}_2) \cong V(\mathfrak{a}_1)$ as $V(\mathfrak{a}_1)$-modules. Hence $\hat{M}|_v = a_r.M|_v$ is projective for all $v \in \mathbb{A}^{r-1} \setminus \{0\}$. This tells us that $\mathcal{V}_{\mathfrak{a}}^{\mathrm{rank}}(\hat{M}|_{\hat{V}}) = \{0\}$ and so $\hat{M}$ is a projective $\hat{V}$-module.

The projective indecomposable modules in the principal block are the projective covers (or equivalently injective hulls) of the trivial modules k_{ev} and k_{od}. Consequently, the simple, one dimensional socle of $\hat{V} \cong V(\mathfrak{a}_{r-1})$ is generated by $\tilde{\mathfrak{a}}_{r-1} = a_1 \otimes \cdots \otimes a_{r-1}$. Thus, $\dim \tilde{\mathfrak{a}}_{r-1}.\hat{M}$ counts the number of summands of projective $V(\mathfrak{a}_r)$-modules in $\hat{M}$ and so

$$\dim \hat{M} = 2^{r-1} \dim \tilde{\mathfrak{a}}_{r-1}.\hat{M}.$$

Also, $\tilde{\mathfrak{a}}_r = \tilde{\mathfrak{a}}_{r-1} \otimes a_r$ is a generator for the socle of $V(\mathfrak{a}_r)$ and $\tilde{\mathfrak{a}}_{r-1}.\hat{M} = \tilde{\mathfrak{a}}_r.M$ by construction. Therefore, M has a projective submodule, Q of dimension $2^r \dim \tilde{\mathfrak{a}}_r.M = 2^r \dim \tilde{\mathfrak{a}}_{r-1}.\hat{M}$. Thus,

$$2 \dim \hat{M} = \dim M - 1$$

and we conclude that $M \cong k \oplus Q$. Note, since this is a direct sum decomposition, as super vector spaces, $k = k|_A$. Thus k has the same parity as $k|_A$ (which was uniquely determined by M) and the claim is proven. $\qquad \square$

We can now classify endotrivial $\mathfrak{a}_r$-modules for all r.

Theorem 7.3. *Let $\mathfrak{a}_r$ be a rank r detecting subalgebra where $r \geq 2$, then $T(\mathfrak{a}_r) \cong \mathbb{Z} \times \mathbb{Z}_2$ and is generated by $\Omega^1(k_{ev})$ and k_{od}.*

Proof. The proof is exactly the same as in Theorem 6.11. $\qquad \square$

A FINITENESS THEOREM FOR $T(\mathfrak{g})$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra. Just as in the case of finite group schemes, it is not known if $T(\mathfrak{g})$ is finitely generated, but we can show that in certain cases, there are finitely many endotrivial modules of a fixed dimension. This is done by extending a proof in [12] to superalgebras.

In order to achieve a situation which is analagous to that in [12] (the main goals working with modules over a finitely generated algebra), we must work in a category which is Morita equivalent to $U(\mathfrak{g})$-modules.

8.1   MORITA EQUIVALENCE

First, consider the following category.

Definition 8.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and let Y^+ denote a set which indexes the simple $\mathfrak{g}$ modules. Let $L(\lambda)$ denote the simple module corresponding to $\lambda \in Y^+$ and $P(\lambda)$ its projective cover. Then define

$$A_{\mathfrak{g}} := \mathrm{End}_{U(\mathfrak{g})}^{fin} \left(\bigoplus_{\lambda \in Y^+} P(\lambda)^{\oplus \dim L(\lambda)} \right)^{\mathrm{op}}$$

which we will call the Khovanov algebra associated to $\mathfrak{g}$. The notation $\mathrm{End}_{U(\mathfrak{g})}^{fin}$ is to indicate that the endomorphisms are $U(\mathfrak{g})$-module endomorphisms which are supported on a finite number of summands.

By [6], the category of $U(\mathfrak{g})$-modules is Morita equivalent to the category of $A_{\mathfrak{g}}$-modules and finite dimensional modules correspond to finite dimensional modules of the same dimension. Shifting to the representation theory of $A_{\mathfrak{g}}$ is still not a sufficient reduction since $A_{\mathfrak{g}}$

is not finitely generated. A restriction to a finite subset Γ of Y^+ must be made in order to construct the variety of all n dimensional representations. In order to give this desired condition, some restrictions must be put on $\mathfrak{g}$. These restrictions will be considered in Section 8.3, so for now we will just assume there is such a finite set Γ.

With this in mind, consider the following subalgebra of $A_{\mathfrak{g}}$

$$A_\Gamma := \mathrm{End}_{U(\mathfrak{g})} \left(\bigoplus_{\lambda \in \Gamma} P(\lambda)^{\oplus \dim L(\lambda)} \right)^{\mathrm{op}}$$

which is constructed by taking $e := e_{\lambda_1} + \cdots + e_{\lambda_t}$ for each $\lambda_i \in \Gamma$ where e_{λ_i} is the idempotent associated to the identity endomorphism of $P(\lambda_i)$. Then $A_\Gamma = eA_{\mathfrak{g}}e$. We will show that this algebra is finitely generated and still has the desired idempotent decompositions of A_Γ.

Lemma 8.2. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra and let $A_{\mathfrak{g}}$ be the Khovanov algebra associated to $\mathfrak{g}$ and let Γ be a finite subset of Y^+. Then A_Γ is finitely generated as an algebra.*

Proof. The approach is to show that A_Γ is a quotient of $U(\mathfrak{g})$, which is finitely generated by the PBW theorem.

First, let M be a module such that $M = U(\mathfrak{g}).v$ for some $v \in M$. Then there is a map

$$\pi : U(\mathfrak{g}) \to \mathrm{End}_{U(\mathfrak{g})}(M) = \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}).v, U(\mathfrak{g}).v)$$

given by $u \mapsto u.\mathrm{id}_M$. Since these maps are $U(\mathfrak{g})$-invariant, any $f \in \mathrm{End}_{U(\mathfrak{g})}(M)$ is completely determined by the image of v. Thus, if $f \in \mathrm{End}_{U(\mathfrak{g})}(M)$, then for some $u \in U(\mathfrak{g})$,

$$f(v) = u.v = u.\mathrm{id}_M(v) = \pi(u)(v)$$

and so π is surjective.

Now it remains to show that $M = \bigoplus_{\lambda \in \Gamma} P(\lambda)^{\oplus \dim L(\lambda)}$ is cyclically generated by one element.

As a module $\mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}), L(\lambda)) \cong L(\lambda)$ and so $\dim \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}), L(\lambda)) = \dim L(\lambda)$ and so there are $\dim L(\lambda)$ homomorphic images of $U(\mathfrak{g})$ so $U(\mathfrak{g}) \twoheadrightarrow L(\lambda)^{\dim L(\lambda)}$ and so for a

finite subset of Y^+, $U(\mathfrak{g}) \twoheadrightarrow \bigoplus_{\lambda \in \Gamma} L(\lambda)^{\dim L(\lambda)}$ and consequently $U(\mathfrak{g}) \twoheadrightarrow \bigoplus_{\lambda \in \Gamma} P(\lambda)^{\oplus \dim L(\lambda)}$ as well. Then, $\bigoplus_{\lambda \in \Gamma} P(\lambda)^{\oplus \dim L(\lambda)}$ is generated as an algebra by the image of 1 and the claim has been shown. $\qquad \square$

Next, a lemma similar to that of [11, Lemma 2.1] is proven for the algebra A_Γ which will be required to prove certain properties about the variety of n dimensional representations.

