The Reduction Map for the Moduli Spaces of Weighted Stable Hyperplane Arrangements

by

Jae Ho Shin

(Under the direction of Valery Alexeev)

Abstract

Abstract. Alexeev constructed moduli spaces of weighted stable hyperplane arrangements generalizing the Hasset's moduli space of curves of genus 0 with weighted n points. For curves, the reduction map $\overline{M}_{\beta}(2,n) \to \overline{M}_{\beta'}(2,n)$ is surjective for any weights $\beta \geq \beta'$. We study first a combinatorial statement about tilings which is related to the surjectivity of the reduction map for the Alexeev's space when $n = 5, 6, 7, 8, 9$. We will show there is a counterexample to the combinatorial statement when $n = 10$, which works as a counterexample to the surjectivity of the reduction map for the Alexeev's space when $n = 10$.

Index words: Hyperplane Arrangements, Weighted Stable Hyperplane Arrangements, Moduli Spaces, Reduction Map, Surjectivity, Matroids, Base Polytopes, Puzzle-pieces, Flakes, Puzzles, Quilts, Regular Quilts, β-puzzles, Extension of Regular Quilts.

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Introduction

Deligne and Mumford introduced the moduli space $\overline{M}_{g,n}$ of stable n-pointed curves of genus g, which proved to be very useful in many fields of mathematics. The genus 0 case is already rich and interesting, and Hassett gave a generalization to the moduli space $\overline{M}_{0,\beta}$ by assigning weights to n points where $\beta = (b_1, ..., b_n)$ is a weight vector with $0 < b_i \leq 1$ for each $1 \leq i \leq n$ [Has03]. Hacking, Keel and Tevelev gave another generalization $\overline{M}(k, n)$, the moduli of stable hyperplane arrangements, by considering its higher dimensional case [HKT06]. Then, Alexeev introduced the moduli of weighted stable hyperplane arrangements $\overline{M}_\beta(k, n)$ which can be thought of as a generalization of both moduli spaces. $\overline{M}_\beta(k, n)$ is shown to be a fine moduli space [Ale08]. We can regard Hassett's space as a special case of Alexeev's space when $k = 2$ and write $\overline{M}_\beta(2, n)$ instead of $\overline{M}_{0,\beta}$.

Definition 0.1. For a connected equidimensional projective variety X and n Weil divisors B_i, a pair $(X, B = \sum_{i=1}^n b_i B_i)$ is called a stable pair if

1. X is reduced, and the pair is semi log canonial, and

2. $K_X + B$ is ample.

Weighted stable hyperplane arrangements are a particular case of this definition. Fix $n \geq 4$. Define the *weight domain*

$$\mathcal{D}(k, n) = \left\{ (b_i) \in \mathbb{Q}^n \mid 0 < b_i \leq 1, \sum b_i > k \right\}$$

There is a partial order on $\mathcal{D}(k,n)$: $\beta > \beta'$ if for all $1 \leq i \leq n$, one has $b_i \geq b_i'$ with at least one strict inequality. For any weights $\beta > \beta'$, there is a natural *reduction morphism* $\rho_{\beta,\beta'} : \overline{M}_{\beta}(k,n) \to \overline{M}_{\beta'}(k,n)$. In the curve case $(k=2)$, $\overline{M}_{0,\beta}$ and $\overline{M}_{0,\beta'}$ are smooth irreducible projective varieties of dimension $n-3$, and $\rho_{\beta,\beta'}$ is birational. Hence, $\rho_{\beta,\beta'}$ is surjective. However, for the higher dimensional case $(k \geq 3)$, the surjectivity of the morphism is a much harder problem. This is because $\overline{M}_{\beta}(r,n)$ has a matroid structure and when $r \geq 3$ the matroid geometry may be arbitrarily complicated, which is predicted by Mnev's universality theorem, c.f. [Laf03].

A hyperplane arrangement has a *loopless matroid structure* with a ground set $S = \{1, 2, ..., n\}$ and a rank function $r : 2^S \to \mathbb{Z}_{\geq 0}$ such that $r(I) = \operatorname{codim} \cap_{i \in I} B_i$ for $I \subset S$ [GGMS87]. A *matroid* is a generalization of a spanning set of a vector space, and a matroid is *loopless* if any singleton set has rank 1. There are several ways to define a matroid, which will be briefly introduced in Chapter 1. Any matroid gives a polytope which is called a *base polytope*, and this correspondence is one-to-one. The base polytope for a loopless matroid (S, r) is given as $\{(x_i) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\, x(S) = k,\, x(I) \leq r(I)\}$, where $x(I)$ denotes the sum $\sum_{i \in I} x_i$. Hypersimplex $\Delta(k,n)$ is defined to be the base polytope that corresponds to the *uniform matroid* U_n^k, that is, its bases are all subsets $I \subset S$ with $|I| = k$. Explicitly $\Delta(k,n)$ is given as $\Delta(k,n) = \{(x_i) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\, x(S) = k\}$.

Over a complex field $\mathbb{C}$, a hyperplane arrangement $(\mathbb{P}^{k-1}, (B_1, ..., B_n))$ can be identified with its embeded image $\mathbb{P}V$ into $\mathbb{P}^{n-1}$ in which B_i appear as intersections of $\mathbb{P}V$ and the coordinate hyperplanes of $\mathbb{P}^{n-1}$, where V is a k-dimensional vector space. An algebraic torus $T = (\mathbb{C}^*)^n / \operatorname{diag}\mathbb{C}^*$ acts on the Grassmannian $\mathrm{G}(r,n)$, and let Y be the closure of the orbit of $[\mathbb{P}V] \in \mathrm{G}(k,n)$, then Y is a toric variety. In addition, it is known that the strata of Y induce the strata of the subdivision of $\Delta(k,n)$ into base polytopes [HKT06], which we call a *tiling* or a *complete cover* of $\Delta(k,n)$. Similarly, given a weight $\beta = (b_1, ..., b_n)$, define the weighted hypersimplex to be $\Delta_\beta(k,n) = \{(x_i) \in \mathbb{R}^n \mid 0 \le x_i \le b_i,\ x(S) = k\}$. Consider a subdivision of $\Delta_\beta = \Delta_\beta(k,n)$ into the polytopes $\cup_j (\Delta_\beta \cap Q_j)$ where $\Delta_\beta^\circ \cap Q_j \neq \emptyset$ and Q_j are loopless representable matroid polytopes in $\Delta(k,n)$ forming a face-fitting tiling. $\cup Q_j$ is called a *partial tiling* or a *partial cover* of Δ. By [Ale08], the following combinatorial statement is closely related to the surjectivity of the reduction map.

Question 0.2. *Can every partial tiling be extended to a complete tiling? (Question 2.1 in [Ale08])*

$k = 3$ is the first case we are interested in. $\Delta(3,4)$ has no non-trivial tiling, and we assume $n \ge 5$. Until Chapter 6, we will develop combinatorial arguments to solve this question. Recall that any matroid gives a base polytope. For a face of the base polytope that is not contained in $\cup_{i=1}^n \{x_i = 0\}$, one can construct a matroid from the given loopless matroid. A *puzzle-piece* for a loopless matroid M is defined to be the collection of M and the matroids that correspond to the faces of the base polytope of M that are not contained in $\cup_{i=1}^n \{x_i = 0\}$. Then, gluing of base polytopes is translated into the gluing of puzzle-pieces, which is a backbone idea of this paper. The dimension that one has to work with for base polytopes remarkably drops down to $3-1=2$ for puzzle-pieces. Moreover, because it is 2-dimensional, we can use visualization, which cuts down much computation and also helps our understanding of the gluing. We define a *puzzle* to be the collection of puzzle-pieces that correspond to face-fitting base polytopes. A *quilt* is a weaker notion than a puzzle so that a flake is a puzzle and a puzzle is a quilt. We will see that a quilt that has *regular shape*, which we call a *regular quilt* for $n \le 7$ is a puzzle; we will define the regular shape and a regular quilt in Chapter 5. The author conjectures that every regular quilt for $n \le 9$ is a puzzle. Since a puzzle is a quilt, a *regular puzzle* is defined to be a puzzle that is a regular quilt at the same time. A *complete puzzle* is a puzzle that corresponds to a complete cover of $\Delta(k,n)$. A *β-puzzle* is a puzzle that comes from a partial cover of Δ_β for some weight β. Every β-puzzle is a sub-quilt of a regular quilt. Then, Question 0.2 is translated into the following question.

Question 0.3. *Can every regular puzzle be extended to a complete puzzle?*

Consider again $Y = \overline{T.[\mathbb{P}V]} \subset \mathrm{G}(k,n)$, and let U be the universal family over $\mathrm{G}(r,n)$ whose fibers are isomorphic to $\mathbb{P}V \cong \mathbb{P}^{r-1}$. Consider the fiber product $U_Y := U \underset{G(r,n)}{\times} Y$ and the GIT quotient $U_Y//_{\mathbb{1}}T$, where $\mathbb{1} = (1,1,...,1)$.

Theorem 0.4 ([Ale08])**.** *$U_Y//_{\mathbb{1}}T$ is the log canonical model of the given pair $(\mathbb{P}^{k-1}, (B_1, ..., B_n))$.*

A matroid (S, r) is called *inseparable* or *connected* if there is no nonempty proper subset A of S such that $r(A) + r(A^c) = r(S)$. If a loopless inseparable matroid is *representable*, i.e., isomorphic to a matroid defined by the set of columns of a matrix, then its corresponding puzzle-piece can be geometrically realized as the log canonical model of the hyperplane arrangement. Then, there is a theorem due to Alexeev that the log canonical model of any hyperplane arrangement on $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_{\mathrm{pts}}\mathbb{P}^2$, [Ale13] Theorem 5.7.2.

In Chapter 6 and 7, we give a partial answer to Question 0.3.

Theorem 0.5. *Every regular quilt for $\Delta(3,n)$ with $4 \leq n \leq 7$ is a puzzle and can be extended to a complete puzzle.*

Conjecture 0.6. *Every regular quilt for $\Delta(3,n)$ with $n = 8, 9$ is a puzzle and can be extended to a complete puzzle.*

Theorem 0.7. *When $n = 10$, there exists a weight β such that the reduction map $\rho_{\mathbb{1},\beta} : \overline{M}_{\mathbb{1}}(3,10) \to \overline{M}_{\beta}(3,10)$ is not surjective.*

This paper is organized as follows. In Chapter 1, we give basic definitions and general facts about matroids and base polytopes. Chapter 2 is devoted to base polytopes and their gluing. We give an equivalent condition for when two base polytopes glue to another base polytope, which is an interesting combinatorial problem. This will tell us about the decomposition of a puzzle-piece. In addition, it will be studied when the base polytope comes from a hyperplane arrangement in Chapter 3, which says that its corresponding matroid is representable. In Chapter 3 and 4, we study hyperplane arrangements and puzzle-pieces as preparation for the remaining chapters. Chapter 5 and 6 are assigned for puzzles and β-puzzles, where we will see theorems and conjectures about the completing quilts and puzzles. In the last chapter, we construct a counter-example of the surjectivity of the reduction map for Alexeev's space when $n = 10$.

Chapter 1

Matroids and polytopes

1.1 Characterizing matroids

The notion of matroid can be defined by several axiom systems. The characterization of matroid by circuits plays an important role in graph theory, but we do not pay attention to that description in this paper. Instead, we list below characterizations of matroid in terms of independent sets, dependent sets, bases, rank function, span function, and flats. Unless separately mentioned, S denotes $\{1, 2, ..., n\}$ for some natural number n.

Independent sets, dependent sets and bases

A pair $M = (S, \mathcal{I})$ is called a *matroid* if S is a finite set and $\mathcal{I}$ is a collection of subsets of S satisfying:

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,

(I3) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I \cup \{z\} \in \mathcal{I}$ for some $z \in J \backslash I$.(*exchange property*)

S is called the *ground set* of M. A subset $I \subseteq S$ is called *independent* if $I \in \mathcal{I}$, and *dependent* otherwise. For $U \subseteq S$, a subset B of U is called a *base* or *basis* of U if B is an inclusionwise maximal independent subset of U. Under condition (I2), condition (I3) is equivalent to:

for any subset U of S, any two bases of U have the same size.

Rank function

The common size of the bases of a subset U of S is called the *rank* of U, denoted by $r_M(U)$. We define a *rank function* r_M mapping 2^S into the set of non-negative integers $\mathbb{Z}_{\geq 0}$ by assigning $r_M(U)$ to $U \in 2^S$. We write $r(U)$ for $r_M(U)$ if the matroid is clear from the context. In addition, we usually write $r(M)$ for $r(S)$. Then r has the following properties:

(R1) if $U \subseteq S$, then $0 \leq r(U) \leq |S|$,

(R2) if $U \subseteq T \subseteq S$, then $r(U) \leq r(T)$,

(R3) if $U, T \subseteq S$, $r(U) + r(T) \geq r(U \cup T) + r(U \cap T)$. (*sub-modularity*)

Conversely, if f is a function mapping 2^S into $\mathbb{Z}_{\geq 0}$ that satisfies (R1)-(R3), then f is the rank function of a matroid.

Independent sets and bases are characterized in terms of the rank function:

(R4) U is independent if and only if $|U| = r(U)$,

(R5) U is a basis of M if and only if $|U| = r(U) = r(M)$,

Span function

The *span function* $\mathrm{span}_M : 2^S \to 2^S$ is defined as follows:

$$\mathrm{span}_M(T) := \{s \in S \mid r_M(T \cup \{s\}) = r_M(T)\}$$

for $T \subset S$. $\text{span}_M(T)$ is called the *span* of T or the *closure* of T, and we say that T spans $\text{span}_M(T)$. The span function is also called the *closure operator* of a matroid M and denoted by cl_M. If the matroid is clear from the context, we drop M from $\text{span}_M(T)$ or $\text{cl}_M(T)$. $\overline{T}$ also denotes the closure of T. The span function has the following properties:

(C1) if $T \subseteq S$, then $T \subseteq \overline{T}$,

(C2) if $T, U \subseteq S$ and $U \subseteq \overline{T}$, then $\overline{U} \subseteq \overline{T}$,

(C3) if $T \subseteq S$, $t \in S \backslash T$, and $s \in \overline{T \cup \{t\}} \backslash \overline{T}$, then $t \in \overline{T \cup \{s\}}$.
(*Mac Lane-Steinitz exchange property*)

Conversely, if a function $f : 2^S \to 2^S$ satisfies (C1)-(C3), then f is the span function of M. Note the following properties:

(C4) $r(T) = r\left(\overline{T}\right)$,

(C5) T is a spanning set of S, i.e. $\overline{T} = S$ if and only if $r(T) = r(S)$.

(C6) T is a basis if and only if it is a minimal spanning set.

Flats

A *flat* is a subset F of S with $\overline{F} = F$. Note that F is a flat if and only if $r(F \cup \{a\}) > r(F)$ for all $a \in F^c$.

$\mathcal{F} \subset 2^S$ is the collection of flats of a matroid M if and only if:

(F1) $S \in \mathcal{F}$,

(F2) if $T, U \in \mathcal{F}$, then $T \cap U \in \mathcal{F}$,

(F3) if $F \in \mathcal{F}$ and $t \in S \backslash F$, and T is the smallest flat containing $F \cup \{t\}$, then there is no flat U with $F \subsetneq U \subsetneq T$.

$\mathcal{F}$ is also called a *geometric lattice*. Note that since every independent subset is contained in a flat, it suffices to list all dependent flats for describing a matroid.

Remark. It is known that conditions (I1)-(I3), (R1)-(R3), (C1)-(C3) and (F1)-(F3) are all equivalent. We may use different descriptions of a matroid: $(S, \mathcal{I})$, (S, r), (S, span) and $(S, \mathcal{F})$; when needed, we list more information like $(S, r, \mathcal{I})$ and $(S, r, \mathcal{F})$. Since the ground set S is finite, we may assume that $S := \{1, ..., n\}$ without loss of generality from now on.

1.2 More about matroids

Dual matroid M^* of a matroid M

For a matroid $M = (S, \mathcal{I}, r)$, its dual matroid $M^* = (S, \mathcal{I}^*, r^*)$ is defined as follows.

$$\mathcal{I}^* = \{I \subset S \,|\, S \backslash I \text{ is a spanning set of } M\}$$

Its rank function $r^* = r_{M^*}$ is given as follows: for $U \subset S$,

$$r^*(U) = |U| + r(S \backslash U) - r(S)$$

Restriction

The *restriction* $M|_T$ of M to $T \subset S$ is a matroid defined on T by the rank function $r_{M|_T} : 2^T \to \mathbb{Z}_{\geq 0}$ given by: for $U \subset T$,

$$r_{M|_T}(U) = r_M(U)$$

Deletion

The *deletion* $M \backslash Z$ of $Z \subset S$ from M is defined to be $M|_{S \backslash Z}$.

Contraction

The *contraction* M/T of M over $T \subset S$ is a matroid defined on T^c by the rank function $r_{M/T} : 2^{T^c} \to \mathbb{Z}_{\geq 0}$ given by: for $U \subset T^c$,

$$r_{M/T}(U) = r_M(U \cup T) - r_M(T)$$

Note that deletion and contraction commute! One can check the following properties.

(RC1) $[M|_A]|_B = M|_B$ for $B \subset A \subset S$.

(RC2) $[M \backslash A] \backslash B = M \backslash (A \cup B)$ for $A, B \subset S$ with $A \cap B = \emptyset$.

(RC3) $[M/A]/B = M/(A \cup B)$ for $A, B \subset S$ with $A \cap B = \emptyset$.

(RC4) $[M|_J]/F = [M/F]|_{J \backslash F}$ for $F \subset J \subset S$.

(RC5) $[M/J]|_F = [M|_{J \cup F}]/J$ for $F, J \subset S$ with $F \cap J = \emptyset$.

Remark. Contraction is the operation dual to deletion: contracting T means replacing M by $(M^* \backslash T)^*$; see [Sch03] Chapter 39, for more information.

Loops

An element $s \in S$ is called a *loop* if $\{s\}$ is dependent, equivalently if $r(\{s\}) = 0$. Note that:

(M1) If T consists of loops, $r(T) = 0$. Hence, the set of loops is denoted by $\overline{\emptyset}$.

(M2) $r\left(T \cup \overline{\emptyset}\right) = r(T)$ and $r\left(T \backslash \overline{\emptyset}\right) = r(T)$.

(M3) $T \subset S$ is a flat if and only if M/T is loopless.

Separators

A subset $T \subset S$ is called a *separator* of M if $r(T) + r(T^c) = r(M)$. $\emptyset$ and S are always separators. The followings are equivalent:

(S1) T is a separator.

(S2) $M|_{T^c} = M/T$.

(S3) $M = M|_T \oplus M/T$.

(S4) $T \cup \overline{\emptyset} = \overline{T}$ and $\overline{T}$ is a separator.

Note the following properties:

(S5) For a loopless matroid, every separator is a flat.

(S6) The family of separators is closed under the complement, union and intersection.

Inseparable matroids and inseparable subsets

M is called *inseparable* or *connected* or *non-separable* if M has no separators other than $\emptyset$ and S. A subset $T \subset S$ is called *inseparable* if $M|_T$ is inseparable. Then,

(S7) A matroid M has the unique decomposition into inseparable nonempty submatroids $M|_{T_i}$ where T_i, $i \in \Lambda$, are minimal nonempty separators of M:

$$M \cong \oplus_{i\in\Lambda} M|_{T_i}$$

Each $M|_{T_i}$ is called a connected component of M. Let $\kappa(M)$ denote the number of the connected components of M. If M is a loopless separable matroid, the number of the minimal nonempty separators is $\kappa(M)$, which is not true if M is inseparable.

Note the following properties:

(M4) If M is inseparable, it is loopless.

(M5) If M is loopless and $r(M) = 1$, M is inseparable. Such matroid is isomorphic to the uniform matroid U_n^1, which is defined in the next page.

(M6) Let M be a loopless matroid of rank 2. The ground set S is a disjoint union of rank 1 flats. Moreover, M is separable if and only if $M \cong M|_T \oplus M|_{T^c}$, where T and T^c are only two rank 1 flats.

(M7) Let M be a loopless matroid of rank 3. M is separable if and only if

 (a) $M \cong M|_T \oplus M|_{T^c}$, where T is only one inseparable flat of rank 2 and T^c is a flat of rank 1, or

 (b) $M \cong M|_{T_1} \oplus M|_{T_2} \oplus M|_{T_3}$, where T_1, T_2, T_3 are only three flats of rank 1 and their union is a partition of S.

(M8) For a subset F of T^c, F is a flat of M/T if and only if $F \cup T$ is a flat of M.

(M9) Let F be a flat, $T \subset S$ subset of rank 1. Then $r(F \cup T) = r(F) + 1$.

Also note that for a loopless matroid, every flat is a direct sum of inseparable flats. Hence, if two loopless matroids have the same family of inseparable flats, they are identically equal.

Non-degenerate subsets

Let M be an inseparable matroid. A non-empty proper subset $J \subset S$ is called a *non-degenerate* subset of S if M/J and $M|_J$ are inseparable.

Lemma 1.1. *Let $M = (S, r)$ be a loopless matroid. Then non-degenerate subsets of S are flats. Hence, $\emptyset \neq J \subsetneq S$ is a non-degenerate subset if and only if J is an inseparable flat such that M/J is inseparable as well.*

Proof. Let $\emptyset \neq J \subsetneq S$ be a non-degenerate subset. By definition, M/J is inseparable, so loopless by (M4). Hence J is a flat by (M3). $\square$

By Lemma 1.1, we say *non-degenerate flats* for non-degenerate subsets from now on.

Lemma 1.2. *Let $M = (S, r)$ be a loopless matroid with $r(M) = 1$. Then there are no non-degenerate flats of S.*

Proof. Let $\emptyset \neq J \subsetneq S$ be a flat. Since M is loopless, $\emptyset$ is only one flat with rank 0. In addition, S is only one flat with rank $r(S)$. So, one has $0 < r(F) < r(S)$. But, $r(S) = 1$ implies that $\emptyset$ and S are only two flats. By Lemma 1.1, there are no non-degenerate flats. $\square$

Lemma 1.3. *Let $M = (S, r)$ be a loopless matroid with $r(M) = 2$. Then non-degenerate flats of S are exactly nontrivial flats, and they have rank 1.*

Proof. Let $\emptyset \neq J \subsetneq S$ be a flat. One has $0 < r(J) < r(S) = 2$, which implies $r(J) = 1$. By (M5), J is inseparable. In addition, M/J is loopless by (M3). So, M/J is inseparable by (M5) since it has rank 1: $r_{M/J}(J^c) = r(J^c \sqcup J) - r(J) = 2 - 1 = 1$. Hence, J is a non-degenerate flat of S. Lemma 1.1 completes the proof. $\square$

Lemma 1.4. *Let $M = (S, r)$ be a loopless matroid with $r(M) = 3$. Then non-degenerate flats of S are exactly those nontrivial flats such that:*

(a) $r(J) = 1$ *and* M/J *is inseparable, or*

(b) $r(J) = 2$ *and* J *is inseparable.*

Proof. Let $\emptyset \neq J \subsetneq S$ be a non-degenerate flat. $0 < r(J) < r(S) = 3$ implies that $r(J) = 1$ or 2. Lemma 1.1 shows that J satisfies (a) and (b).

Conversely, let $\emptyset \neq J \subsetneq S$ be a flat. (a) Suppose that $r(J) = 1$ and M/J is inseparable. Since J is inseparable by (M5), J is non-degenerate. (b) Suppose that J is inseparable with $r(J) = 2$. Then, M/J has rank 1: $r_{M/J}(J^c) = r(J^c \cup J) - r(J) = 3 - 2 = 1$. So, M/J is loopless by (M3), hence inseparable by (M5), which means M/J is a non-degenerate flat. $\square$

Uniform matroids

Let $\mathcal{I}$ be the collection of all those subsets I of S such that $|I| \leq k$ where k is a fixed natural number with $1 \leq k \leq n$. Then $(S, \mathcal{I})$ is a matroid which is called a *k-uniform matroid* and denoted by U_n^k.

Representable matroids

Let A be an $m \times n$ matrix over a field $\mathbb{F}$. For a subset I of S, denote by $A(I)$ the submatrix of A consisting of the columns with index in I. Let $r(I) := \operatorname{rank}(A(I))$, then (S, r) is a matroid. Note that $I \subset S$ is an independent set if and only if the columns of $A(I)$ are linearly independent. Any matroid obtained in this way, or isomorphic to such a matroid, is called a *representable matroid* or a *linear matroid* over the given field $\mathbb{F}$. Note that every matroid with $1 \leq |S| \leq 7$ is representable.

Graphic matroids

Let G be a graph with the set of edges $S = \{1, ..., n\}$. A subset $I \subset S$ is independent if I forms a *forest*, i.e., a maximal subset of edges that has no cycles. Then $(S, \mathcal{I})$ is a matroid, and we call it a *graphic matroid.* Note that a graphic matroid is *regular*, i.e., representable over any field.

1.3 Polytopes

The notations IP_M, SP_M and BP_M are due to Alexeev.

Compact convex polytopes

A *compact convex polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points, which is necessarily compact. Alternatively, it can be defined to be the intersection of a finite number of half-spaces that is compact at the same time. In this paper, we assume compactness of a convex polytope and say simply a *convex polytope* unless separately mentioned.

Polytopes

We define a *n-dimensional (compact) polytope* Q in $\mathbb{R}^m$ with $n \leq m$ to be the face-fitting union of a finite number of (compact) convex polytopes that is homeomorphic to a closed Euclidean n-ball. Note that any codimension 2 face P of a polytope Q is the intersection of exactly 2 facets Q_1 and Q_2 of Q.

Incidence vectors

For a subset $I \subseteq S$, the incidence vector x^I of I in $\mathbb{R}^n$ is defined by

$$x^I(i) := \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$$

Independent set polytopes

For a subset $I \subseteq S$ and a vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we use the shortcut $x(I) = \sum_{i \in I} x_i$. The *independent set polytope* IP_M of a matroid $M = (S, r)$ is the convex hull of the incidence vectors x^I of the independent sets I of M. IP_M is fully determined by the following linear inequalities; see [Sch03] Section 40.2:

$$\begin{aligned} x_s &\geq 0 && \text{for } s \in S, \\ x(U) &\leq r(U) && \text{for } U \subset S. \end{aligned}$$

Since $x(U) \leq r(U)$ is satisfied by $x\left(\overline{U}\right) \leq r\left(\overline{U}\right) = r(U)$, above describing inequalities of an independent set polytope can be replaced with a unique minimal collection of inequalities:

$$\begin{aligned} x_s &\geq 0 && \text{for } s \in S, \\ x(F) &\leq r(F) && \text{for nonempty inseparable flats } F. \end{aligned}$$

Spanning set polytopes

The spanning set polytope SP_M of a matroid $M = (S, r)$ is the convex hull of the incidence vectors of the spanning sets of M. Recall that by definition, a subset $U \subset S$ is a spanning set of M if and only if $S \backslash U$ is independent in M^*. So, $x \in \mathrm{SP}_M$ if and only if $\mathbf{1} - x \in \mathrm{IP}_{M^*}$, where $\mathbf{1} = (1, ..., 1)$. Hence, SP_M is fully determined by the following inequalities:

$$\begin{aligned} 0 \leq x_s \leq 1 \qquad & \text{for } s \in S, \\ x(U) \geq r(U) - r(S \backslash U) \qquad & \text{for } U \subset S. \end{aligned}$$

Base polytopes

The *base polytope* BP_M of a matroid $M = (S, r)$ is the convex hull of the incidence vectors x^B of bases B of M. BP_M is fully determined by the equality $x(S) = r(S)$ and the following linear inequalities:

$$\begin{aligned} x_s \geq 0 \qquad & \text{for } s \in S, \\ x(F) \leq r(F) \qquad & \text{for nonempty inseparable flats } F. \end{aligned}$$

Remark. In this paper, we pay attention only to base polytopes.

Hypersimplices

We define a partial order $\leq$ on $\mathbb{R}^n$ as follows: for two vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, ..., n$. For a hypercube $[0, 1]^n$, $\mathbf{0} = (0, ..., 0)$ is the smallest element and $\mathbf{1} = (1, ..., 1)$ is the largest element.

The base polytope BP_M of a uniform matroid $M = U_n^k$ is called the *hypersimplex* and denoted by $\Delta(k, n)$ or Δ_n^k:

$$\Delta(k, n) = \mathrm{Conv}\left(x^I \mid I \subset S,\ |I| = k\right) = \{x \in \mathbb{R}^n \mid \mathbf{0} \leq x \leq \mathbf{1},\ x(S) = k\}$$

A *weight* β is a vector $\beta = (b_1, ..., b_n) \in [0,1]^n \setminus \{\mathbf{0}\}$. A *weighted cut hypersimplex* $\Delta_\beta(k,n)$ or Δ_β^k is defined to be:

$$\Delta_\beta(k,n) = \{x \in \mathbb{R}^n \,|\, \mathbf{0} \leq x \leq \beta,\, x(S) = k\}$$

If n and k are clear from the context, we use simply Δ or Δ_β. For linear inequalities $f_i(x) \leq c_i$ and equalities $g_j(x) = d_j$, $\{f_i(x) \leq c_i, g_j(x) = d_j\}$ denotes the sub-polytope of Δ satisfying them.

U_n^k-polytope

An *edge* of a polytope Q is a (bounded) face of dimension 1. An edge necessarily connects two distince vertices of Q, and we say two vertices of Q are *adjacent* if they are connected by an edge of Q.

A polytope Q in $\mathbb{R}^n$ is called an U_n^k*-polytope* if all its edges and vertices are edges and vertices of Δ_n^k. Note that every edge of Δ_n^k is parallel to a vector $x^{\{i\}} - x^{\{j\}}$ for some $i, j \in S$.

Let $\mathcal{B}$ be a set of bases of U_n^k, and Q be a convex hull of the incidence vectors of $B \in \mathcal{B}$, i.e., the vertices of Q are vertices of Δ_n^k. Let $\mathcal{I} = \{I \subset S \,|\, I \subset B \text{ for some } B \in \mathcal{B}\}$, then $\mathcal{I}$ satisfies (I1) and (I2).

Theorem 1.5 ([GS87]). *The exchange property* (I3) *is equivalent to the condition that* BP_M *is a* U_n^k*-polytope.*

In other words, if Q is a U_n^k-polytope, then $(S, \mathcal{I}, \mathcal{B})$ is a matroid, and Q is its corresponding base polytope.

Corollary 1.6 ([GS87]). *A convex polytope* Q *in* $\mathbb{R}^n$ *is a base polytope if and only if* Q *is a* U_n^k*-polytope.*

The following theorem says that a base polytope in Δ_n^k with $2k < n$ is determined by its intersection with $\cup_{i=1}^n \{x_i = 0\}$.

