THE IMPACT OF STUDENTS' UNDERSTANDING OF DERIVATIVES ON THEIR PERFORMANCE WHILE SOLVING OPTIMIZATION PROBLEMS

by

BRIAN SCOTT SWANAGAN

(Under the direction of James W. Wilson)

ABSTRACT

Students regularly struggle with applications, particularly those concerning optimization problems, in calculus. Although teachers would like their students to learn to transfer their knowledge to nonroutine and real-life situations, students run into a number of difficulties, including using their understanding of the derivative when finding optimal solutions. In this qualitative study, I interviewed five AP Calculus students and observed them while they solved three optimization problems. I focused on their problem solving, points of struggle, use of the derivative, and understanding of the derivative. Each student participated in a 2- to 3-hour interview, during which I asked about their background, gave them three problems to solve, questioned them about their solution attempts, and inquired into their understanding of the derivative. Only one of the students actually constructed a complete solution, whereas the others varied in their progress. Their limited use and understanding of the derivative was surprising, with the most capable student showing by far the greatest understanding. In their solutions, they struggled with variables, substitution, optimization, objective functions, progress assessment, verification of results, recall of previous learning, calculator use, word problems, and derivative use. Based on these findings, I make recommendations for future research regarding why these students struggle and some teaching methods that may be tried to help them.

INDEX WORDS: Problem solving, conceptual understanding, derivative, optimization problems, calculus, mathematics education

THE IMPACT OF STUDENTS' UNDERSTANDING OF DERIVATIVES ON THEIR PERFORMANCE WHILE SOLVING OPTIMIZATION PROBLEMS

by

BRIAN SCOTT SWANAGAN

B.S., Georgia Institute of Technology, 2006

M.Ed., University of Georgia, 2008

Ed.S., University of Georgia, 2008

A Dissertation Submitted to the Graduate Faculty of The University of Georgia in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

ATHENS, GEORGIA

2012

THE IMPACT OF STUDENTS' UNDERSTANDING OF DERIVATIVES ON THEIR

PERFORMANCE WHILE SOLVING OPTIMIZATION PROBLEMS

by

BRIAN SCOTT SWANAGAN

Major Professor: James W. Wilson

Committee: Thomas Hebert
Jeremy Kilpatrick

Electronic Version Approved:

Maureen Grasso
Dean of the Graduate School
The University of Georgia
May 2012

DEDICATION

To God, who has always loved me and helped me grow; to Diana, the love of my life, who has always provided loving encouragement, support, and feedback; to Tobias and Theodore, who motivate me to become an example of everything a loving, faithful man should be.

ACKNOWLEDGEMENTS

There are many deserving thanks for all their efforts in helping me along my journey toward earning my doctoral degree:

My dissertation committee, Thomas Hebert and Jeremy Kilpatrick, provided regular encouragement and productive criticism throughout my time at the University of Georgia. They also were always ready to lend an attentive ear toward my thoughts and trials concerning teaching and research.

My family provided untold support, consistently thinking the best of me. My wife, Diana, regularly turned me from frustration and discouragement toward enthusiasm and productivity. She provided an additional critical eye and some invaluable insight into the entire process. All this, she carried out while at times planning a wedding, looking into buying and improving our house, enduring two pregnancies, and throughout many other events, both wonderful and stressful. My sons, Tobias and unborn Theodore, gave up precious time with me and supplied regular stress relieving moments and laughter. My mother, Nancy, constantly looked for ways to help and encourage me, lending an ear whenever I needed to vent. My father, Jeffery, regularly rained down praises and supplied a generally positive outlook, leading me to remember to focus on the most important things. My four siblings, Jennifer, Christina, David, and Kevin, have helped me grow in many ways, particularly spiritually, through a variety of trials and special moments together. My grandparents, Evelyn, Otho, Jeffery, and Jean, have always provided an exceptional example to follow, especially concerning persevering academically to the highest levels and in continual growth.

Several teachers and friends have helped me along the way to get over various humps, particularly Dennis Hembree and Cathy Franklin who helped me realize my potential in mathematics and education. They listened and seriously considered my wild mathematical conjectures and explorations, providing rare insight and silly encouragement that unleashed an unquenchable thirst for understanding. The McClure family provided childcare and other means of support that were invaluable in helping a busy, overcommitted high school teacher finish in a reasonable amount of time. My fellowship cohort and others involved with the Knowles Science Teaching Foundation sustained me through difficult teaching trials and repeatedly inspired me to become the most excellent teacher possible.

Jim Wilson, my major professor, deserves special thanks for his regular encouragement in mathematics and education, commitment to my growth, time spent reading and editing to meet tight deadlines, exemplary dedication to his students, and supply of challenging, enjoyable problems.

My Savior, Jesus Christ, gave me a hope, through His incomprehensible sacrifice, that allowed me to persevere through the most stressful and darkest moments and grew within me, through the Holy Spirit, a continually expanding love and understanding for others.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Having an interest in mathematically gifted students and how they solve problems, I decided to speak to some AP Calculus teachers about their students. Several calculus teachers mentioned they had concerns about how well their students solve application problems. One teacher, whose students routinely shine on the AP Calculus examination, noted that students struggle with optimization problems in particular. In my own classroom, I noticed I often had talented students who could work common classroom problems, such as those found in textbooks, quite well but would become stymied when given something that they had not seen before or had a different structure than usual. As situations in the real world can vary widely in structure and presentation, so can applied problems. Because students' difficulties with solving application problems appeared to have critical implications for classroom teachers, I chose to research this topic further, especially concerning calculus students.

Researchers have determined that conceptual understanding plays an important role in students' performance while solving application problems. When students have a conceptual understanding of a mathematical topic, they have insight into the underlying principles behind the procedures, relationships, and connections to other ideas that surround that topic (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993). At a high school stressing an approach that led to conceptual understanding, Boaler (1998) found students could apply and transfer their knowledge to new situations much more readily than students at a high school that focused on the development of procedural knowledge. Students who have obtained a conceptual understanding of mathematical topics then find themselves more equipped to handle new and difficult problems.

Calculus is typically considered to be a difficult subject for high school students. Although students can often solve procedural calculus problems, they struggle to solve problems that are dissimilar to problems solved in class (Selden, Selden, Hauk, & Mason, 1999) or problems that require them to apply their calculus knowledge to real-world problems (Tall, 1992). Researchers have explored why students struggle with applied problems (Craig, 2002) and how students perceive their difficulties with these types of problems (Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010).