Lemma 8.3. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a classical Lie superalgebra and let* $A_\mathfrak{g}$ *be the Khovanov algebra associated to* $\mathfrak{g}$ *and let* Γ *be a finite subset of* Y^+. *Let* M *be an* $A_\mathfrak{g}$-*module such that all the composition factors of* M *and the composition factors of the projective covers of* M *lie in* Γ. *Then there exist elements* $u_\lambda \in A_\Gamma$ *such that, if* $u_\lambda M \neq \{0\}$ *then* M *has a direct summand isomorphic to* $P_\Gamma(\lambda)$. *Moreover, if* $d_\lambda = \dim(u_\lambda P_\Gamma(\lambda))$ *is the rank of the operator of left multiplication by* u_λ *on* $P_\Gamma(\lambda)$ *and* $a_\lambda = \dim(u_\lambda M)/d_\lambda$, *then* $M \cong P_\Gamma(\lambda)^{d_\lambda} \oplus N$ *where* N *has no direct summands isomorphic to* $P_\Gamma(\lambda)$.

Proof. Since $A_\mathfrak{g}$ is an algebra with idempotent decomposition, we can assume that each projective module $P(\lambda) = A_\mathfrak{g} e_\lambda$ for an idempotent e_λ. Since $A_\mathfrak{g}$ is self injective, each projective indecomposable module has a simple socle and is thus generated by any nonzero element $u_\lambda \in \mathrm{Soc}(P(\lambda))$, thus $A_\mathfrak{g} u_\lambda = \mathrm{Soc}(P(\lambda))$ and $u_\lambda = u_\lambda e_\lambda$.

By assumption, there exists an $m \in M$ such that $u_\lambda m \neq 0$. Define $\psi_\lambda : P(\lambda) \to M$ by $\psi_\lambda(a e_\lambda) = a e_\lambda m$ for $a \in A_\mathfrak{g}$. Note that $\psi_\lambda(u_\lambda e_\lambda) = u_\lambda e_\lambda m = u_\lambda m \neq 0$ and since u_λ generates the socle of $P(\lambda)$, the map ψ_λ is injective. Furthermore, since $A_\mathfrak{g}$ is self injective, this projective module is injective and so $M \cong P(\lambda) \oplus N$. The multiplicity of the projective module follows from an inductive argument applied to the module N.

Next, this argument is extended to apply to the smaller algebra $A_\Gamma = eAe$ for some idempotent $e \in A_\mathfrak{g}$. This is done by realizing that each step of the above proof actually occurred in A_Γ.

First we consider what functor which gives the Morita equivalence between $U(\mathfrak{g})$ modules and $A_{\mathfrak{g}}$ modules. According to [6, Section 5], the functor is given by

$$F(-) := \operatorname{Hom}_{U(\mathfrak{g})}\left(\bigoplus_{\lambda \in Y^+} P(\lambda)^{\oplus \dim L(\lambda)}, -\right) \tag{8.1}$$

and then $A_{\mathfrak{g}}$ now acts by precomposition. Because we are only considering modules whose composition factors correspond to the λ which occur in Γ, then the only nonzero action of $A_{\mathfrak{g}}$ will be in A_{Γ}. Since we chose u_λ to act nontrivially on M, then the results also hold for A_{Γ}, which proves the result. $\qquad\square$

8.2 The Variety of n Dimensional Representations

Now we turn to the variety of n dimensional A_{Γ} representations. This is done by using a construction of Dade, as introduced in [15]. The goal is to show that the subvariety of endotrivial modules is open and that each component has a finite number of isomorphism classes of endotrivial modules in it. Since there is a correspondence between n dimensional A_{Γ}-modules and n dimensional $U(\mathfrak{g})$-modules, the same result holds for endotrivial $\mathfrak{g}$-modules.

Now, we actually construct the variety of n dimensional representations, with some additional structure to account for the $\mathbb{Z}_2$ grading of the representations. The variety of all representations of a fixed dimension n is denoted $\mathcal{V}_n$ and is defined by considering a set of homogeneous generators $g_1, \ldots, g_r$ for the superalgebra A_{Γ}. A representation is a homomorphism of superalgebras $\varphi : A_{\Gamma} \to \operatorname{End}_k(V)$ where $\dim(V) = n$, and if a homogeneous basis for V is fixed, we can think of this homomorphism as a superalgebra homomorphism $\varphi : A_{\Gamma} \to M_n(k)$. Since $g_1, \ldots, g_r$ generate A_{Γ}, the map φ is completely determined by $\varphi(g_i)$ which is an $n \times n$ matrix with entries $(g_{i,st})$ in k where $1 \le i \le r$ and $1 \le s, t \le n$.

Consider the polynomial ring $R = k[x_{i,st}]$, where $1 \le i \le r$ and $1 \le s, t \le n$, which has rn^2 variables. The information of each representation can be encoded in the form of a variety by defining a map $\overline{\varphi} : A_{\Gamma} \to M_n(R)$ by $(\overline{\varphi}(g_i))_{st} = x_{i,st}$ for $1 \le i \le r$. Since $A_{\Gamma} = \langle g_1, \ldots, g_r \rangle / \mathcal{I}$, the relations in $\mathcal{I}$ must be imposed on $M_n(R)$ by constructing the following ideal of R. By

using $\overline{\varphi}$, a relation in $\mathcal{I}$ is transferred the same relation in $M_n(R)$ by creating a relation on the rows and columns in the corresponding matrix multiplication. For example, take the relation $g_1 g_2 = 0$ in A_Γ. This would correspond to the relation $\sum_{u=1}^{n} x_{1,su} x_{2,ut} = 0$ for each $1 \leq s, t \leq n$. Furthermore, the fact that a representation is defined by a superalgebra homomorphism will introduce further relations to ensure that $\overline{\varphi}$ is homogeneous. Thus, if $g_i \in (A_\Gamma)_j$ for $j \in \mathbb{Z}_2$, then $(\overline{\varphi}(g_i))_{st} = x_{i,st} = 0$ for s and t which correspond to $(M_n(R))_{j+1}$ where the grading is inherited from $GL_n(k)$. With these relations, the matrices have the same algebra structure as A_Γ does (and now $\overline{\varphi}$ is actually a homomorphism), but they are expressed as zero sets in the polynomial ring R. Thus, the ideal $\mathcal{I}$ uniquely corresponds to an ideal $\mathcal{J} \subseteq R$.