Theorem 1.7. *Let $2k < n$ and $Q \subset \Delta_n^k$ be a convex hull of incidence vectors x^I where $I \subset S$ has cardinality k. Then, Q is a base polytope if and only if $Q \cap \{x_i = 0\}$ is a base polytope or an empty set for all $i = 1, ..., n$.*

Proof. ($\Leftarrow$) Since Q is a convex polytope, $Q \cap \{x_i = 0\}$ is a convex polytope. By Corollary 1.6, $Q \cap \{x_i = 0\} \subset \Delta_n^k \cap \{x_i = 0\} \cong \Delta_{n-1}^k$ is a U_{n-1}^k-polytope. Any incidence vector that gives a vertex of Q also gives a vertex of $Q \cap \{x_i = 0\}$ for some i since $k < n$. So, it is a vertex of $\Delta_n^k \cap \{x_i = 0\}$, hence a vertex of Δ_n^k. Take two distinct incidence vectors x^{I_1} and x^{I_2} that give vertices of Q. Since $2k < n$, x^{I_1} and x^{I_2} is contained in $Q \cap \{x_i = 0\}$ for some i, by pigeon hole principle. If $x^{I_1} - x^{I_2}$ is an edge of Q, it is also an edge in $Q \cap \{x_i = 0\}$, which is an edge of $\Delta_n^k \cap \{x_i = 0\}$, hence an edge of Δ_n^k. Therefore, all vertices and egdes of Q are vertices and edges of Δ_n^k, which means that Q is a U_n^k-polytope. Since Q is convex, Q is a base polytope by Corollary 1.6.

($\Rightarrow$) Since Q is a base polytope, it is a U_n^k-polytope by Corollary 1.6, i.e., all of its verices and edges are verices and edges of Δ_n^k. So, for any $i = 1, ..., n$, $Q \cap \{x_i = 0\}$ is either an empty set or a convex polytope such that all of its verices and edges are verices and edges of $\Delta_n^k \cap \{x_i = 0\} \cong \Delta_{n-1}^k$. By Corollary 1.6 again, $Q \cap \{x_i = 0\}$ is a base polytope. □

Theorem 1.5 is about the adjacency of vertices of a base polytope, which is generalized to Theorem 1.8 that is a generalized version for an independent set polytope; see [Sch03] Theorem 40.6.

Theorem 1.8. *Let $M = (S, r)$ be a loopless matroid and let I and J be distinct independent sets. Then x^I and x^J are adjacent vertices of IP_M if and only if $|I \triangle J| = 1$, or $|I \backslash J| = |J \backslash I| = 1$ and $r(I \cup J) = |I| = |J|$, where $I \triangle J$ denotes the symmetric difference of I and J, i.e., $I \triangle J = (I \backslash J) \cup (J \backslash I)$.*

Remark. If B and B' are two distinct bases of M, $|B \triangle B'| \neq 0, 1$ and one always has $r(B \cup B') = |B| = |B'|$. Hence, $x^B - x^{B'}$ is an edge of BP_M if and only if $|B \backslash B'| = |B' \backslash B| = 1$.

Facets of base polytopes

For a loopless matroid M, we have an important correspondence theorem as follows, which implies that nonempty inseparable flats in the describing inequalities of BP_M are actually non-degenerate flats.

Theorem 1.9 ([GS87]). *Let $M = (S, r)$ be an inseparable matroid. Non-degenerate flats of S are in 1-1 correspondence with the facets of BP_M that are not contained in $\cup_{j=1}^{n} \{x_j = 0\}$. To the non-degenerate flat J there corresponds the matroid $M|_J \oplus M/J$ and the facet $\mathrm{BP}_M(J) := \mathrm{BP}_{M|_J \oplus M/J} = \mathrm{BP}_{M|_J} \times \mathrm{BP}_{M/J}$.*

We denote by $S_M(\mathrm{BP}_M(J)) = S_{\mathrm{BP}_M}(\mathrm{BP}_M(J)) := J$ the non-degenerate flat corresponding to the facet $\mathrm{BP}_M(J)$ or the matroid $M|_J \oplus M/J$. If the matroid is clear from the context, we sometimes drop M or BP_M and write simply $S(\mathrm{BP}_M(J))$. For a face Q of BP_M, we denote by $M(Q)$ the matroid corresponding to Q. Remark that $M(Q)$ does not depend on which base polytope Q is a face of. Note the following properties:

(B1) If M is inseparable, there corresponds an inequality of its associated unique minimal collection of inequalites to a facet of BP_M.

(B2) The dimension of BP_M is $|S| - \kappa(M)$. By (S7), BP_M has dimension $n-1$ if and only if M is inseparable.

Intersection/union of two base polytopes/matroids

The intersection of two base polytopes BP_{M_1} and BP_{M_2} in Δ_n^k is not necessarily a base polytope. The following theorem describes $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ in terms of the common bases of M_1 and M_2, which can be found in [Sch03] Corollary 41.12d.

Theorem 1.10. *Let M_1 and M_2 be two matroids with the same ground set. The intersection of two base polytopes BP_{M_1} and BP_{M_2} is the convex hull of the incidence vectors of the common bases of M_1 and M_2.*

Corollary 1.11. *The edges of* $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ *are edges of* BP_{M_1} *or* BP_{M_2} *if and only if* $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ *is another base polytope.*

Proof. The vertices of $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ are the common vertices of BP_{M_1} and BP_{M_2} by Theorem 1.10. Suppose that the edges of $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ are edges of BP_{M_1} or BP_{M_2}. Then, since BP_{M_1} and BP_{M_2} are U_n^k-polytopes and $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ is a convex polytope, by Corollary 1.6, $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ is a U_n^k-polytope, hence a base polytope.

Conversely, suppose that $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ is a base polytope. Let $x^B - x^{B'}$ be any edge of $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$, where B and B' are two distinct common bases of both M_1 and M_2. Then, by Theorem 1.8, $|B \backslash B'| = |B' \backslash B| = 1$, which is true for both BP_{M_1} and BP_{M_2}. By Theorem 1.8 again, $x^B - x^{B'}$ is a common edge of both BP_{M_1} and BP_{M_2}. □

If $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ satisfies Corollary 1.11, it is a base polytope of some matroid, which we denote by $M_1 \wedge M_2$ and call it the *intersection of the matroids* M_1 and M_2. By Theorem 1.10, its bases is $\mathcal{B}_1 \cap \mathcal{B}_2$ where $M_1 = (S, \mathcal{B}_1)$ and $M_2 = (S, \mathcal{B}_2)$.

If $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$, the *union of the matroids* $M_1 \vee M_2$ is defined as follows: $M_1 \vee M_2 = (S_1 \cup S_2, \mathcal{I}_1 \vee \mathcal{I}_2)$ where $\mathcal{I}_1 \vee \mathcal{I}_2$ is defined to be $\{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. Note that for any two matroids, their union always exists. However, $M_1 \vee M_2$ is not what we want in this paper as the counterpart of the intersection of matroids, since it doesn't say much about the gluing of two base polytopes BP_{M_1} and BP_{M_2}. In the next chapter, we will see when a glued base polytopes becomes another base polytope, and define a *gluing of matroids* $M_1 \# M_2$ that will work as the counterpart of the intersection of matroids.

Chapter 2

Base polytopes and gluing theorems

2.1 The facets of base polytopes and non-degenerate flats

Lemma 2.1. *Let $J \subset S$ be a non-degenerate flat of an inseparable matroid M. By Theorem 1.9, there correspond a matroid $M|_J \oplus M/J$ and a facet $Q_J = \mathrm{BP}_{M|_J} \times \mathrm{BP}_{M/J}$ of BP_M to J. Suppose that $Q_1 \nsubseteq \cup_{j=1}^{n} \{x_j = 0\}$ is a codimension 2 face of BP_M that is a facet of Q_J at the same time. Then, Q_1 is contained in either $\mathrm{BP}_{M|_J} \times \left(\text{a facet of } \mathrm{BP}_{M/J}\right)$ or $\left(\text{a facet of } \mathrm{BP}_{M|_J}\right) \times \mathrm{BP}_{M/J}$. In other words, there corresponds a non-degenerate flat F of M/J or $M|_J$ to Q_1. Call $Q_1 = Q_F$. By the convexity of BP_M, Q_F is the intersection of exactly two facets of BP_M; write $Q_F = Q_J \cap Q_{J'}$ where J' is a non-degenerate flat of M.*

- *(a) In case that F is a non-degenerate flat of M/J, $J' = J \cup F$ if $J \cup F$ is inseparable, $J' = F$ otherwise.*
- *(b) In case that F is a non-degenerate flat of $M|_J$, $J' = F$ if M/F is inseparable, $J' = (J \backslash F)^c = J^c \cup F$ otherwise.*

I.e., there are 4 cases for J' as follows.

	F is a nondegenerate flat of	$M\|_{J\cup F}$	M/F	J'
(i)	M/J	inseparable		$J \cup F$
(ii)	M/J	separable		F
(iii)	$M\|_J$		inseparable	F
(iv)	$M\|_J$		separable	$J^c \cup F$

Table 2.1:

Proof. (a) Suppose that F is a non-degenerate flat of M/J. Then, its corresponding matroid $M(F)$ is:

$$M(F) = M|_J \oplus [M/J]|_F \oplus [M/J]/F.$$

Since $F \cap J = \emptyset$, the ground sets of $M|_J$, $[M/J]|_F$ and $[M/J]/F$ are J, F and $(J \cup F)^c$, respectively. Considering $M(F)$ in $M|_{J'} \oplus M/J'$, each of $M|_J$, $[M/J]|_F$ and $[M/J]/F$ can be identified with $M|_{J'}$ or M/J'. But, $M|_J \neq M/J'$ since otherwise $J' = J^c$ and $r_{M|_J}(J) = r_{M/J'}(J)$ implies that $r(J) = r(J \cup J') - r(J') = r(S) - r(J^c)$, i.e., $r(S) = r(J) + r(J^c)$, which is a contradiction since M is inseparable and J is a nonempty proper subset. Hence, only possibility for $M|_J$ is to be $M|_{J'}$, but this means that $J = J'$, which is a contradiction. Now, $[M/J]/F = M/(J \cup F)$ by (RC3), and $M/(J \cup F)$ cannot be $M|_{J'}$ by the same reason. So, the remaining possibility for $M/(J \cup F)$ is to be M/J', in which case $J' = J \cup F$.

(i) Suppose that $J \cup F$ is inseparable. Since $[M/J]/F = M/(J \cup F)$ is inseparable, $J \cup F$ is a non-degenerate flat of M. Moreover, $M|_J = [M|_{J\cup F}]|_J$ by (RC1) and $[M/J]|_F = [M|_{J\cup F}]/J$ by (RC5) imply that:

$$M(F) = [M|_{J\cup F}]|_J \oplus [M|_{J\cup F}]/J \oplus M/(J \cup F).$$

Hence, $J' = J \cup F$.

(ii) Suppose that $J \cup F$ is separable. Let A be a nontrivial separator of $M|_{J\cup F}$:

$$r_{M|_{J\cup F}}(J) = r_{M|_{J\cup F}}(J \cap A) + r_{M|_{J\cup F}}(J \backslash A).$$

But, J is inseparable, so one has either $J \cap A = \emptyset$ or $J\backslash A = \emptyset$. Since $B := (J \cup F) \backslash A$ is also a nontrivial separator of $M|_{J\cup F}$, without loss of generality, assume $J \subset A$ and $F \supset B$. Let $F_1 = F \cap A$ and $F_2 = F \cap B = B$. Note that:

$$r_{M/J}(F_2) = r(F_2 \cup J) - r(J) = [r(F_2) + r(J)] - r(J) = r(F_2)$$
$$r(F \cup J) = r(F_1 \cup F_2 \cup J) = r([F_1 \cup J] \cup F_2) = r(F_1 \cup J) + r(F_2)$$

Then,

$$\begin{aligned} r_{M/J}(F_1) + r_{M/J}(F_2) &= [r(F_1 \cup J) - r(J)] + r(F_2) \\ &= r(F_1 \cup J) + r(F_2) - r(J) \\ &= r(F \cup J) - r(J) \\ &= r_{M/J}(F) \end{aligned}$$

Since $F = F_1 \cup F_2$ is an inseparable flat of M/J, one has either $F_1 = \emptyset$ or $F_2 = \emptyset$. But, $F_2 = B$ is a nonempty separator, so one has $F_1 = F \cap A = \emptyset$. Hence, $F = F_2 = B$ and $J = A$. Moreover, J and F are only two nontrivial separators of $M|_{J\cup F}$. Now, by (S2), $[M/J]|_F = M|_F$ and $[M/F]|_J = M|_J$. Recall that as other matroids $M|_J$ and $[M/J]/F$, $[M/J]|_F$ has possibility to be identified with $M|_{J'}$ or M/J'. Now, $[M/J]|_F = M|_F$ cannot be identified with M/J' since otherwise the inseparablility of M would be violated. Hence, the remaining possibility for $[M/J]|_F = M|_F$ is $[M/J]|_F = M|_{J'}$, i.e., $J' = F$. Furthermore, Theorem 1.9 forces $J' = F$. The corresponding matroid $M(F)$ of F is written as follows:

$$M(F) = M|_F \oplus [M/F]|_J \oplus [M/F]/J$$

(b) Suppose that F is a non-degenerate flat of $M|_J$. Then, its corresponding matroid $M(F)$ is:

$$M(F) = [M|_J]|_F \oplus [M|_J]/F \oplus M/J$$

Since $F \subset J$, the ground sets of $[M|_J]|_F$, $[M|_J]/F$ and M/J are F, $J \backslash F$ and J^c, respectively. Observe that each of $[M|_J]|_F$, $[M|_J]/F$ and M/J can be identified with $M|_{J'}$ or M/J'. Similarly as in (a), M/J has no chance to be $M|_{J'}$ or M/J' by inseparability of M. Also, for $[M_J]|_F = M|_F$, only possibility is that $J' = F$.

(iii) Suppose that M/F is inseparable. Since $M|_F = [M_J]|_F$ is inseparable, F is a non-degenerate flat of M. By (M8), $J \backslash F$ is a flat of M/F since $J = (J \backslash F) \cup F$ is a flat of M. Moreover, $[M|_J]/F = [M/F]|_{J \backslash F}$ by (RC4) and $M/J = [M/F]/(J \backslash F)$ by (RC3) are inseparable, which means that $J \backslash F$ is a non-degenerate flat of M/F. Hence, $J' = F$ and one has:

$$M(F) = M|_F \oplus [M/F]|_{J \backslash F} \oplus [M/F]/(J \backslash F)$$

(iv) Suppose that M/F is separable. Let A be a nontrivial separator of M/F such that $A \cap (J \backslash F) \neq \emptyset$. Since $[M|_J]/F = [M/F]|_{J \backslash F}$ is inseparable, $A \supset J \backslash F$. Let $T := J \backslash F$, then $J = T \cup F$ and $A \supset T$. Let $B := F^c \backslash A$, then B is a separator of M/F:

$$r_{M/F}(A) + r_{M/F}(B) = r_{M/F}(A \cup B) = r_{M/F}(F^c) = r(S) - r(F).$$

Note that $B \cap J = \emptyset$, $B \cup (A \backslash J) = J^c$ and $A \backslash J = A \backslash T$. Then,

$$r_{M/J}(B) = r_{[M/F]/T}(B) = r_{M/F}(B \cup T) - r_{M/F}(T) = r_{M/F}(B)$$
$$r_{M/J}(A \backslash J) = r_{[M/F]/T}(A \backslash T) = r_{M/F}(A) - r_{M/F}(T)$$

The sum of the right hand side formulas becomes:

$$\begin{aligned} r_{M/F}(B) + r_{M/F}(A) - r_{M/F}(T) &= r_{M/F}(A \cup B) - r_{M/F}(T) \\ &= [r(S) - r(F)] - [r(J) - r(F)] \\ &= r(S) - r(J) \\ &= r_{M/J}(J^c) \end{aligned}$$

By equating this with the sum of the left hand side formulas, one has:

$$r_{M/J}(B) + r_{M/J}(A/J) = r_{M/J}(J^c)$$

which means that $A \backslash J = \emptyset$ since M/J is inseparable and B is nonempty. Since $A \supset J \backslash F$, one has $J = A \cup F$ and $A = J \backslash F$. So, $J \backslash F = A$ and $J^c = B$ are separators of M/F. Note that by (M3) M/F is loopless since F is a flat. Then, by (S5) two separators J/F and J^c are flats of M/F. Now, $[M/F]/J^c = [M/F]|_{J \backslash F}$ by (S2) and $M/(J \backslash F)^c = [M/F]/J^c$ by (RC3). Hence $M/(J \backslash F)^c = [M/F]|_{J \backslash F}$. Moreover, $\left[M|_{(J \backslash F)^c}\right]/F = M/J$ because $\left[M|_{(J \backslash F)^c}\right]/F = [M/F]|_{J^c}$ by (RC4), $M/J = [M/F]/(J \backslash F)$ and $[M/F]|_{J^c} = [M/F]/(J \backslash F)$ by (S2). Since $\left[M|_{(J \backslash F)^c}\right]|_F = M|_F$ by (RC1), one has:

$$M(F) = \left[M|_{(J \backslash F)^c}\right]|_F \oplus \left[M|_{(J \backslash F)^c}\right]/F \oplus M/(J \backslash F)^c$$

Now, by (M8) $(J \backslash F)^c = J^c \cup F$ is a flat of M since J^c is a flat of M/F. By Theorem 1.9, $J' = (J/F)^c$ is one and only one choice for J'.

Thus, the lemma is proved. □

Corollary 2.2. *Let M be an inseparable matroid, J a non-degenerate flat of M.*

(a) Let F be a non-degenerate flat of M/J. Then, $M|_{J \cup F}$ is separable if and only if F is a non-degenerate flat of M if and only if M/F is

inseparable.

(b) *Let F be a non-degenerate flat of $M|_J$. Then, M/F is separable if and only if $(J\backslash F)^c$ is a non-degenerate flat of M if and only if $M|_{(J\backslash F)^c}$ is inseparable.*

Proof. (a) Suppose that $M|_{J\cup F}$ is separable. By Lemma 2.1(a), F is non-degenerate. Suppose that $M|_{J\cup F}$ is inseparable. By Lemma 2.1(a), $J \cup F$ is a non-degenerate flat of M. Then J and $J \cup F$ are only two non-degenerate flats whose corresponding facets contain Q_F. By Theorem 1.9, F must be degenerate. So, we proved that $M|_{J\cup F}$ is separable if and only if F is a non-degenerate flat. Moreover, in the proof of Lemma 2.1(a), we see that $M|_F = [M/J]\,|_F$ is inseparable. Hence, F is a non-degenerate flat of M if and only if M/F is inseparable.

(b) The proof for the first part is similar. Note that in the proof of Lemma 2.1(b), $M/\left(J\backslash F\right)^c = [M/F]\,|_J$ is inseparable. So, $(J\backslash F)^c$ is a non-degenerate flat of M if and only if $M|_{(J\backslash F)^c}$ is inseparable. □

Corollary 2.3. *Let M be an inseparable matroid. Let P be any codimension 2 face of BP_M that is not contained in $\cup_{j=1}^{n}\{x_j = 0\}$. By Theorem 1.9, P is the intersection of two facets Q_1 and Q_2 of BP_M with non-degenerate flats J_1 and J_2, respectively. Let $M_A \oplus M_B \oplus M_C$ be the corresponding matroid of P. Then, there are 3 cases for J_1 and J_2, up to symmetry, as follows.*

1. $J_1 = A$, $J_2 = A \cup C$
2. $J_1 = A$, $J_2 = C$
3. $J_1 = A \cup B$, $J_2 = B \cup C$, *where M/B is separable.*

Proof. Lemma 2.1 (i) and (iii) give the same case (1). Lemma 2.1 (ii) and (iv) give the case (2) and (3), respectively. □

2.2 Gluing base polytopes

We say two base polytopes BP_{M_1} and BP_{M_2} *face-fit* or simply *fit* if $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ is empty or a common face of both polytopes. We say BP_{M_1} and BP_{M_2} *meet nicely* if they fit in $\Delta_n^k \backslash \cup_{i=1}^n \{x_i = 0\}$ or $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \subset \cup_{i=1}^n \{x_i = 0\}$. Denote $\Delta_+ := \Delta_n^k \backslash \cup_{i=1}^n \{x_i = 0\}$ when k, n are clear from the context. If BP_{M_1} and BP_{M_2} fit in Δ_+ with the common facet $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$, we can glue them through the common facet. The glued one $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a polytope, but may not be another base polytope. If $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope, there corresponds a loopless matroid which we denote by $M_1 \# M_2$ such that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2} = \mathrm{BP}_{M_1 \# M_2}$. This matroid $M_1 \# M_2$ is different from the union of matroids $M_1 \vee M_2$.

It is an interesting question when the gluing of base polytopes gives another base polytope, which we will see an equivalent condition in terms of matroids in Theorem 2.6. For its proof, we need Lemma 2.4. Recall that if $Q \subset \mathbb{R}^n$ is a full dimensional polytope, each codimension 2 face P is the intersection of exactly 2 facets Q_1 and Q_2. We say Q is *convex at a codimension 2 face* P if near the interior of P, Q is the intersection of two half-spaces that are determined by Q_1 and Q_2.

Lemma 2.4. *Let $Q \subset \mathbb{R}^n$ be a full dimensional (compact) polytope. Then, Q is a convex polytope if and only if Q is convex at every codimension 2 face.*

Proof. If Q is a convex polytope, near the interior of any codimension 2 face, Q appears as the intersection of two half-spaces.

Suppose that the converse statement is not true. Then there is a hyperplane $L_0 \subset \mathbb{R}^n$ determined by a facet of Q such that $L_0 \cap Q$ is disconnected. Indeed, let L_0 be a hyperplane in $\mathbb{R}^n$ determined by any facet Q_0 of Q. Let Q_1 be one of its neighboring facet, i.e., $Q_0 \cap Q_1$ is a codimension 2 face of Q. Let L_1 be the hyperplane determined by Q_1. Then, either $L_0 = L_1$ or $L_0 \neq L_1$. Collect the facets $Q_1, ..., Q_m$ such that their corresponding hyperplanes are L_0 and $Q_0 \cup Q_1 \cup \cdots \cup Q_m$ is connected, which is connected in codimension 2. Let $\{Q_0, Q_1, ..., Q_m\}$ be the maximal family of those facets. Then, since Q is convex at every codimension 2 face, $Q_0 \cup Q_1 \cup \cdots \cup Q_m$ is a nonempty connected component of $Q \cap L_0$. If $Q_0 \cup Q_1 \cup \cdots \cup Q_m = L_0 \cap Q$ for all Q_0, then Q is an intersection of half-spaces determined by Q_0. So, we may assume that there is a pair of a facet Q_0 and its associated hyperplane $L_0 \subset \mathbb{R}^n$ such that $L_0 \cap Q$ is disconnected.

Fix a normal vector $\vec{n}$ of L_0 and consider the translation of L_0 by $\epsilon\vec{n}$ for $\epsilon \in \mathbb{R}$ which we denote by $t_\epsilon(L_0)$. Since Q is connected and convex at any codimension 2 face, by translating L_0, one can find two distinct facets of Q whose intersection is a codimension 2 face of Q such that the intersection of $t_\epsilon(L_0)$ for some ϵ and the two corresponding half-spaces, say $t_\epsilon(L_0) \cap (H_1 \cap H_2)$ is disconnected. But, this is impossible since $H_1 \cap H_2$ is convex. □

Glued base polytopes at a facet

Theorem 2.5. *Fix $k = 2$. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two inseparable matroids of rank 2 such that BP_{M_1} and BP_{M_2} glue through a common facet. Then, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope.*

Proof. Since both BP_{M_1} and BP_{M_2} are base polytopes, they are U_n^k-polytopes by Corollary 1.6. For any codimension 2 face P of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$, P is contained in $\cup_{i=1}^n \{x_i = 0\}$. Since P is contained in the boundary of Δ_n^k which is a convex polytope, Δ_n^k is convex at P. So, the union of two convex polytopes $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2} \subset \Delta_n^k$ is convex at P. By Lemma 2.4, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a convex polytope. By Corollary 1.6, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope. □

Theorem 2.6. *Fix $k \geq 3$. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two inseparable matroids of rank k such that BP_{M_1} and BP_{M_2} glue through a common facet. Let J_1 and J_2 be the corresponding non-degenerate flats of the common facet in M_1 and M_2, respectively, with a partition of $S = J_1 \cup J_2$. Take any $i_1 \neq i_2$ from $\{1, 2\}$. Let F be an arbitrary non-degenerate flat of $M_{i_1}/J_{i_1} = M_{i_2}|_{J_{i_2}}$; see Theorem 1.9. Then, for any pair (i_1, i_2) not* *both $M_{i_1}|_{J_{i_1}\cup F}$ and M_{i_2}/F are inseparable for every such F if and only if $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope.*

Proof. ($\Rightarrow$) Since both BP_{M_1} and BP_{M_2} are base polytopes, they are U_n^k-polytopes by Corollary 1.6. So, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a U_n^k-polytope. If we prove that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex, it is a base polytope by Corollary 1.6 again. By Lemma 2.4, we will prove that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex at every codimension 2 face P. Suppose that P is not contained in the common facet $Q_0 := \mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$. Then, P is contained solely in BP_{M_1} or BP_{M_2}, not in both. Since both polytopes are convex, by Lemma 2.4, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex at P.

Now, suppose that P is contained in Q_0. Since $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is homeomorphic to a full-dimensional ball, P is the intersection of exactly two facets $Q_1, Q_2 \neq Q_0$ of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$. We can say without loss of generality that Q_i is contained in BP_{M_i} for $i = 1, 2$. Note that $M_1/J_1 = M_2|_{J_2}$ and $M_1|_{J_1} = M_2/J_2$ by Theorem 1.9. Write the associated matroid of Q_0 to be $M_1|_{J_1} \oplus M_2|_{J_2}$. Recall Lemma 2.1 and we know that there corresponds a non-degenerate flat F of $M_1|_{J_1}$ or $M_2|_{J_2}$ to P. Since the argument is symmetric, assume that F is a non-degenerate flat of $M_1/J_1 = M_2|_{J_2}$, in which case $P = \mathrm{BP}_{M_1|_{J_1}} \times (\text{a facet of } \mathrm{BP}_{M_1/J_1}) = \mathrm{BP}_{M_2/J_2} \times (\text{a facet of } \mathrm{BP}_{M_2|_{J_2}})$. Then, there are 4 cases as follows.

	$M_1\|_{J_1\cup F}$	M_2/F	$S_{M_1}(Q_1)$	$S_{M_2}(Q_2)$
(i)	separable	separable	F	$J_1 \cup F$
(ii)	separable	inseparable	F	F
(iii)	inseparable	separable	$J_1 \cup F$	$J_1 \cup F$
(iv)	inseparable	inseparable	$J_1 \cup F$	F

Table 2.2:

We will show that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex at $P = Q_1 \cap Q_2$ for the cases (i),(ii) and (iii). Note that if $M_1|_{J_1\cup F}$ is separable, J_1 and F are its separators, and if M_2/F is separable, J_1 and $J_2 \backslash F$ are its separators; see the proof of Lemma 2.1.

Case (i)(ii). Suppose that $M_1|_{J_1 \cup F}$ is separable. $[M_1|_{J_1 \cup F}]/J_1 = M_1|_F$ by (S2). Then, one has:

$$r_{M_2}(F) = r_{M_2|_{J_2}}(F) = r_{M_1/J_1}(F) = r_{[M_1|_{J_1 \cup F}]/J_1}(F) = r_{M_1|_F}(F) = r_{M_1}(F)$$

Since F is a flat of M_2, BP_{M_2} is contained in the half space determined by the inequality $x(F) \leq r_{M_2}(F) = r_{M_1}(F)$ while Q_1 has the the same describing inequality. Hence, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex at P.

Case (iii). Since M_2/F is separable, $r_{M_2/F}(J_1 \cup (J_2 \backslash F)) = r_{M_2/F}(J_1) + r_{M_2/F}(J_2 \backslash F)$ implies that:

$$r_{M_2}(J_1 \cup F) = r_{M_2}(S) - r_{M_2}(J_2) + r_{M_2}(F)$$

$r_{M_1/J_1}(F) = r_{M_2|_{J_2}}(F)$ implies that:

$$r_{M_1}(J_1 \cup F) = r_{M_1}(J_1) + r_{M_2}(F) \qquad (*)$$

$r_{M_1/J_1}(J_1) = r_{M_2|_{J_2}}(J_1)$ implies that:

$$r_{M_1}(J_1) + r_{M_2}(J_1) = r_{M_1}(S)$$

Then, one has:

$$\begin{aligned} r_{M_1}(J_1 \cup F) - r_{M_2}(J_1 \cup F) &= r_{M_1}(J_1) + r_{M_2}(J_2) - r_{M_2}(S) \\ &= r_{M_1}(S) - r_{M_2}(S) \\ &= 0 \end{aligned}$$

So, $r_{M_1}(J_1 \cup F) = r_{M_2}(J_2 \cup F)$ and two facets Q_1, Q_2 have the same describing inequality, which means that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is convex at P.