Considering the difficulties students have with applied problems in calculus, researchers have begun to look at students' conceptual understanding of several topics in the subject. Students tend to misunderstand differences between slope, rate of change, and steepness, which results in errors on some problems (Teuscher & Reys, 2010). Also, Martin (2000) explored university students' understanding of related rates, finding their performance to be poor, particularly during steps requiring conceptual understanding. Shepherd (2007) found that while attempting to solve a problem concerning surface area, students typically preferred a procedural approach to a conceptual one and were not comfortable with a conceptual argument.

An important category of applied problems in calculus is optimization. Optimization problems require one to find the "best" of a given set of possible solutions. The "best" solution is typically judged by an objective function that is often minimized or maximized on a *feasible region* defined by decision variables and constraints (Carvallo, Figueiredo, Gomez, & Velho, 2003; Derigs, 2001; Perkins, 2010; Rockafellar, 2007; Villegas, Castro, & Gutierrez, 2009). Optimization problems and strategies vary widely; they may have objective functions and can also have constraints that require decision variables to manipulate (Derigs, 2001; Mebarak, 2003; Rockafellar, 2007). Optimization problems also differ in the nature of the feasible region (continuous, discrete, etc.), the types of constraints (equalities, inequalities, etc.), and the types of objective functions (linear, quadratic, separable, etc.) (Kallrath, 2000). In this study, I explore the following research questions:

- How do Advanced Placement (AP) Calculus students understand and solve optimization problems?
- How does students' understanding of the concept of derivative affect their approach to solving optimization problems in AP Calculus?

Rationale

According to the National Council of Teachers of Mathematics (NCTM), "a major goal of high school mathematics is to equip students with knowledge and tools that enable them to formulate, approach, and solve problems beyond those that they have studied" (p. 335, 2000). It stands to reason, therefore, that mathematics educators need to prepare students to apply their mathematical knowledge to a wide range of problem contexts; it is insufficient to require students to demonstrate procedural knowledge. Within their learning principle, NCTM furthermore uses Bransford, Brown, and Cocking's argument that "one of the most robust findings of research is that conceptual understanding is an important component of proficiency, along with factual knowledge and procedural facility" (as cited in NCTM, 2000, p. 20). Specific to calculus, the College Board (2010) lists "students should be able to use derivatives to solve a variety of problems" (p. 6) as one of the AP Calculus goals. The College Board's philosophy emphasizes conceptual knowledge over procedural knowledge while maintaining that the latter has utility in the classroom:

> Broad concepts and widely applicable methods are emphasized. The focus of the courses is neither manipulation nor memorization of an extensive taxonomy of functions, curves,

> theorems or problem types. Thus, although facility with manipulation and computational competence are important outcomes, they are not the core of these courses. (p. 5)

Optimization problems, both global and local, are also specifically listed as one of the applications of derivatives that students should cover in an AP Calculus course. It is important, therefore, to consider how students' understanding of derivatives enables them to solve problems modeling real-world situations, including optimization problems.

Not only are optimization problems part of an AP Calculus course, but they are also important in a variety of fields. Businesses use optimization strategies to their benefit in resource allocation, production scheduling, warehousing, telecommunication networks, distribution systems, logistics, marketing, and so forth (Carvallo et al., 2003; Derigs, 2001; Kallrath, 2000; Perkins, 2010; Rockafellar, 2006). Operations research for optimization led to hundreds of millions of dollars in annual savings of major companies like Blue Bell Creameries and Delta Airlines in the 1990s (Carvallo et al., 2003; Derigs, 2001). Financial investments rely heavily on optimization of differential equations as well (Derigs, 2001; Kallrath, 2000). In computer graphics, designers use optimization to create realistic, detailed images (Carvallo et al., 2003). Optimization also plays a key role in engineering design, chemistry (blending, refining, process design), and robotics as well as a variety of other fields (Kallrath, 2000; Mebarak, 2003; Rockafellar, 2007).

Overview of Theoretical Framework

Because I wanted to analyze students' solution processes and determine what causes them to struggle when solving optimization problems, I decided to look for a structured framework that would help me organize the students' solution attempts meaningfully and make sense of key points in their processes that keep them from easily solving the problems. Useful for this goal, Schoenfeld (1985) developed an efficient and effective framework for following and analyzing a student's problem-solving process broken down into six types of macroscopic episodes and transitions between them. His framework involves asking questions within identified episodes that focus on students' decisions, or lack thereof, during the solution process. The framework lends itself well to analyzing the work of students, whom I expected to struggle during the problem-solving process.

Optimization problems in calculus typically make use of the concept of derivative because it can help one pinpoint maximums and minimums of an objective function with relative ease. For this reason, I wanted to determine if students' understanding of the concept of derivative had some influence on their performance. Additionally, I hoped to get as complete a view of their understanding as possible, not only conceptually but also in practice. Zandieh (2000) developed a framework for categorizing students' understanding of derivative into four representations, of which there are three layers, after one asks students a number of probing questions. Building off Zandieh's framework, Roorda, Vos, and Goedhart (2007) included a means to represent a student's flow through the representations and layers while problem solving as well as additional representations regarding applications of derivative and another layer before the ratio layer consisting of the original function. Using a combination of these frameworks allowed me to analyze how students solve nonroutine optimization problems and what role their concept of the derivative plays in their solution process.

CHAPTER 2

LITERATURE REVIEW

Problem Solving

Mathematically gifted students have a talent for problem solving, and indeed developing this skill along with increasing knowledge of mathematics is important. Wilson, Fernandez, and Hadaway (1993) describe mathematics as synonymous with problem solving. Romberg (1994) describes mathematics as "a vast collection of ideas derived as a consequence of searching for solutions to social problems" (p. 297). In their "Problem Solving Standard," NCTM (2000) claims that "most mathematical concepts or generalizations can be effectively introduced using a problem solving situation" and one of the major goals of mathematics education is to "equip students with knowledge and tools that enable them to formulate, approach and solve problems beyond those they have studied" (pp. 334–335). Additionally, Polya (1962) describes problem solving as one of mankind's greatest accomplishments and says that those who do it well are above others who cannot.