If $\mathcal{V}_n := \mathcal{V}(\mathcal{J}) \subseteq k^{rn^2}$, then each point of $\mathcal{V}_n$ uniquely defines a representation of A_Γ. Furthermore, the orbit of any point under conjugation by an element of $(GL_n(k))_{\overline{0}}$ will yield an isomorphic representation, so the n dimensional representations of A_Γ are in one to one correspondence with the orbits of the points of $\mathcal{V}_n$ under the action of $(GL_n(k))_{\overline{0}}$. Thus, $\mathcal{V}_n$ contains all n dimensional representations of A_Γ.

Lastly, in order to achieve the results here, we artificially introduce a tensor structure on $A_{\mathfrak{g}}$. Let F' denote the equivalence from $A_{\mathfrak{g}}$-modules to $U(\mathfrak{g})$-modules since they are Morita equivalent. Then for M and N $A_{\mathfrak{g}}$-modules, define

$$M \otimes N := F(F'(M) \otimes F'(N)) \tag{8.2}$$

for modules in the image of $U(\mathfrak{g})$ under F. Note that by constructions, F is now a tensor functor.

Given this setup, we may now use the variety of n dimensional A_Γ-modules and the functor F (Equation 8.1), to prove corresponding versions of [12, Lemmas 2.2 and 2.3] and [12, Theorem 2.4], since A_Γ-modules satisfy a similar result to that of [12, Lemma 2.1].

Lemma 8.4. *Let M be a A_Γ-supermodule, $P(\lambda)$ a projective indecomposable A_Γ-supermodule and $m \in \mathbb{N}$. Let $\mathcal{U}$ be the subset of $\mathcal{V}_n$ of all representations, $\overline{\phi}$, of A_Γ such that $M \otimes L_{\overline{\phi}}$ has*

no submodule isomorphic to $P(\lambda)^m$, *where* $L_{\overline{\phi}}$ *is the module given by the representation* $\overline{\phi}$. *Then* $\mathcal{U}$ *is closed in* $\mathcal{V}_n$.

Proof. Using a similar idea as in the proof of Theorem 7.1, consider the rank of the matrix of u_λ (see the discussion preceding Lemma 6.3 for the definition) on $M \otimes L_{\overline{\phi}}$. Denote this matrix by M_{u_λ} and let r be the rank of the matrix. Since u_λ is a polynomial in the generators of A_Γ and the matrix of the action of $\overline{\phi}(g_i)$ on $M \otimes L_{\overline{\phi}}$ has entries in the polynomial ring R, then for a fixed representation M, the entries of M_{u_λ} are all polynomials in R.

By the same reasoning in Theorem 7.1, the condition that the rank of M_{u_λ} be less than r is the same condition that any $r \times r$ submatrix have determinant zero which we can then translate to a condition that certain polynomials in R be zero. Hence, the subset $\mathcal{U}$ is closed in $\mathcal{V}_n$. $\qquad\square$

Lemma 8.5. *Let M be an endotrivial $U(\mathfrak{g})$-supermodule of dimension n. Let $\mathcal{U}$ be the subset of representations $\overline{\phi}$ of $\mathcal{V}_n$ such that $L_{\overline{\phi}}$ is not isomorphic to $F(M) \otimes \lambda$ for any one dimensional module λ, where $L_{\overline{\phi}}$ is the module given by $\overline{\phi}$. Then $\mathcal{U}$ is closed in $\mathcal{V}_n$.*

Proof. Let μ be a one dimensional $U(\mathfrak{g})$-supermodule. Since M is endotrivial, so is $M \otimes \mu$. Then since F is a tensor functor,

$$F((M \otimes \mu) \otimes (M \otimes \mu)^*) \cong k \oplus \bigoplus_{i=1}^{l} P(\lambda_i)^{n_i}$$

where $P(\lambda_i)$ is a projective indecomposable A_Γ-module and $n_i \in \mathbb{N}$. For each i, let $\mathcal{U}_i \subseteq \mathcal{V}_n$ where

$$\mathcal{U}_i := \{\overline{\phi} \in \mathcal{V}_n \mid L_{\overline{\phi}} \otimes F(M^*) \otimes \mu^* \text{ does not contain a submodule isomorphic to } P(\lambda_i)^{n_i}\}.$$

By the previous lemma, each $\mathcal{U}_i$ is closed and so is $\mathcal{U}_\mu = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_l$.

Clearly, for any $\overline{\phi} \in \mathcal{U}$, $L_{\overline{\phi}}$ is not isomorphic to $F(M) \otimes \mu$ since they have different projective indecomposable summands. Now we will consider some $\overline{\phi} \notin \mathcal{U}_\mu$ and show that $L_{\overline{\phi}} \cong F(M) \otimes \lambda$ for some one dimensional module λ. Since $\overline{\phi} \notin \mathcal{U}_\mu$,

$$L_{\overline{\phi}} \otimes F(M^*) \otimes \mu^* \cong \nu \oplus \bigoplus_{i=1}^{l} P(\lambda_i)^{n_i}$$

for some supermodule ν. However, $\dim L_{\overline{\phi}} \otimes F(M^*) \otimes \mu^* = n^2 = \dim (F(M) \otimes \mu) \otimes (F(M^*) \otimes \mu^*)$ so ν is one dimensional. Since ν is one dimensional,

$$\nu^* \otimes L_{\overline{\phi}} \otimes F(M^*) \otimes \mu^* \cong k \oplus \bigoplus_{i=1}^{l} (\nu^* \otimes P(\lambda_i)^{n_i}).$$

Tensoring both sides by $\nu \otimes F(M) \otimes \mu$ yields

$$L_{\overline{\phi}} \oplus \bigoplus_{i=1}^{l} P_i^{n_i} \cong (F(M) \otimes \mu \otimes \nu) \oplus \bigoplus_{i=1}^{l} (P_i^{n_i} \otimes F(M) \otimes \mu)$$

and so $L_{\overline{\phi}} \cong F(M) \otimes \nu \otimes \mu \cong F(M) \otimes \lambda$ where $\lambda = \nu \otimes \mu$ by comparing the nonprojective summands. Then $\mathcal{U}_\mu$ is exactly the subset of $\mathcal{V}_n$ such that $L_{\overline{\phi}}$ is not isomorphic to $F(M) \otimes \mu$ where μ is a fixed one dimensional module. The proof is concluded by observing that

$$\mathcal{U} = \bigcap_\lambda \mathcal{U}_\lambda$$

where the intersection is over all one dimensional modules λ. Thus, $\mathcal{U}$ is closed. $\qquad \square$

Theorem 8.6. *Let* $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ *be a classical Lie superalgebra and let* Γ *be a finite subset of* Y^+*. Then there are only finitely many isomorphism classes of endotrivial* $U(\mathfrak{g})$*-modules of dimension* n*.*

Proof. Let M be an indecomposable endotrivial $U(\mathfrak{g})$-supermodule of dimension n. Let $\mathcal{U}_M$ be the subset of $\mathcal{V}_n$ of representations $\overline{\phi}$ such that the associated module $L_{\overline{\phi}}$ is isomorphic to $F(M) \otimes \lambda$ for some one dimensional module λ.