($\Leftarrow$) It is enough to show that in Case (iv), $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is not convex at P. Since $J_1 \cup F$ is inseparable, one has:

$$\begin{aligned} r_{M_1}(J_1) + r_{M_2}(F) = r_{M_1}(J_1 \cup F) &< r_{M_1}(J_1) + r_{M_1}(F) \\ r_{M_2}(F) &< r_{M_1}(F) \end{aligned}$$

The describing inequalities for the facets Q_1 of BP_{M_1} and Q_2 of BP_{M_2} are, respectively,

$$\begin{aligned} x(J_1) + x(F) = x(J_1 \cup F) &\leq r_{M_1}(J_1 \cup F) = r_{M_1}(J_1) + r_{M_2}(F) \quad \text{by } (*) \\ x(F) &\leq r_{M_2}(F) \end{aligned}$$

Consider the half-spaces H_1, H_2 determined by these two inequalities. Then the intersection of H_1 and the plane $\{x(F) = r_{M_2}(F)\}$ is:

$$L := \{x(F) = r_{M_2}(F), x(J_1) \leq r_{M_1}(J_1)\} \subset \Delta_n^k$$

This plane L is contained in the half-space $\{x(J_1) \leq r_{M_1}(J_1)\}$, where $x(J_1) \leq r_{M_1}(J_1)$ is the facet inducing inequality of Q_0 in BP_{M_1}. Hence, the plane L intersects BP_{M_1} in codimension 1. But, since $L \cap \mathrm{BP}_{M_1}$ is not a facet of BP_{M_1}, we conclude that $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is not convex at $P = Q_1 \cap Q_2$. □

Assume the same settings as in Theorem 2.6. Consider nonzero vectors $v_0, v_1, v_2 \in \mathbb{R}^{n-1}$ such that $v_i \perp Q_i$ for $i = 0, 1, 2$. Since $P = Q_0 \cap Q_1 \cap Q_2$ is a common face of BP_{M_1} and BP_{M_2} that has codimension 2, v_0, v_1, v_2 span a 2-dimensional vector space of $\mathbb{R}^{n-1}$ that is a normal section of P, which can be visualized since it has dimension 2. Figure 2.1 gives its visualization, where each picture corresponds to the case (i), (ii), (iii), and (iv) as in the proof of Theorem 2.6. A black dot in the middle represents P. A red line and the gray dashed line facing each other represent $Q_0 \subset \mathrm{BP}_{M_1}$ and $Q_0 \subset \mathrm{BP}_{M_2}$, respectively. Here, P is a common facet of $Q_0 \subset \mathrm{BP}_{M_1}$ and $Q_0 \subset \mathrm{BP}_{M_2}$: $P = \mathrm{BP}_{M_1|_{J_1}} \times (\text{a facet of } \mathrm{BP}_{M_1/J_1}) = \mathrm{BP}_{M_2/J_2} \times (\text{a facet of } \mathrm{BP}_{M_2|_{J_2}})$, and we represent $Q_0 \subset \mathrm{BP}_{M_1}$ and $Q_0 \subset \mathrm{BP}_{M_2}$ as a solid line and a dashed

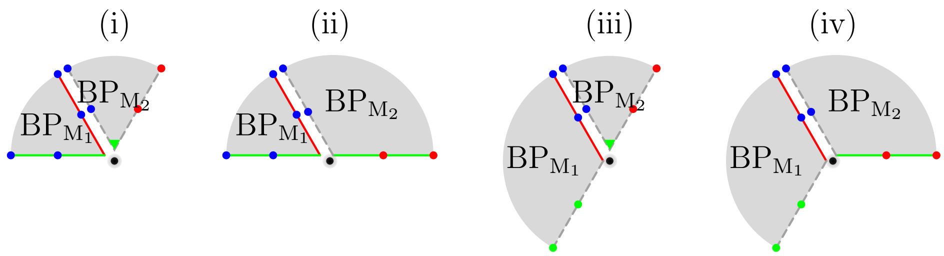


Figure 2.1:

line, respectively. Similarly, in the picture for the case (i), P is a facet of Q_1: $P = \mathrm{BP}_{M_1|_F} \times$ (a facet of $\mathrm{BP}_{M_1/F}$), and Q_1 is represented as a solid line. In the picture for the case (ii), P is a facet of Q_2: $P = \mathrm{BP}_{M_2/J_1 \cup F} \times$ (a facet of $\mathrm{BP}_{M_2|_{J_1 \cup F}}$), and Q_2 is represented as a dashed line. Draw 60° for the angle between the same kind of lines, 120° otherwise. Dots on the lines represent codimension 2 faces of the base polytopes that are intersections of appropriate facets. The colors used in the picture play a role of labeling and tracking, which is useful especially for the case of rank 3 matroids.

Remark 2.7. The pictures tell us the convexity of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ at the codimension 2 face $P \subset \mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$that is not contained in $\cup_{i=0}^{n} \{x_i = 0\}$.

Glued matroids

Suppose that M_1 and M_2 are two distinct inseprable matroids such that $M_1/J_1 = M_2|_{J_2}$ where J_1 and J_2, respectively are non-degenerate flats of M_1 and M_2 with rank 1 and 2, respectively. Then, their base polytopes BP_{M_1} and BP_{M_2} fit through their common facet $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} = \mathrm{BP}_{M(J_1)} = \mathrm{BP}_{M(J_2)}$. In this case, we say that M_1 and M_2 *fit through* J_1 *and* J_2.

Now, one can glue BP_{M_1} and BP_{M_2} to a polytope $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$. If $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is another base polytope, there corresponds a matroid $M_1 \# M_2$, which is an inseprable matroid since $\mathrm{BP}_{M_1 \# M_2} = \mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ has full dimension. In this case, we say that the matroids M_1 and M_2 *glue to a matroid* $M_1\#M_2$ *through* J_1 *and* J_2. Since $M_1\#M_2$ is a matroid, by Theorem 1.9, $M_1\#M_2$ has a unique set of non-degenerate flats. Observe the following:

1. When BP_{M_1} and BP_{M_2} glue, all the facets of both polytopes except $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ remain the same.

2. Near their intersection $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$, the cases (i)(ii)(iii) in Figure 2.1 are possible. As in Theorem 2.6, let $Q_1 \in \mathrm{BP}_{M_1}$ and $Q_2 \in \mathrm{BP}_{M_2}$ be any pair of two adjacent facets of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$, $F_1 := S_{M_1}(Q_1)$ and $F_2 := S_{M_2}(Q_2)$ their corresponding non-degenerate flats in M_1 and M_2, respectively. In cases (ii)(iii), $F_1 = F_2$, and in case (i), $F_1 \subsetneq F_2$.

In other words, the facets of $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ are exactly all the facets of BP_{M_1} and BP_{M_2} except their common facet, and the non-degenerate flats of $M_1\#M_2$ is the union of all non-degenerate flats of both M_1 and M_2 minus J_1 and J_2.

In addition to Theorem 2.6, it is also an interesting combinatorial problem when the glued matroids $M_1\#M_2$ becomes a representable matroid, which is true for $k = 2$ case by Corollary 3.8. Theorem 3.1 gives a nice criterion for that: if one can find a hyperplane arrangement corresponding to $M_1\#M_2$, it is a represenable matroid. At the end of Chapter 3, we will deal with this topic briefly. However, it still remains as a very difficult problem to give an equivalent condition to the representability of $M_1\#M_2$.

Glued base polytopes at a codimension 2 face

Theorem 2.8. *Fix $k \geq 3$. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two inseparable matroids such that $P \subset \mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ is a common face of BP_{M_1} and BP_{M_2} with codimension 2. Let $M_P = M_R \oplus M_G \oplus M_B$ be the corresponding matroid of P, where $R \cup G \cup B$ is a partition of S and $M_R = (R, r_R)$, $M_G = (G, r_G)$ and $M_B = (B, r_B)$ are inseparable matroids. For $i = 1, 2$,* *Let Q_{i1}, Q_{i2} be two facets of BP_{M_i} that contain P. Then, there are 6 cases for the quadruple $((Q_{11}, Q_{12}), (Q_{21}, Q_{22}))$, up to symmetry and isomorphism. Let J_{ij}, $i, j \in \{1, 2\}$, be the corresponding non-degenerate flats of Q_{ij} in M_i, then those 6 cases are given below.*

(J_{11}, J_{12})	$(R, R \cup G)$	$(R, R \cup G)$	$(R, R \cup G)$
(J_{21}, J_{22})	(B, G)	$(B, B \cup G)$	$(B \cup R, B \cup G)$
(J_{11}, J_{12})	(R, G)	(R, G)	$(R \cup G, R \cup B)$
(J_{21}, J_{22})	(B, G)	$(B \cup R, B \cup G)$	$(B \cup R, B \cup G)$

Table 2.3:

Proof. Suppose that M_1 has the pair $(R, R \cup G)$; see Corollary 2.3. Recall that an appropriate pair of a non-degenerate flat J of M_1 and a non-degenerate flat F of $M_1|_J$ or M_1/J determines a facet of BP_{M_1} that contains P. So, G is a non-degenerate flat of M_1/R for Q_{11} and R is a non-degenerate flat of $M_1|_{R\cup G}$ for Q_{12}.

$$\begin{aligned} M_P &= M_1|_R \oplus [M_1/R]|_G \oplus M_1/(R \cup G) \quad \text{for } Q_{11} \\ &= M_1|_R \oplus [M_1|_{R\cup G}]/R \oplus M_1/(R \cup G) \quad \text{for } Q_{12} \end{aligned}$$

There are 12 cases for a facet Q_2 of BP_{M_2} that contains P as follows.

1. R is a non-degenerate flat of M_2/B.

2. R is a non-degenerate flat of $M_2|_{R\cup G}$.

3. R is a non-degenerate flat of M_2/G.

4. R is a non-degenerate flat of $M_2|_{R\cup B}$.

5. G is a non-degenerate flat of M_2/R.

6. G is a non-degenerate flat of $M_2|_{B\cup G}$.

7. G is a non-degenerate flat of M_2/B.
8. G is a non-degenerate flat of $M_2|_{R\cup G}$.
9. B is a non-degenerate flat of M_2/G.
10. B is a non-degenerate flat of $M_2|_{R\cup B}$.
11. B is a non-degenerate flat of M_2/R.
12. B is a non-degenerate flat of $M_2|_{B\cup G}$.

In each case, the matroid M_P is expressed as follows.

1. $M_P = [M_2/B]\,|_R \oplus M_2/\,(R\cup B) \oplus M_2|_B$.
2. $M_P = M_2|_R \oplus [M_2|_{R\cup G}]\,/R \oplus M_2/\,(R\cup G)$.
3. $M_P = [M_2/G]\,|_R \oplus M_2|_G \oplus M_2/\,(R\cup G)$.
4. $M_P = M_2|_R \oplus M_2/\,(R\cup B) \oplus [M_2|_{R\cup B}]\,/R$.
5. $M_P = M_2|_R \oplus [M_2/R]\,|_G \oplus M_2/\,(R\cup G)$.
6. $M_P = M_2/\,(B\cup G) \oplus M_2|_G \oplus [M_2|_{B\cup G}]\,/G$.
7. $M_P = M_2/\,(B\cup G) \oplus [M_2/B]\,|_G \oplus M_2|_B$.
8. $M_P = [M_2|_{R\cup G}]\,/G \oplus M_2|_G \oplus M_2/\,(R\cup G)$.
9. $M_P = M_2/\,(B\cup G) \oplus M_2|_G \oplus [M_2/G]\,|_B$.
10. $M_P = [M_2|_{R\cup B}]\,/B \oplus M_2/\,(R\cup B) \oplus M_2|_B$.
11. $M_P = M_2|_R \oplus M_2/\,(R\cup B) \oplus [M_2/R]\,|_B$.
12. $M_P = M_2/\,(B\cup G) \oplus [M_2|_{B\cup G}]\,/B \oplus M_2|_B$.

We will check actually which case can be chosen for the facets Q_{21} and Q_{22}.

(1) $L_2 := \{x(B) = r_2(B)\}$ is the hyperplane in $\mathbb{R}^{n-1}$ that contains the facet Q_2, while $L_{12} := \{x(R \cup G) = r_1(R \cup G)\}$ is the hyperplane that contains Q_{12}. Using the expression of M_P, we have $r_1(R) = r_2(R \cup B) - r_2(B)$, $r_1(G \cup R) - r_1(R) = r_2(S) - r_2(R \cup B)$, $r_1(S) - r_1(R \cup G) = r_2(B)$. Since $x(R \cup G) + x(B) = x(S) = r_1(S) = r_2(S)$, by the last equality, $x(B) = r_2(B)$ implies that $r_1(S) - x(R \cup G) = r_1(S) - r_1(R \cup G)$, $x(R \cup G) = r_1(R \cup G)$. Hence, $L_2 = L_{12}$. Now, $\{x(R) = r_1(R)\}$ divides the hyperplane $L_2 = L_{12}$ into two halves, and Q_2 is contained in the half $\{x(R) \leq r_1(R)\} \cap L_2$. Similarly, $\{x(R) = r_2(R)\}$ divides $L_2 = L_{12}$ into two halves, and Q_{12} is contained in $\{x(R) \leq r_2(R)\} \cap L_{12}$. But, by the first equality and submodularity of the rank function, $x(R) \leq r_1(R) = r_2(R \cup B) - r_2(B) \leq r_2(R)$. Hence, $x(R) \leq r_1(R)$ implies $x(R) \leq r_2(R)$, so Q_2 and Q_{12} share P and are located in the same side. This means that $Q_2 \cap Q_{12}$ has codimension 1, which contradicts that $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \supset Q_2 \cap Q_{12}$ has codimension 2.

(2) Q_2 is contained in $\{x(R \cup G) = r_2(R \cup G)\}$. We have $r_1(R) = r_2(R)$, $r_1(G \cup R) - r_1(R) = r_2(R \cup G) - r_2(R)$, $r_1(S) - r_1(R \cup G) = r_2(S) - r_2(R \cup G)$ from the expression of M_P. Excluding one redundant equation, one can simplify the above equations into $r_1(R) = r_2(R)$, $r_1(R \cup G) = r_2(R \cup G)$. Hence, Q_2 is contained in

$$L_{12} = \{x(R \cup G) = r_1(R \cup G) = r_2(R \cup G)\}$$

In addition, $\{x(R) = r_1(R) = r_2(R)\}$ divides L_{12} into two halves, but Q_2 and Q_{12} are in the same side $\{x(R) \leq r_1(R) = r_2(R)\}$. This means that $Q_2 \cap Q_{12}$ has codimension 1, a contradiction.

(5) and (6) are not the cases in the same way.

(3) We have $r_1(R) = r_2(R \cup G) - r_2(G)$, $r_1(G \cup R) - r_1(R) = r_2(G)$, $r_1(S) - r_1(R \cup G) = r_2(S) - r_2(R \cup G)$. After simplifying, we get $r_1(R \cup G) = r_1(R) + r_2(G) = r_2(R \cup G)$, and by submodularity, $r_2(G) \leq r_1(G)$ and

$r_1(R) \leq r_2(R)$. Q_2 is contained in $L_2 := \{x(G) = r_2(G)\}$. The facet inducing inequalities of Q_{11} and Q_{12} are $x(R) \leq r_1(R)$ and $x(R \cup G) \leq r_1(R \cup G)$, respectively. The facet inducing inequality of Q_2 is $x(G) \leq r_2(G)$, and every point satisfying $x(G) \leq r_2(G)$ also satisfies $x(G) \leq r_1(G)$. In addition, $L_2 \cap \{x(R \cup G) \leq r_1(R \cup G)\}$ is $\{x(G) = r_2(G), x(R) \leq r_1(R)\}$, which has codimension 1 since $G \cap R = \emptyset$. $L_2 \cap \{x(R \cup G) \leq r_1(R \cup G)\}$ is contained in the intersection of two half spaces

$$\{x(R) \leq r_1(R)\} \cap \{x(R \cup G) \leq r_1(R \cup G)\}$$

Since originially Q_2 is contained in $L_2 \cap \left\{x(R) \leq r_{M_2/G}(R \cup G) = r_1(R)\right\}$, Q_2 intersects BP_{M_1} in codimension 1, which is a contradiction.

(4) From the expression of M_P, we get $r_1(R) = r_2(R)$, $r_1(G \cup R) - r_1(R) = r_2(S) - r_2(R \cup B)$, $r_1(S) - r_1(R \cup G) = r_2(R \cup B) - r_2(R)$, which are simplified into $r_1(R) = r_2(R)$, $r_1(R \cup G) + r_2(R \cup B) = r_1(R) + r_1(S)$. Q_2 is contained in $L_2 \cap \{x(R) \leq r_2(R)\}$ where $L_2 := \{x(R \cup B) = r_2(R \cup B)\}$. So, Q_2 is contained in $\{x(R) \leq r_1(R) = r_2(R)\}$ which is a facet inducing inequality for Q_{11}. For the points on L_2, the facet inducing inequality $x(R \cup G) \leq r_1(R \cup G)$ of Q_{12} becomes $x(R) + x(S) = x(R \cup G) + x(R \cup B) \leq r_1(R \cup G) + r_2(R \cup B) = r_1(R) + r_1(S)$, i.e., $x(R) \leq r_1(R)$, which is already satisfied by Q_2. Hence Q_2 intersects BP_{M_1} in codimension 1, which is a contradiction.

(7) We have $r_1(R) = r_2(S) - r_2(G \cup B)$, $r_1(G \cup R) - r_1(R) = r_2(G \cup B) - r_2(B)$, $r_1(S) - r_1(R \cup G) = r_2(B)$, which are simplified into $r_1(R \cup G) + r_2(B) = r_1(S) = r_2(S) = r_1(R) + r_2(G \cup B)$. The facet inducing inequality of Q_2 is $x(B) \leq r_2(B)$. For every point in the intersection of $\mathrm{BP}_{M_1} \cap Q_2$, one has $x(S) = x(R \cup G) + x(B) \leq r_1(R \cup G) + r_2(B) = r_1(S)$, hence the equality should hold in the intermediate inequality. The same thing for the inequality $x(S) = x(R) + x(G \cup B) \leq r_1(R) + r_2(G \cup B)$. Therefore, $\mathrm{BP}_{M_1} \cap Q_2$ has codimension ≥ 2. By assumption, the codimension of $\mathrm{BP}_{M_1} \cap Q_2$ is ≤ 2. So, Q_2 intersects BP_{M_1} in codimension 2.

(8) We have $r_1(R) = r_2(R \cup G) - r_2(G)$, $r_1(G \cup R) - r_1(R) = r_2(G)$, $r_1(S) - r_1(R \cup G) = r_2(S) - r_2(R \cup G)$, which are simplified into $r_1(R) + r_2(G) = r_1(R \cup G) = r_2(R \cup G)$. The facet inducing inequality of Q_2 is

$$x(R \cup G) \leq r_2(R \cup G) = r_1(R \cup G)$$

By assumption $L_{12} = \{x(R \cup G) = r_1(R \cup G)\}$ already intersects $L_{11} := \{x(R) = r_1(R)\}$. But, Q_{12} is contained in $L_{12} \cap \{x(R) \leq r_1(R)\}$ and Q_2 is contained in $L_{12} \cap \{x(G) \leq r_2(G)\}$. For every point in $Q_{12} \cap Q_2$, $x(R \cup G) = x(R) + x(G) \leq r_1(R) + r_2(G) = r_1(R \cup G)$, so one has equality in the intermediate inequality since $x(R \cup G) = r_1(R \cup G)$ for such a point. Hence $Q_{12} \cap Q_2$ has codimension 2, and by assumption $\mathrm{BP}_{M_1} \cap Q_2$ has codimension 2.

(9)(10)(11)(12) are possible cases which can be checked in the similar way.

Hence, for the facet Q_{21} and Q_{22}, (7)-(12) cases are possible. In case that Q_{21} has the non-degenerate flat B as in (7), by Lemma 2.1, G as in (9) or $B \cup G$ as in (12) is the non-degenerate flat for Q_{22}. In case that Q_{21} has the non-degenerate flat $R \cup G$ as in (8), by Lemma 2.1 again, Q_{22} has non-degenerate flat G as in (3) or $B \cup G$ as in (6), which we already know is not possible. In case that Q_{21} has the non-degenerate flat G as in (9), the non-degenerate flat for Q_{22} is B as in (7) which we already counted, or $B \cup G$ as in (6) which is not a case. In case Q_{21} has the non-degenerate flat $R \cup B$ as in (10), $B \cup G$ as in (12) or B as in (1) which is not possible corresponds to Q_{22}. Hence, (B, G), $(B, B \cup G)$, and $(B \cup R, B \cup G)$ are possible pairs of non-degenerate flats for Q_{21}, Q_{22}. So, the first line in the table can be checked. The remaining cases do not add more possibilities.

Likewise, one can compute by hands the possible quadruples of non-degenerate flats for $((Q_{11}, Q_{12}), (Q_{21}, Q_{22}))$, which are given as in the table. The section that is normal to P can be visualized as in Figure 2.2. □

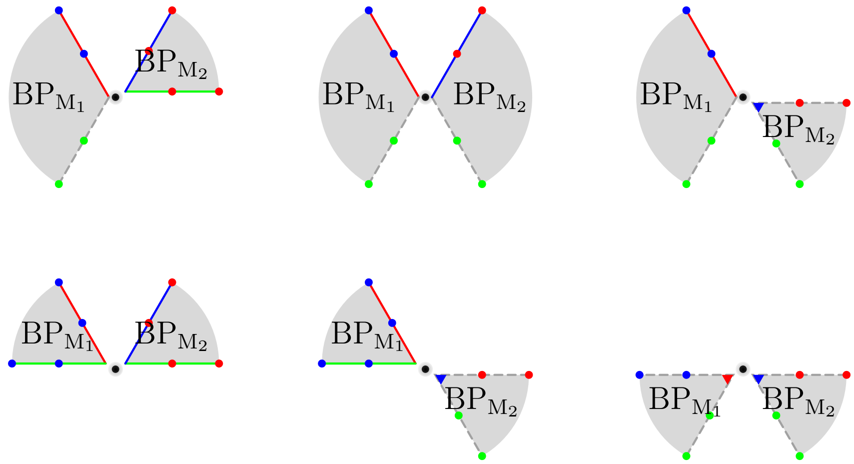


Figure 2.2:

Using Lemma 2.1, Theorem 2.6 and Theorem 2.8, we get a useful theorem that classifies the local pictures of the glued base polytopes at P.

Corollary 2.9. *Fix $(k \geq 3, n)$. Let $P \subset \mathbb{R}^{n-1}$ be a codimension 2 base polytope that is not contained in $\cup_{j=1}^{n} \{x_j = 0\}$. Suppose that one has base polytopes that contains P and are all face-fitting. Then, the maximal unions of them are given up to symmetry in the table below, and any such union is a part of one of them. The pictures of the normal section at P are given in Figure 2.3.*

	M_1	M_2	M_3	M_4
	M_5	M_6		
(i)	(R, G)	$(G \cup B, G \cup R)$	(G, B)	$(B \cup R, B \cup G)$
	(R, B)	$(R \cup G, R \cup B)$		
(ii)	$(R, R \cup G)$	$(G \cup B, G \cup R)$	(G, B)	$(B \cup R, B \cup G)$
	(R, B)			
(iii)	$(R, R \cup G)$	$(G \cup B, G \cup R)$	$(B, B \cup G)$	(R, B)
(iv)	$(R, R \cup G)$	$(G, G \cup B)$	$(B \cup R, B \cup G)$	(R, B)
(v)	$(R, R \cup G)$	$(G, G \cup B)$	$(R, R \cup B)$	

Table 2.4:

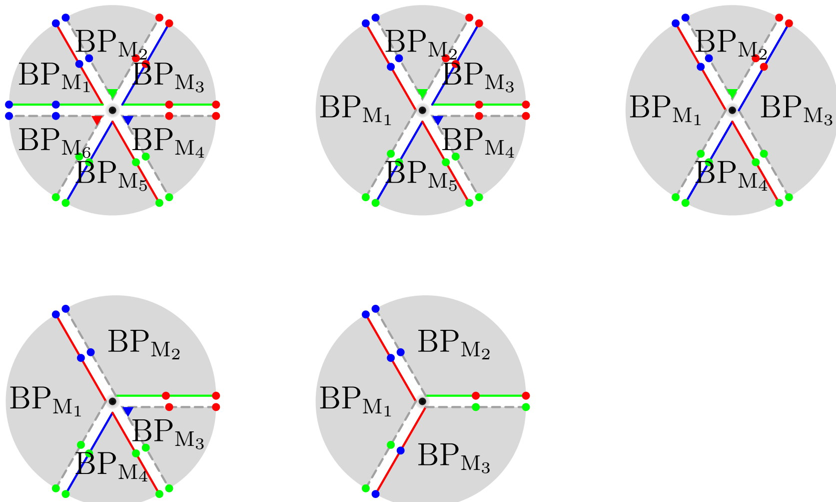


Figure 2.3:

Chapter 3

Hyperplane arrangements

3.1 Hyperplane arrangements

Let $S := \{1, ..., n\}$, $\mathbb{F}$ a field and $V \cong \mathbb{F}^k$ for some $k \geq 2$. A *hyperplane arrangement* over a field $\mathbb{F}$ is a pair $(\mathbb{P}V, (B_1, ..., B_n))$ where B_i, $i = 1, ..., n$, are hyperplanes in a projective space $\mathbb{P}V \cong \mathbb{P}^{k-1}$ such that $\cap_{i \in S} B_i = \emptyset$.

Theorem 3.1 ([GGMS87]). *A hyperplane arrangement* $(\mathbb{P}V, (B_1, ..., B_n))$ *with* $V \cong \mathbb{F}^k$*, gives a loopless representable matroid of rank* $k \geq 2$*. In addition, for any loopless representable matroid* M *of rank* $k \geq 2$*, there exists a hyperplane arrangement whose corresponding matroid is* M*.*

Proof. Let $f_i \in V^*$ be a linear equation defining B_i, i.e.,

$$B_i = \{u \in \mathbb{P}V \mid f_i(u) = 0\}$$

Defining $\operatorname{codim}_{\mathbb{P}V}(\emptyset) = r$ one has:

$$\dim_{V^*} \operatorname{span}\{f_i \mid i \in I\} = \operatorname{codim}_{\mathbb{P}V}(\cap_{i \in I} B_i)$$

If $\cap_{i \in S} B_i = \emptyset$, then the map $(\mathbb{F}^n)^* \to V^*$ by $x_i \mapsto f_i$ is surjective, which induces an injective map $\iota : V \hookrightarrow \mathbb{F}^n$, where x_i, $i = 1, ..., n$, are the standard coordinate functions of $\mathbb{F}^n$. Then B_i are indentified as the intersections of $\iota(\mathbb{P}V) \subset \mathbb{P}^{n-1}$ with $\{x_i = 0\}$. Hence, a hyperplane arrangement $(\mathbb{P}V, (B_1, ..., B_n))$ defines a representable matroid $M = (S, r)$ with the rank function $r(I) = \dim \operatorname{span}\{f_i \mid i \in I\}$ for a subset $I \subset S$. In addition, it is obvous that M is loopless.

Now, let $M = (S, r)$ be a loopless matroid of rank k which is representable over $\mathbb{F}$. Then, there is a set of non-zero vectors $\left\{f_i \in \mathbb{F}^k \mid i \in S\right\}$ which is a spanning set of $\mathbb{F}^k$. Let $V = \left(\mathbb{F}^k\right)^* \cong \mathbb{F}^k$ and consider the hyperplanes $B_i = \{u \in \mathbb{P}V \mid f_i(u) = 0\}$. Then, one has:

$$r(I) = \dim_{V^*} \operatorname{span}\{f_i \mid i \in I\} = \operatorname{codim}_{\mathbb{P}V}(\cap_{i \in I} B_i)$$

Since $k = r(S) = \operatorname{codim}_{\mathbb{P}V}(\cap_{i \in S} B_i)$ implies $\cap_{i \in S} B_i = \emptyset$, $(\mathbb{P}V, (B_1, ..., B_n))$ is a hyperplane arrangement. □

Remark 3.2. The dimension of the family of hyperplane arrangements in general linear position is $(k-1)(n-k-1)$. So, the correspondence

$$\text{(hyperplane arrangements)} \to \text{(loopless representable matroids)}$$

in Theorem 3.1 is not one-to-one, in general. However, we can define an equivalence relation on hyperplane arrangements that two hyperplane arrangements are equivalent if they give the same matroid. We say they have the same *type*.

For a hyperplane arrangement $(\mathbb{P}V, (B_1, ..., B_n))$ with $\dim V = k$, let $M = (S, r)$ be its corresponding matroid. Then, $\operatorname{Aut}(\mathbb{P}V, (B_1, ..., B_n)) = (\mathbb{F}^\times)^{\kappa(M)-1}$, where $\kappa(M)$ is the number of the connected components of M as in (S7).

Definition 3.3. Let M be a loopless matroid of rank k. We say that for a subset $J \subset S$ with $|J| \geq k$, the hyperplanes B_j, $j \in J$, are in *general linear position* if $M|_J \cong U^k_{|J|}$.

Lemma 3.4. *Let $M = (S, r)$ be a loopess matroid of rank k. If there is $J \subset S$ such that $M|_J \cong U^k_{k+1}$, then M is an inseparable matroid.*

Proof. Suppose that such M is separable, and let T and T^c be two nonempty separators of M, then $r(T) + r(T^c) = k$. By (S7) $r(T \cap J) + r(T^c \cap J) = r(J)$. Since $M|_J \cong U^k_{k+1}$ is an inseparable matroid, $r(T \cap J) = 0$ or $r(T^c \cap J) = 0$. Let $r(T^c \cap J) = 0$ without loss of generality, i.e., $T^c \cap J = \emptyset$, $J \subset T$ since M is loopless. Then, $k = r(T) + r(T^c) = k + r(T^c)$ implies that $r(T^c) = 0$, $T^c = \emptyset$, a contradiction. □

Remark 3.5. For $k = 3$, $(\mathbb{P}V, (B_1, ..., B_n))$ has $3 + 1$ lines in general linear position if and only if M is inseparable. For it suffices to prove the converse of Lemma 3.7 for $k = 3$. Since M has rank 3, there is a basis, say $\{1, 2, 3\}$. Let $T_j = \overline{\{j\}}$, $j = 1, 2, 3$. Then $T_1 \cup T_2 \cup T_3$ is a disjoint union that is a proper subset of S since M is inseparable. So, there exists an element of $S - T_1 \cup T_2 \cup T_3$ and consider its closure, say 4 and $T_4 = \overline{\{4\}}$. Note that the lines $Z(T_i)$, $i \in \{1, 2, 3\}$ are in general linear position. If $Z(T_i)$, $i \in \{1, 2, 3, 4\}$ are in general linear position, $J = \{1, 2, 3, 4\}$. If not, T_4 is contained in one of $\overline{T_1 \cup T_2}$, $\overline{T_1 \cup T_3}$, $\overline{T_2 \cup T_3}$ since T_i, $i \in \{1, 2, 3, 4\}$ are distinct rank 1 flats. Say $T_4 \subset \overline{T_1 \cup T_2}$ then $M|_{T_1 \cup T_2 \cup T_3 \cup T_4}$ is separable. Since M is inseparable, $S - T_1 \cup T_2 \cup T_3 \cup T_4$ is not empty, and there is a rank 1 flat T_5 different from T_i, $i \in \{1, 2, 3, 4\}$. If T_5 is not contained in $\overline{T_1 \cup T_2}$, either $T_5 \not\subseteq \overline{T_1 \cup T_3}$ or $T_5 \not\subseteq \overline{T_2 \cup T_3}$. Suppose not, then $\overline{T_1 \cup T_3}$ and $\overline{T_2 \cup T_3}$ are two minimal flats containing T_3 by (F3). Since $\overline{T_1 \cup T_3} \cap \overline{T_2 \cup T_3}$ is a flat containing T_3, one has $\overline{T_1 \cup T_3} \cap \overline{T_2 \cup T_3} = T_3$, which implies that $T_5 \subset T_3$, a contradiction. So, suppose that $T_5 \not\subseteq \overline{T_1 \cup T_3}$. Then $Z(T_i)$, $i \in \{2, 3, 4, 5\}$ are in general linear position. If T_5 is contained in $\overline{T_1 \cup T_2}$, there is a rank 1 flat T_6 different from T_i, $i \in \{1, 2, 3, 4, 5\}$. If T_6 is not contained in $\overline{T_1 \cup T_2}$, similarly one can find a $3 + 1$ lines in general linear position. Otherwise, there is a rank 1 flat T_7 different from T_i, $i \in \{1, 2, 3, 4, 5, 6\}$. Likewise, we keep this process, which will terminate with $3 + 1$ lines in general linear position since our matroid M is finite.

We use the shorthand notation $Z_M(I) := \cap_{i\in I} B_i$. If the matroid is clear from the context, we write simply $Z(I)$ without M.