In order to understand problem solving clearly, one should first define what a problem is. Polya (1962) defines a problem as a desire "to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable aim" (p. 117). Defining a problem in a similar manner but with an intentional emphasis on the individual, Henderson and Pingry (1953) point out three essential parts: the problem solver must have identified a desired goal, recognizes some type of block exists which he or she does not immediately know how to remove or circumvent to reach the goal, and considers a number of possible methods for attaining the goal despite the block. In this way, problems may exist for some but not for others. Schoenfeld (1985) also describes a problem as having a desired goal under pursuit by an individual who does not have ready access to a method for achieving that goal.

Despite that problems may differ depending on the individual, they do have some common characteristics. Polya (1962) divides problems into two categories: “problems to find” and “problems to prove” (p. 119). In problems to find, a problem solver should note the unknown, conditions, and data within the problem. The unknown is the object to be found or constructed. Conditions put restrictions on what properties that the object must satisfy while data in the problem provide some given, specific information that can be used as a starting place to begin the search. In problems to prove, one should note the hypothesis and condition and attempt to find a link between them. To these categories, Henderson and Pingry (1953) add another dimension: pre-formulated problems and life problems. In an attempt to teach problem-solving techniques or generalizations of mathematical principles, teachers create pre-formulated problems for students who still must accept a problem for themselves in order for it to become a real problem. Hartung claims that life problems emerge from realistic situations and have the following characteristics: undefined questions that must be formed; necessary data that must be collected, a much more complicated analysis to be done, and potentially no solution (as cited in Henderson & Pingry, 1953).

Problems come from an assortment of sources and have a variety of structures. Among the sources are people such as teachers, other problems such as when one problem is partitioned into smaller ones, and situations such as those that can be modeled mathematically (Kilpatrick, 1987). Frederickson (1984) divides problems into three types: “well-structured” where all the needed information is clear including the solution procedure; “structured…requiring productive thinking,” which require the problem solvers to generate much of the solution method themselves possibly by adding something to or restructuring the problem; and "ill-structured," which may not be clear and may lack a definite solution method (p. 392). Hiebert, Carpenter, Fennema, Fuson, Human, et al. (1996) argue that although problems that are motivational or have real-life applications may be good ones, problems should be judged primarily on whether students make them their own and on the potential residues. Van de Walle (2003) suggests an effective creation or choice of tasks considers what mathematics the students should learn, what they can bring to the tasks in terms of knowledge and understanding, and how the task might be modified to engage students in different ways and at various levels.

The traits of the problem solver play an important role in their ability to solve problems. Emphasizing a variable for researchers to consider—namely, the subject variable—Kilpatrick (1978) lists a number of specific traits typical in successful problem solving such as "the ability to generalize a relationship from a small number of instances, the ability to classify problems according to their mathematical structure, the ability to recall structural features of a problem" (p. 8). Of the dimensions pertaining to the problem solver, four arise as the most prominent: knowledge, strategies, skills, and disposition. These align relatively well with Schoenfeld's (1985) four categories within his framework for analyzing mathematical behavior: resources, heuristics, control, and belief systems. The knowledge dimension includes a number of "resources" such as factual knowledge of definitions, for example, as well as conceptual understanding of mathematical objects and systems. However, the category "disposition" includes elements such as learning style, memory, and various preferences that may exist separately from an individual's beliefs.

Of the four dimensions of the problem solver, knowledge appears to be most important. According to Hatfield (1978), "student's background knowledge of mathematics appears to be a dominant factor in successful problem solving performance" (p. 33). Without the knowledge to understand a problem, the problem solver may not evoke appropriate strategies and skills to solve it. Yet, at the same time, when students engage in problem-solving activities, they construct knowledge and gain a personal sense of mathematics (Cai, 2003). After taking a month-long problem-solving course that enriched their knowledge base, novices changed their categorization of problems from similarities on the surface to similarities in solution method (Schoenfeld & Herrmann, 1982). Understanding—a deeper aspect of knowledge such as a familiarity with how to derive a formula and with its general use—also affects the disposition component, which consists in part of the attitudes and beliefs of the student while additionally promoting further understanding and development in problem-solving ability (Lambdin, 2003). In turn, problem solving leads to understanding, especially when teachers engage students in challenging problems (Hiebert, 2003). Shimizu (2003) also emphasizes that teachers should choose the numbers in, and the context of, the problem carefully, considering the students' current understanding and knowledge base.

Although problem solvers require knowledge to understand and interpret a given problem, they must also familiarize themselves with general strategies that might be useful when solving any problem such as working backward. They should also become acquainted with specific strategies useful in particular domains or with certain problem structures such as problems with an inductive nature that might lend themselves well to proof by induction. Some problem strategies, known as heuristics, appear relatively often in a number of problems such as "trial and error, successive approximation, working backwards, drawing a pattern or representation, and inductive pattern searching" (Hatfield, 1978, p. 33). When taught heuristics such as working backward or simplifying the problem using different representations, eighth graders improved their mathematical problem-solving performance (Perels, Gürtler, & Schmitz, 2005). Lester (1978) also lists some strategies and skills identified by authors such as Seymour who emphasize that students may benefit from learning them from their teachers: "The strategies [Seymour] considers appropriate for intermediate grades include: analogy, pattern recognition, deduction, [and] trial and error" (p. 63).

Although some authors group strategies and skills together, skills differ in that the problem solver makes choices about how to implement the strategies mentioned above and must evaluate when to abandon them or continue forward and when to implement additional strategies. A problem solver needs proficiency in both to work through problems successfully; each is hardly useful without the other. Perels, Gürtler, & Schmitz (2005) found eighth graders' self-regulatory skills in mathematics increased when they were taught a combination of these skills and problem-solving heuristics but not when they were taught either one singularly. This result may have occurred because students' skills require both knowledge of strategies to use and how to monitor oneself while using them. Knowing what strategies might be useful while not knowing how to implement them will not get you much farther than knowing how to implement certain strategies but not knowing when they should be used. For example, you might notice that a problem has an inductive nature but not know how to go about structuring it so that it would fit well into an inductive proof, such as establishing Euler's formula $V - E + F = 2$. You might realize that adding an edge results in adding either a vertex or a face but may not know how to structure it neatly into a rigorous proof. You might also know how to solve algebraic equations but might not recognize that you can work backward by substituting known values into variables typically considered dependent, such as when given the formula $a_n = d(n - 1) + a_1$ and the values of a_1, a_n, and d so that you can determine the number of terms in the sequence. Students also need to understand the difference between a proof and a guess as well as the difference between a reasonable approach and one that is unreasonable (Polya, 1954). In other words, students need to understand how to evaluate how well the chosen strategy is faring while constructing the solution:

> You should not trust any guess too far, neither usual heuristic assumptions nor your own conjectures. To believe without proof that your guess is true would be foolish. Yet to undertake some work in the hope that your guess *might* be true, may be reasonable. Guarded optimism is the reasonable attitude. (p. 200)

In order to evaluate the strategy and the overall proof, a student needs to maintain an accurate understanding of the problem and goals when solving it; children sometimes reconceptualize the problem repeatedly while attempting to solve it because of either a poor conceptual system or a poor procedural system (Schoenfeld, 1985). On the other hand, good problem solvers maintain a stable conceptual model while developing and implementing strategies.