By the previous lemma, $\mathcal{U}_M$ is open in $\mathcal{V}_n$ and so $\overline{\mathcal{U}_M}$ is a union of components in $\mathcal{V}_n$. There are finitely many components of $\mathcal{V}_n$, and any endotrivial module will be contained in some $\mathcal{U}_M$ for some endotrivial $U(\mathfrak{g})$-module M. Since A_Γ has only finitely many isomorphism classes of one dimensional modules, there are finitely many isomorphism classes in each such $\mathcal{U}_M$. Hence, we conclude that there are only finitely many endotrivial $U(\mathfrak{g})$-modules of dimension n. $\qquad \square$

8.3 Conditions for Finiteness

Now that the main theorem of the chapter has been established, we consider conditions on a classical Lie superalgebra $\mathfrak{g}$ that will yield the finite subset Γ of Y^+, the set which indexes the simple modules of $\mathfrak{g}$.

The first case considered to guarantee this is when $\mathfrak{g}_{\bar{0}}$ has finitely many simple modules of dimension $\leq n$. The condition on $\mathfrak{g}_{\bar{0}}$ can be extended to all of $\mathfrak{g}$.

Lemma 8.7. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ has finitely many simple modules of dimension $\leq n$. Then $\mathfrak{g}$ has finitely many simple modules of dimension $\leq n$ and consequently, there is a finite set $\Gamma \subseteq Y^+$ such that any $U(\mathfrak{g})$ module of dimension $\leq n$ has composition factors $L(\lambda)$ such that $\lambda \in \Gamma$.*

Proof. Let S be a simple $\mathfrak{g}$-module of dimension $\leq n$. Then $S|_{\mathfrak{g}_{\bar{0}}}$ has a simple $\mathfrak{g}_{\bar{0}}$-module in its socle, call it T. Then,

$$0 \neq \mathrm{Hom}_{U(\mathfrak{g}_{\bar{0}})}(T, S|_{\mathfrak{g}_{\bar{0}}}) = \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} T, S)$$

and so there is a surjection $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} T \twoheadrightarrow S$ for some simple $\mathfrak{g}_{\bar{0}}$-module T.

Consider the set

$$\mathscr{C} = \{U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} T \mid T \text{ is a simple } \mathfrak{g}_{\bar{0}}\text{-module of dim} \leq n\}$$

Note that each element of $\mathscr{C}$ is finite dimensional since T is finite dimensional. Furthermore, by the previous observation, a module from this set surjects onto any simple $\mathfrak{g}$-module, and since the set $\mathscr{C}$ is finite by assumption, the result is proven. $\square$

Now the Lie algebras which satisfy this condition are considered. Since $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is classical, $\mathfrak{g}_{\bar{0}}$ is reductive. Note that if $\mathfrak{g}_{\bar{0}}$ has any toral elements, then there will be infinitely many one dimensional modules where the toral elements act via scalars. Thus, it must be that $\mathfrak{g}$ is semisimple.

Then $\mathfrak{g}_{\overline{0}} \cong \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_s$ where h_i is a simple Lie algebra. Then any simple finite dimensional $\mathfrak{g}_{\overline{0}}$-module $L(\lambda) \cong L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_s)$ where $L(\lambda_i)$ is a simple $\mathfrak{h}_i$ module.

Since there are finitely many weights λ_i of $X(\mathfrak{h}_i)$ such that $L(\lambda_i)$ is of dimension $\leq n$, then the same result holds for $L(\lambda)$ and consequently $\mathfrak{g}_{\overline{0}}$.

Corollary 8.8. *Let $\mathfrak{g}$ be a classical Lie superalgebra such that $\mathfrak{g}_{\overline{0}}$ is a semisimple Lie algebra. Then there are finitely many isomorphism classes of endotrivial $\mathfrak{g}$-modules of dimension n for any $n \in \mathbb{Z}$.*

Proof. Since $\mathfrak{g}_{\overline{0}}$ is semisimple, there are only finitely many simple modules of dimension $\leq n$ for $n \in \mathbb{Z}$. The result follows from Lemma 8.7 and Theorem 8.6. $\square$

The conditions given in the previous theorem are sufficient but not necessary for all classical Lie superalgebras. Some interesting cases are Lie superalgebras whose even component contains a torus, and thus is not semisimple. In this case, it may be possible to conclude the result assuming that there are only finitely many *one* dimensional modules, regardless of what n may be. The condition of having only finitely many one dimensional representations is explored here.

By direct computations, there are only finitely many endotrivial modules of a fixed dimension n for detecting subalgebras–which only have *two* one dimensional modules, k_{ev} and k_{od}.

Now consider an example where this condition fails. When $\mathfrak{g} = \mathfrak{gl}(1|1)$, there are infinitely many one dimensional $\mathfrak{g}$-modules. The matrix realization $\mathfrak{gl}(1|1)$ has basis vectors x and y as in $\mathfrak{sl}(1|1)$, but has two toral basis elements t_1 and t_2 which are given by

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the weights of the simple modules in $\mathfrak{g}$ are given by $(\lambda|\mu)$ where $\lambda, \mu \in k$ and t_1 and t_2 act on k via multiplication by λ and μ respectively. If $\lambda = -\mu$ then the representation of the simple $\mathfrak{g}$-module is one dimensional. Thus there are infinitely many one dimensional modules given by the representations $(\lambda| - \lambda)$.

In general, the condition that $\mathfrak{g}$ has finitely many one dimensional modules in $\mathcal{F}$ is equivalent to the condition that $\mathfrak{g}_{\overline{0}}/([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\overline{0}})$ has finitely many one dimensional modules in $\mathcal{F}$.

Proposition 8.9. *Let $\mathfrak{g}$ be a classical Lie superalgebra. Then there are finitely many one dimensional $\mathfrak{g}$-modules in $\mathcal{F}$ if and only if $\mathfrak{g}_{\overline{0}} \subseteq [\mathfrak{g}, \mathfrak{g}]$.*

Proof. Assume that $\mathfrak{g}$ is a Lie superalgebra such that $\mathfrak{g}_{\overline{0}} \subseteq [\mathfrak{g}, \mathfrak{g}]$ A one dimensional representation of $\mathfrak{g}$ corresponds to a Lie superalgebra homomorphism $\phi : \mathfrak{g} \to k_{ev}$ since $\text{End}_k(k)$ (for k either even or odd) is always isomorphic to k_{ev}. Since k_{ev} is concentrated in degree $\overline{0}$ and ϕ is an even map, any element of $\mathfrak{g}_{\overline{1}}$ necessarily maps to 0. Furthermore, since k_{ev} is abelian as a Lie superalgebra, then $[\mathfrak{g}, \mathfrak{g}]$ must be mapped to zero and so by assumption $\mathfrak{g}_{\overline{0}}$ maps to 0 as well and ϕ is the 0 map. This forces the one dimensional module to be either k_{ev} or k_{od}.