Lemma 3.6. *Consider a hyperplane arrangement $\left(\mathbb{P}^{k-1}, (B_1, ..., B_n)\right)$, $k \geq 2$, and its associated matroid $M = (S, r)$. Let J be a flat of M with $r(J) = s$. Then, $Z(J) \cong \mathbb{P}^{k-s-1}$ and*

(a) M/J gives a hyperplane arrangement in $\mathbb{P}^{k-s-1}$.

(b) $M|_J$ gives a hyperplane arrangement in $\mathbb{P}^{s-1}$.

Proof. (a) Since J is a flat, any other hyperplane B_i, $i \in J^c$, intersects $Z(J) \cong \mathbb{P}^{k-s-1}$ such that $\dim B_i \cap Z(J) = k - s - 2$. Write

$$\pi(I) := \cap_{i\in I} (B_i \cap Z(J)) = Z(I) \cap Z(J) = Z(I \cup J)$$

for $I \subset J^c$. Evidently, $\pi(J^c) = Z(S) = \emptyset$. Moreover,

$$\begin{aligned} \operatorname{codim}_{Z(J)} \pi(I) &= \operatorname{codim}_{\mathbb{P}^{k-1}} \pi(I) - \operatorname{codim}_{\mathbb{P}^{k-1}} Z(J) \\ &= r(I \cup J) - r(J) \\ &= r_{M/J}(I) \end{aligned}$$

Hence, M/J defines a hyperplane arrangement $\left(\mathbb{P}^{k-s-1}, (\pi(i))_{i\in J^c}\right)$.

(b) For $B_j = \{u \in \mathbb{P}V \mid f_j(u) = 0\}$ with $j \in J$, let $\tilde{B}_j := \{u \in V \mid f_j(u) = 0\}$ and consider $W := \left(\cap_{j\in J} \tilde{B}_j\right)^{\perp}$. W has dimension $k - (k - s) = s$. Let $\rho(j) := \mathbb{P}\left(W \cap \tilde{B}_j\right)$, then $\rho(j)$ are hyperplanes in $\mathbb{P}W \cong \mathbb{P}^{s-1}$. Since

$\cap_{j\in I}\left(W\cap\tilde{B}_j\right)=W\cap W^{\perp}=\{\mathbb{0}\}$, $\rho(J)=\emptyset$. Moreover, for $I\subset J$,

$$\begin{aligned}\operatorname{codim}_{\mathbb{P}W}\rho(I) &= \operatorname{codim}_W\cap_{j\in I}\left(W\cap\tilde{B}_j\right)\\ &= \operatorname{codim}_V\cap_{j\in I}\tilde{B}_j \qquad \text{since } W:=\left(\cap_{j\in J}\tilde{B}_j\right)^{\perp}\\ &= \dim_{V^*}\operatorname{span}\{f_j\mid j\in I\}\\ &= r_{M|_J}(I)\end{aligned}$$

Hence, $M|_J$ defines a hyperplane arrangement $\left(\mathbb{P}^{s-1},(\rho(i))_{i\in J}\right)$. □

Hyperplane arrangements on $\mathbb{P}^{2-1}$

Consider a loopless representable matroid M of rank 2, then $1\le\kappa(M)\le 2$; see (S7). By Theorem 3.1, there exists a hyperplane arrangement on $\mathbb{P}^1$. Recall that all non-degenerate flats are exactly those flats of rank 1; see Lemma 1.3. In addition, since M is a loopless matroid of rank 2, S has a partition into rank 1 flats by (M6).

Figure 3.1:

1. If $\kappa(M)=1$, M is inseparable. So, there are at least 3 point loci in general linear position, which looks like the first panel of Figure 3.1.

2. If $\kappa(M)=2$, M is separable, and there are exactly two point loci on $\mathbb{P}^1$; see the second panel of Figure 3.1.

Moreover, the following two lemmas say that every rank 2 matroid is representable.

Lemma 3.7. *Let $M = (S, r)$ be a loopless matroid with $r(M) = 2$. Then, there exists a hyperplane arrangement on $\mathbb{P}^1$ whose corresponding matroid is M. Hence M is representable.*

Proof. We construct a hyperplane arrangement from M. Since M is a loopless matroid of rank 2, S has a partition into rank 1 flats by (M6): $S = \cup_{i=1}^{m} F_i$. Assign to F_i distinct points $\psi(F_i) := P_i$ on $\mathbb{P}^1$, which is possible if $|\mathbb{F}| \geq m$ for any field $\mathbb{F}$. Then ψ induces a map $\tilde{\psi} : S \to \mathbb{P}^1$ defined by $j \mapsto P_i$ where $j \in F_i$ for some i. This map defines a hyperplane arrangement $(\mathbb{P}^1, (\tilde{\psi}(1), ..., \tilde{\psi}(n)))$. Note that ψ is a 1-1 correspondence between non-trivial flats and point loci on $\mathbb{P}^1$. It is easy to check that matroid conditions (F1)-(F3) are satisfied, and M is the associated matroid of the constructed hyperplane arrangement. Hence, M is a representable matroid by Theorem 3.1. □

Since loops do not impact the representability of a matroid, we have the following corollary.

Corollary 3.8. *Any matroid of rank 2 is representable.*

Hyperplane arrangements on $\mathbb{P}^{3-1}$

Consider a loopless representable matroid M of rank 3, then $1 \leq \kappa(M) \leq 3$. By Theorem 3.1, there exists a hyperplane arrangement on $\mathbb{P}^2$ whose corresponding matroid is M.

1. If $\kappa(M) = 1$, M is inseparable. So, by Remark 3.5, there are at least 4 line loci in general linear position, which looks like the first panel of Figure 3.2.

2. If $\kappa(M) = 2$, there exists an inseparable flat J of rank 2 such that $M = M|_J \oplus M|_{J^c}$ by (M6), where $M/J = M|_{J^c}$ by (S2) and $r_M(J^c) = 1$. Since $M|_J$ is an inseparable matroid of rank 2, J is a disjoint union of

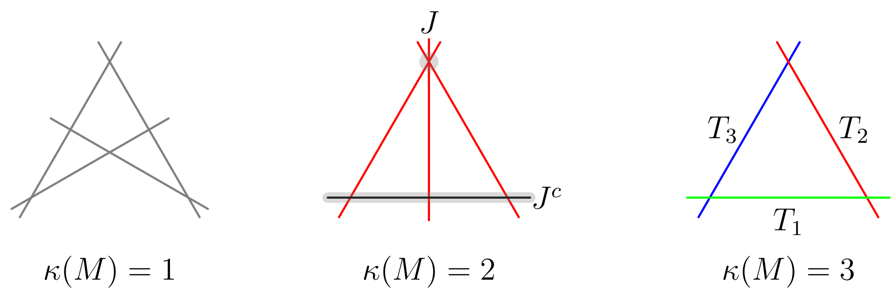


Figure 3.2:

rank 1 flats F_i such that the number of rank 1 flats is ≥ 3 by (M6). Then F_i and J^c are all rank 1 flats of M, so $Z(F_i)$ and $Z(J^c)$ are all line loci of the given hyperplane arrangement, which looks like the second panel of Figure 3.2.

3. If $\kappa(M) = 3$, $M = M|_{T_1} \oplus M|_{T_2} \oplus M|_{T_3}$ by (M6), where T_1, T_2, T_3 are only three rank 1 flats. So, $Z(T_1), Z(T_2), Z(T_3)$ are only three line loci on $\mathbb{P}^2$, which looks like the third panel of Figure 3.2.

Lemma 3.9. *Fix $k = 3$. Consider a hyperplane arrangement $(\mathbb{P}^2, (B_1, ..., B_n))$ and its associated matroid M. Then the facets of BP_M are in 1-1 correspondence with those flats $\emptyset \neq F \subsetneq S$ such that:*

(a) $r(F) = 1$ and M/F is inseparable, or

(b) $r(F) = 2$ and F is inseparable.

Proof. Apply Lemma 1.4 and Theorem 1.9. □

Lemma 3.10. *Fix $k = 3$. Consider a hyperplane arrangement $(\mathbb{P}^2, (B_1, ..., B_n))$ and its associated matroid M. Let F be a non-degenerate flat.*

(a) If $r(F) = 1$, $Z(F) \cong \mathbb{P}^1$ has more than 2 point loci.

(b) If $r(F) = 2$, $Z(F) \cong \mathbb{P}^0$ is the intersection of more than 2 line loci.

Proof. (a) Let $r(F) = 1$. M/F is an inseparable matroid of rank 2: $r_{M/F}(F^c) = r(F^c \cup F) - r(F) = 3 - 1 = 2$. By (M6), the number of nontrivial flats of M/F is > 2. By Lemma 3.6(a), M/F defines a hyperplane arrangement $\left(\mathbb{P}^1, (\pi(i))_{i \in F^c}\right)$ with rank function $r_{M/F}(I) = \operatorname{codim}_{Z(F)} \pi(I)$ for $I \subset F^c$. Note that for any nontrivial flat I of M/F, $Z_{M/F}(I)$ is a point on $Z(F)$. Hence, $Z(F) \cong \mathbb{P}^1$ has more than 2 point loci.

(b) Let $r(F) = 2$. $M|_F$ is an inseparable matroid of rank 2. By (M6), the number of nontrivial flats of $M|_F$ is > 2. By Lemma 3.6(b), $M|_F$ defines a hyperplane arrangement $(\mathbb{P}^1, (\tilde{B}_j \cap E)_{j \in F})$ with rank function $r_{M|_F}(I) = \operatorname{codim}_E \cap_{j \in I} (\tilde{B}_j \cap E)$ for $I \subset F$. Since any nontrivial flat I of $M|_F$ has rank 1, $Z_{M|_F}(I)$ is a point on E and $Z_M(I)$ is a line locus on $\mathbb{P}^2$, since $r_{M|_F}(I) = r(I)$. Therefore, $Z(F) \cong \mathbb{P}^0$ is the intersection of more than 2 line loci. □

Lemma 3.11. *Fix $k = 3$. Let M be an inseparable matroid that has a flat F of rank 1 such that M/F is separable. Then*

(a) F is one and only one flat of rank 1 such that M/F is separable.

(b) Any flat J of rank 2 with $J \cap F = \emptyset$ is separable.

(c) There are exactly two flats of rank 2 that are inseparable, which contain F.

Proof. Suppose that F is a rank 1 flat such that M/F is separable. Consider the associated hyperplane arrangement on $\mathbb{P}^2$ of the matroid M. Then, $Z(F)$ is a line with only two intersection points; see Figure 3.1. Since $Z(F)$ meets any line $Z(i)$ with $i \in S \backslash F$, the hyperplane arrangement looks like the Figure 3.3. It is easy to see all three statements are true using Lemma 3.10. □

Corollary 3.12. *Fix $k = 3$. Let M be an inseparable matroid that has a degenerate flat F of rank 1. Then F is one and only one rank 1 degenerate flat. Moreover, there are exactly two rank 2 non-degenerate flats, which contain F.*

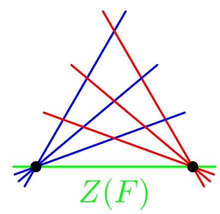


Figure 3.3:

Proof. Apply Lemma 3.9 and 3.8. □

Remark 3.13. Lemma 3.11 is not generalized to higher rank. Indeed, consider a matroid $M = (S, \mathcal{F})$ of rank 4 such that $S = \{1, ..., 7\}$ and all dependent flats of M are given in Table 3.1. M is a graphic matroid as seen in Figure

rank	Dependent flats of M
1	None
2	$\{1,4,7\}$, $\{2,4,6\}$, $\{3,5,7\}$
3	$\{2,3,4,6\}$, $\{2,3,5,7\}$, $\{2,4,5,6\}$, $\{3,5,6,7\}$, $\{1,2,4,6,7\}$, $\{1,3,4,5,7\}$
4	S

Table 3.1:

3.4, hence a regular matroid. Moreover, M is inseparable with rank 4. For

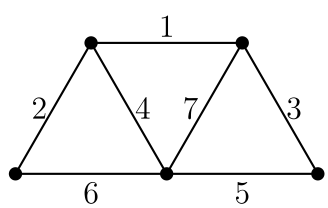


Figure 3.4:

convenience, we mean $\{a_1, ..., a_l\}$ by $a_1 \cdots a_l$. Then,

$$\begin{aligned} r_{M/\{7\}}(1246) + r_{M/\{7\}}(35) &= [r(12467) - r(7)] + [r(357) - r(7)] \\ &= [3-1] + [2-1] = 3 \\ &= r_{M/\{7\}}(123456) \end{aligned}$$

$$\begin{aligned} r_{M/\{4\}}(1357) + r_{M/\{4\}}(26) &= [r(13457) - r(4)] + [r(246) - r(4)] \\ &= [3-1] + [2-1] = 3 \\ &= r_{M/\{4\}}(123567) \end{aligned}$$

So, $\{4\}, \{7\}$ are inseparable flats of rank 1 such that $M/\{4\}$ and $M/\{7\}$ are separable.

Construction of hyperplane arrangements when $k = 3$

Fix $k = 3$. We introduce three special hyperplane arrangements below; see Figure 3.5.

(a) Let $S = A \sqcup B \sqcup C$. Suppose that $M_1 = (A^c, r_1)$ and $M_2 = (B^c, r_2)$ are inseparable matroids of rank 2 such that B is a nontrivial flat of M_1 and A is a nontrivial flat of M_2. Recall that M_1 and M_2 are representable over some field $\mathbb{F}$ with $|\mathbb{F}| \gg 1$ by Lemma 3.7. By Lemma 3.10, M_1 and M_2 define hyperplane arrangements $\mathcal{H}_1 := \left(\mathbb{P}^1, \left(P_1, ..., P_{n-|A|}\right)\right)$ and $\mathcal{H}_2 := \left(\mathbb{P}^1, \left(Q_1, ..., Q_{n-|B|}\right)\right)$, respectively. Since B is a flat of M_1 of rank 1, $Z_{M_1}(B)$ is a point locus. $Z_{M_2}(A)$ is also a point locus by the same reason. Embed those two hyperplane arrangements into $\mathbb{P}^2$ as two distinct lines $\cong \mathbb{P}^1$ that intersect each other at $Z_{M_1}(B) = Z_{M_2}(A)$. For each $i \in A$, let L_i be the embedded image of $\mathbb{P}^1$ for $\mathcal{H}_1$. Similarly for each $i \in B$, let L_i be the embedded image of $\mathbb{P}^1$ for $\mathcal{H}_2$. Draw a line passing through $Z_{M_1}(i)$ and $Z_{M_2}(i)$ on $\mathbb{P}^2$ for $i \in C$ and denote it by L_i. Hence, we get a hyperplane arrangement $(\mathbb{P}^2, (L_1, ..., L_n))$

which gives a representable loopless matroid M by Theorem 3.1. This hyperplane arrangement is not necessarily unique, but if the underlying field is large enough, for instance infinite, we can construct one such that no non-trivial incidence relations are made outside of $Z_M(A) \cup Z_M(B)$. Note that $M/A = M_1$ and $M/B = M_2$. Since M_1 and M_2 are inseparable, $r_1(C) = 2 = r_2(C)$. Then, $Z_M(A)$ and $Z_M(B)$ have at least 3 distinct point loci. So, there are at least 4 lines in general linear position, and M is inseparable.

(b) Let $S = A \sqcup B \sqcup C$. Suppose that $M_1 = (A^c, r_1)$ and $M_2 = (B^c, r_2)$ are inseparable matroids of rank 2 such that C is a nontrivial flat of both M_1 and M_2. Pick two distinct points $O_1, O_2 \in \mathbb{P}^2$. Draw a line passing through O_1 and O_2, and denote it by L_C. For $i \in C$, assign a line $L_i = L_C$ to i. Let $A_1, ..., A_{m_1}$ be all nontrivial flats of M_1 that are contained in A. Since M_1 is a loopless rank 2 matroid, A is their disjoint union. For each A_j, draw a distinct line passing through only O_1 but not O_2, and denote it by L_{A_j}. Assign the line L_{A_j} to any $l \in A_j$. Similarly, let B be the disjoint union of flats $B_1, ..., B_{m_2}$ of M_2 of rank 1. For each B_j, draw a distinct line passing through only O_2 but not O_1, and denote it by L_{B_j} which should be assigned to any $l \in B_j$. Hence, we construct a unique hyperplane arrangement $(\mathbb{P}^2, (L_1, ..., L_n))$, and a loopless representable matroid M by Theorem 3.1. Furthermore, $M|_{A^c} = M_1$ and $M|_{B^c} = M_2$. Since M_1 and M_2 are inseparable, there are at least 3 distinct rank 1 flats for each. So, without counting the line L_C, there are at least 4 lines in general linear position, which means that M is inseparable. Note that this matroid M satisfies Lemma 3.11, where C is one and only one degenerate flat of rank 1, and A, B are only two inseparable flats of rank 2. Also, note that this hyperplane arrangement can be obtained in (a).

Remark 3.14. The corresponding matroid to the hyperplane arrangement constructed in (b) satisfies Lemma 3.11 and Corollary 3.12, where C is the degenerate flat of rank 1, and A^c and B^c are the two non-degenerate flats of rank 2 (which contains C).

(c) Let $S = A \sqcup B \sqcup C \sqcup D$. Suppose that $M_1 = (A^c \backslash D, r_1)$ and $M_2 = (B^c \backslash D, r_2)$ are inseparable matroids of rank 2 such that B is a nontrivial flat of M_1 and A is a nontrivial flat of M_2. Using Construction (a), there is a hyperplane arrangement with the associated matroid on $A \sqcup B \sqcup C$ being inseparable. Now, for a partition of $D = \cup_{j=1}^{m} D_j$, draw a line passing through $Z(A) \cap Z(B)$ for each D_j. As in (a), we can draw D_j without generating extra nontrivial incidence relations except at $Z(A) \cap Z(B)$. In this way, we construct another hyperplane arrangement, and the corresponding matroid M which is inseparable, since it already has 4 lines in general position. In this construction, $M_3 := M|_{A \cup B \cup D}$ is a rank 2 inseparable matroid, and we see that M_1, M_2, M_3 detemines a matroid.
If M_1 and M_3 are given first, we can consider a separable rank 2 matroid M_2 whose nontrivial flats are $\{A \cup D\}$ and $\{C\}$. Then, draw a hyperplane arrangement of M_1 on $\mathbb{P}^2$. For a partition of $D = \cup_{j=1}^{m} D_j$ into rank 1 flats, draw a line for each D_j that passes through the point $Z_{M_1}(B) \subset \mathbb{P}^2$. Draw a line for B that passes through $Z_{M_1}(B)$ too, then pick a point $\neq Z_{M_1}(B)$ for C on this line and draw lines that connect $Z(C)$ and the points on the line $Z(A)$. This way of construction gives an alternative for the construction (b).

To summarize, given rank 2 inseparable matroids M_1 and M_3, there are two types of hyperplane arrangements that can be constructed, which depends on the degeneracy of the rank 2 matroid M_2.

3.2 Glued matroids for $k = 3$ and its representability

Theorem 2.6 gives an equivalent condition phrased in terms of matroids for when the union of two full dimensional base polytopes $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_1}$ becomes another base polytope, where M_1 and M_2 are inseparable matroids of rank

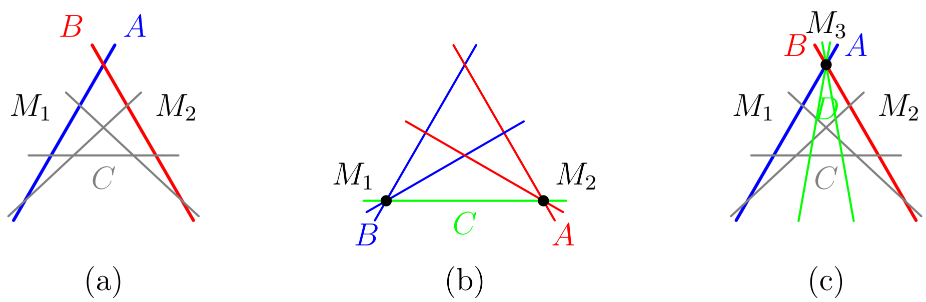


Figure 3.5:

$k \geq 3$. As in Theorem 2.6, let J_1 and J_2 be the non-degenerate flats of M_1 and M_2 with $r_1(J_1) = 1$ and $r_2(J_2) = 2$, respectively, that correspond to $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$. In addition, suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are hyperplane arrangements whose corresponding matroids are M_1 and M_2, respectively. By Theorem 3.1, if one can find a hyperplane arrangement corresponding to $M_1\#M_2$, $M_1\#M_2$ is a representable matroid. In other words, if one can draw all together in $\mathbb{P}^{k-1}$ the hyperplanes of $\mathcal{H}_1$ except $Z_{M_1}(J_1)$ and the hyperplanes of $\mathcal{H}_2$ except $Z_{M_2}(J_2)$ in a way that gives $M_1\#M_2$, $M_1\#M_2$ is the corresponding matroid of the resulting hyperplane arrangement, hence representable. If the hyperplanes $Z_{M_i}(j)$, $j \in J_1$, and $Z_{M_i}(j')$, $j' \in J_2$, for fixed $i = 1, 2$ behave independently in both hyperplane arrangements $\mathcal{H}_i$, such a drawing is always possible, and $M_1\#M_2$ is represnentable. When $k = 3$, we can slightly weaken this condition as in Theorem 3.17. We review first the gluing of matroids for $k = 3$.

Lemma 3.15. *Fix $k = 3$. Let $M_1 = (S, r_1, \mathcal{F}_1)$ and $M_2 = (S, r_2, \mathcal{F}_2)$ be rank 3 inseparable matroids such that $M_1/J_1 = M_2|_{J_2}$, where J_1 and J_2, respectively are non-degenearte flats of M_1 and M_2 with $r_1(J_1) = 1$, $r_2(J_2) = 2$. Then, M_1 and M_2 glue to a matroid $M_1\#M_2$ if and only if there is at most one rank 1 flat $F \subset J_2$ of M_2 that is not a rank 1 flat of M_1, in which case F is a degenerate flat of M_2.*

Proof. By (M8), T is a flat of M_1/J_1 if and only if $J_1 \cup T$ is a flat of M_1. Rank 1 flats $T \subset J_2$ of M_2 are exactly rank 1 flats of $M_2|_{J_2}$. Since $M_2|_{J_2} = M_1/J_1$ by assumption, $T \subset J_2$ is a rank 1 flat of M_2 if and only if $J_1 \cup T$ is a flat of M_1. Since J_1 is a rank 1 flat of M_1, $M_1|_{J_1 \cup T}$ is separable if and only if T is a rank 1 flat in M_1.

($\Leftarrow$) If every rank 1 flat $T \subset J_2$ of M_2 is a rank 1 flat of M_1, $M_1|_{J_1 \cup T}$ is separable, hence by Theorem 2.6, no matter M_2/T is separable or not, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope; see Table 2.2. Therefore, $M_2|_{J_2} = M_1/J_1$. Since $M_2/J_2 = M_1|_{J_1} \cong U^1_{|J_1|}$, M_1 and M_2 glue to a matroid $M_1 \# M_2$.

Else if there is a rank 1 flat $F \subset J_2$ of M_2 that is not a rank 1 flat of M, by assumption F is only one such rank 1 flat and F is a degenerate flat of M_2, i.e., M_2/F is separable. By Theorem 2.6 again, $\mathrm{BP}_{M_1} \cup \mathrm{BP}_{M_2}$ is a base polytope which implies that M_1 and M_2 glue to a matroid $M_1 \# M_2$.

($\Rightarrow$) Suppose that $F \subset J_2$ is a rank 1 flat of M_2 that is not a rank 1 flat of M_1. Then, $M_1|_{J_1 \cup F}$ is inseparable. By Theorem 2.6, M_2/F is separable, i.e., F is a degenearte flat. But, by Lemma 3.11, F is one and only one such flat. □

Definition 3.16. Fix $k = 3$. For a loopless representable matroid $M = (S, r)$, we say two disjoint subsets $A, B \subset S$ *behave independently* in M if for any pair of elements $(a \in A, b \in B)$, one has $\overline{\{a\}} \neq \overline{\{b\}}$ and $\overline{\{a, b\}} = \overline{\{a\}} \cup \overline{\{b\}}$. In other words, in its associated hyperplane arrangement, the lines $Z(a)$ and $Z(b)$ are distinct, and no other lines pass through their intersection point $Z(a) \cap Z(b) = Z(a, b)$.

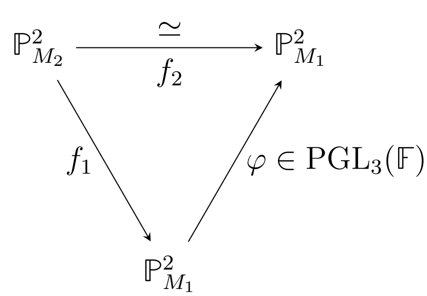
Theorem 3.17. *Fix $k = 3$. Let $M_1 = (S, r_1, \mathcal{F}_1)$ and $M_2 = (S, r_2, \mathcal{F}_2)$ be rank 3 inseparable matroids that are representable. Suppose that M_1 and M_2 glue to a matroid $M_1 \# M_2$ through $J_1 \in \mathcal{F}_1$ and $J_2 \in \mathcal{F}_2$ with $r_1(J_1) = 1$, $r_2(J_2) = 2$. By Lemma 3.15, there is at most one rank 1 flat $F \subset J_2$ of M_2 that is not a rank 1 flat of M_1, in which case F is a degenerate flat of M_2. Suppose that:*

1. *J_1 and J_2 behave independently in M_2 if every rank 1 flat $T \subset J_2$ of M_2 is non-degenerate.*

2. *J_1 and $J_2 \backslash F$ behave independently in M_2 if F is a degenerate rank 1 flat of M_2.*

Then, $M_1 \# M_2$ is representable.

Proof. Let $\mathcal{H}_1 = (\mathbb{P}^2_{M_1}, (Z_{M_1}(i))_{i \in S})$ and $\mathcal{H}_2 = (\mathbb{P}^2_{M_2}, (Z_{M_2}(i))_{i \in S})$ be two hyperplane arrangements whose corresponding matroids are M_1 and M_2, respectively. Let T be any rank 1 flat of $M_1/J_1 = M_2|_{J_2}$. Suppose that (1) every rank 1 flat $T \subset J_2$ of M_2 is non-degenerate. Then, T and J_1 behave independently in M_1, i.e., $Z_{M_1}(T)$ is a line in $\mathcal{H}_1$ that makes only trivial incidence relation with the line $Z_{M_1}(J_1)$ in $\mathcal{H}_1$. Then, we can choose an isomorphism $f_2 = \varphi \circ f_1 : \mathbb{P}^2_{M_2} \to \mathbb{P}^2_{M_1}$ where $f_1 : \mathbb{P}^2_{M_2} \to \mathbb{P}^2_{M_1}$ and $\varphi : \mathbb{P}^2_{M_1} \to \mathbb{P}^2_{M_1}$ are isomorphisms; see the following diagram. For a generic



choice of $\varphi \in \mathrm{PGL}_3(\mathbb{F})$,

- $(\varphi \circ f_1)(Z_{M_2}(j_1))$ with $j_1 \in J_1$ are different from $Z_{M_1}(j_2)$ with $j_2 \in J_2$, and

- $(\varphi \circ f_1)(Z_{M_2}(j_1)) \cap Z_{M_2}(j_2)$ is a point and no other lines pass through it.

Take the lines $f_2(Z_{M_2}(i))$ for $i \in J_1$ and $Z_{M_1}(i)$ for $i \in J_2$ on $\mathbb{P}^2_{M_1}$, which gives a new hyperplane arrangement on $\mathbb{P}^2$, say $\mathcal{H}_3$. By Theorem 3.1, there corresponds a loopless representable matroid M_3. $\mathcal{H}_3$ has 4 lines in general linear position, since $\mathcal{H}_1$ already has four. So, M_3 is an inseparable matroid. By construction of $\mathcal{H}_3$, all nontrivial incidence relations but J_1 and J_2 in both $\mathcal{H}_1$ and $\mathcal{H}_2$ remain the same, while J_1 and J_2 are discarded. So, $M_1 \# M_2 = M_3$ which is representable.

Suppose that (2) F is a degenerate rank 1 flat of M_2. Similarly as in the case (1), we choose φ such that

- $(\varphi \circ f_1)(Z_{M_2}(j_1))$ with $j_1 \in J_1$ are different from $Z_{M_1}(j_2)$ with $j_2 \in J_2$,
- $(\varphi \circ f_1)(Z_{M_2}(j_1)) \cap Z_{M_2}(j_2)$ for $j_2 \in J_2 \backslash F$ is a point and no other lines pass through it, and
- $(\varphi \circ f_1)(Z_{M_2}(J_1 \cup F)) = Z_{M_1}(J_1 \cup F)$, which is a point.

Take the lines $f_2(Z_{M_2}(i))$ for $i \in J_1$ and $Z_{M_1}(i)$ for $i \in J_2$ on $\mathbb{P}^2_{M_1}$, which gives a new hyperplane arrangement on $\mathbb{P}^2$, say $\mathcal{H}_3$. By the same argument, $M_1 \# M_2 = M_3$ which is representable. $\square$

Chapter 4

Puzzle-pieces and their gluing

Assume $S = \{1, ..., n\}$ unless separately mentioned.

4.1 Puzzle-pieces

Abstract hyperplane arrangements

Let $M = (S, r, \mathcal{F})$ be a loopless matroid of rank k, $\mathcal{F}$ the geometric lattice of M. For any flat $F \in \mathcal{F}$, M/F is a loopless matroid by (M3). Consider the set of loopless matroids $\mathcal{H}(M) = \{M/F \mid F \in \mathcal{F}\}$ and define a partial order on it: for any two flats $F \subsetneq J \in \mathcal{F}$, $M/J \prec M/F$. Define the dimension of M/F in $\mathcal{H}(M)$ to be $\dim_{\mathcal{H}(M)}(M/F) = k - 1 - r(F)$. Now, assign to $i \in S$ $\psi(i) := M/\overline{\{i\}}$. We call $\left(\mathcal{H}(M), (\psi(i))_{i \in S}\right)$ an *abstract hyperplane arrangement*. In other words, $\psi(i)$, $i \in S$, are abstract hyperplanes, and flats of M other than S give local incidence relations. Since there corresponds a loopless matroid to every hyperplane arrangement, the abstract hyperplane arrangement of a loopless representable matroid M can be thought of as the type of the hyperplane arrangements; see Remark 3.2.

Remark 4.1. For a loopless matroid M of rank 3, any two distinct lines, say $\psi(1)$ and $\psi(2)$, meet at a point. Indeed, $\psi(1) = M/\overline{\{1\}} \neq M/\overline{\{2\}} = \psi(2)$

implies $\overline{\{1\}} \cap \overline{\{2\}} = \emptyset$. Write $F_1 := \overline{\{1\}}$, $F_2 := \overline{\{2\}}$, and $F_3 := \overline{F_1 \cup F_2}$. Then, F_3 is a unique rank 2 flat that contains F_1 and F_2 by (M9). Hence, $\psi(1)$ and $\psi(2)$ meet at a point.