Even when a student fails to choose and implement strategies appropriately or fails to understand a problem as may even happen when gifted mathematicians first approach a problem, he or she may still persevere because of a productive disposition. A portion of a student's disposition consists of his or her beliefs, such as about how long a person should spend on a problem before giving up or whether the average person can engage in mathematics (Schoenfeld, 1985). Other relatively stable and long-term individual factors, which I include in disposition, relate to attributes such as "a student's aptitudes, preferences, cognitive structures, memory, learning styles, or personality" (Hatfield, 1978, p. 34), and may be extremely important factors to discern in problem solving. Furthermore, anxiety and fears of success may hinder a student when solving problems, and effecting changes in beliefs can improve performance as well as enable students to use more of the tools that they already have when working on problems (Schoenfeld, 1985). In fact, Schoenfeld considers some students "*naïve empiricists* who do not use the mathematical tools potentially at their disposal, because it does not occur to them that such tools might be useful" (p. 43), which he believes is affected primarily by their belief systems. Strabala (2003) discovered that sixth-grade students worked with mathematics at a deeper level and persisted much longer with difficult problems when she stressed to them that their focus should be on their thinking and how they solved the problem rather than on their answer. Students who already solve problems well typically have a number of important traits related to disposition that should be considered during research, such as "a resistance to fatigue in performing mathematical tasks, a sensitivity to problem situations, a preference for elegance in problem solutions, a reflective cognitive style, and a field independent cognitive style" (Kilpatrick, 1978, p. 9).

Independently of the problem solver, several researchers have developed models for the activity of problem solving all of which include some form of problem comprehension, solution construction, solution implementation, and solution evaluation (Lester, 1978). Similarly, Polya (1945/2004) outlines these four stages that fit directly into each of these categories: understanding the problem, devising a plan, carrying out the plan, and looking back. In their separate models, Johnson and Webb, however, combine solution construction and implementation into a category named production (Lester, 1978). In the paper by Lester, only the three models (of seven) include components related to problem posing under headings such as hypothesizing, setting a goal, and problem awareness.

Problem posing appears separately from and within the other four components of problem-solving activity. The three models that include problem posing include it as a component that comes before the others. Brown and Walter (1990) suggest that it appears implicitly within the problem-solving process as well, and thus in the other components. For example, they suggest that problem solvers might pose new problems related to the original when lacking givens, or in general when reformulating the original problem as required when attempting to solve it as would occur during the problem comprehension component. Additionally, they submit that problem posing exists in the solution construction and implementation phases when problem solvers step back and consider other focuses, as well as in the solution evaluation phase when problem solvers ask new questions, such as to look into surprising results or to clear up any dissatisfaction. In a study of 509 middle school students, Silver and Cai (1996) found that generally those with good problem-solving skills also were able to pose more problems in order to help solve a given story problem.

When thinking about problem solving, other factors arise that researchers should take into account. These variables, task and teacher, and some others are stressed by Kilpatrick (1978). While investigating research questions that require an exploration of these variables, however, the researcher should strongly emphasize the student variable foremost. For example, task variables such as context matter only in that the student may or may not have the necessary knowledge, in the sense of familiarity with the context, to solve it, a subject variable. Kilpatrick also stresses dependent variables, emphasizing the process variables that parallel the problem-solving activity aspect I have outlined above. In particular, his concomitant variable relates to problem comprehension, the process variable to solution construction, product variable to solution implementation, and evaluation variable to solution evaluation. At the same time, researchers should consider that gifted problem solvers may have a difficult time explaining how they arrived at an answer because they are decontextualists in their processing, stressing further the importance of considering subject variables (Rogers, 1999).

Because of the benefits that problem solving has for their students, teachers should make problem solving a priority in the classroom. Problem solving and problem posing can increase creativity for students that may lead to improvement in mathematical activity (Silver & Cai, 1996). Wilson, Fernandez, and Hadaway (1993) argue that teachers should "include problem solving in school mathematics because it can stimulate the interest and enthusiasm of the students." Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray et al. (1997) found that students familiar with inquiry learning and the problem-solving process could "reason mathematically and construct important understandings without teacher intervention" (p. 127). This is perhaps one of the biggest goals for our students: entering into the world able to handle mathematical situations independently or with their peers and continuing to further develop their own understanding.

Problem solving plays an important role in areas other than mathematics. After asking some engineers in the field to describe some typical problems they handled recently and how they went about it, Jonassen, Strobel, and Lee (2006) recommended that educators include more ill-structured problems and teach students a number of techniques generally associated with good problem solving such as solving problems in a number of ways, representing problems in different ways, breaking problems down into smaller ones, analyzing solutions, and focusing on interpreting and analyzing the goals and conditions of a problem. When asked to find meaningful life problems and engage in problem-based learning, preservice science teachers became much more enthusiastic toward teaching and felt that problem-based learning gave life to the science they were to teach (Etherington, 2011). Chieu, Luengo, Vadcard, and Tonetti (2010) argue that problem-solving plays an important role in orthopedic surgery training, especially in the control processes, where surgeons have to make decisions about what strategies to use and evaluate strategies in progress.