Now, assume that $\mathfrak{g}$ has only finitely many one dimensional modules but that $\mathfrak{g}_{\overline{0}} \not\subseteq [\mathfrak{g}, \mathfrak{g}]$. Let $g \in \mathfrak{g}_{\overline{0}} \setminus [\mathfrak{g}, \mathfrak{g}]$. As noted, since k_{ev} is abelian, if ϕ is a representation of $\mathfrak{g}$, then $\overline{\phi} : \mathfrak{g}_{\overline{0}}/([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\overline{0}}) \to k_{ev}$ yields another representation which agrees on nonzero elements. They are also equivalent in the sense ϕ can be obtained uniquely from $\overline{\phi}$ and vice versa. Since $\mathfrak{g}$ is classical, $\mathfrak{g}_{\overline{0}}$ is a reductive Lie algebra and given that $g \notin [\mathfrak{g}, \mathfrak{g}]$, g must be in the center of $\mathfrak{g}_{\overline{0}}$ and therefore in the torus of $\mathfrak{g}_{\overline{0}}$ as well. Thus, g is a semisimple element, and in $\mathcal{F}$, g must act diagonally on any one dimensional module. If $\overline{g}$ is the image of g in $\mathfrak{g}_{\overline{0}}/([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\overline{0}})$, then $\overline{g}$ is nonzero and $\langle \overline{g} \rangle$ is a one dimensional abelian Lie superalgebra. Since $\overline{g}$ acts diagonally and k is infinite, this yields infinitely many distinct one dimensional modules resulting from the diagonal action of $\overline{g}$. These one dimensional modules lift (possibly non-uniquely) to $\mathfrak{g}_{\overline{0}}/([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\overline{0}})$ and consequently $\mathfrak{g}$ as well. This is a contradiction and the assumption that $\mathfrak{g}_{\overline{0}} \not\subseteq [\mathfrak{g}, \mathfrak{g}]$ is false. $\square$

Corollary 8.10. *Let $\mathfrak{g}$ be a simple classical Lie superalgebra. Then there are finitely many one dimensional $\mathfrak{g}$-modules in $\mathcal{F}$.*

Proof. By the necessary condition of simplicity given in Proposition 1.2.7 of [20] that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$, the lemma is proven. $\square$

Note that in the case where there are finitely many one dimensional modules in $\mathcal{F}$, there are in fact only two, k_{ev} and k_{od}, as in the case of the detecting subalgebras.

In regard to the $\mathfrak{g} = \mathfrak{gl}(1|1)$ example, $[\mathfrak{g}, \mathfrak{g}] = \langle x, y, t_1 + t_2 \rangle$. Then $\mathfrak{g}_{\bar{0}}/([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\bar{0}}) = \langle t_1, t_2 \rangle/(t_1 + t_2)$ and it is clear that if t_1 has any weight λ, then μ is determined to be $-\lambda$. Thus, there are infinitely many one dimensional representations resulting from the free parameter λ.

It may be the case that the conditions in Theorem 8.6 may be relaxed to a condition that there only be finitely many one dimensional modules, as these examples suggest, however, the proof technique fails in this case, so other approaches must be considered.

CHAPTER 9

ENDOTRIVIAL MODULES FOR $\mathfrak{gl}(n|n)$

9.1 CONSTRUCTION OF THE RELATIVE CATEGORY

The focus of this chapter is to give a reduction for computing the endotrivial modules for the Lie superalgebra $\mathfrak{gl}(n|n)$. The result obtained is that the group $T(\mathfrak{g})$ injects into the group of endotrivial modules for a specific intermediate parabolic, which will be denoted as $\mathfrak{p}$.

First, several definitions and constructions are given in order to reach this goal.

9.2 PROPERTIES OF THE RELATIVE CATEGORY

Now, a few results are given to further understand the category $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. The first proposition gives a concrete description of all the relatively projective modules as direct summands of induced modules.

Proposition 9.1. *A $\mathfrak{g}$-module M where $\mathfrak{g}$ is a Lie superalgebra, is $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projective if and only if it is a direct summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} N$ for some $U(\mathfrak{g}_{\bar{0}})$-module N.*

Proof. First, assume that M is projective in $\mathcal{F}$. Then the $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-exact sequence

$$0 \longrightarrow \ker \mu \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M|_{\mathfrak{g}_{\bar{0}}} \overset{\mu}{\longrightarrow} M \longrightarrow 0$$

is split by using the $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projectivity of M to extend the identity map on M, in the standard way.

Now, let M be a direct summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} N$ for some $U(\mathfrak{g}_{\bar{0}})$-module N. Then

$$\mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, R) \hookrightarrow \mathrm{Ext}^1_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} N, R) = \mathrm{Ext}^1_{(\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}})}(N, R) = 0$$

for any module R in $\mathcal{F}$. Thus, M is $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projective and thus projective in $\mathcal{F}$. $\quad\square$

Proposition 9.2. *Let M be a module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ where $\mathfrak{g}$ is a Lie superalgebra, then M is projective in $\mathcal{F}$ if and only if it is an injective module in $\mathcal{F}$.*

Proof. The proof given in [3, Propositions 2.2.2] holds for the category $\mathcal{F}$. □

Corollary 9.3. *Let M be a module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ where $\mathfrak{g}$ is a Lie superalgebra, then there exists a projective module P in $\mathcal{F}$ such that there is a homomorphism of $\mathcal{F}$ modules $\pi : P \twoheadrightarrow M$ and an injective module I in $\mathcal{F}$ such that $\iota : M \hookrightarrow I$.*

Proof. If M is any $\mathfrak{g}$-module, then $M|_{\mathfrak{g}_{\bar{0}}}$ is a $\mathfrak{g}_{\bar{0}}$-module and so by Proposition 9.1, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M|_{\mathfrak{g}_{\bar{0}}}$ is a $(U(\mathfrak{g}), U(\mathfrak{g}_{\bar{0}}))$-projective module which surjects on to M via the "multiplication map", $u \otimes m \mapsto u.m$ (since $U(\mathfrak{g})$ is a unital superalgebra).

A dual argument to this completes the proof. □

With a better understanding of projective modules in $\mathcal{F}$, we now define the object of interest in this paper.

Definition 9.4. A module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ where $\mathfrak{g}$ is a stable Lie superalgebra, is called endotrivial if $\text{End}_k(M) \cong k_{ev} \oplus P$ as $U(\mathfrak{g})$-modules for some projective module P in $\mathcal{F}$.

9.3 THE $\mathfrak{gl}(n|n)$ CASE

The desired injection depends on a specific parabolic subalgebra and endotrivial modules for this subalgebra which are addressed here.

Define the maximal torus $\mathfrak{t}_{\bar{0}}$ to be the subalgebra of diagonal matrices of $\mathfrak{g} = \mathfrak{gl}(n|n)$ and $\mathfrak{p}$, referred to as the *distinguished parabolic subalgebra*, to be the subalgebra defined as follows. Let $\mathfrak{p}_{\bar{0}} \subseteq (\mathfrak{gl}(n|n))_{\bar{0}} \cong \mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$ be generated by the upper triangular matricies of each $\mathfrak{gl}(n)$ and $\mathfrak{p}_{\bar{1}} \subseteq (\mathfrak{gl}(n|n))_{\bar{1}}$ be generated by $n \times n$ matricies whose entries are all on or above the odd diagonal.

Note that if $\mathfrak{f}$ is defined to be the subalgebra of $\mathfrak{p}$ generated by elements which are strictly on the diagonal (either the even diagonal or the odd diagonal), then $\mathfrak{f} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ and $\mathfrak{t}_{\mathfrak{f}} \subseteq \mathfrak{t}_{\mathfrak{p}} = \mathfrak{t}_{\bar{0}}$. Given this set up, we can relate $T(\mathfrak{f})$, $T(\mathfrak{p})$, and $T(\mathfrak{g})$.