Puzzle-pieces

Let $M = (S, r)$ be an inseparable matroid of rank k. For any face Q of BP_M that is not contained in $\cup_{i=1}^n \{x_i = 0\}$, there exists a sequence of faces $Q_1 \succ \cdots \succ Q_c$ that are also not contained in $\cup_{i=1}^n \{x_i = 0\}$, where Q_j is a facet of Q_{j-1}, $c = n - 1 - \dim Q$, $Q_c = Q$ and $\dim Q_j = n - 1 - j$ for $j = 1, ..., c$. Let $F_1 := S_M(Q_1)$ and $M_0 = M_{0,1} := M$. By Theorem 1.9, as in Lemma 2.1, there exists a sequence of matroids $M_1, ..., M_c$ that correspond to $Q_1, ..., Q_c$, respectively, and a sequence of subsets $F_1, ..., F_c$ of S such that:

1. $M_j = M_{j,1} \oplus \cdots \oplus M_{j,j+1}$ where $M_{j,l}$, $l = 1, ..., j+1$, are inseparable matroids.

2. Each F_j is a non-degenerate flat of $M_{j-1,l}$ for some l.

3. M_j is obtained by replacing $M_{j-1,l}$ with $M_{j-1,l}|_{F_j} \oplus M_{j-1,l}/F_j$ in the direct sum decomposition of M_{j-1}.

The set $\mathcal{P}(M)$ of such matroids is called a *puzzle-piece*.

We define a puzzle-piece for a separable loopless matroid M as follows. M can be written as a direct sum of inseparable matroids by (S7), and $\mathcal{P}(M)$ is defined to be the set of the direct sum of elements of $\mathcal{P}(L)$ where L are summands of the given direct sum decomposition of M. We say that two puzzle-pieces $\mathcal{P}(M)$ and $\mathcal{P}(M')$ are *isomorphic* as puzzle-pieces if $\mathcal{P}(M) \cong \mathcal{P}(M') \oplus N$ or $\mathcal{P}(M') \cong \mathcal{P}(M) \oplus N$ for some matroid N. $\mathcal{P}(M) \cong \mathcal{P}(M') \oplus N$ means that for every matroid $L \in \mathcal{P}(M)$ appears $L' \oplus N$ for a matroid $L' \in \mathcal{P}(M')$ and vice versa. We say that $\mathcal{P}(M')$ is a *sub-puzzle-piece* of $\mathcal{P}(M)$ if there is an element N of $\mathcal{P}(M)$ that is isomorphic to M' as puzzle-pieces. We call a sub-puzzle-piece of $\mathcal{P}(M)$ a *face* of $\mathcal{P}(M)$. We call sub-puzzle-pieces of M with dimension d *strata of M with dimension d.*

We mean by $S_{\mathcal{P}(M)}(\mathcal{P}(M'))$ or $S_M(M')$ the ordered sequence of flats $(F_1, ..., F_c)$. If $S_M(M')$ is a singleton, we write it without parentheses. In this notation we may replace the matroid, puzzle-piece, or non-degenerate flat as before with each other as long as it makes consistent sense.

Let $\mathcal{Q}(M)$ be the set of BP_M itself and the faces of BP_M that are not contained in $\cup_{i=1}^{n}\{x_i = 0\}$. Let $\mathcal{P}_+(M)$ be the set of faces of $\mathcal{P}(M)$, i.e., $\mathcal{P}_+(M) = \mathcal{P}(M) \setminus \{M\}$, and $\mathcal{Q}_+(M) = \mathcal{Q}(M) \setminus \{\mathrm{BP}_M\}$ which is the set of the faces of BP_M that are not contained in $\cup_{i=1}^{n}\{x_i = 0\}$. There is a 1-1 correspondence between $\mathcal{P}_+(M)$ and $\mathcal{Q}_+(M)$ by Theorem 1.9. Define a partial order on $\mathcal{P}_+(M)$ such that for $L_1, L_2 \in \mathcal{P}_+(M)$, $L_1 \prec L_2$ if $Q_1 \prec Q_2$ where $Q_1, Q_2 \in \mathcal{Q}_+(M)$ are the faces of BP_M that correspond to L_1, L_2, respectively. We can give a geometric structure of a puzzle-piece as follows:

1. For any inseparable matroid N of rank s, define the dimension of N to be $\dim N := s - 1$.

2. For $L = N_1 \oplus \cdots \oplus N_{\kappa(L)} \in \mathcal{P}(M)$ where N_i are inseparable matroids, define $\dim L := \dim N_1 + \cdots + \dim N_t = k - \kappa(L)$.

Define $\dim \mathcal{P}(M) := \dim M$. A 0-dimensional puzzle-piece is called a *point*, and 1-dimensional puzzle-piece is called a *line*. Observe that

$$\dim L + \mathrm{codim}_{\mathrm{BP}_M} Q = k - 1 \quad \text{or} \quad \dim Q - \dim L = n - k$$

where $Q \in \mathcal{Q}_+(M)$ is the corresponding facet of $L \in \mathcal{P}_+(M)$.

Fix $\mathbb{F} = \mathbb{C}$. Consider a hyperplane arrangement $(\mathbb{P}V, (B_1, ..., B_n))$ and its associated matroid M of rank k. Suppose that M is inseparable. B_i, $i = 1, ..., n$, are thought of as the intersections of $\mathbb{P}V \subset \mathbb{P}^{n-1}$ with $\{x_i = 0\}$, where x_i are the standard coordinate functions of $\mathbb{P}^{n-1}$. Since the torus $T = (\mathbb{C}^\times)^n / \mathrm{diag}\mathbb{C}^\times$ acts on $\mathbb{P}^{n-1}$, T also acts on the grassmanian $\mathrm{G}(k, n)$.

For $[\mathbb{P}V] \in \mathrm{G}(r,n)$, let $Y := \overline{T.[\mathbb{P}V]}$ be the closure of its orbit, U the universal family over $\mathrm{G}(r,n)$ whose fibers are isomorphic to $\mathbb{P}^{k-1}$. Consider the fiber product $U_Y := U \times_{\mathrm{G}(k,n)} Y$ and the GIT quotient $U_Y//_{\mathbf{1}}T$. For the dimensions, $\dim V = k$, $\dim U_Y = n+k-2$, $\dim T = n-1$, and $\dim U_Y//_{\mathbf{1}}T = k-1$. The automorphism group $\mathrm{Aut}\left(\mathbb{P}V, (B_1, ..., B_n)\right)$ is trivial. (Recall that we assumed M is inseparable.)

Theorem 4.2 ([Ale08]). *(a) $U_Y//_{\mathbf{1}}T$ is the log canonical model of the hyperplane arrangement.*

(b) $Y \cap \mathrm{G}_e(r-1, n-1) = U_Y//_{\mathbf{1}}T$. (For the notion of $Y \cap \mathrm{G}_e(r-1,n-1)$, see the Hacking-Keel-Tevelev's paper [HKT06].)

The *strata of codimension $c > 0$ of a puzzle-piece* $\mathcal{P}(M)$ is defined to be the set of matroids $L \in \mathcal{P}_+(M)$ such that $\kappa(L) = c+1$.

Theorem 4.3 ([HKT06]). *Let $X = \cup X_i$ be a stable variety, $\Delta = \cup \mathrm{BP}_i$ the polyhedral decomposition of Δ into the base polytopes BP_i that are associated to X_i. Then the strata of $\cup X_i$ are in 1-1 correspondence with the strata of $\cup \mathrm{BP}_i \backslash \cup_{j=1}^{n} \{x_j = 0\}$.*

Combining Theorem 4.1(a) and 4.2(a), we obtain the following theorem.

Corollary 4.4. *The strata of $\mathcal{P}(M)$ are in 1-1 correspondence with the strata of $U_Y//_{\mathbf{1}}T$ which is a variety.*

Hence, if M is a loopless representable matroid over $\mathbb{C}$, the strata of the puzzle-piece $\mathcal{P}(M)$ are in one-to-one correspondence of the log canonical model $U_Y//_{\mathbf{1}}T$ of the associated hyperplane arrangement.

Puzzle-pieces when $k = 2$

Let M be an inseparable matroid of rank 2, then $\mathcal{P}(M)$ has dimension 1, hence a line. Codimension 1 strata of $\mathcal{P}(M)$ are points. Evidently, there are no codimension 2 strata of $\mathcal{P}(M)$. So, $\mathcal{P}(M)$ can be identified with the abstract hyperplane arrangement of M. M is representable over $\mathbb{C}$ by Corollary 3.8. If M is separable, $\mathcal{P}(M)$ has dimension 0, hence a point.

Puzzle-pieces when $k = 3$

Assume that M is an inseparable representable matroid of rank 3, then $\mathcal{P}(M)$ has dimension 2. Codimension 1 strata of $\mathcal{P}(M)$ are lines, and codimension 2 strata are points. If $\mathcal{P}(N)$ is a point or a line, we sometimes say simply N is a point or a line as long as the meaning is clear. We can identify the set of lines of $\mathcal{P}(M)$ with the set of non-degenerate flats of M. Indeed, take a line $\mathcal{P}(M|_J \oplus M/J)$ for a non-degenerate flat J of M. If J has rank 1, $M|_J$ is an inseparable matroid of rank 1, M/J is an inseparable matroid of rank 2. So, $\mathcal{P}(M|_J)$ is a point, and $L := \mathcal{P}(M/J)$ is a line that is isomorphic to $\mathcal{P}(M|_J \oplus M/J)$ as a puzzle-piece. Similarly, if J has rank 2, $\mathcal{P}(M/J)$ is a point and $L := \mathcal{P}(M|_J)$ is a line isomorphic to $\mathcal{P}(M|_J \oplus M/J)$ as a puzzle-piece. So, there is a bijection between the lines of $\mathcal{P}(M)$ and the non-degenerate flats of M.

Recall that $\dim L + \operatorname{codim}_{\mathrm{BP}_M} Q = k - 1$ where $Q \in \mathcal{Q}_+(M)$ is the corresponding facet of $L \subset \mathcal{P}_+(M)$. Every facet $Q \in \mathcal{Q}(M)$ corresponds to a line $\in \mathcal{P}(M)$ and every codimension 2 face $P \in \mathcal{Q}(M)$ corresponds to a point $\in \mathcal{P}(M)$, and vice versa.

Now, any line L is identified with an abstract hyperplane arrangement. We can visualize the puzzle-piece M by giving the pictures of local incidence of lines at points. Draw a solid line for $L = M/J$, a dashed line for $L = M|_J$. For two lines L_1, L_2, draw 60° for the angle between them if both are obtained from M by the same operation (restriction or contraction), draw 120° otherwise.

If M is a separable representable matroid of rank 3, $\mathcal{P}(M)$ has dimension 0 or 1; see Figure 4.1, where the first line pictures are hyperplanes in $\mathbb{P}^2$ and the second line pictures are their corresponding puzzle-pieces.

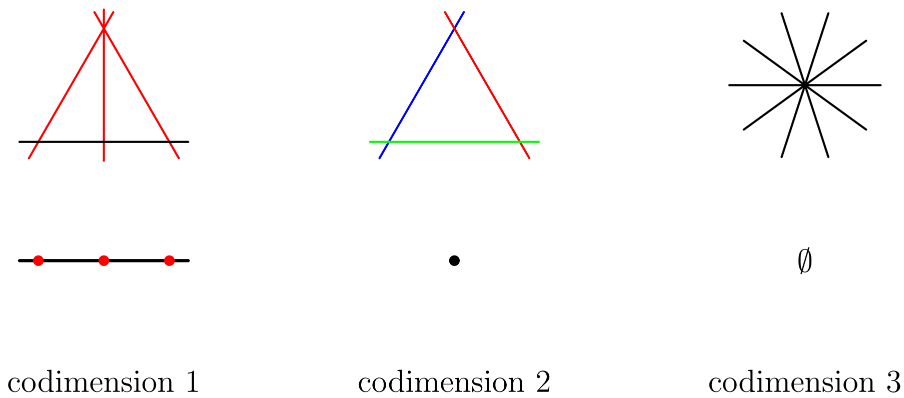


Figure 4.1:

Remark 4.5. Since codimension 2 face of a base polytope is the intersection of exactly two facets, any point of a 2-dimensional puzzle-piece is the intersection of exactly two lines. Then, any two distinct lines of $\mathcal{P}(M)$ can be represented as two sides of the connected part of the boundary of one of the polygons in Figure 4.2, up to symmetry, where each side of the connected part represents a line in $\mathcal{P}(M)$ and the vertices on it represent points in $\mathcal{P}(M)$.

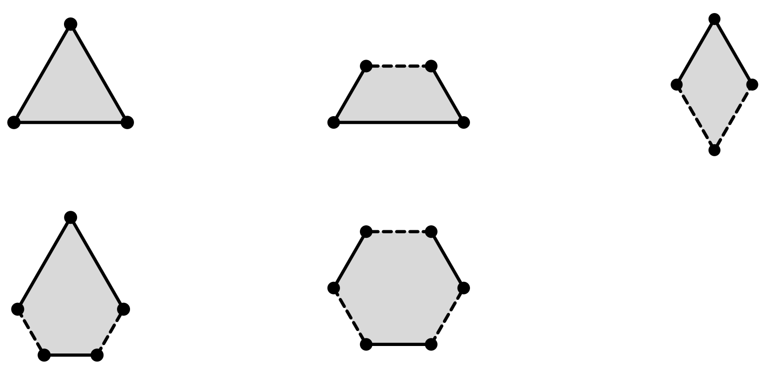


Figure 4.2:

Indeed, let L_1 and L_2 be any two distinct lines of $\mathcal{P}(M)$. For each $i = 1, 2$, its non-degenerate flat F_i has rank 1 or 2. If $r(F_i) = 1$, L_i is isomorphic to

$\mathcal{P}(M/F_i)$ as puzzle-pieces. If $r(F_i) = 2$, L_i is isomorphic to $\mathcal{P}(M|_{F_i})$.

Assume that there is no degenerate flat of rank 1.

1. Suppose that $r(F_1) = r(F_2) = 1$, then L_1 and L_2 are lines in $\mathcal{H}(M)$. By Remark 4.1, L_1 and L_2 meet at a point M/J of $\mathcal{H}(M)$ with $J = \overline{F_1 \cup F_2}$. If J is inseparable, by Lemma 1.4, J is non-degenerate flat. So, $L_3 := \mathcal{P}(M|_J \oplus M/J) \cong \mathcal{P}(M|_J)$ is a line in $\mathcal{P}(M)$, and by Remark 4.5, L_1 and L_2 meet L_3 once, but they do not meet each other. The line segment for L_3 is drawn to be a dashed line with length $|J| - |F_1| - |F_2|$. The line segment for L_i, $i = 1, 2$, is drawn to be a solid line such that the angle between L_i and L_3 is $120°$. For the length we need to choose a point on it, equivalently a non-degenerate flat T_i of M/F_i. Its length is the number of indices of the points on L_i except two vertices, i.e., $|(F_i)^c| - |J - F_i| - |T_i| = n - |J| - |T_i|$. If J is separable, L_1 and L_2 are two lines of $\mathcal{P}(M)$ that meet at the point $\mathcal{P}\left(M|_{F_1} \oplus M|_{F_2} \oplus M|_{(F_1 \cup F_2)^c}\right)$. The line segments are drawn in the same way. By Remark 4.5, we can keep this process at the vertices as long as the lines are forming a connected part of the boundary of a polygon.

2. Suppose $r(F_1) = r(F_2) = 2$, then $\mathcal{P}(M/F_i)$, $i = 1, 2$, are two distinct points on $\mathcal{H}(M)$. If those two points are connected on $\mathcal{H}(M)$, call it L'_0. Choose a line $L'_i = M/J_i \neq L_0$ on $\mathcal{H}(M)$ that passes through the point M/F_i, where J_i are flats. L'_1 and L'_2 meet at a point on $\mathcal{H}(M)$, and if there corresponds a non-degenerate flat to this point, there corresponds a line L'_3 of $\mathcal{P}(M)$. Consider those lines of $\mathcal{H}(M)$ lying on $\mathcal{P}(M)$. Draw the lines $L_1, L_2, L'_1, L'_2, L'_0, L'_3$ of $\mathcal{P}(M)$ following the same directions as above.

3. Suppose that $r(F_1) = 1$, $r(F_2) = 2$ without loss of generality. Choose a line $L'_3 \neq L_1$ on $\mathcal{H}(M)$. Do the same thing as above two cases.

Assume that there is a degenerate flat F of rank 1. By Lemma 3.11, F is only one such flat.

4. Suppose $r(F_1) = r(F_2) = 2$, then F_1 and F_2 are only two non-degenerate flats of rank 2 by Lemma 3.11. Choose a line $L'_i = M/J_i$ on $\mathcal{H}(M)$ such that $J_i \neq F$ are non-degenerate flats of rank 1. Draw those lines L_1, L_2, L'_1, L'_2 of $\mathcal{P}(M)$ in the same way as before, which will form a rhombus as in the third panel of Figure 4.2.

For the other cases, the process is essentially the same as before, so we omit those. Figure 4.3 gives pictorial examples of Remark 4.5.

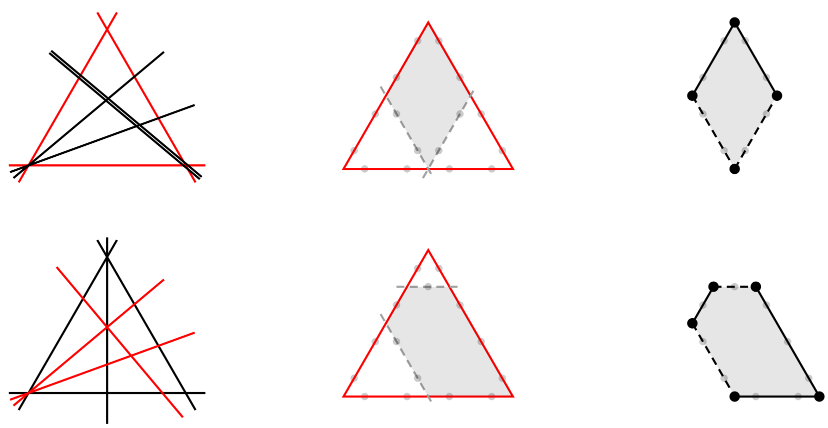


Figure 4.3:

The polygons in Figure 4.2 can be drawn in a triangular guide grid as in Figure 4.4, where the unit length of line segment is assumed to be 1. Figure 4.5 gives the classification of polygons that appear in a grid, up to symmetry. For fixed n, the boundary of a guide grid is a regular triangle with side length $n-3$. Figure 4.6 is a guide grid when $n = 6$. This triangular guide grid with boundary is coordinatized by the following discrete set up to permutation group S_3:

$$\left\{(x, y, z) \in \mathbb{Z}^3 \,|\, x, y, z \geq 1 \text{ and } x + y + z = n\right\}$$

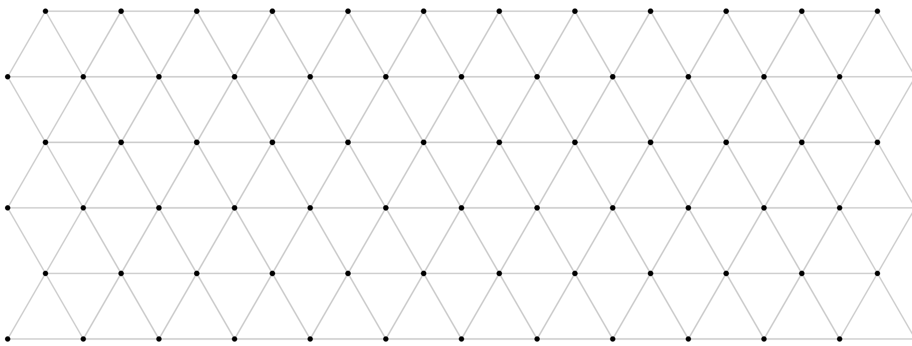


Figure 4.4:

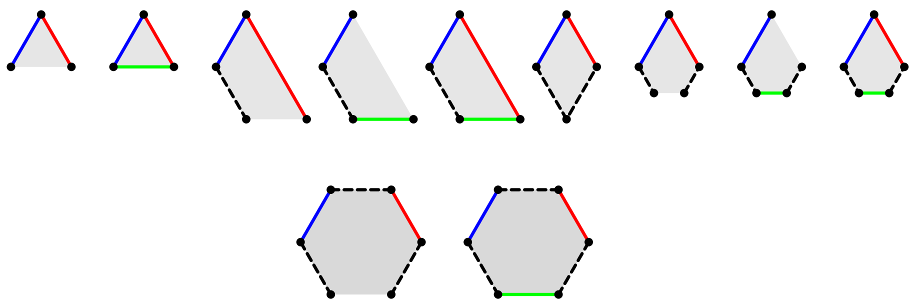


Figure 4.5:

Then, a center $P := \mathcal{P}(M_A \oplus M_B \oplus M_C)$ can take a point $(|A|, |B|, |C|)$ in a guide grid up to S_3, where $M_D \cong U^1_{|D|}$ denotes a loopless matroid of rank 1 with ground set D. Figure 4.7 explains how much information we have when moving from one point in a grid to another point: if two centers $P_1 := \mathcal{P}(M_{A_1} \oplus M_{B_1} \oplus M_{C_1})$ and $P_2 := \mathcal{P}(M_{A_2} \oplus M_{B_2} \oplus M_{C_2})$ take points in a grid that lie on the same line, exactly one of $|A_1| = |A_2|$, $|B_1| = |B_2|$, $|C_1| = |C_2|$ is true, say without loss of generality $|A_1| = |A_2|$, $|B_1| \neq |B_2|$, $|C_1| \neq |C_2|$. In addtion, if P_1 and P_2 are connected by a line, i.e., by a 1-dimensional puzzle-piece, then their points in a grid are connected by a line segment, and exactly one of $A_1 = A_2$, $B_1 = B_2$, $C_1 = C_2$ is true, and the other two equalites change to strict containments in opposite way, say without loss of generality $A_1 = A_2$, $B_1 \supsetneq B_2$, and $C_1 \subsetneq C_2$.

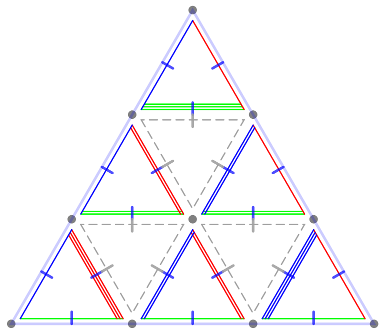


Figure 4.6:

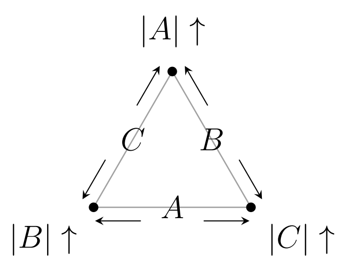


Figure 4.7:

Log canonical model of a hyperplane arrangement on $\mathbb{P}^2$

The following two theorems are due to Alexeev; see [Ale13] Theorem 5.7.2.

Theorem 4.6. *Fix $k = 3$ and $\mathbb{F} = \mathbb{C}$. Consider a hyperplane arrangement and its log canonical model $U_Y//_{\mathbf{1}}T$ (by Theorem 4.2(a)). Then, $U_Y//_{\mathbf{1}}T$ is obtained by successive blowups of $\mathbb{P}^2$ at a certain number of points and at most one contraction of a curve.*

Corollary 4.7. *The log canonical model of any hyperplane arrangement on $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_{\mathrm{pts}}\mathbb{P}^2$.*

Then, puzzle-pieces when $k = 3$ work as *type* for log canonical models of hyperplane arrangements on $\mathbb{P}^2$.

4.2 Gluing puzzle-pieces

Since there is a 1-1 correspondence between $\mathcal{P}(M)$ and $\mathcal{Q}(M)$, we can define the *gluing of puzzle-pieces* that have the same rank and the same ground set as the counterpart of gluing of base polytopes. If two base polytopes BP_{M_1} and BP_{M_2} glue to $\mathrm{BP}_{M_1 \# M_2}$ through the common facet $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \in \mathcal{Q}_+(M_1) \cap \mathcal{Q}_+(M_2)$, we say that two puzzle-pieces $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ *glue* to $\mathcal{P}(M_1 \# M_2)$ through the corresponding facet $\mathcal{P}_+(M_1) \cap \mathcal{P}_+(M_2)$. Gluing of puzzle-pieces is just combinatorial translation of the topological gluing of base polytopes. For $k = 2, 3$ cases, this translation is very useful since one can draw 1- or 2-dimensional local pictures for the gluing.

1-dimensional puzzle-pieces

For $k = 2$ case, the gluing is extremely simple, since all strata are points. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be rank 2 inseparable matroids, J_1 and J_2 be nontrivial flats of M_1 and M_2, respectively, then J_i, $i = 1, 2$, are non-degenerate flats of M_i by Lemma 1.3. If $J_1 \cup J_2$ is a partition of S, one has $M_1|_{J_1} = M_2/J_2$ since both are loopless matroids of rank 1 with ground set J_1 by (M3), hence isomorphic to $U^1_{|J_1|}$ by (M5). Similarly, one has $M_1/J_1 = M_2|_{J_2}$, so BP_{M_1} and BP_{M_2} glue to a base polytope by Theorem 2.5.

2-dimensional puzzle-pieces

For $k = 3$ case, suppose that $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ are inseparable flats of rank 3. We say that two full dimensional puzzle-pieces $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ *face-fit* or simply *fit* if their corresponding base polytopes BP_{M_1} and BP_{M_2} meet nicely, i.e., $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \in \mathcal{Q}_+(M_1) \cap \mathcal{Q}_+(M_2)$ or $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \subset \cup_{i=1}^{n} \{x_i = 0\}$, which counts the case $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} = \emptyset$. We say BP_{M_1} and

BP_{M_2} *fit in* Δ_+ if $\emptyset \neq \mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \in \mathcal{Q}_+(M_1) \cap \mathcal{Q}_+(M_2)$. Note the following.

- BP_{M_1} and BP_{M_2} fit in Δ_+ and $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2}$ has codimension $c \leq k-1$ if and only if $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ fit and $\mathcal{P}(M_1) \cap \mathcal{P}(M_2)$ has codimension $c \leq k-1$.

- If $\mathrm{BP}_{M_1} \cap \mathrm{BP}_{M_2} \subset \cup_{i=1}^{n} \{x_i = 0\}$, then $\mathcal{P}(M_1) \cap \mathcal{P}(M_2)$ has codimension k, i.e., $\mathcal{P}(M_1) \cap \mathcal{P}(M_2) = \emptyset$.

If two puzzle-pieces $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ fit and $\mathcal{P}(M_1) \cap \mathcal{P}(M_2)$ has codimension $c \leq k-1$, those can be depicted in a grid as polygons such that:

(G1) The polygons of two puzzle-pieces share a line segment and lie on the different sides if and only if those puzzle-pieces fit through the common facet, which is a common line in both puzzle-pieces.

(G2) The polygons of two puzzle-pieces share only a point and do not overlap except the point if and only if those puzzle-pieces fit through the common face with codimension 2, which is a common point in both puzzle-pieces.

Then, Figure 2.1 and 2.2 work as local pictures of two face-fitting puzzle-pieces whose intersection is not empty. So, we take Figure 2.3 as the classification of the local pictures at a center Z of the face-fitting puzzle-pieces whose intersection is Z. For an example, see Figure 4.8. $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ have full dimension since each associated hyperplane arrangement has 4 lines in general linear position. Figure 4.9 illustrate two face-fitting puzzle-pieces $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ in a grid (for $n = 8$).

Now, if BP_{M_1} and BP_{M_2} glue to a base polytope $\mathrm{BP}_{M_1 \# M_2}$, the puzzle-pieces $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ glue to a puzzle-piece $\mathcal{P}(M_1 \# M_2)$. Hence, using gluing of puzzle-pieces, we can construct a new puzzle-piece.

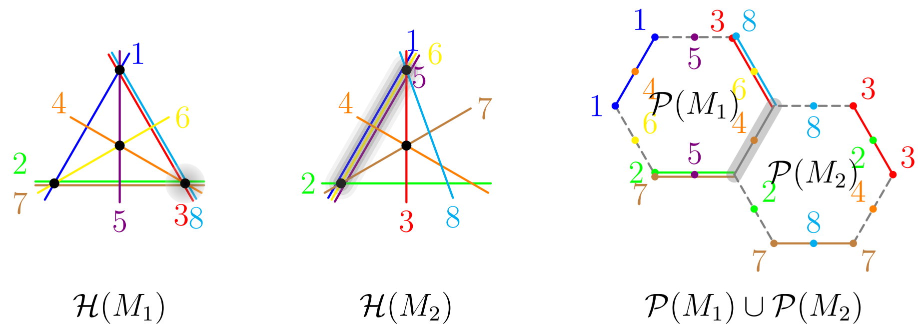


Figure 4.8:

Construction of puzzle-pieces when $k = 3$

Consider the hyperplane arrangements in Figure 3.5. For cases (a) and (b), let Z be the 0-dimensional puzzle-piece that is the intersection of two lines $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$. For case (c), note that $M|_{A\cup B\cup D}$ is a rank 2 inseparable matroid, say M_3. Recall that M_1, M_3 determines a hyperplane arrangement as in the second paragraph of 3.2.(b). Let Z be a point that is the intersection of two lines $\mathcal{P}(M_1)$ and $\mathcal{P}(M_3)$. Then, Figure 4.10 shows the local pictures of the puzzle-pieces at Z obtained from those hyperplane arrangements in Figure 3.5. For the explicit pictures, the puzzle-pieces of (a) look like those in Figure 4.11. If a puzzle-piece is obtained in the way of (c) such that M_2 is inseparable, it looks like one of Figure 4.12. Otherwise, it looks like ⑤ of Figure 4.11 but the opposite vertex, with two dashed lines.

If two puzzle-pieces glue in such a way as their base polytopes glue to another base polytope, their union becomes another puzzle-piece.