Even though problem solving benefits students in mathematics and other fields, teachers face a number of challenges when teaching it. Working with two seventh-grade classes Lester, Garofalo, and Kroll (1989) found that teaching problem solving is difficult; teachers have to be prepared (a) to teach basic skills while also teaching problem-solving strategies; (b) to work through the difficulties of weak students in communicating, completing work, and reflecting; and (c) to expect to have trouble modeling problem solving because real problems for them may be over the heads of their students. Students also struggle with real-world and nonroutine problems, sometimes because they do not have the necessary background (Peter-Koop, 2004; Verschaffel & de Corte, 1997). Additionally, teachers have pressures to cover all the content concerning their courses, and they work with textbooks that generally do not contain many nonroutine problems (McIntosh & Jarrett, 2000). To combat some of these difficulties, Van de Walle (2003) suggests that teachers choose one important lesson each week to work on, use problems from various sources such as NCTM publications, encourage students to make conjectures, and take activities from textbooks or other teachers and make changes so that students provide explanations usually given by the teachers themselves.

Conceptual and Procedural Knowledge

Perhaps one of the most influential ways teachers help students solve problems well is to help them develop both conceptual and procedural knowledge. Indeed, a student needs to have some understanding of the basic ideas in a problem and knowledge of how to proceed correctly in order to solve problems (Rittle-Johnson & Alibali, 1999). Students may not develop one or the other first, and as each increases, the other may increase as a result (Hallett, Nunes, & Bryant, 2010; Rittle-Johnson & Alibali, 1999). Hallett, Nunes, and Bryant (2010) found different profiles of students exist that rely on one or the other, implying that students who gain one type of knowledge may not gain in the other. Not surprisingly, researchers have also shown that having both conceptual and procedural knowledge are associated with lower mathematics anxiety, a construct that makes future learning and engagement in mathematics difficult (Rayner, Pitsolantis, & Osana, 2009).

As mentioned in Chapter 1, Eisenhart et al. (1993) describe conceptual understanding as having insight into the underlying principles behind the procedures, relationships, and connections to other ideas that surround that topic. Rittle-Johnson and Alibali (1999) define it "as explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (p. 175). After reviewing definitions submitted by a number of authors, Hallet, Nunes, and Bryant (2010) describe "conceptual knowledge not as the memorization of separate nuggets of information but as the ability to see interconnections between knowledge" (p. 396). In a commonly cited article concerning types of understanding, Skemp (1976/2006) describes a related concept that he calls relational understanding regarding "knowing both what to do and why" (p. 89).

Researchers generally support teaching conceptual understanding because of the overwhelming benefits to students. When describing the advantages of relational understanding with examples, Skemp (2006) outlines four benefits: "It is more adaptable to new tasks, . . . is easier to remember, . . . can be effective as a goal in itself, [and is] organic in quality" (p. 92) in that it may cause a student to seek out further understanding, motivating him or her to continually grow. Students who learn conceptual understanding in mathematics could transfer their knowledge more easily to other situations and settings, both in school and outside of school

(Boaler, 1998). Additionally, conceptual knowledge allows students to determine whether their results make sense and to develop new appropriate procedural strategies from previously learned strategies (Kotsopoulos, 2007).

Procedural knowledge is a useful and necessary type of knowledge that differs from conceptual knowledge. Rittle-Johnson and Alibali (1999) describe procedural knowledge as "action sequences for solving problems" (p. 175). In a similar manner, Hallet, Nunes, and Bryant (2010) characterize procedural knowledge as "'knowing how' to do something … that they are linearly executed and independent of meaning: An individual using a procedure should not need to reflect on what the elements implemented in the procedure mean" (p. 396). Skemp (2006) also defines a closely related concept called *instrumental understanding* as "rules without reasons" (p. 89), which is something that he might not have considered at all until he realized that many teachers and students often refer to such rules when they discuss understanding.

Procedural knowledge has some advantages over conceptual knowledge. While attempting to determine why teachers focus primarily on instrumental understanding, Skemp (2006) notes three advantages: providing an easy way to teach students to get the right answers, giving students a more readily accessible and noticeable reward when they find the right answers to mathematics problems, and helping students get the right answers more quickly and reliably with less knowledge than otherwise necessary. Procedural fluency frees the mind to contemplate complexities within a problem and allows students to handle problems that they may not be able to visualize (Wu, 1999). Basson (2002) argues that students often do not learn procedures in mathematics concerning definitions of concepts that may benefit them in physics and that teachers should provide students with operational definitions that allow them to better apply their knowledge.

Although both kinds of knowledge have their advantages, conceptual knowledge benefits students more in mathematics than procedural knowledge does. Although instruction concerning each type of knowledge may lead to benefits in the other, Rittle-Johnson and Alibali (1999) found that conceptual instruction had a greater influence on growth and transfer in procedural knowledge than the other way around. While studying different clusters of children varying in their conceptual and procedural knowledge, Hallett, Nunes, and Bryant (2010) found that students who rely on one or the other do perform differently, with evidence that those relying on conceptual knowledge had an advantage. It should be noted that those students who relied on both performed the best in the study. If a teacher has to choose, teaching conceptual understanding provides students with the greatest advantage mathematically but evidence shows that teaching both provides the largest gains.

Students provide evidence of their understanding a number of ways that teachers can look for. McIntosh and Jarrett (2000) claim that "students demonstrate conceptual understanding by interpreting the mathematical principles in a problem and translating those ideas into a coherent mathematical representation using the important facts of the problem" (p. 15). In order to discover what ideas students have and their understanding of those ideas, teachers and researchers must listen to each student, ask them about their work and to explain their reasoning, and examine their written work (Van de Walle, 2003). In order to determine students' understanding, direct observation of their work with problems and questioning of their approaches and processes must take place.

Concept of Derivative

Within calculus there are a number of broad important concepts that students take with them to higher level mathematics explorations and other fields such as engineering and the sciences. The central concepts to calculus are "limits and continuity, differentiation, integration, series, [and] differential equations" (Seltzer, Hilbert, Maceli, Robinson, & Schwartz, 1996, p. 83). Similarly, College Board (2010) identifies the central topics for their AP Calculus courses as functions, graphs, limits, derivatives, integrals, polynomial approximations and series (pp. 9–12). Physics courses use calculus techniques to derive necessary formulas and to represent phenomena in a convenient way (Redish & Steinberg, 1999). Rossman and Chance (2002) developed a new post-calculus introductory statistics course that attempts to provide a more informative and balanced approach using concepts such as optimization. A particular area of calculus, known as fractional calculus, which dates back to the creation of differential calculus has recently become applicable within the past couple decades in engineering and science for "modeling and controlling dynamic systems" (Machado et al., 2010, p. 33). Calculus also plays an integral role in determining optimum marginal costs in industrial production (Man, Modrak, & Garbara, 2011).