First, it must be established that restriction to each of these subalgebrs takes projectives to projectives in order to have well defined maps between these groups which live in the stable module category.

9.3.1 Restriction Preserves Projectivity

Because $\mathfrak{g}$ is a Type I Lie superalgebra, $\mathfrak{g}$ has a $\mathbb{Z}$ grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is consistent with the standard $\mathbb{Z}_2$ grading. This gives a consistent $\mathbb{Z}$ grading on $\mathfrak{p} \subseteq \mathfrak{g}$ by defining $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{g}_i$ for $i \in \mathbb{Z}$, and so $\mathfrak{p} = \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1$. Given this grading, define $\mathfrak{p}^+ := \mathfrak{p}_0 \oplus \mathfrak{p}_1$ and $\mathfrak{p}^- := \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$.

Following the work in [3], define $\mathcal{F}(\mathfrak{p}_{\pm 1})$ to be the category of finite dimensional $\mathfrak{p}_{\pm 1}$-modules. For the objects in $\mathcal{F}(\mathfrak{p}_{\pm 1})$, define the support variety $\mathcal{V}_{\mathfrak{p}_{\pm 1}}(M)$ as in [3] and the rank variety

$$\mathcal{V}_{\mathfrak{p}_{\pm 1}}^{\mathrm{rank}}(M) = \{x \in \mathfrak{p}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\}.$$

Since $\mathfrak{p}_1$ and $\mathfrak{p}_{-1}$ are both abelian Lie superalgebras, both are well defined and identified by a canonical isomorphism as discussed in [2].

As before, let $X(\mathfrak{t}_{\bar{0}}) \subseteq \mathfrak{t}_{\bar{0}}^*$ be the set of weights relative to a fixed maximal torus $\mathfrak{t}_{\bar{0}} \subseteq \mathfrak{p}_{\bar{0}}$. For $\lambda \in X(\mathfrak{t}_{\bar{0}})$, consider the simple finite dimensional $\mathfrak{p}_{\bar{0}}$-module of weight λ. An important property of $\mathfrak{p}_{\bar{0}}$ is that is constructed to be solvable as a Lie algebra and so by Lie's theorem, irreducible $\mathfrak{p}_{\bar{0}}$-modules are one dimensional modules where $t \in \mathfrak{t}_{\bar{0}}$ acts by $t.v = \lambda(t)v$. Typically, such one dimensional modules are usually just denoted by k_λ and may be concentrated in either even or odd degree.

It will be very useful to have a partial ordering on these weights. Let $d = \dim \mathfrak{t}_{\bar{0}}$. The weights $X(\mathfrak{t}_{\bar{0}})$ can be parametrized by the set k^d so any $\lambda \in X(\mathfrak{t}_{\bar{0}})$ can be though of as an ordered d-tuple, $(\lambda_1, \ldots, \lambda_d)$. For two weights $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$, we say that $\lambda \geq \mu$ if and only if for each $k = 1, \ldots, d$,

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$$

and equality holds if and only if $\lambda = \mu$. This ordering will allow the use of highest weight theory.

A module $M \in \mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$ is called a *highest weight module* if in the weight decomposition $M \cong \bigoplus_{\lambda \in X(\mathfrak{t}_{\bar{0}})} M_\lambda$, there exists a weight λ_0 such that $\lambda_0 \geq \mu$ for each nonzero weight space M_μ of M.

Proposition 9.5. *If $S \in \mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$ is a simple $\mathfrak{p}$-module, then S is a highest weight module in $\mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$.*

Proof. Because S is finite dimensional, there exists a weight $\lambda_0 \in X(\mathfrak{t}_{\bar{0}})$ such that $\mu \not> \lambda_0$ for all nonzero weight spaces S_μ of S. Note that this means all weights are either less than or equal to or not comparable to λ_0.

For any element u of $\mathfrak{u} = \mathfrak{p}/\mathfrak{t}_{\bar{0}}$, $u.S_\lambda \subseteq S_\mu$ implies that $\lambda > \mu$ in $X(\mathfrak{t}_{\bar{0}})$. This yields that $u.S_{\lambda_0} = 0$ for any $u \in \mathfrak{u}$. Since S is simple (because S is and this does not depend on the torus), for $v \in S_{\lambda_0}$, v generates S and $u.v = 0$.

Thus, $S = U(\mathfrak{p}_{-1})U(\mathfrak{p}^+).v$ but since any element of $\mathfrak{p}^+$ either stabilizes or kills v, it follows that $S = U(\mathfrak{p}_{-1}).v$. It is now clear, because $v \in S_{\lambda_0}$ that any element of $S_\mu \neq 0$ is equal to $cy.v$ for some $y \in U(\mathfrak{p}_{-1})$ and $c \in k$, that $\lambda_0 \geq \mu$. Thus S is a highest weight module. $\square$

Because $\mathfrak{p}_1 \subseteq \mathfrak{p}^+$ and $\mathfrak{p}_{-1} \subseteq \mathfrak{p}^-$ are ideals, the module k_λ can be considered as a simple $\mathfrak{p}^\pm$-module by inflation via the canonical quotient map $\mathfrak{p}^\pm \twoheadrightarrow \mathfrak{p}_{\bar{0}}$. By construction, $\mathfrak{p}_1$ and $\mathfrak{p}_{-1}$ act by 0 on k_λ. Define

$$K(\lambda) = U(\mathfrak{p}) \otimes_{U(\mathfrak{p}^+)} k_\lambda \quad \text{and} \quad K^-(\lambda) = \operatorname{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{p}), k_\lambda)$$

to be the Kac module and the dual Kac module, respectively.

The Kac module $K(\lambda)$ has several useful properties. First, by construction it is a highest weight module in $\mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$. Since $K(\lambda)$ is generated by one element, it has a simple head. Also, if S is any simple module in $\mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$ where S has highest weight λ for some weight λ of S, and S is generated by v, and $w \in k_\lambda$, there is a homomorphism from $K(\lambda) \twoheadrightarrow S$ given by

$1 \otimes w \mapsto v$. This homomorphism also lifts uniquely to a map $K(\lambda) \twoheadrightarrow S$ by extending the action given by the previous map to the full torus.

Furthermore, $K(\lambda)/\operatorname{Rad}(K(\lambda)) \cong S$ and is denoted $L(\lambda)$. Note that this surjective homomorphism is in fact valid for any highest weight module and in this sense, the Kac module is universal.

Dually, $K^-(\lambda)$ has a simple socle which is isomorphic to $L(\lambda)$ as well and λ is the lowest weight of $K^-(\lambda)$.

Now we define two useful filtrations of a module M in $\mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$. M is said to admit a Kac filtration if there is a filtration

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$$

of the module M such that for $i = 1, \ldots t$, $M_i/M_{i-1} \cong K(\lambda_i)$ for some $\lambda_i \in X(\mathfrak{t}_{\bar{0}})$. Similarly, if M has a filtration as above such that $i = 1, \ldots t$, $M_i/M_{i-1} \cong K^-(\lambda_i)$, then M is said to admit a dual Kac filtration.