1. For consider an inseparable matroid N_1 as in Figure 3.5(a) that is constructed by N_1/A and N_1/B_1 where A and B_1 are non-degenerate flats of N_1 with rank 1 such that the flats of N_1/A are flats of N_1. Consider another inseparable matroid N_2 as in Figure 3.5(b) that is constructed by $N_2|_{A^c}$ and $N_2|_{B_2^c}$, where A^c and B_2^c are rank 2 non-

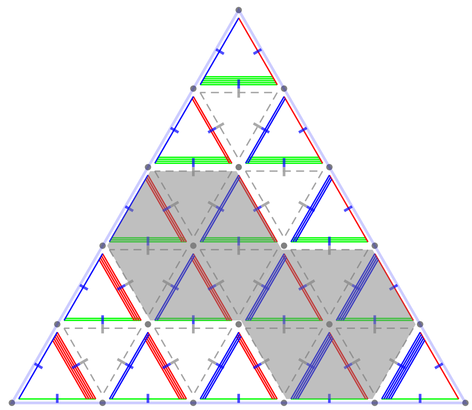


Figure 4.9:

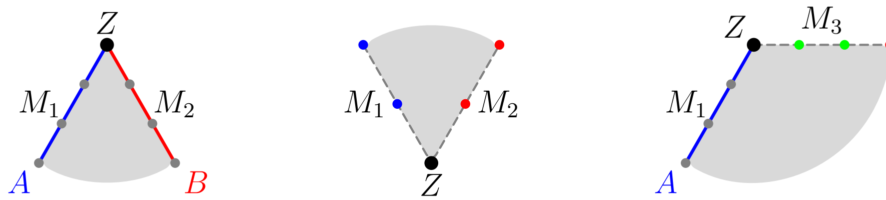


Figure 4.10:

degenerate flats of N_2. Then, the flats of $N_2|_{A^c}$ are flats of N_2. Suppose that $N_1/A = N_2|_{A^c}$. Then, by Lemma 3.15, N_1 and N_2 glue to $N_1 \# N_2$, hence we obtain a new puzzle-piece $\mathcal{P}(N_1 \# N_2)$. Moreover, N_1 and N_2 satisfies the premises of Theorem 3.17, hence $N_1 \# N_2$ comes from a hyperplane arrangement. For the local pictures, see Figure 4.13.

2. Suppose that N_2 is given the same, but N_1 is given differently: N_1 is constructed by N_1/A and $N_1|_{B^c}$ as in 3.2.(c) where A, B^c are non-degenerate flats of N_1 with rank 1,2, respectively such that the flats of N_1/A are flats of N_1. Suppose that $N_1/A = N_2|_{A^c}$. Then, N_1

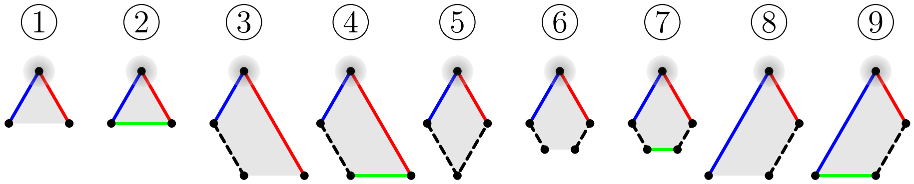


Figure 4.11:

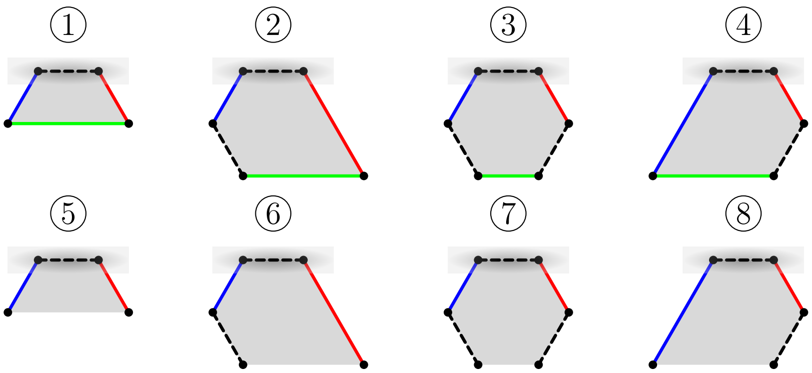


Figure 4.12:

and N_2 glue to $N_1 \# N_2$ by Lemma 3.15. $N_1 \# N_2$ is also representable by Theorem 3.17. The difference with above case is that the flat F mentioned in Theorem 3.17 is allowed. The pictures for the polygons of $\mathcal{P}(N_1)$ that contain the line $N_1|_F$ are given in Figure 4.14. The pictures of the polygons that do not contain $N_1|_F$ are already given in Figure 4.13.

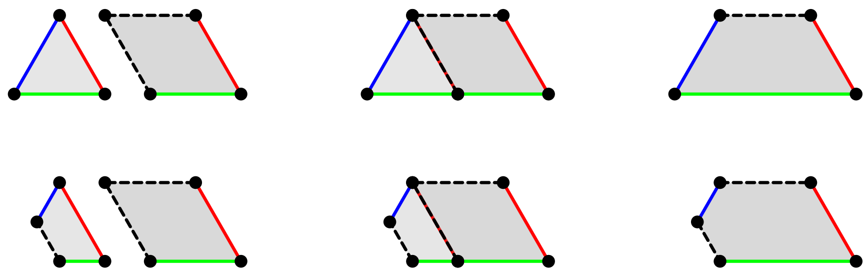


Figure 4.13:

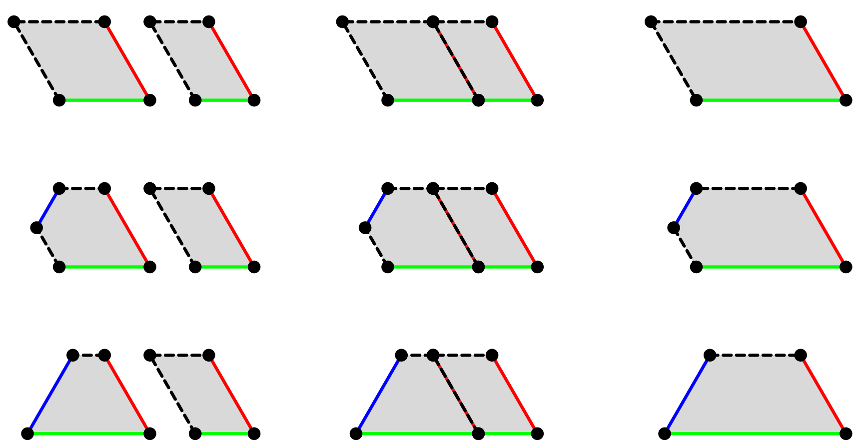


Figure 4.14:

Chapter 5

Flakes, puzzles, quilts and β-puzzles

We fix $k = 3$ throughout this chapter.

5.1 Flakes, puzzles and quilts

Definition 5.1. A *flake centered at a point* $Z = \mathcal{P}(M_Z)$ is a collection of full dimensional puzzle-pieces $X_i = \mathcal{P}(M_i)$ with $\cap X_i = Z$ such that BP_{M_i} fit in Δ_+. We say that the center $Z = \cap X_i$ is an *interior center* if BP_{M_Z} is not contained in the boundary of Δ_n^k. In other words, $M_Z \cong U^1_{|J_1|} \oplus U^1_{|J_2|} \oplus U^1_{|J_3|}$ with a partition of $S = \cup_{i=1}^3 J_i$ such that $|J_i| > 1$. We say that two distinct flakes X and X' are *compatible* if for any point Z of X or X', the collection of puzzle-pieces of X and X' that contains Z is again a flake.

Recall that Figure 2.3 classifies the local pictures of a flake at the center, up to symmetry. There are two generalizations of the notion of flake: puzzles and quilts. A *tiling* or *complete cover* of Δ is a face-fitting subdivision of Δ into base polytopes. A *partial tiling* is a union of face-fitting base polytopes that are contained in Δ.

Definition 5.2. A *puzzle* is a collection of full dimensional puzzle-pieces $X_i = \mathcal{P}(M_i)$ such that $\cup \mathrm{BP}_{X_i} \backslash \cup_{j=1}^{n} \{x_j = 0\}$ is a partial tiling of Δ_+ that is connected in Δ_+.

If $X = \{X_i \,|\, i \in \Omega\}$ and $X' = \{X'_i \,|\, i \in \Omega'\}$ are two puzzles such that $\{\mathrm{BP}_{X_i} \,|\, i \in \Omega\}$ refines $\left\{\mathrm{BP}_{X'_i} \,|\, i \in \Omega'\right\}$, we say that X is a *refinement* of X' or X is a *decomposition* into puzzle-pieces that is *finer* than X'.

The notion of quilt is weaker than that of puzzle.

Definition 5.3. A *quilt* is a collection of full dimensional puzzle-pieces $X_i = \mathcal{P}(M_i)$ such that for any point Z of X_i for any i, the collection of those puzzle-pieces that contain Z, which is denoted by $\mathcal{F}_X(Z)$, is a flake centered at Z. In other words, a *quilt* is a collection of compatible flakes. A quilt X' is called a *sub-quilt* of a quilt X if $X' \subset X$.

Remark. A flake is a puzzle, and a puzzle is a quilt.

For a quilt X, we define a *local chart* at a center Z to be a grid such that

(i) $\mathcal{F}_X(Z)$ is depicted as a collection of polygons in the grid,

(ii) each point in a grid occupied by at most one center of the quilt, and

(iii) those points occupied by centers in the grid are connected by line segments.

Remark. The local chart is for local computations, not global ones. Nevertheless, the guide grid can be used to track puzzle-pieces that are connected, but we do not require any point in it to be occupied by at most one center.

Let $Y = \mathcal{P}(M_Y)$ be a 1-dimensional sub-puzzle-piece with an inseparable matroid $M_Y = (S_Y, r_Y)$ with rank 2. Y is called *open* in a quilt $X = \{X_i \,|\, i \in \Omega\}$ if Y is a line of some puzzle-piece X_i such that $|S_Y| < n-1$ and Y is not an intersection of two distinct 2-dimensional puzzle-pieces of X. In other words, $\mathrm{BP}_{M_Y} \times \mathrm{BP}_{U^1_{n-|S_Y|}}$ is not contained in $\cup_{i=1}^{n} \{x_i = 1\}$ and is not a common facet of two distinct full dimensional base polytopes BP_1 and BP_2, where BP_1 and BP_2, respectively, are base polytopes that correspond to some puzzle-pieces X_{i_1} and X_{i_2} of X.

If Y is not open, we say Y is *saturated* or *closed* in X. Y is called a *boundary puzzle-piece of* X if either Y is open in X or $|S_Y| = n-1$, i.e., $\mathrm{BP}_{M_Y} \times \mathrm{BP}_{U^1_{n-|S_Y|}}$ is contained in $\cup_{i=1}^{n} \{x_i = 1\}$.

We say that a quilt is *complete* if it has no open puzzle-pieces. We say that a flake X with center Z is *saturated at* Z if X has no open puzzle-pieces containing Z.

For every center Z of a quilt X, $\mathcal{F}_X(Z)$ can be expressed in a local chart. The family of such local charts not only visualizes the gluing of puzzle-pieces of X, but also describes X itself.

Lemma 5.4. *Any flake X with center Z can be saturated at Z.*

Proof. It suffices to consider Figure 2.2 for X. For the first panel of the first line pictures, let $X = \{\mathcal{P}(M_1), \mathcal{P}(M_2)\}$ as seen in Figure 5.1. Let $\mathcal{P}(N_1)$ be the line represented by a red line segment and $\mathcal{P}(N_2)$ the line represented by a blue line segment. With N_1 and N_2, we can construct a hyperplane arrangement (b) in Figure 3.5, hence a puzzle-piece $\mathcal{P}(M_3)$ of type (b) in Figure 4.10 fits to X through both $\mathcal{P}(N_1)$ and $\mathcal{P}(N_2)$ by (G1) and (G2). Now, let $\mathcal{P}(N_3)$ be the line represented by the dashed line segment with green points and $\mathcal{P}(N_4)$ the line represented by the green line segment. With N_3 and N_4, we can construct a hyperplane arrangement as in (c) of Figure 3.5 such that its associated puzzle-piece $\mathcal{P}(M_4)$ looks like (c) in Figure 4.10. $\mathcal{P}(M_4)$ fits through $\mathcal{P}(N_3)$ and $\mathcal{P}(N_4)$, and we obtain a new flake that has no open puzzle-pieces containing Z, i.e., X is saturated at Z; see Figure 5.1. X also can be saturated as the second line of pictures of Figure 5.1. The other cases are all similar.

□

Let X be a quilt, Z one of its centers. $\mathcal{F}_X(Z)$ is depicted in a local chart.

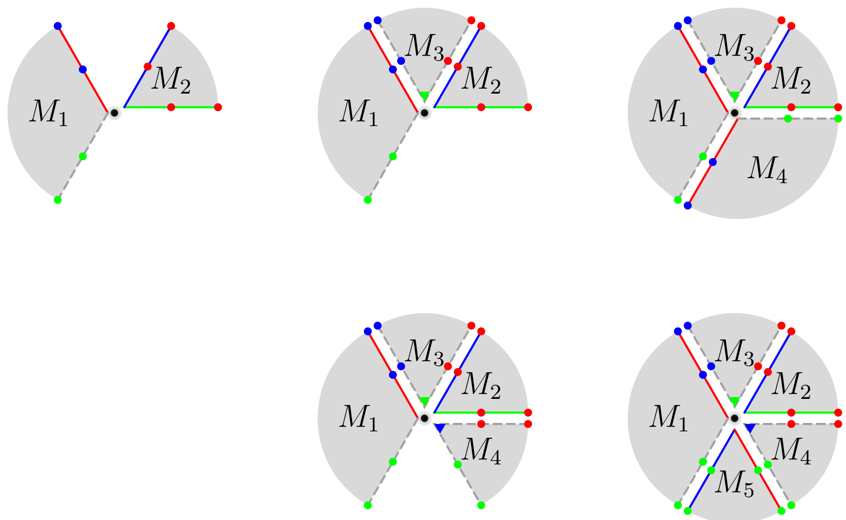


Figure 5.1:

Extend this picture as far as possible in the same local chart such that no two distinct centers occupy the same point on the chart. Then, the collection of those puzzle-pieces that are drawn in the chart that way is a quilt and a subquilt of X. This extension need not be unique, since the local chart is only for local computations.

Definition 5.5. A quilt X is called *planar* if the puzzle-pieces of X can be expressed as polygons in *one* grid according to (G1) and (G2). This grid works as a local chart for each point in the grid. We define a $\mathrm{PlanarSupport}\,(X)$ to be the union of the polygons in the grid.

Quilts connected in codimension 1

Definition 5.6. For a quilt $X = \{X_i \,|\, i \in \Omega\}$, we say that X is *connected in codimension 1* if $\cup X_i$ is connected in codimension 1.

Let $X = \{X_i \,|\, i \in \Omega\}$ be a flake with center Z that is connected in codimension 1. Each full dimensional puzzle-piece of X is expressed in a local chart as a polygon with a vertex representing Z with two neighboring sides with angle $60°$ or $120°$. Since X_i are connected in codimension 1, their polygons are also connected in codimension 1, so we define the *angle of X at Z* to be the angular defect of a vertex corresponding to Z in a local chart (see Figure 2.3) and denote it by $\angle_X Z$. In other words, $\angle_X Z$ is defined to $360°$ minus (the sum of angles of two neighboring sides of the polygons at the vertex representing Z). $\angle_X Z$ takes its value $0°, 60°, 120°, 180°, 240°, 300°$.

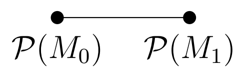
A quilt X is called *locally connected in codimension 1* if for any center Z of X, $\mathcal{F}_X(Z)$ is a flake that is connected in codimension 1.

If a quilt X is connected in codimension 1 and locally connected in codimension 1, we define *the angle of X at Z* to be the angle of the flake $\mathcal{F}_X(Z)$ at Z and denote it by $\angle_X Z$.

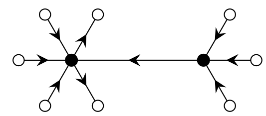
The *dual graph of a quilt X* is a graph that has a vertex corresponding to each full dimensional puzzle-piece, and an edge joining two full dimensional puzzle-pieces that fit through their common facet.

If X is connected in codimension 1, its dual graph is connected. We can add more information to the dual graph by attaching to a vertex an edge for each open sub-puzzle-piece of the corresponding full dimensional puzzle-piece, and by marking an arrow for each edge such that the arrow goes from $X_1 = \mathcal{P}(M_1)$ to $X_2 = \mathcal{P}(M_2)$, where $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ fit through J_1 and J_2 with $r_1(J_1) = 1$ and $r_2(J_2) = 2$. We call the graph obtained in this way the *extended dual graph of a quilt*. For an example, the dual graph and the extended one of the quilt $\{\mathcal{P}(M_1), \mathcal{P}(M_2)\}$ in Figure 4.8 are given in Figure 5.2.

Lemma 5.7. *Let X be a quilt that is connected in codimension 1 and locally connected in codimension 1. Then for any two distinct puzzle-pieces X_0 and X_1 of X, there is a sequence of full dimensional puzzle-pieces $X_1, X_2, ..., X_f = X_0$ of X such that $\{X_i \mid i = 1, ..., f\}$ is a quilt and its dual graph is a simple path in the dual graph of X.*



Dual graph of $\mathcal{P}(M_1) \cup \mathcal{P}(M_2)$



Extended dual graph of $\mathcal{P}(M_1) \cup \mathcal{P}(M_2)$

Figure 5.2:

Proof. Since X is connected in codimension 1, take a sub-quilt X' of X that is also connected in codimension 1 and contains X_0 and X_1 such that the number of full dimensional puzzle-pieces is the smallest. Then, the dual graph of X' is a simple path. For start with X_1. There is a center Z_1 such that $\mathcal{F}_{X'}(Z) \setminus X_1$ is not empty and connected in codimension 1 by the construction of X'. Write $\mathcal{F}_{X'}(Z) = \{X_i \,|\, i = 1, ..., m_1\}$ such that X_i and X_{i+1} fit through a line. By Figure 2.3 and the minimality of X', $\angle_{X'} Z > 0$ and $m_1 < 6$. The dual graph of $\mathcal{F}_{X'}(Z)$ is a simple path.

Now, suppose that $\mathcal{F}_{X'}(Z) = \{X_{j_i} \,|\, i = 1, ..., m\}$ with $j_1 < \cdots < j_m$ such that X_{j_i} and $X_{j_{i+1}}$ fit through a line. By the Figure 2.3 and the minimality of X', $\angle_{X'} Z > 0$ and $m < 6$. Then, no X_l with $l > j_m$ intersects $X_{l'}$ with $l' \leq j_1$. Indeed, suppose that X_l with $l > j_m$ intersects $X_{l'}$ with $l' \leq j_1$. Recall that any two distinct lines in a full dimensional puzzle-piece can be reprsented as two sides of the boundary of a polygon of Figure 4.5; see Remark 4.5. So, $X_l \cap X_{l'}$is a point or line, either way $X_l, X_{l'}$ are contained in a flake $\mathcal{F}_{X'}(Z')$ that is connected in codimension 1 for some center Z'. Then, cast away the puzzle-pieces $X_{l'+1}, ..., X_{l-1}$ and construct a new quilt connected in codimension 1 that contains X_0 and X_1 such that the number of full dimensional puzzle-pieces is smaller than that of X', a contradiction. Thus, finally we end up with a sequence of full dimensional puzzle-pieces $X_1, ..., X_f = X_0$ such that $\{X_i \,|\, i = 1, ..., f\}$ is a quilt and its dual graph is a simple path in the dual graph of X; see Figure 5.3.

□

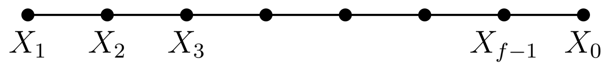


Figure 5.3:

Regular quilts

Suppose that X is a quilt that is connected in codimension 1 and locally connected in codimension 1. Then, $\mathcal{F}_X(Z)$ is connected in codimension 1 for any center Z of X. We say that X is *regular at a center* Z if $\angle_X Z \neq 60°$. For two distinct codimension 2 puzzle-pieces Z, Z', we say that X is *regular at the pair* (Z, Z') if the following properties are satisfied:

1. X is regular at both Z and Z'.

2. If Z and Z' are connected by open lines $\mathcal{P}(M_i/J)$ of X where M_i, $i \in \Lambda$ are inseparable rank 3 matroids with the same non-denerate flat J of rank 2, then either one of $\angle_X Z$, $\angle_X Z'$ is bigger than 120°.

Pictorially, X is not allowed to have the part like Figure 5.4.

Figure 5.4:

Definition 5.8. A quilt $X = \{X_i \mid i \in \Omega\}$ that is connected in codimension 1 and locally connected in codimension 1 is called a *regular quilt* if it is regular at all of its centers and at all pairs of its centers. A *regular puzzle* is a puzzle that is a regular quilt at the same time.

Let X be a regular quilt, and Y_0 one of its open lines. Since Y_0 is an open puzzle-piece of X, there is exactly one full dimensional puzzle-piece of X that contains Y_0, say $X_0 = \mathcal{P}(M_0)$.

Suppose that $S_{X_0}(Y_0)$ has rank 1 in M_0. Take a local chart for Y_0, where one needs to choose 2 distinct centers Z_1, Z_2 of Y_0. Take any full dimensional puzzle-piece X_a of X. Then, as in Lemma 5.7, there is a shortest path in the dual graph of X, say $\{X_0, X_1, ..., X_{m-1}, X_m = X_a\}$ with $X_i = \mathcal{P}(M_i)$, $i = 0, 1, ..., m$, where X_{i+1} is the immediate successor of X_i such that X_1 contains one of Z_1, Z_2, and two puzzle-pieces X_i, X_{i+1} fit through their common facet $X_i \cap X_{i+1}$. Suppose that X_1 contains Z_1, in which case X_1 does not contain Z_2. Then, in the given grid, the polygons of X_i will be depicted according to (G1) and (G2), and we see that there is an area of the given grid such that no puzzle-piece X_i has its polygon that intersecting inside of the area. We call this area the *safe zone* for Y_0 with Z_1, Z_2, which is depicted in the first panel of Figure 5.5.

Suppose that $S_{X_0}(Y_0)$ has rank 2 in M_0, and $\mathcal{P}(M_1/J_1 \oplus M_1|_{J_1})$ is an open puzzle-piece where J_1 is a rank 1 non-degenerate flat of M_1 such that $\{X_0, X_1\}$ is a quilt with angle 120°. The safe zone for Y_0 is either the second panel picture or the third panel picture.

We list below several conectures on the regular quilts with the sketch of possible proofs.

Conjecture 5.9. *Every planar regular quilt is a puzzle.*

Proof. (Sketch of a possible proof) Let $X = \{X_i \mid i \in \Omega\}$ be a planar regular quilt. Since X is planar, there is one grid that contains PlanarSupport(X); see Definition 5.5. Sometimes in the grid the boundary of a polygon is broken. But, such broken part happens only in the boundary of PlanarSupport(X) because X is planar. Cut off the safe zones along the line segments in the boundary of PlanarSupport(X) which represent boundary puzzle-pieces of X. Since X is regular, observe that a safe zone of the second or the third panel of Figure 5.5 is removed when the safe zones of first panel are cut off from the grid. So, cutting off the safe zones of first panel in Figure 5.5 is enough. Except the broken part of the boundary of PlanarSupport(X), the shape of PlanarSupport(X) is obtained by cutting off all the safe zones along

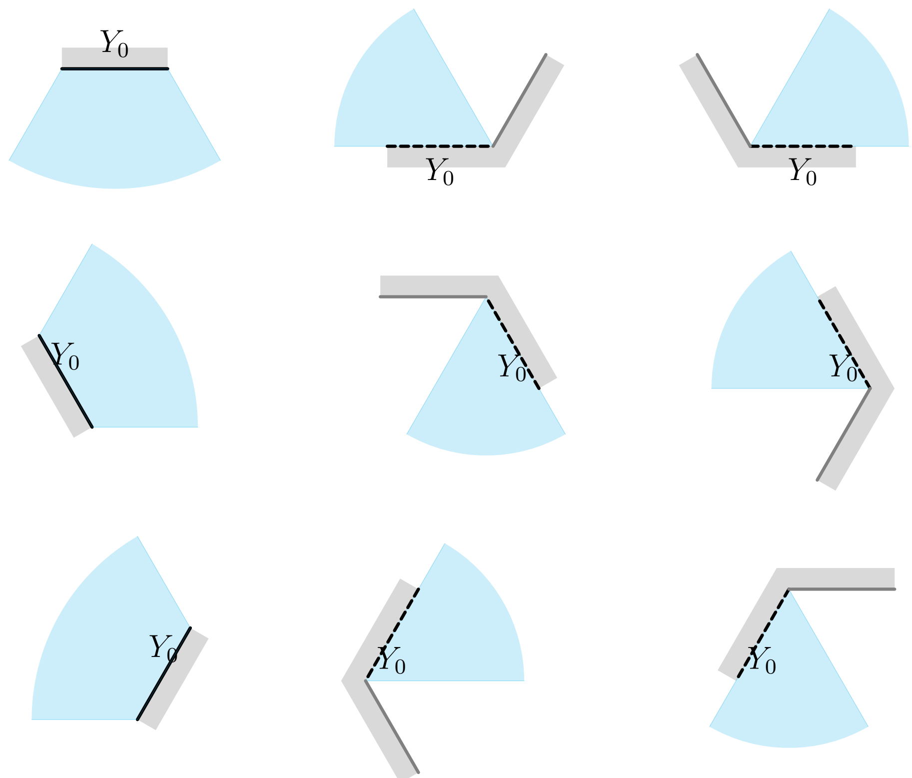


Figure 5.5:

the boundary of PlanarSupport (X), which is regular; see Figure 5.6 for an example where a black shaded echelon represents a removed safe zone.

Let X_1 and X_2 be two distinct full dimensional puzzle-pieces of X. We need to show that X_1 and X_2 fit, i.e., BP_{X_1} and BP_{X_2} meet nicely. If X_1 and X_2 are contained in a flake at the same time, then BP_{X_1} and BP_{X_2} meet nicely by definition of a flake. So, suppose not. Then, because the grid has triangular shape, there exist two distinct parallel separating lines in a grid such that the polygons of X_1 and X_2 do not intersect the middle area that those two parallel lines make. Without loss of generality, assume that those two lines in the grid are $x(B) = 1$ and $x(B') = 1$ with $|B_1| > |B_2|$ and let Z and Z', respectively be two centers of X_1 and X_2 whose points

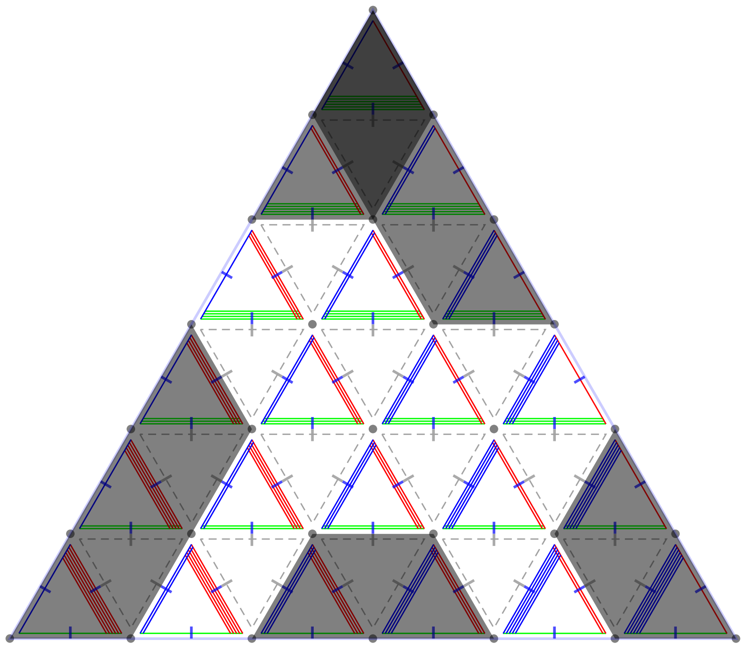


Figure 5.6:

in the grid are one the lines $x(B_1) = 1$ and $x(B_1) = 1$ with coordinates (G, R, B) and (G', R', B') respectively; see Figure 5.7. Then, the polygons of the puzzle-pieces of X that are contained in the middle area are connected in codimension 1, and there exists a simple path consisting of line segments that connect points contained in the middle area starting from the point for Z ending at the point for Z'. In addition, such a path that does not increase back the B-coordinates can be found because the broken part of $\text{PlanarSupport}(X)$ happens only in the boundary of $\text{PlanarSupport}(X)$ not inside, and the shape of $\text{PlanarSupport}(X)$ is regular; see the paragraph of Figure 4.7 for the direction.

Let $Z = Z_0, Z_1, ..., Z_{f-1}, Z_f = Z'$ be the centers whose points are in the simple path such that the coordinates Z_i are (G_i, R_i, B_i), and for a fixed

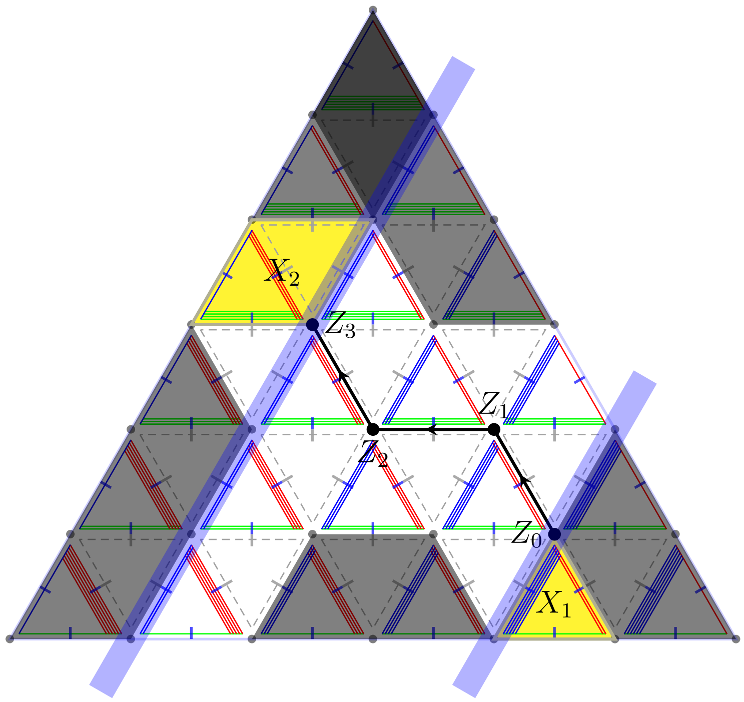


Figure 5.7:

$i < f$, Z_{i+1} is the immediate successor of Z_i. Then, we see that $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_{f-1} \supseteq B_f$ and $B = B_0 \supsetneq B_f = B'$. Moreover, we have $\mathrm{BP}_{X_1} \subset \{x(B) \leq 1\}$ and $\mathrm{BP}_{X_2} \subset \{x(S \backslash B') \leq 2\}$. Then,

$$\mathrm{BP}_{X_1} \cap \mathrm{BP}_{X_2} \subset \{x(B) \leq 1, x(S \backslash B') \leq 2\}$$

But, since $B \supsetneq B'$, one has $x(S) + x(B \backslash B') = x(B) + x(S \backslash B') \leq 1 + 2 = 3$, which means that $x(B \backslash B') \leq 0$, $x(B \backslash B') = 0$ since $x(S) = 3$. Now, $B \backslash B' \neq \emptyset$ implies that $\mathrm{BP}_{X_1} \cap \mathrm{BP}_{X_2} \subset \cup_{i=1}^{n} \{x_i = 0\}$. Hence, BP_{X_1} and BP_{X_2} meet nicely. $\square$

We will see that every regular quilt for $n \leq 7$ is a puzzle in Theorem 6.4. For $n = 8, 9$, it remains as a conjecture.