Among the central topics and common applications of calculus, the concept of derivative plays a key role. Authors of textbooks in introductory calculus might define "the derivative of a function f [as] the function f' with value at x given by: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists" (Salas, Hille, & Etgen, 2003, p. 122), or in a similar manner, such as by defining the derivative at a point initially and then expanding to the derivative as a function (Rogawski & Cannon, 2012). Derivatives can be thought of in a number of different ways, such as formally using the epsilon-delta definition, graphically in relation to steepness, symbolically as a formula, numerically as a gradient, and visuo-spatially as velocity (Tall, 1997).

Not surprisingly, the concept of derivative also plays a key role in a number of applications that students learn in a calculus course and in their performance while attempting to make use of those applications. Because of students' difficulty visualizing rates of change with two different quantities, they struggle to solve related rates problems (Hausknecht & Kowalczyk, 2008). Gundersen and Steihaug (2010) explored some techniques for using the second derivative in Newton's method, which they recognized as commonly useful in practical applications, and the third derivative in Halley's method. Black (1997) looked at different approaches to using the derivative in powerful ways to help students understand and solve optimization problems.

In particular, I am interested in the struggle that students have with optimization problems, which have powerful applications in the real world. In Chapter 1, I discuss and include literature about how students struggle with applications in general, optimization problems specifically, and the benefits of understanding and using optimization.

Theoretical Framework

Attempting to determine how a student understands the concept of derivative can be complicated. While developing a framework useful toward this end, Zandieh (2000) reviewed literature concerning conceptual frameworks regarding concept image and concept definition (Tall & Vinner, 1981), concept of function (Vinner & Dreyfus, 1989), multiple representations (Hart, 1991; Krussel, 1995; Tall & Vinner, 1981; Vinner & Dreyfus, 1989), and process-object pairs (Sfard, 1992). Additionally, she examined textbooks and listened to a variety of experts such as mathematics educators and mathematicians concerning the concept of derivative, interviewed nine AP Calculus students concerning their understanding of the derivative over the course of a year (Zandieh, 1997), and two students concerning the use of the formal definition (Zandieh, 1998). Zandieh (1997) demonstrates that students do not necessarily develop their understanding regarding the representations or process-object pairs in any particular order, due in part to the construction of pseudo-objects, such as an awareness of steepness before the formal ratio of differences used to calculate slope. Notably, Zandieh (2000) argues that her framework is not useful for determining the development of understanding of derivative nor in what ways students use their understanding, in part because students may not display all their knowledge in a given setting.

Zandieh (2000) found students might refer to four common different representations or contexts when asked about derivatives: graphical, verbal, physical, and symbolic. The graphical approach involves a visual understanding that the derivative is a function giving the slope of a tangent line for particular points on a graph. Students invoke verbal representations when they describe derivatives as functions for determining instantaneous rates of change. A common physical concept familiar to most students before they enter calculus, velocity proves integral in some students' understanding of derivative. Students may also recall the symbolic representation of derivative involving the limit of the ratio of two differences, one involving range and the other domain. Notably, Zandieh recognizes that other representations or contexts may exist, but that these four were particularly prominent in her search. Of these four representations, students tend to use a graphical context most often as part of their description of the concept of derivative even when solving application problems (Roorda, Vos, & Goedhart, 2010; Zandieh, 2000).

Within these four contexts, Zandieh (2000) specifies three different "layers of process-objects pairs" (p. 105), namely, ratio, limit, and function. Ratio involves the division of the difference of two variables, such as the process students often perform when finding an average rate of change or the slope of a secant line. The limit layer refers to students' understanding of instantaneous rate of change or a tangent line resulting from repeatedly calculating a secant line by choosing progressively closer points. Lastly, students display a comprehension of the function layer by referring to the derivative as a calculation of limits over an infinite set of values or an infinite set of tangent lines of a number of points on the graph. Furthermore, students may have constructed “pseudo-objects” that allow them to solve problems without understanding the problems’ underlying features (p. 107). For example, a student may bring up and use the idea of tangent lines for determining the instantaneous slope of a graph at a maximal point, without referring at all to the underlying process for calculating such a line. Zandieh mentions that students may not show more than a pseudo-object understanding because either they do not need greater understanding to solve a given problem or they actually may not have the underlying understanding.

Expanding upon Zandieh’s framework, Roorda, Vos, and Goedhart (2007a) include application contexts for understanding the concept of derivative. Based on another framework developed by Kendal and Stacey (2003) that also emphasizes the use of multiple representations by students to gauge their understanding, Roorda et al. chose to use the following representations: numeric, graphic, and function. Kendal and Stacey’s numeric representation involves using values from a table and calculating rates of change over values. Roorda et al. also chose to eliminate Zandieh’s verbal representation altogether. This change stemmed from a realization that the verbal context tends to be redundant because students can verbalize their understanding of the other contexts. Zandieh’s physical representation was converted to an application representation involving physics.

While studying how nine high school juniors transfer their knowledge of the derivative to modeling situations and applications, Roorda, Vos, and Goedhart (2007b) discovered a need for a framework that allows translations between and within representations to be visualized. In a chapter that influenced Roorda et al.’s formulation of this framework, Dreyfus (1991) discusses the importance of translations between and transformations within representations. Roorda et al. (2007a) validated their framework while interviewing six students in a task-based interview.

Essentially, Roorda et al. (2007a) added four other domains that involve: general applications, physics concepts related to acceleration, chemistry concepts concerning reaction time, and economics concepts with respect to marginal costs. Additionally, the augmented framework included a new layer at the initial functions level. Most importantly, Roorda et al. specified students' progressions within and between different representations; the progressions are indicated by arrows in the model. Along with the arrows, a star was used to indicate when a student mentioned a representation layer but did not use it in his or her work toward a solution. Roorda et al. admitted, however, that their framework has some weaknesses in that the absence of a particular context or layer in a student's solution does not mean that the context or layer is not known to the student. Realized in a longitudinal study concerning the development of student's development of the derivative concept from Grades 10 to 12, another admitted weakness resides in an inability to display a student's misstatements that appear to hinder the student's ability to solve problems (Roorda, Vos, & Goedhart, 2010).