By the same reasoning in [3], modules in $\mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$ satisfy the following.

Theorem 9.6. *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be the distinguished parabolic subalgebra as defined at the beginning of Section 9.3 and let M be a module in $\mathcal{F}_{\mathfrak{p}} = \mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$. Then the following are equivalent.*

1. *M has a Kac filtration;*

2. *$\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{p}}}(M, K^-(\mu)) = 0$ for all $\mu \in X(\mathfrak{t}_{\bar{0}})$;*

3. *$\operatorname{Ext}^1_{\mathcal{F}(\mathfrak{p}_{-1})}(M, k) = 0$;*

4. *$\mathcal{V}_{\mathfrak{p}_{-1}}(M) = 0$.*

A dual version of this theorem can be proved in a similar way.

Theorem 9.7. *Let $\mathfrak{p} \subseteq \mathfrak{g}$ be the distinguished parabolic subaglebra as defined at the beginning of Section 9.3 and let M be a module in $\mathcal{F}_{\mathfrak{p}} = \mathcal{F}_{(\mathfrak{p},\mathfrak{p}_{\bar{0}})}$. Then the following are equivalent.*

1. *M has a dual Kac filtration;*

2. $\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{p}}}(K(\mu), M) = 0$ *for all* $\mu \in X(\mathfrak{t}_{\overline{0}})$;

3. $\operatorname{Ext}^1_{\mathcal{F}(\mathfrak{p}_1)}(k, M) = 0$;

4. $\mathcal{V}_{\mathfrak{p}_1}(M) = 0$.

These two theorems can be used to show the following powerful condition relating projectivity in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$ and the support varities of $\mathfrak{p}_{\pm 1}$.

Theorem 9.8. *Let M be in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$. Then M is projective in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$ if and only if $\mathcal{V}_{\mathfrak{p}_1}(M) = \mathcal{V}_{\mathfrak{p}_{-1}}(M) = \{0\}$.*

Corollary 9.9. *A projective module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$ is also projective in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$ and thus, there is a well defined map*

$$\operatorname{res}^{T(\mathfrak{g})}_{T(\mathfrak{p})} : T(\mathfrak{g}) \to T(\mathfrak{p})$$

given by $M \mapsto M|_{\mathfrak{p}}$.

Proof. Let P be a projective module in $\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$. Then by, [3, Theorem 3.5.1], $\mathcal{V}_{\mathfrak{g}_1}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$. Since $\mathcal{V}_{\mathfrak{p}_1}(M) \subseteq \mathcal{V}_{\mathfrak{g}_1}(M) = \{0\}$, and $\mathcal{V}_{\mathfrak{p}_{-1}}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$, then by Theorem 9.8, $M|_{\mathfrak{p}}$ is projective in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$.

With this conclusion, the restriction map now descends to a well defined map on each of the respective stable module categories, and in particular, if for $M \in \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\overline{0}})}$, $M \otimes M^* \cong k_{ev} \oplus P$, then $(M \otimes M^*)|_{\mathfrak{p}} \cong k_{ev} \oplus P|_{\mathfrak{p}}$ □

With these tools established, the final maps follow easier.

Proposition 9.10. *Let M be a projective module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\overline{0}})}$. Then $M|_{\mathfrak{p}^+}$ and $M|_{\mathfrak{p}^-}$ are projective in their respective categories.*

Proof. First, assume that M is projective in $\mathcal{F}$. Then, by Proposition 9.1, M is a summand of $U(\mathfrak{p}) \otimes_{U(\mathfrak{p}_{\overline{0}})} N$ for some $U(\mathfrak{p}_{\overline{0}})$-module N. Because

$$U(\mathfrak{p}) \otimes_{U(\mathfrak{p}_{\overline{0}})} N \cong U(\mathfrak{p}^+) U(\mathfrak{p}_{-1}) \otimes_{U(\mathfrak{p}_{\overline{0}})} N \cong U(\mathfrak{p}^+) \otimes_{U(\mathfrak{p}_{\overline{0}})} [U(\mathfrak{p}_{-1}) \otimes N]$$

where the second isomorphism is given on basis elements by

$$u^+ u_{-1} \otimes_{U(\mathfrak{p}_{\bar{0}})} n \mapsto u^+ \otimes_{U(\mathfrak{p}_{\bar{0}})} [u_{-1} \otimes n],$$

any summand of $U(\mathfrak{p}) \otimes_{U(\mathfrak{p}_{\bar{0}})} N$ is also a summand of $U(\mathfrak{p}^+) \otimes_{U(\mathfrak{p}_{\bar{0}})} N'$ for some $\mathfrak{p}_{\bar{0}}$-module N'. Thus, if M is projective in $\mathcal{F}_{(\mathfrak{p}, \mathfrak{p}_{\bar{0}})}$, M is also $(U(\mathfrak{p}^+), U(\mathfrak{p}_{\bar{0}}))$-projective as well, or projective in $\mathcal{F}_{(\mathfrak{p}^+, \mathfrak{p}_{\bar{0}})}$. By a similar argument, M is also projective in $\mathcal{F}_{(\mathfrak{p}^-, \mathfrak{p}_{\bar{0}})}$. □

9.4 Restriction from $T(\mathfrak{g})$ to $T(\mathfrak{p})$

Let $\mathfrak{g} = \mathfrak{gl}(n|n)$ and $\mathfrak{p} \subseteq \mathfrak{g}$ be the distinguished parabolic. Now that restriction from $T(\mathfrak{g})$ to $T(\mathfrak{p})$ is well defined, properties of this map can be exploited to relate a classification of one to the other. An important step in understanding the relationship between these two groups is an induction functor from $\mathfrak{p}$ to $\mathfrak{g}$.

In [19, Section 3], the geometric induction functor Γ_0 is defined. The functor Γ_0 is from $\mathfrak{p}$-modules to $\mathfrak{g}$-modules and will be denoted $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ since the geometric structure will not be emphasized in this paper. This functor is of particular interest because it will allow us to show that restriction map is injective.

Since $T(\mathfrak{g})$ and $T(\mathfrak{p})$ are groups and the restriction map is a homomorphism (restriction commutes with the tensor product over k), we can show that the restriction map

$$\operatorname{res}_{T(\mathfrak{p})}^{T(\mathfrak{g})} : T(\mathfrak{g}) \to T(\mathfrak{p})$$

given by $M \mapsto M|_{\mathfrak{p}}$ is injective by checking that $\ker \left(\operatorname{res}_{T(\mathfrak{p})}^{T(\mathfrak{g})} \right) = \{k_{ev}\}$ since k_{ev} is the identity in $T(\mathfrak{g})$.

In order to show this map is injective, we need to show that $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} k_{ev} = k_{ev}$. This is done by considering [19, Lemma 3] and the proof of the lemma. In particular, the authors observe that if L_μ (respectively $L_\mu(\mathfrak{a})$) is the simple $\mathfrak{g}$-module (respectively $\mathfrak{a}$-module) with highest weight μ, then if L_μ occurs in $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} k_\lambda$, then $L_\mu(\mathfrak{g}_{\bar{0}})^*$ occurs in $H^0(G_0/B_0, \mathcal{L}_\lambda^*(\mathfrak{p}) \otimes S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^*)$.