Conjecture 5.10. *Every regular quilt when $n = 8, 9$ is a puzzle.*

Proof. (Sketch of a possible proof) Assume $n = 9$. Let X be a regular quilt. Take any two distinct full dimensional puzzle-pieces X_0 and X_1 of X. Take a shortest path $X' := \{X_1, ..., X_f\}$ connecting X_1 and $X_f = X_0$. Draw first a polygon of X_0 in a grid and keep locating a polygon of each X_i in order. The size of the grid is 6 which is too small for X' to be not planar: wherever the polygon of X_0 is located, X' should have regular shape, otherwise its minimality is violated. Indeed, locate a polygon of X_0 in the leftmost corner of a grid as in Figure 5.8. Since X is a regular quilt, X_i cannot make a turn with $60°$ in view of the inner boundary of PlanarSupport$\,(X')$, but a turn with $120°$, since otherwise the minimality of X' would be violated. Once a $120°$ turn is made, by the same reason, X_i cannot make a turn even with $120°$ anymore. Hence, X' is a regular planar quilt. Then, by Conjecture 5.9, X' is a puzzle, which means that X_0 and X_1 fit. Therefore, X is a regular quilt. The cases for $n = 8$ are similar. □

Conjecture 5.11. *Every complete quilt for $n = 8, 9$ is a puzzle.*

Proof. Let X be a complete quilt. Z is an interior center of X if and only if $\angle_X Z = 0°$. Since X has no open puzzle-piece, if Z is not an interior center, then $\angle_X Z = 180°$ or $240°$. So, X is regular at every center and every pair of centers. Hence, X is a regular quilt, and a puzzle by Theorem 5.10. □

5.2 β-puzzle

Definition 5.12. We say that a partial tiling $\cup \mathrm{BP}_{M_i}$ is a *partial cover of* Δ_β if $\cup \mathrm{BP}_{M_i} \supset \Delta_\beta$ and $\mathrm{BP}_{M_i} \cap \mathrm{int}\Delta_\beta \neq \emptyset$ for all BP_{M_i}. A *β-puzzle* is a puzzle $X = \cup X_i$ that comes from a partial cover of Δ_β for some weight vector β.

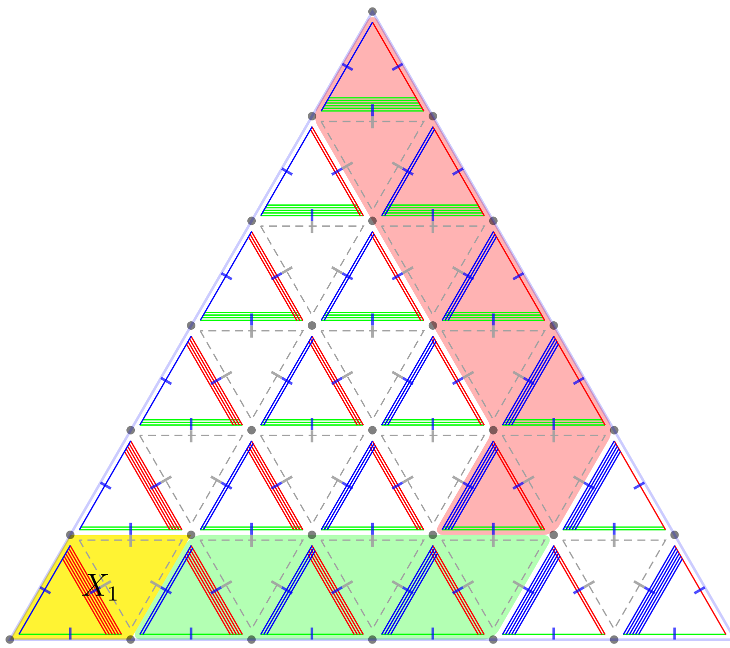


Figure 5.8:

The following theorem is a corollary to Theorem 4.3.

Theorem 5.13. *Let $\cup V_i$ be a stable variety, $\cup \mathrm{BP}_i$ the polyhedral decomposition of Δ into the base polytopes BP_i that are associated to V_i, $\cup X_i$ the corresponding β-puzzle. Then there is a 1-1 correspondence between the strata of $\cup V_i$, the strata of $\cup \mathrm{BP}_i \setminus \cup_{j=1}^{n} \{x_j = 0\}$, and the strata of $\cup X_i$.*

Lemma 5.14. *Any β-puzzle is a sub-quilt of a regular quilt.*

Proof. (Sketch of a possible proof) Let $X = \{X_i \,|\, i \in \Omega\}$ be a β-puzzle. X is connected in codimension 1 since $\cup \mathrm{BP}_{X_i}$ covers Δ^3_β which is a convex polytope. Suppose that $X_0 = \mathcal{P}(M_0)$ for $M_0 = (S, r_0)$ is a 2-dimensional puzzle-piece of X and a line $Y_0 = \mathcal{P}(M_0/A \oplus M_0|_A)$ is a open puzzle-piece of X with $r_0(A) = 1$, i.e., $S_{X_0}(Y) = A$ with $|A| \geq 2$ and $r_0(A) = 1$. We will separate two cases and prove that X is a regular quilt for each case, which is a long argument. After that, for the case $r_0(A) = 2$, we will show that X is a part of a regular quilt.

Let $Z = \mathcal{P}(M_A \oplus M_B \oplus M_C)$ be a point on Y_0 such that $S_{Y_0}(Z) = B$, $A \cup B \cup C = S$ is a partition of S, where M_D denotes a matroid $\cong U^1_{|D|}$ with ground set D. Then, $r_0(C) > 1$ since M_0 is inseparable and $r_0(A \cup B) = 2$. One has $\beta(C) > 1$, otherwise $\mathrm{BP}_{X_0} \subset \{x(C) \le \beta(C) \le 1, x(A \cup B) \le 2\}$ has codimension 1.

Moreover, not both $\beta(A) > 1$ and $\beta(B) > 1$ hold true at the same time. Suppose not: $\beta(A) - 1, \beta(B) - 1, \beta(C) - 1 > 0$. Note that there exists a point $P_0 = (p_1, ..., p_n) \in \mathrm{BP}_Z \backslash \cup_{i=1}^n \{x_i = 0\}$ that is not contained in $\cup_{a \in A} \{x_a = 1\}$. Consider a point $P_\epsilon = (q_1, ..., q_n)$ such that $q_a = p_a + \frac{2\epsilon}{|A|}$ for $a \in A$, $q_b = p_b - \frac{\epsilon}{|B|}$ for $b \in B$, and $q_c = p_c - \frac{\epsilon}{|C|}$. There exists $0 < \epsilon \ll 1$ such that $q_a \le \beta(a)$, $q_b > 0$, $q_c > 0$, hence $0 < q_i \le \beta(i)$ for all $i = 1, ..., n$. By the following equality, one has $P_\epsilon \in \Delta_\beta$:

$$\begin{aligned}
\sum_{i=1}^n q_i &= \sum_{a \in A} q_a + \sum_{b \in B} q_b + \sum_{c \in C} q_c \\
&= \left(\sum_{a \in A} p_a + 2\epsilon\right) + \left(\sum_{b \in B} p_b - \epsilon\right) + \left(\sum_{c \in C} p_c - \epsilon\right) \\
&= \sum_{a \in A} p_a + \sum_{b \in B} p_b + \sum_{c \in C} p_c \\
&= 3
\end{aligned}$$

Also, note that:

$$\begin{aligned}
\sum_{a \in A} q_a &> \sum_{a \in A} p_a = r_0(A) \\
\sum_{b \in B} q_b &< \sum_{b \in B} p_b = r_0(B) \\
\sum_{c \in C} q_c &< \sum_{c \in C} p_c = r_0(C)
\end{aligned}$$

which implies that $P_\epsilon \notin \mathrm{BP}_{X_0}$. Now, using Corollary 2.9, one can check that P_ϵ should be contained in the interior of BP_{X_1} where X_1 is a 2-dimensional puzzle-piece of the β-puzzle X that contains Y_0. This contradicts that Y is an open puzzle-piece of X. Hence, one of the inequalities $\beta(A) > 1$ and $\beta(B) > 1$ is not true.

(a) Suppose that $\beta(A) > 1$, then $\beta(B) \leq 1$ for any rank 1 non-degenerate flat $B \subset A^c$ of M_0. The line $\mathcal{P}(M_0/B \oplus M_0|_B)$ is an open puzzle-piece of X, since otherwise $Y_1 = X_0 \cap X_1$ for some 2-dimensional puzzle-piece X_1 where $\mathrm{BP}_{X_1} \subset \{x(B^c) \leq 2\}$, but $\beta(B) \leq 1$, so we have:

$$\mathrm{BP}_{X_1} \subset \{x(B^c) \leq 2, x(B) \leq \beta(B) \leq 1\}$$

If $\beta(B) < 1$, BP_{X_1} is a empty set, otherwise BP_{X_1} is contained in a codimension 1 polytope, which is a contradiction.

(i) For any non-degenerate flat J of M_0 that strictly contains A, since $r_0(J) \geq r_0(A) = 1$ and A is a flat, one has $r_0(J) = 2$. Then, the puzzle-piece $Y_1 = \mathcal{P}(M_0|_J \oplus M_0/J)$ is an open puzzle-piece. Indeed, if $Y_1 = X_0 \cap X_1$ for a 2-dimensional puzzle-piece $X_1 = \mathcal{P}(M_1)$ with $M_1 = (S, r_1)$, one has $r_1(J^c \cup A) = 2$ since $M_1/J^c = M_0|_J$ and $r_1(J^c \cup A) = r_{M_1/J^c}(A) + r_1(J^c) = r_{M_0|_J}(A) + 1 = 1 + 1 = 2$. But, $J \backslash A = (J^c \cup A)^c$ is a rank 1 flat of $M_0/A \oplus M_0|_A$ and not both $\beta(A) > 1$ and $\beta(J \backslash A) > 1$ hold true at the same time. Then, $\beta(J \backslash A) \leq 1$ since we already have $\beta(A) > 1$. Now, BP_{X_1} is contained in $\{x(J \backslash A) \leq \beta(J \backslash A) \leq 1, x(J^c \cup A) \leq 2\}$ which has at least codimension 1. This contradicts that BP_{X_1} has full dimension, hence Y_1 is an open puzzle-piece of X. Moreover, at the point $Y_0 \cap Y_1$, the angle $\angle_{X_0} Y_0 \cap Y_1$ is 240°.

(ii) For any non-degenerate flat J of M_0 with $r_0(J) = 1$ such that $J \neq A$, i.e., $J \cap A = \emptyset$, $Y_1 = \mathcal{P}(M_0|_J \oplus M_0/J)$ is an open puzzle-piece. For suppose that $X_0 \cap X_1 = Y_1$ for some 2-dimensional puzzle-piece $X_1 = (S, r_1)$ of X. $\beta(J) \le 1$ since J is a non-degenerate flat of $M_0|_J \oplus M_0/J$ and not both $\beta(A) > 1$ and $\beta(J) > 1$ hold true at the same time. Then, BP_{X_1} is contained in $\{x(J) \le \beta(J) \le 1, x(J^c) \le 2\}$ since $r_1(J^c) = 2$. This is again a contradiction, so Y_1 is an open puzzle-piece of X. The angle $\angle_{X_0} Y \cap Y_1$ is $300°$.

Remark. As a result of (ii), for any rank 1 flat F of M_0, $\beta(F) \le 1$. In addition, if Y_1 is not an open puzzle-piece of X, $J := S_{X_0}(Y_1)$ has rank $r_0(J) = 2$ and does not intersect A.

Suppose $Y_1 = X_0 \cap X_1$ for some 2-dimensional puzzle-piece $X_1 = \mathcal{P}(M_1)$ of X with $M_1 = (S, r_1)$. Then, $A_1 := J^c$ is a rank 1 flat of M_1. Consider any line Y_2 of X_1 that intersects Y_1 and let $J_1 := S_{X_1}(Y_2)$. By Lemma 2.1, either $J_1 \supsetneq A_1$ with $r_1(J_1) = 2$ or $J_1 \cap A_1 = \emptyset$ with $r_1(J_1) = 1$.

(iii) Suppose that $J_1 \supsetneq A_1$ with $r_1(J_1) = 2$. $J_1 \backslash A_1$ is a non-degenerate flat of $M(Y_1) = M_0|_J \oplus M_0/J$, and also a flat of M_0 such that $r_0(J_1 \backslash A_1) = 1$ and $J_1 \backslash A_1 \ne A$. So, $\beta(J_1 \backslash A_1) \le 1$ by the previous argument. If $Y_2 = \mathcal{P}(M_1|_{J_1} \oplus M_1/J_1)$ is not an open puzzle-piece of X, write $Y_2 = X_2 \cap X_3$ for some 2-dimensional puzzle-piece $X_3 = \mathcal{P}(M_3)$ with $M_3 = (S, r_3)$. The point $Y_1 \cap Y_2$ of X_3 is the intesection of two lines of X_3, say Y_2 and Y_3, where Y_3 is different from Y_1 since Y_1 is an open puzzle-piece of X. Then $S_{Y_2}(Y_2 \cap Y_3) = A_1$. Let $J_2 := S_{X_3}(Y_2)$. Then, $J_2 = A_1$ or $J_2 = J_1^c \cup A_1$ by Lemma 2.1. In either case $\mathrm{BP}_{X_3} \subset \{x(J_1^c \cup A_1) \le 2\}$. But, $\beta(J_1 \backslash A_1) \le 1$ forces BP_{X_3} to have a positive codimension, which is a contradiction. So, Y_2 is a open puzzle-piece of X. Then, the angle of the flake $X_1 \cup X_2$ at $Y_1 \cap Y_2$ is $180°$ if $J_1 \backslash A_1$ is a degenerate flat of M_0, $120°$ otherwise.

(iv) Suppose that $J_1 \cap A_1 = \emptyset$ with $r_1(J_1) = 1$. J_1 is a non-degenerate flat of $M(Y_1) = M_0|_J \oplus M_0/J$, and also a flat of M_0 such that $r_0(J_1) = 1$ and $J_1 \ne A$. So, $\beta(J_1) \le 1$ by the previous argument, which means that Y_2 is a open puzzle-piece of X. The angle of the flake $X_1 \cup X_2$ at $Y_1 \cap Y_2$ is 240° if $J_1 \backslash A_1$ is a degenerate flat of M_0, 180° otherwise.

Remark. For any rank 1 flat $F \neq A_1$ of M_1 , $\beta(F) \leq 1$. In addition, if Y_2 is not an open puzzle-piece of X, $J_1 := S_{X_1}(Y_2)$ has rank $r_1(J_1) = 2$ and does not intersect A_1.

If X_{j+1} glues to X_j, X_{j+1} inherits the properties of above two remarks. This makes the β-puzzle X regular at every point and every pair of points. The dual graph of X is a tree and in the extended dual graph of X, X_0 is one and only one sink; see Figure 5.9.

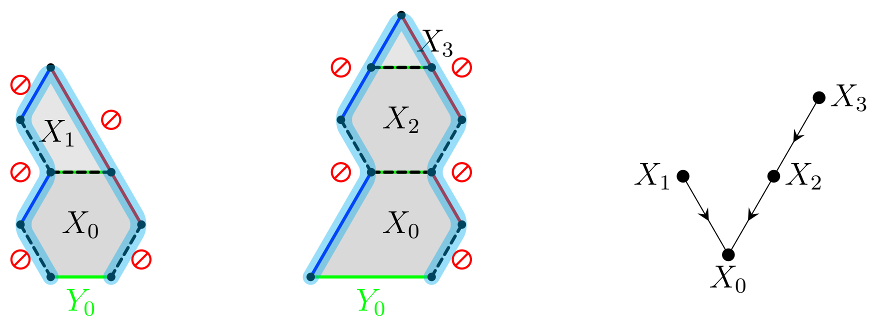


Figure 5.9:

(b) Recall the setting given in the early part of this proof. If there is a center Z such that $\beta(A) > 1$, the argument (a) says everything for that. So, we may suppose that $\beta(A) \leq 1$ for all such A. Now, fix Z. Similarly as in (a), X is regular at every point. To prove that X is regular at every pair of points, suppose that there is a 2-dimensional puzzle-piece $X_1 = \mathcal{P}(M_1)$ with $M_1 = (S, r_1)$ and a line $Y_1 = \mathcal{P}(M_1|_{J_1} \oplus M_1/J_1)$ where $J_1 = B^c$ is a non-degenerate flat of M_1 with rank $r_1(B^c) = 2$, such that $Y_1 \cap Y_0 = Z = \mathcal{P}(M_A \oplus M_B \oplus M_C)$. Then any non-degenerate flat F of $M_1|_{J_1} \oplus M_1/J_1$ is a rank 1 flat of M_1 which is contained in A, for which we use Figure 2.3. So, we have $\beta(F) < \beta(A) \leq 1$, which implies that $\mathcal{P}(M_1/F \oplus M_1|_F)$ is an open puzzle-piece; see Figure 5.10. Therefore, X is

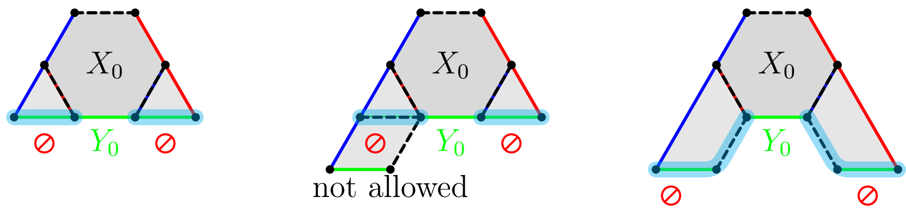


Figure 5.10:

regular at any pair of points (Z, Z'). Since Z is arbitrary, X is regular at every pair of points. Thus, any β-puzzle is a regular quilt.

Now, go back to the early stage of this proof and suppose that Y_0 is an open puzzle-piece of X such that Y_0 is contained in a full dimensional puzzle-piece of X, say X_0 and $r_0(A) = 2$. Let $Z = \mathcal{P}\left(M_{A \backslash B} \oplus M_B \oplus M_{A^c}\right)$ be a center on Y_0. Similarly as in above argument, there are two cases: either $\beta(B) \leq 1$ or $\beta(A \backslash B) \leq 1$ while $\beta(C) > 1$ is always true.

(c) Suppose that $\beta(B) \leq 1$. Then, near Z, X looks like Figure 5.11. So,

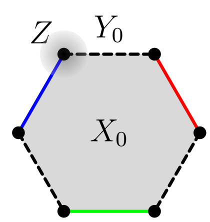


Figure 5.11:

X is regular at Z and every pair (Z, Z') for any center Z'.

(d) Suppose that $\beta(A \backslash B) \leq 1$. Near Z, X looks like the first panel of Figure 5.12. Then, X can be extended to a quilt that is regular at Z and every pair (Z, Z') by saturating with the puzzle-pieces obtained in Figure 4.11 as seen in the second panel of Figure 5.12. □

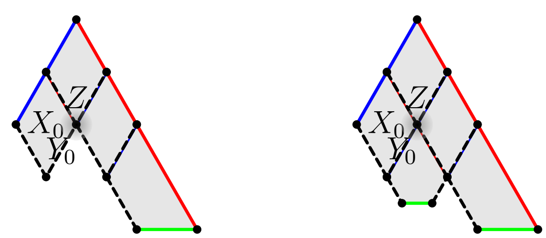


Figure 5.12:

Chapter 6

Extension of a regular quilt

We fix $k = 3$ throughout this chapter.

Definition 6.1. Let $X_0 = \mathcal{P}(M_0)$ with $M_0 = (S, r_0)$ be a full dimensional puzzle-piece. X_0 or M_0 is called *simple* if $r_0(F) = 1$ implies $|F| = 1$, i.e, there is no flat F of rank 1 with $|F| > 1$. This is equivalent to saying that every singleton set is a flat since we assume the matroids of puzzle-pieces to be loopless. For a flat J of M_0, we say that J is a *simple incidence relation* if $M_0|_J$ is simple.

X_0 is called *irrelevant* or *almost simple* if there is at most one flat F of rank 1 with $|F| > 1$.

Irrelevancy is defined the same way for 1-dimensional puzzle-pieces.

X_0 is called *planar* if all of its open puzzle-pieces can be depicted in one local chart as a part of the boundary of a polygon in Figure 4.5. X_0 is called *planar up to irrelevancy* if X_0 is planar after ignoring irrelevant lines.

Lemma 6.2. *Fix $n \leq 9$. Let $X = \{X_i \,|\, i \in \Omega\}$ be a regular quilt, Y_0 an open puzzle-piece of X. If Y_0 is irrelevant, it can be saturated with a full dimensional irrelevant puzzle-piece X_{00} so that $\{X_{00}\} \cup \{X_i \,|\, i \in \Omega\}$ is a regular quilt.*

Proof. Y_0 is a line of some full dimensional puzzle-piece $X_0 = \mathcal{P}(M_0)$ of

X with $M_0 = (S, r_0)$. Then, $J := S_{X_0}(Y_0)$ has rank 1 or 2 in M_0. If J has rank 1, consider hyperplanes B_i, $i \in S$, on $\mathbb{P}^2$ such that only nontrivial incidence relation is $\operatorname{codim} \bigcap_{j \in J} B_j = 2$. In other words, only nontrivial flat of the corresponding matroid M_1 is J. Then, $M_0/J = M_1|_J$ and two puzzle-pieces $\mathcal{P}(M_0)$ and $\mathcal{P}(M_1)$ glue through the line $Y_0 = \mathcal{P}(M_0/J \oplus M_0|_J) = \mathcal{P}(M_1|_J \oplus M_1/J)$. Observe that there are at least 4 lines in general linear position since $|J| \geq 2$, so M_1 is inseparable and X_{00} has full dimension. Moreover, X_1 is irrelevant. At every point Z on Y_0, the angle $\angle_{X'} Z = 180°$, where $X' = \{X_{00}\} \cup \{X_i \,|\, i \in \Omega\}$. X' has no open line passing through Z. Hence, X' is regular.

If J has rank 2, consider hyperplanes B_i, $i \in S$, on $\mathbb{P}^2$ such that only nontrivial incidence relation is $\operatorname{codim} \bigcap_{j \in J} B_j = 1$. By the similar argument, there is an irrelevant full dimensional puzzle-piece X_{00} so that X' is a regular puzzle. □

Corollary 6.3. *Fix $n \leq 9$. If a regular quilt X contains only irrelevant puzzle-pieces, X can be extended to a complete quilt.*

Remark. Lemma 6.2 says that irrelevant puzzle-pieces are irrelevant to gluing puzzle-pieces to a regular puzzle. Therefore, it makes sense to consider gluing of puzzle-pieces *up to irrelevancy.*

Theorem 6.4. *Every regular quilt X for $n \leq 7$ is a puzzle and can be extended to a complete puzzle.*

Proof. Fix $n \leq 7$, and let X be a regular quilt. In the following pictures of puzzle-pieces M, a solid line segment for the line $\cong \mathcal{P}(M/J)$ is drawn doubled as many times as the cardinality $|J|$ of J. The colors play a role of labelling.

(a) If $n = 4$, note that U_4^3 is only one inseparable matroid with cardinality 4 ground set. Equivalently, there is only one hyperplane arrangement that has 4 lines in general linear position. Also, Δ_4^3 is only one full dimensional

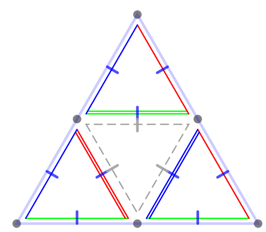


Figure 6.1:

base sub-polytope of itself. So, there is only one quilt, which is obviously a puzzle.

(b) If $n = 5$, see Figure 6.1 for a grid. Any interior center $\mathcal{P}(M_A \oplus M_B \oplus M_C)$ takes a point in a grid with coordinate $(|A|, |B|, |C|)$ up to the permutation group S_3. So, there is no interior center by Figure 6.1. Using Figure 2.3, one can check that X is a subquilt of one of the three quilts given in Figure 6.2 (up to symmetry) that are actually puzzles.

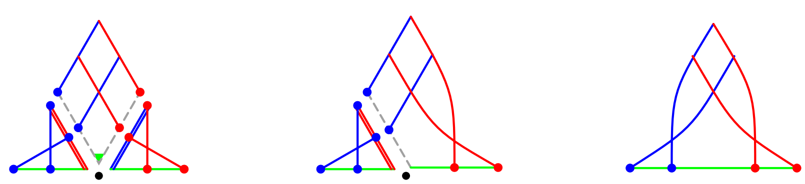


Figure 6.2:

(c) If $n = 6$, see Figure 4.6 for a grid. Observe that there is at most one interior center.

(i) If there is one interior center for X, using Figure 2.3 one can check that X is, up to decomposition and up to symmetry, a subquilt of one of the five complete quilts $\tilde{X}$ given in Figure 6.3. In other words, there exists a quilt X' that is obtained by decomposing and gluing puzzle-pieces of one of the quilts given in Figure 6.3 such that X is a sub-quilt of X'. Those five quilts $\tilde{X}$ are actually puzzles since all of them are, up to symmetry, decompositions of one of the flakes given in Figure 6.4; see Figure 4.13 and 4.14 for the decomposition of a puzzle-piece that is in use. If follows that X is a puzzle and can be extended to a complete puzzle. Maximally decomposed puzzles are given in Figure 6.5.

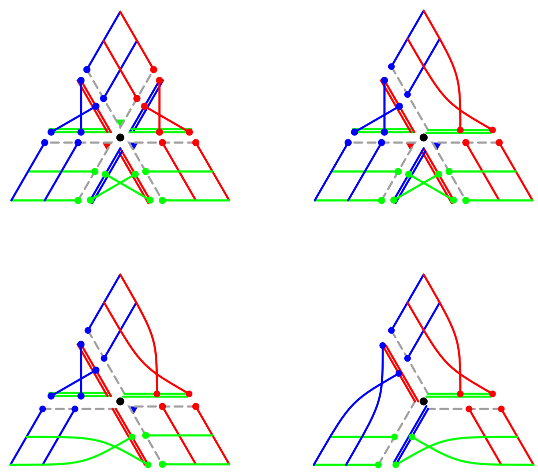


Figure 6.3:

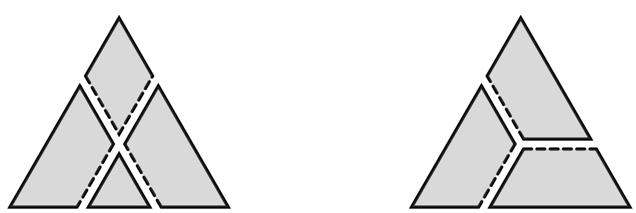


Figure 6.4:

(ii) Assume that there is no interior center for X. Then, all puzzle-pieces of X are irrelevant puzzle-pieces. By Corollary 6.3, X can be extended to

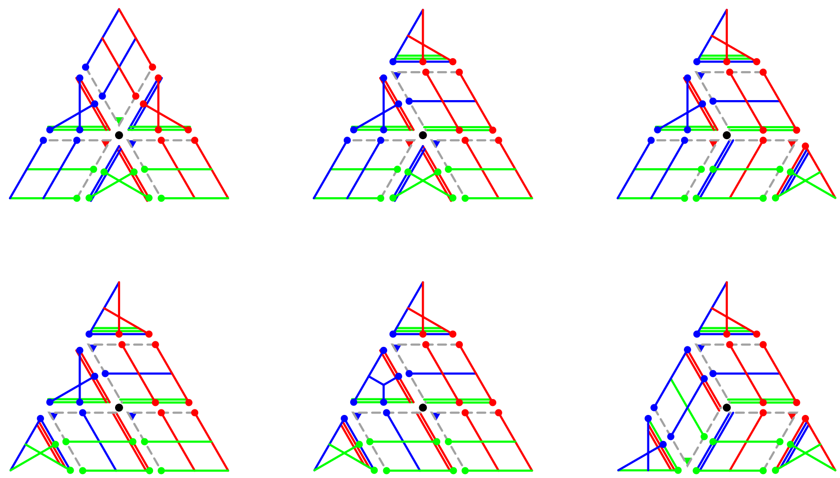


Figure 6.5:

a complete quilt. Indeed, there is only one maximally decomposed quilt $\tilde{X}$ for this case. Its local pictures are depicted in Figure 6.6 together with its dual graph. Also $\tilde{X}$ is a decomposion of a flake, hence it is a

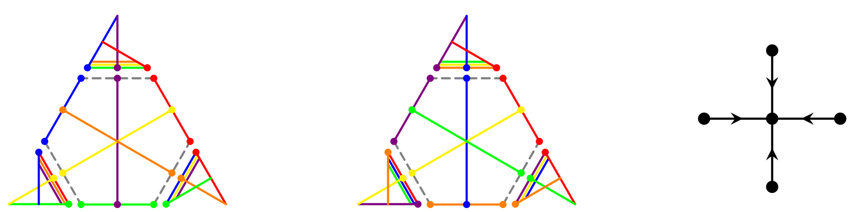


Figure 6.6:

puzzle. So, X is a puzzle and can be extended to a complete puzzle.

(d) If $n = 7$, see Figure 6.7 for a grid. Observe that X has at most 3 interior centers. If X has 3 interior centers, a triangle or a rhombus would take those 3 interior centers. Figure 6.8 classifies such regular quilts as sub-quilts of complete quilts up to decomposition and up to symmetry. Moreover, each complete quilt $\tilde{X}$ in Figure 6.8 is a decomposition of a flake centered at some interior center, hence $\tilde{X}$ is a puzzle. Therefore, X is a puzzle that can be extended to a complete puzzle.

If X has 2 interior centers, a rhombus, a pentagon, or a hexagon can take those 2 interior centers as in Figure 6.9. In this case, the quilt X has simpler form than above case. By computation by hands, one can see that Figure 6.8 classifies such X up to decomposition and up to symmetry. If X has 1 interior center, similarly one has the same result. It follows that X is a puzzle and can be extended to a complete puzzle.

If X has no interior centers, by Corollary 6.3, X can be extended to a complete quilt. Similarly as above, by computation by hands, one can check that such a complete quilt that has no interior centers is a puzzle.

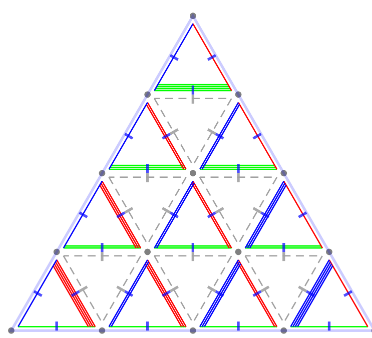


Figure 6.7:

□

For $n = 8, 9$, we have a conjecture that is a weaker statement than Theorem 6.4.