Protocol analysis has been used by various researchers as a tool for studying problem-solving processes. Some researchers created codes to analyze problem-solving behavior at a macroscopic level, which became rather cumbersome and unaesthetic (Schoenfeld, 1983). While studying the problem-solving processes and their change after some development in nine eighth-grade students, Kantowski (1977) focused on a subset of heuristics, pointing out the need for an efficient protocol analysis. Within these protocols that focused at the macroscopic level, behaviors had to be explicit in order to be coded, which did not allow for researchers to note absences of certain decisions or behaviors (Schoenfeld, 1985).

In an attempt to focus on decisions made during the problem-solving process, Schoenfeld (1985) developed a framework that involves "[partitioning protocols] into macroscopic chunks of consistent behavior called episodes" (p. 292). He identified six common behaviors in problem solving: reading, analyzing, exploring, planning, implementing, and verifying. Reading episodes include any actions involved in reading the problem and the silence that may occur afterwards as a student processes the reading or silently rereads the problem. As a student orients himself or herself to the problem and breaks it down, the analyzing episode takes place. When students are less familiar with the problem, they may begin an exploring episode that sometimes lacks structure as they begin to search for strategies or facts that may have some relevance. During a planning episode, the student constructs a set of procedures that has potential to solve the problem. Afterwards, the student attempts to follow the plan, and the implementation episode occurs. Oftentimes, however, students will plan and implement simultaneously, typically evaluating their progress toward the goal in some structured way. Once students have implemented the plan and have a result, they may verify their answer or solution in some way. When a student switches between these behaviors, one episode ends and another begins, marking key periods in the student's problem-solving decision-making. In addition to identifying and defining the six types of episodes, Schoenfeld (1985) included key questions typically focusing on decisions made for the researcher to ask while analyzing the episodes. Students may not switch between these behaviors in any particular order. Realistically though, students will presumably begin by reading the problem. Furthermore, Schoenfeld includes methods for analyzing transition periods between episodes as well as moments when a student realizes new information or performs assessments within episodes.

CHAPTER 3

METHODOLOGY

Patton (2002) says, “a qualitative case study seeks to describe … in depth and detail, holistically, and in context” (p. 55). In order to determine how AP Calculus students solve optimization problems, I needed to explore in detail their solution process in response to these types of questions, looking at strategic decisions during the solution process that might lead to an understanding of their struggles. By studying individual students, I gained a general feel for their experience in AP Calculus and their dispositions toward mathematics that also influenced their problem solving. Furthermore, because I believed that the concept of derivative plays a role and I wanted to gain a full view of tools at their disposal, I asked probing questions concerning their knowledge of the concept and their solution process.

Participants

After receiving approval from the University of Georgia’s Institutional Review Board and a Northwest Georgia county board of education to conduct my study, I visited three AP Calculus classrooms at two high schools. Two of the classes were AP Calculus AB, and one was AP Calculus BC. Quite a few students showed an interest in the study and took home permission forms to discuss the possibility of participating with their parents. Unfortunately, by this time, only one and a half weeks remained of the school year. Most of the potential participants were seniors, some of whom were moving out of town or traveling with their families on a final vacation before entering for college. I accepted any willing volunteers from these classes. Luckily, seven students each arranged and participated in an interview during the month of June.

I interviewed the first two students using the protocols to be discussed below in order to determine if any adjustments needed to be made. I detailed these adjustments in the Pilot section. The interviews of the final five students were analyzed and included in this study. All seven students had successfully completed an AP Calculus AB course. Two of the students, one of whom was in the initial two participating in the pilot, had completed 2 years of AP Calculus, AB and BC. I have given the five students included in the analysis of this study the following pseudonyms: Alice, Brenda, Chris, Donna, and Ethan.

Alice decided to take AP Calculus AB her senior year because she did well in honors precalculus and liked where the mathematics was going. She enjoyed the course, especially since she continued to use what she learned throughout the course such as finding derivatives. In particular, she liked how her teacher laid out the step-by-step procedures. During the course, she struggled with the unit circle, which she apparently had less trouble with in precalculus. Confident she could earn a passing score, she decided to take the AP examinations. When asked how she thought she did, Alice responded that she felt okay but that there were some things that were difficult, a result of some scheduling issues that had given her limited time to learn the concepts. According to her, several students were struggling in the 90-minute block class every other day, so they had opted to add a calculus support class on the days they did not have calculus.

Other than Calculus, Alice took AP Literature and Honors Economics but did not feel confident enough to take the AP Literature examinations. She did take both the ACT and SAT, preferring the former. She scored around a 25 or 26 in mathematics on the ACT but could not remember her scores on the SAT. After high school, Alice planned to attend a South Georgia university while majoring in mathematics because she thought that it was the one thing she understood. In the future, she hoped to either teach or enter an engineering field.

Brenda took AP Calculus AB her senior year hoping for a challenge. She found the class easy, however, but enjoyed following set procedures given by her teacher and using the various rules in different ways. She did not take the AP examinations because she forgot to turn in her money and because her intended major did not require it. She did find one of the concepts near the end of the course challenging but could not remember what it was.

Throughout high school and middle school, Brenda had taken honors classes, including AP U.S. History, AP Language, and AP Literature. She took the history and language examinations the previous year but scored 1 (out of 5) in both. She earned high grades in the courses, however. When asked about other examinations, Brenda mentioned that she had scored 1290 on the SAT and 27 on the ACT. In the mathematics portions, she scored in the 600s and a 27 or 28, respectively. She planned to attend a large South Georgia university in athletic training and eventually become a physical training assistant or physical therapy assistant.

During his senior year, Chris chose AP Calculus AB because it was the only option besides AP Statistics. He had already struggled in precalculus and then taken discrete mathematics, which turned out to be essentially advanced precalculus. He did not enjoy any of the mathematics courses he had taken and struggled with any use of formulas or memorization, particularly concerning trigonometry. When asked whether he had enjoyed anything in the calculus course, Chris mentioned that calculus seemed to have more applications than other courses but that the teacher did not give many examples. Not feeling he would pass, he decided not to take the AP examination.

In addition to AP Calculus, he had taken AP Literature that year and AP Language the year before. He had started out in AP Biology, his favorite subject area, but had to drop it for Honors Anatomy/Physiology after getting a job. Chris took the SAT, scoring high in mathematics, but he could not remember his score and had not taken the ACT. Although he had not decided on his major, he had interests in biology-related fields and planned to attend a Northwest Georgia university. He had no particular career plans.