The case when k_λ is the trivial module k is of particular interest in this section. Thus we consider $H^0(G_0/B_0, S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^*)$, and more specifically, the dominant weights in

$S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^{*})$. In order for such a weight to be dominant, it must have positive inner product with $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n$ and $\delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{n-1} - \delta_n$. The weights of $S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^{*})$ are positive linear combinations of the weights of the form $\varepsilon_i - \delta_j$ and $\delta_i - \varepsilon_j$ where $i > j$.

Proposition 9.11. *Let $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{gl}(n|n)$. No weight of $S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^{*}$ is dominant.*

Proof. This will be proven by induction on n. The first case is trivial since when $n = 1$, $\mathfrak{p}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

The first nontrivial base case is when $n = 2$. If the weights $\varepsilon_2 - \delta_1$ and $\delta_2 - \varepsilon_1$ are represented as $(0, 1| - 1, 0)$ and $(-1, 0|0, 1)$ respectively, then a positive linear combination of such weights $r(\varepsilon_2 - \delta_1) + s(\delta_2 - \varepsilon_1)$ is represented as $(-s, r| - r, s)$. We compute

$$\langle (1, -1|0, 0), (-s, r| - r, s) \rangle = -s - r$$

$$\langle (0, 0|1, -1), (-s, r| - r, s) \rangle = -s - r$$

and so any nonzero weight has negative inner product and thus, is not dominant.

Now let $n > 2$. In order for a positive linear combination of weights to be dominant, there are a set of conditions which must be satisfied. Let λ be an arbitrary weight and let $a_{i,j}$ be the coefficient for the weight $\varepsilon_i - \delta_j$ and $b_{k,l}$ be the coefficient of the weight $\delta_k - \varepsilon_l$, where $i > j$ and $k > l$. Then

$$\lambda = \left(\sum_{i>j} a_{i,j}(\varepsilon_i - \delta_j) \right) + \left(\sum_{k>l} b_{k,l}(\delta_k - \varepsilon_l) \right)$$

or if we denote $\alpha_{i,j} = a_{i,j}(\varepsilon_i - \delta_j)$ and $\beta_{k,l} = b_{k,l}(\delta_k - \varepsilon_l)$, then $\lambda = \sum_{i>j}(\alpha_{i,j} + \beta_{i,j})$.

Note that

$$\langle \varepsilon_s - \varepsilon_{s+1}, \alpha_{i,j} \rangle = \delta_{s,i}a_{i,j} - \delta_{s+1,i}a_{i,j}$$

$$\langle \varepsilon_s - \varepsilon_{s+1}, \beta_{i,j} \rangle = -\delta_{s,j}b_{i,j} + \delta_{s+1,j}b_{i,j}$$

$$\langle \delta_s - \delta_{s+1}, \alpha_{i,j} \rangle = -\delta_{s,j}a_{i,j} + \delta_{s+1,j}a_{i,j}$$

$$\langle \delta_s - \delta_{s+1}, \beta_{i,j} \rangle = \delta_{s,j}b_{i,j} - \delta_{s+1,j}b_{i,j}$$

where $\delta_{s,t}$ is the Kronecker delta. We note that the conditions $\langle \varepsilon_s - \varepsilon_{s+1}, \lambda \rangle \geq 0$ and $\langle \delta_s - \delta_{s+1}, \lambda \rangle \geq 0$ for each $s = 1, \ldots, n-1$ gives $2(n-1)$ inequalities which the coefficients $a_{i,j}$ and $b_{i,j}$ must satisfy.

The important step in this proof is to add all the given inequalities together to produce one inequality.

$$\sum_{s=1}^{n-1} \left(\langle \varepsilon_s - \varepsilon_{s+1}, \lambda \rangle + \langle \delta_s - \delta_{s+1}, \lambda \rangle \right) =$$

$$\sum_{s=1}^{n-1} \sum_{i>j} \left(\langle \varepsilon_s - \varepsilon_{s+1}, \alpha_{i,j} + \beta_{i,j} \rangle + \langle \delta_s - \delta_{s+1}, \alpha_{i,j} + \beta_{i,j} \rangle \right) \geq 0.$$

Next, observe that each $a_{i,j}$ and $b_{i,j}$ appears exactly twice as a negative term in the inequality. Furthermore, each term $a_{k,l}$ and $b_{k,l}$ with $1 < k, l < n$ appears twice as a positive term and $a_{i,1}$, $a_{n,j}$, $b_{i,1}$, and $b_{n,j}$ appear at most once as a positive term (with $a_{1,n}$ and $b_{1,n}$ being the terms which do not appear at all). Rearranging the inequality then yields

$$0 \geq \sum_{s=1}^{n} (a_{s,1} + a_{n,s} + b_{s,1} + b_{n,s})$$

and so each coefficient of this form is forced to be zero in order for a weight to be dominant. However, by induction, we have now reduced to a weight whose nonzero coefficients come from a lower diagonal $(n-2) \times (n-2)$ matrix which has no dominant weights by the inductive hypothesis. Thus, the claim is proven. $\square$

Corollary 9.12. *Let* $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{gl}(n|n)$, *then* $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} k \cong k$.

Proof. Since $S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^*$ has no dominant weights by the previous lemma,

$$H^0(G_0/B_0, S^{\bullet}(\mathfrak{g}/(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}))^*) \cong k.$$

Furthermore, note that the induction functor does not change the parity of the module, so the degree (either even or odd) is fixed and the result is proven. $\square$

Corollary 9.13. *The restriction map*

$$\mathrm{res}_{T(\mathfrak{p})}^{T(\mathfrak{g})} : T(\mathfrak{g}) \to T(\mathfrak{p})$$

given by $M \mapsto M|_{\mathfrak{p}}$ *is injective*

Proof. Let $M \in T(\mathfrak{g})$ be an indecomposable endotrivial $\mathfrak{g}$-module such that $M|_{\mathfrak{p}} \cong k_{ev} \oplus P$. Then

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M|_{\mathfrak{p}} \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(k_{ev} \oplus P) \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} k_{ev} \oplus \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} P \cong k_{ev} \oplus \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} P.$$

However, since M is already a $\mathfrak{g}$-module, by the tensor identity given in [19, Lemma 1],

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M|_{\mathfrak{p}} \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M|_{\mathfrak{p}} \otimes k_{ev}) \cong M \otimes \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} k_{ev} \cong M \otimes k_{ev} \cong M$$

and so we have that, as $\mathfrak{g}$-modules, $M \cong k_{ev} \oplus \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} P$. Since M is indecomposable, $M \cong k_{ev}$ as $\mathfrak{g}$-modules and thus the map $\mathrm{res}_{T(\mathfrak{p})}^{T(\mathfrak{g})}$ is injective. $\square$

With this relationship established, the remaining goal is to classify $T(\mathfrak{p})$.

The author is interested in generalizing these results to other Type I Lie superalgebras, particularly $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ as similar techniques may be employed.

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