Conjecture 6.5. *Every regular quilt X for $n = 8, 9$ can be extended to a complete quilt.*

Proof. **(a)** Fix $n = 8$, see Figure 4.9 for a grid. We will show that any open puzzle-piece of a regular quilt X can be saturated to give a new regular

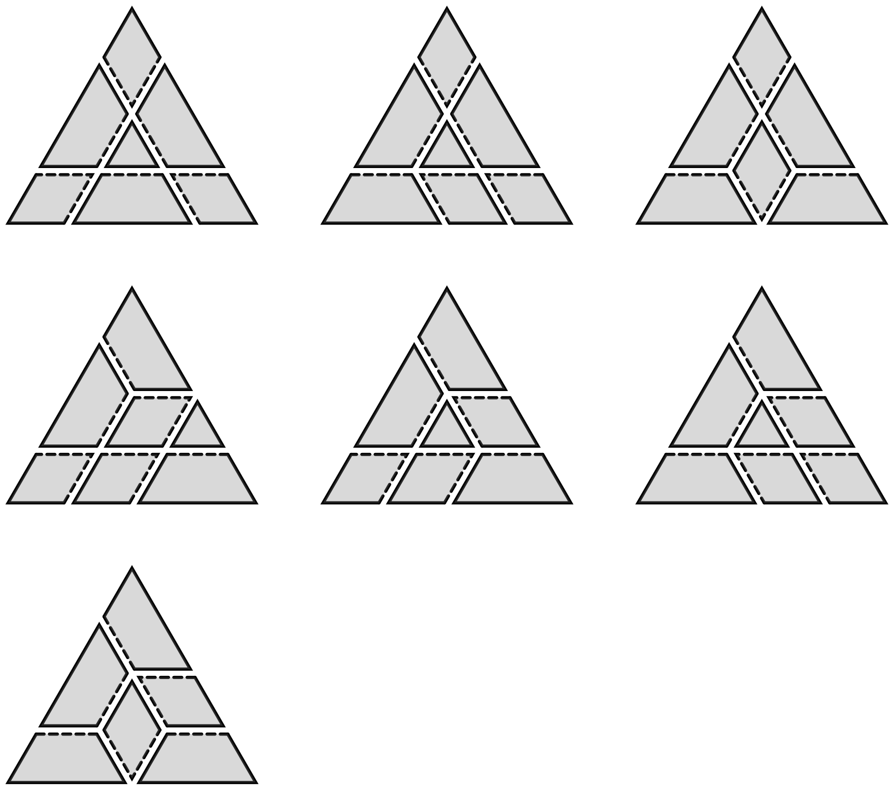


Figure 6.8:

quilt. Then, since there are only finitely many puzzle-pieces, eventually the saturating process must terminate, which means that we will end up with a regular quilt with no open lines, i.e, a complete quilt.

Note that if an open line Y_0 of a regular quilt has less than three interior centers on itself, then there is no compatibility issue when saturating Y_0. Indeed, since Y_0 has at most two interior centers, choose a grid and a line segment for Y_0 so that those interior centers take end points of the line segment of Y_0 in a grid. Then, because no full dimensional puzzle-piece can take a point in the safe zone for Y_0, whatever we fit to Y_0, there is no conflict for Y_0 to be saturated to give a new regular quilt.

Now, see Figure 6.9. We saturate Y_0 with the puzzle-pieces constructed

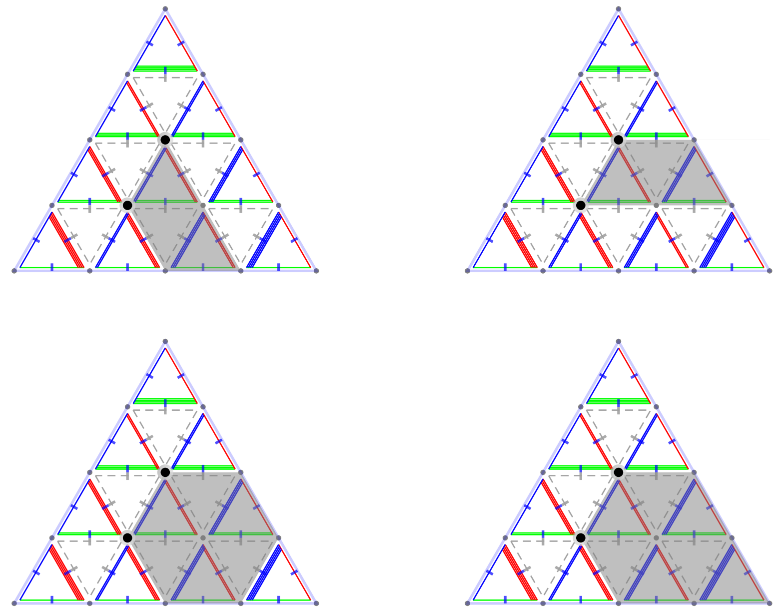


Figure 6.9:

in Figure 4.10. It is easy to see that the new quilt is regular since the number of lines is 8 and there is not enough room for the new quilt to have irregular shape. Observe that actually it suffices to consider the case of the third line pictures of Figure 6.10 up to decomposition and symmetry, since the number of points on each side of X_0 is 2 or less. In Figure 6.12, one of puzzle-pieces has 3 points on its side, but it doesn't make a difference. Figure 6.10 and Figure 6.11 explain enough how the quilt X is extended to a regular quilt by saturating Y_0 for the similar cases.

If an open line Y_0 has at least three interior centers on itself, it may be possible that there is a compatibility issue when saturating Y_0. Now, since $n = 8$, there is at most 3 interior centers on Y_0. Similarly as in above, it suffices to check the case as seen in Figure 6.13. The first line pictures show the local pictures of X, the second pictures are the local pictures of the

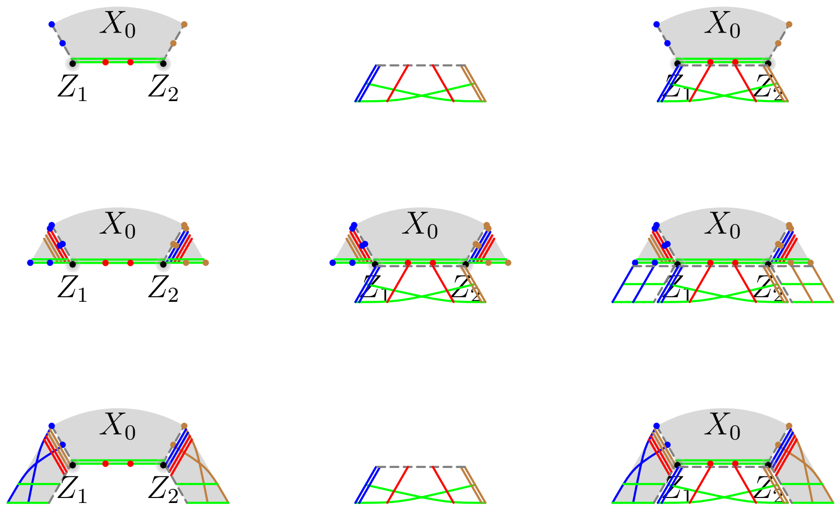


Figure 6.10:

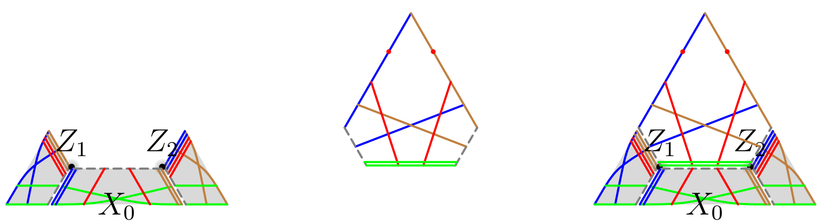


Figure 6.11:

same puzzle-piece that we glue with in order to saturate Y_0. The third line pictures show the local pictures of the resulting puzzle-piece. It is also easy to check that the new one is regular. For our better understanding, we do one more example in a slightly different way. Suppose that X is given with its local pictures as the first line pictures of Figure 6.14. We saturate Y_0 with a puzzle-piece that is slightly different from that of Figure 6.13. In this case the puzzle-piece we glue with has only two green lines, and it is easy to see that there is no compatibility issue as seen in Figure 6.15, and we obtain a new regular quilt as in Figure 6.16.

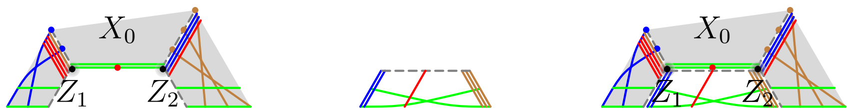


Figure 6.12:

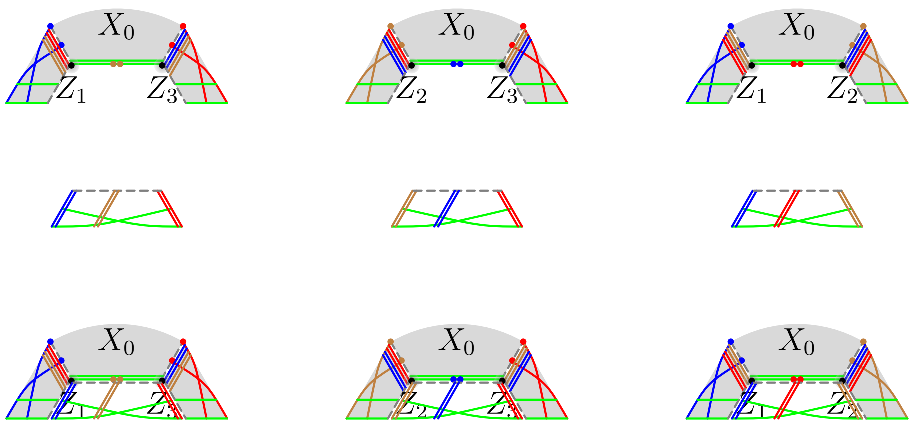


Figure 6.13:

(b) For $n = 9$, see Figure 5.6 for a grid. The number of interior centers that an open line $Y_0 = \mathcal{P}(N_0)$ can have is still at most 3. A newly added case we need to check is when $|N_0| = 6$ as depicted in Figure 6.17. It suffices to consider the case in Figure 6.18 up to decomposition and symmetry, since the number of points on each side of X_0 is 2 or less.

Write $F = \{7,8,9\}$, $J_1 = \{1,2\}$, $J_2 = \{3,4\}$ and $J_3 = \{5,6\}$. Let $X_1 = \mathcal{P}(M_1), X_2 = \mathcal{P}(M_2), X_3 = \mathcal{P}(M_3)$ be puzzle-pieces as seen in Figure 6.18, and Y_1, Y_2, Y_3 open lines of X that are contained in $X_1, X_2 X_3$, respectively. We want to construct a hyperplane arrangement $\mathcal{H}(M_4)$ whose puzzle-piece $X_4 := \mathcal{P}(M_4)$ glues to X_0 through Y_0 without compatibility issue. There are 4 possibilities for the point arrangement for each Y_i, $i = 1,2,3$ as follows: rank 1 flats of $M(Y_i)$, $i = 1,2,3$, are given in Table 6.

If two of $M(Y_i)$ have the same non-trivial rank 1 flat, say $M(Y_1)$ and

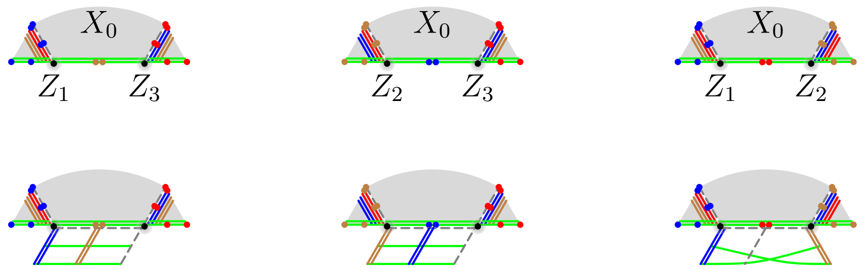


Figure 6.14:

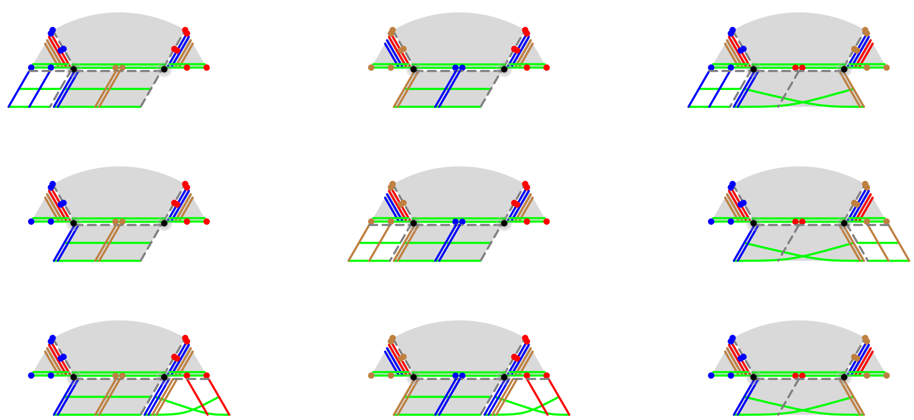


Figure 6.15:

$M(Y_2)$ have a flat $\{8,9\}$, construct a hyperplane arrangement as in Figure 6.19, and glue its puzzle-piece X_4 to X_0 as in Figure 6.20. Now, at Z_3, Y_3 can be saturated with a planar puzzle-piece of type (a) in Figure 3.5 and 4.10. The new quilt is regular.

If at least one of $M(Y_i)$ has a non-trivial rank 1 flat, but none of them have the same non-trivial flat, we can construct a hyperplane arrangement as in Figure 6.21. Glue the puzzle-piece X_4 obtained from this hyperplane arrangement through Y_0. Similarly as above, one can check that the new quilt is regular, and X is extended to a regular quilt.

Else if none of $M(Y_i)$ have a non-trivial rank 1 flat, it is easy to see that

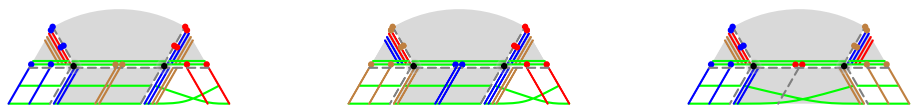


Figure 6.16:

Figure 6.17:

the quilt X is extended to a regular quilt. $\square$

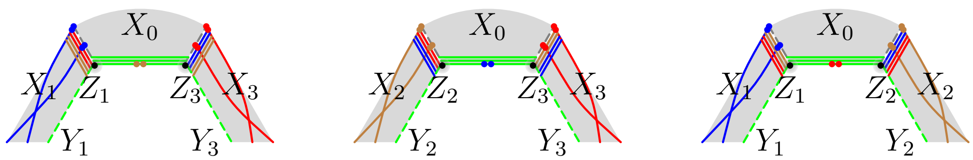


Figure 6.18:

	$M(Y_1)$	$M(Y_2)$	$M(Y_3)$
1	7, 8, 9, 3456	7, 8, 9, 1256	7, 8, 9, 1234
2	7, 89, 3456	7, 89, 1256	7, 89, 1234
3	8, 79, 3456	8, 79, 1256	8, 79, 1234
4	9, 78, 3456	9, 78, 1256	9, 78, 1234

Table 6.1:

Figure 6.19:

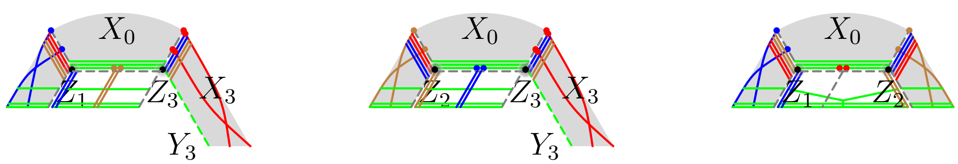


Figure 6.20:

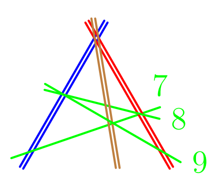


Figure 6.21:

Chapter 7

Surjectivity of the reduction map

Fix $k = 3$ and $\mathbb{F} = \mathbb{C}$. Consider the moduli spaces of weighted stable hyperplane arrangements and reductions maps $\rho_{\mathbb{1},\beta} : \overline{M}_{\mathbb{1}}(3,n) \to \overline{M}_{\beta}(3,n)$ with weights $\beta \leq \mathbb{1}$. We show that there exists a counter-example to the surjectivity of the reduction map $\rho_{\mathbb{1},\beta}$ when $n = 10$.

Theorem 7.1. *There exists a β-puzzle for Δ^3_{10} that is not extended to a complete puzzle.*

Proof. Consider the hyperplane arrangements $\mathcal{H}(M_i)$, $i = 0, 1, 2, 3$ as in Figure 7.1. Their puzzle-pieces $X_i = \mathcal{P}(M_i)$ are depicted with a choice of boundary lines in Figure 7.2. Their non-degenerate flats F with $|F| \geq 2$ are given in Table 7.1 and the describing inequalities of BP_{X_i} are given in Table 7.2, where $x_{c_1 c_2 \cdots c_m}$ with $c_i \in S = \{0, 1, ..., 9\}$ denotes $\sum_{i=1}^m x(c_i)$.

M_0	7890, 127890, 347890, 567890
M_1	78, 3456, 34567890
M_2	78, 90, 1256, 12567890
M_3	90, 1234, 12347890

Table 7.1:

Then, $\{X_0, X_1, X_2\}$, $\{X_0, X_1, X_3\}$ and $\{X_0, X_2, X_3\}$ are puzzles. Indeed, X_0 fits to X_i, $i = 1, 2, 3$. Also, X_1 and X_2 fit since $\mathrm{BP}_{X_1} \subset \{x_{3456} \leq 1\}$,

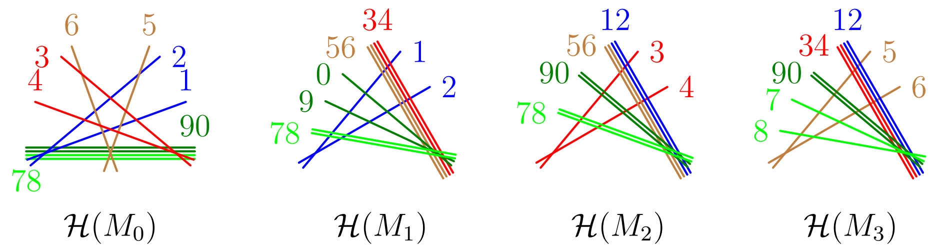


Figure 7.1:

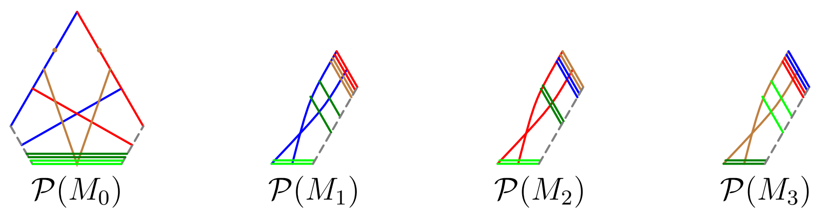


Figure 7.2:

$\mathrm{BP}_{X_2} \subset \{x_{1256\,78\,90} \leq 2\}$ implies $\mathrm{BP}_{X_1} \cap \mathrm{BP}_{X_2} \subset \{x_{34\,56} \leq 1, x_{1256\,78\,90} \leq 2\}$, and the following inequality means that $x_{56} \leq 0$, $x_{56} = 0$:

$$3 + x_{56} = x_S + x_{56} = x_{34\,56} + x_{1256\,78\,90} \leq 1 + 2$$

Hence BP_{M_1} and BP_{M_2} meet nicely, which means that X_1 and X_2 fit together. Similarly, X_1 and X_3 fit together, so do X_2 and X_3. Therefore, $\{X_0, X_1, X_2, X_3\}$ is a puzzle; see Figure 7.3 for the local pictures.

This puzzle X cannot be extended to a complete puzzle. Indeed, consider the open puzzle-piece $Y_0 = \mathcal{P}\left(M_0 / \{7, 8, 9, 0\}\right)$ of X_0 that is represented by multiple green lines and let Z_i, $i = 1, 2, 3$, be three points on Y_0 with $S_{Y_0}(Z_1) = \{1, 2\}$, $S_{Y_0}(Z_2) = \{3, 4\}$, $S_{Y_0}(Z_3) = \{5, 6\}$. If X is extended to a complete puzzle, Y_0 should be saturated with some 2-dimensional puzzle-piece $X_4 = \mathcal{P}(M_4)$. In other words, M_4 is an inseparable matroid of rank 3, and $S \backslash \{7, 8, 9, 0\} = \{1, 2, 3, 4, 5, 6\}$ is a non-degenerate flat of M_4 such

M_0	$\mathrm{BP}_{M_0} = \{x_{7890} \leq 1, x_{12\,7890} \leq 2, x_{34\,7890} \leq 2, x_{56\,7890} \leq 2\}$.
M_1	$\mathrm{BP}_{M_1} = \{x_{78} \leq 1, x_{3456} \leq 1, x_{3456\,78\,90} \leq 2\}$
M_2	$\mathrm{BP}_{M_2} = \{x_{78} \leq 1, x_{90} \leq 1, x_{1256} \leq 1, x_{1256\,78\,90} \leq 2\}$
M_3	$\mathrm{BP}_{M_3} = \{x_{90} \leq 1, x_{1234} \leq 1, x_{1234\,78\,90} \leq 2\}$

Table 7.2:

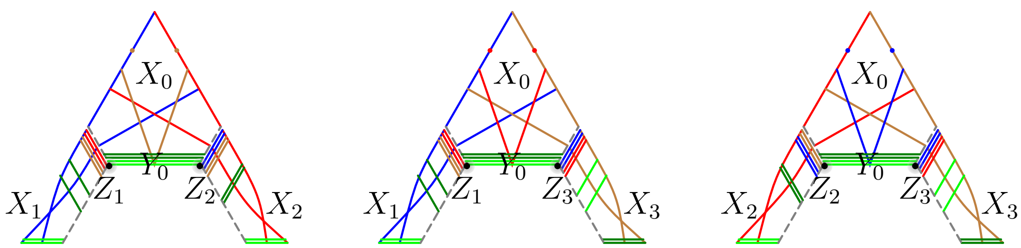


Figure 7.3:

that $M_4|_{\{1,2,3,4,5,6\}} = M_0/\{7,8,9,0\}$. Then, $S_{X_4}(Y_0) = \{1,2,3,4,5,6\}$, and $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ are rank 1 flats of both $M_4|_{\{1,\dots,6\}}$ and M_4. $Z(123456)$ is a point in $\mathcal{H}(M_4)$ which is the intersection of $Z(12)$, $Z(34)$ and $Z(56)$; see the first panel of Figure 7.4.

Figure 7.4:

Suppose that for a flat $F \in \{\{1,2\},\{3,4\},\{5,6\}\}$, $Z(F) \cap Z(789)$ is a point in $\mathcal{H}(M_4)$, say $F = \{1,2\}$, then for other flats $F' = \{3,4\},\{5,6\}$, $Z(F') \cap Z(789) = \emptyset$, since otherwise the lines $Z(7), Z(8), Z(9)$ coincide at two distinct points, hence $Z(7) = Z(8) = Z(9)$. Then, $r_4(S) = 3 = 2 + 1 = r_4(123456) + r_4(789)$, which implies that $M_4 = (S, r_4)$ is separable, a contradiction. So, the lines $Z(34)$ and $Z(56)$ have three distinct points on themselves. Then, $\{3,4\}$ and $\{5,6\}$ are non-degenerate flats of M_4 by Lemma 3.10, and X_4 locally looks like the second panel of Figure 7.4. By the classification theorem of a flake (see Figure 2.3), X_4 and X_2 fit through their common facet, and X_4 and X_3 also fit through their common facet. So, one has $M_2|_{\{1,2,5,6,7,8\}} = M_4/\{3,4\}$ and $M_3|_{\{1,2,3,4,7,8\}} = M_4/\{5,6\}$; see Figure 7.5. $M_2|_{\{1,2,5,6,7,8\}} = M_4/\{3,4\}$ implies that $Z(7) \cap Z(8) \cap Z(34)$ is a point

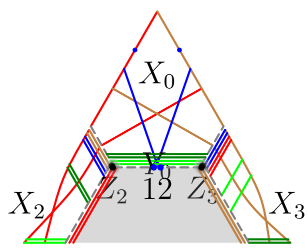


Figure 7.5:

that is different from $Z(12) \cap Z(789)$. Then the lines $Z(7)$ and $Z(8)$ pass through two distinct points at the same time, hence $Z(7) = Z(8)$. However, $M_3|_{\{1,2,3,4,7,8\}} = M_4/\{5,6\}$ implies that $Z(7) \cap Z(8) \cap Z(56) = \emptyset$, which is a contradiction since $Z(7) \cap Z(8) \cap Z(56) = Z(7) \cap Z(56)$ is a point. For other choice of $F = \{3,4\}, \{5,6\}$, we get contradictions in the same way.

Suppose that $Z(F) \cap Z(789)$ is empty for any flat $F \in \{\{1,2\}, \{3,4\}, \{5,6\}\}$. Then, one has:

$$\begin{aligned} M_1|_{\{3,4,5,6,7,8\}} &= M_4/\{1,2\} \\ M_2|_{\{1,2,5,6,7,8\}} &= M_4/\{3,4\} \\ M_3|_{\{1,2,3,4,7,8\}} &= M_4/\{5,6\} \end{aligned}$$

which implies that the lines $Z(7)$ and $Z(8)$ coincide at two distinct points, so $Z(7) = Z(8) = Z(78)$. Then, $r_3(78) = 2$ from Table 6.2. However, the following equation tells that $M_3|_{\{1,2,3,4,7,8\}} \neq M_4/\{5,6\}$, which is a contradiction.

$$r_{M_4/\{5,6\}}(78) = r_4(5678) - r_4(56) = 2 - 1 = 1 \neq 2 = r_3(78)$$

Therefore, X cannot be extended to a complete puzzle.

Now, we will show that X is a β-puzzle. For let:

$$\beta = \left(1, 1, 1, 1, 1, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

then, $\mathrm{BP}_{X_i} \cap \mathrm{int}\Delta_\beta \neq \emptyset$ for $i = 0, 1, 2, 3$. Indeed, for X_0, let:

$$\upsilon = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in \mathrm{BP}_{X_0}$$

One can decrease $\upsilon_7, \upsilon_8, \upsilon_9, \upsilon_0$ by $0 < \epsilon \ll 1$ and increase $\upsilon_1, ..., \upsilon_6$ by $\frac{4\epsilon}{6}$ so that the new point is still contained in BP_{X_0} and also in $\mathrm{int}\Delta_\beta$. For X_1, let:

$$\upsilon = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \in \mathrm{BP}_{X_1}$$

One can decrease $\upsilon_7, \upsilon_8, \upsilon_9, \upsilon_0$ and $\upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6$ by $0 < \epsilon \ll 1$ and increase υ_1, υ_2 by $\frac{8\epsilon}{2}$ so that the new point is contained in both BP_{X_0} and $\mathrm{int}\Delta_\beta$. The other cases are similar.

Moreover, $\cup_{i=0}^{3}\mathrm{BP}_{X_i}$ covers Δ_β. For suppose that there exists a point $\upsilon \in \Delta_\beta \setminus \cup_{i=0}^{3} \mathrm{BP}_{X_i}$. This means that υ violates at least one inequality for each base polytope. Observe that then since $\upsilon_{7890} \leq \beta_{7890} = 1$, one has $\upsilon_{3456} > 1$, $\upsilon_{1256} > 1$, $\upsilon_{1234} > 1$ for $\mathrm{BP}_{X_1}, \mathrm{BP}_{X_2}, \mathrm{BP}_{X_3}$, respectively. Whatever is violated out of 3 inequalities $x_{12\,7890} \leq 2, x_{34\,7890} \leq 2, x_{56\,7890} \leq 2$ for BP_{M_0}, one reaches a contradiction because:

$$\begin{aligned} 3 &= \upsilon_{12\,7890} + \upsilon_{3456} > 2 + 1 = 3 \quad \text{or} \\ 3 &= \upsilon_{34\,7890} + \upsilon_{1256} > 2 + 1 = 3 \quad \text{or} \\ 3 &= \upsilon_{56\,7890} + \upsilon_{1234} > 2 + 1 = 3. \end{aligned}$$

Therefore, $\Delta_\beta \setminus \cup_{i=0}^{3} \mathrm{BP}_{X_i}$ is empty, which means that $\cup_{i=0}^{3}\mathrm{BP}_{X_i}$ covers Δ_β. □

Corollary 7.2. *When $n = 10$, there exists a weight vector β such that the reduction map $\rho_{\mathbb{1},\beta} : \overline{M}_{\mathbb{1}}(3, 10) \to \overline{M}_\beta(3, 10)$ is not surjective.*

Proof. Fix $n = 10$. Let X_0, X_1, X_2, X_3 be the puzzle-pieces obtained in Theorem 7.1.Then, the codimension 1 puzzle-pieces $Y_i := X_i \cap X_0$, $i = 1, 2, 3$ have exactly 3 distinct point loci on themselves. Consider the varieties V_0, V_1, V_2, V_3 that give puzzle-pieces X_0, X_1, X_2, X_3, respectively. Recall that there is a one-to-one correspondence between the strata of the log canonical model of a hyperplane arrangement and that of its corresponding puzzle-piece. Let W_1, W_2, W_3 be the 1-dimensional subvarieties of V_1, V_2, V_3, respectively, that give the puzzle-pieces Y_1, Y_2, Y_3. Let W_1', W_2', W_3' be the 1-dimensional subvarieties of V_0 that give Y_1, Y_2, Y_3, respectively. Observe that W_i, W_i', $i = 1, 2, 3$ are 1-dimensional hyperplane arrangemnts that are all isomorphic to $\mathbb{P}^1$ with 3 distinct points. Because $(\mathbb{P}^1, 3\text{pts})$ has no moduli, V_0, V_i, $i = 1, 2, 3$ uniquely glue to a variety.

Recall that any element of $\overline{M}_\beta(3, n)$ gives a partial cover of Δ_β, and this correspondence is commutative with reduction maps. If $\rho_{\mathbb{1},\beta} : \overline{M}_{\mathbb{1}}(3, 10) \to \overline{M}_\beta(3, 10)$ is surjective, there must exist a tiling of $\Delta = \Delta_{\mathbb{1}}$ such that Δ is an extension of Δ_β, in other words, there must exist a complete puzzle that is an extension of the β-puzzle that corresponds to the given partial cover of Δ_β. But, in Theorem 7.1, we see that the β-puzzle $X_0 \cup X_1 \cup X_2 \cup X_3$ is not extended to a complete puzzle, a contradiction. Hence, $\rho_{\mathbb{1},\beta} : \overline{M}_{\mathbb{1}}(3, 10) \to \overline{M}_\beta(3, 10)$ is not surjective for β given in Theorem 7.1. □

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