Donna took AP Calculus AB her junior year because that was next in the progression. She found the class relatively easy because she could fall back on the basic formulas she had learned at the beginning, which she liked. She did find it challenging to remember all of the different specific formulas, however. Because of the expense, Donna did not take the AP examination but thought that she would have done alright.

The past year, Donna also had taken AP Language, AP U.S. History, and AP European History, but she took no examinations because of the cost. She predicted that she would have done best in AP Language and AP European History. The following year, she planned to take AP Literature, AP Statistics, AP Anatomy, and AP Economics. The last two were probably actually AP Biology and AP Government. She had taken only the ACT, scoring a 23 overall and either a 23 or 24 in mathematics. In college, Donna hoped to study psychology to become either a psychologist or psychiatrist. She would like to take AP Psychology as well, but only if it fit into her schedule.

Ethan began with prealgebra in sixth grade and followed the progression of accelerated mathematics up to AP Calculus BC in his senior year. He took AP Calculus AB his junior year and mentioned that the courses did have a lot of overlap. The BC course, however, had much more advanced problems and progressed quickly. Even though he enjoyed the course, Ethan struggled with trigonometry problems, which had also challenged him in precalculus. On the other hand, he enjoyed the application problems because he hoped to use them in the future. He did not take the AP examination either year because of the cost.

Other than calculus, Ethan took AP European History, AP U.S. History, AP Language, AP Literature, and AP Biology. He took the AP Biology examination and felt pretty good about it. He could recall only his composite scores on the SAT and ACT, which were 2010 and 31, respectively. Admitted to the Naval Academy, he planned to study aerospace engineering. As for career plans, Ethan hoped to go as far as he could in the Navy and then evaluate from there where to go next.

Schools

The five students attended two high schools in the same county in Northwest Georgia: East High School and West High School (both pseudonyms). Alice and Brenda attended East High School, and the other three attended West High School.

East High School had approximately 700 students. Over 85% of juniors in 2009 through 2011 passed the Georgia High School Graduation Test in mathematics. The SAT average in 2010 was about 1550. The race/ethnicity breakdown was about 85% White and 15% nonwhite groups.

At East High School, students have the option to take AP Calculus AB, but not BC. They typically take it in a 90-minute block every day all year. They receive two credits, one for AP Calculus and one for regular calculus. If students need to take another class, they usually have the option to take either Honors Discrete Mathematics (HDM) or AP Statistics.

A few changes had been made in the mathematics curriculum at East High, specific to their school, because of the new Georgia Performance Standards (GPS). With the new curriculum, HDM has been replaced by a similar course called Advanced Decision Making, which includes some personal finance topics. Because the GPS did not originally have as many accelerated options as there had been in the past, fewer students were at the calculus level during these students' senior year. As a result, the school could offer only an every other day course for one credit. During the second semester, however, most of the students found a way to fit in a calculus support elective and were able to gain back some of the extra time previously offered. The teacher of the course typically had a high pass rate, but only two students took the AP Examination, including Alice.

West High School had approximately 800 students. Over 75% of juniors in 2009 through 2011 passed the Georgia High School Graduation Test in mathematics. The SAT average in 2010 was about 1700. The race/ethnicity breakdown was about 75% White, 15% Hispanic, and 10% other groups.

At West High School, students took AP Calculus AB as a 90-minute block class, every other day for one year. The year of the study, three students chose to take AP Calculus BC after having completed the AB course. The class took place in a distance learning lab, which allowed one student from another nearby high school to learn the content as well. The teacher taught through a computer and document projector at each school, and the course could be tuned into from any school in the county. It also was the teacher's first time teaching BC after having taught the AB course for a number of years in the regular classroom.

All subjects participated in the study at a local library during the summer. The library supplied small rooms separated from the rest of the library with a small door. The rooms also contained a table that allowed for two people to sit down side-by-side and had enough depth to easily fit several sheets of paper and recording equipment discussed in the data collection section. The door was generally kept partially ajar to keep from overheating but outside distractions were essentially nonexistent due in part to regularly roaming librarians outside in the main portion of the library. One student, however, felt more comfortable participating at the library of the local school that the student attended. At the center of the library in a large open area full of tables, we sat at a circular table large enough to allow for eight people to sit side-by-side. No one else used or entered the library during the interview, making distractions nonexistent.

Instrument

Each participant was asked to solve a set of three optimization problems that were given one after another on separate pieces of paper as the student attempted each one. I chose 10 optimization problems (see Appendix A) from calculus textbooks and other related sources that were examined by high school AP Calculus teachers, mathematics professors, and mathematics education professors, who selected the three problems that best met the goals of the study. Specifically, I asked them to choose three problems that should be reasonable for the students to solve, that were interesting, and that students might solve in different ways, with the most convenient ways involving the derivative.

The three problems chosen were Number 4, 6, and 9 in Appendix A. From this point forward, I have numbered them as Problem 1, 2, and 3, respectively. I chose these three because all three required students to determine the objective function while being relatively easy to understand and envision. More importantly, the AP teachers and college professors alike found these to be the most interesting which told me not only that these problems had a good chance of being adopted by the participants as problems for themselves but also that AP teachers and college professors would want their students to be able to solve problems similar to these. In fact, these were almost exclusively and unanimously chosen with one professor having an interest in Number 10 because it explicitly involved angles. Also, a student also did not necessarily need to have studied calculus to solve the problems although it would have been helpful. The three chosen problems were the following:

1. A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?
2. Find the point P on the parabola, $y = x^2$ closest to the point (1, 0) (Rogawski, 2008, p. 225).
3. Find the maximum length of a pole that can be carried around a corner joining corridors of widths 8 ft and 4 ft (Rogawski, 2008, p. 229).

In particular, Problem 1 has some similarities to the word problems calculus students typically see in class with a small twist concerning the cost of one side of the fencing being different. When using the derivative of the objective function in Problem 2, one must still use more than basic algebraic techniques to solve it. Problem 3 is simplistic and something these students would likely have encountered before in the real world, but it still requires some ingenuity involving geometry and algebra to solve.

To develop a context concerning the students, their experience in an AP Calculus course, their disposition toward the subject, and future plans, I decided to interview the students concerning these background factors and developed questions in an interview protocol (see Appendix B). I chose first to ask them about their AP Calculus course and their thoughts concerning the course for several reasons: it gave me a sense of how their class was structured and their disposition toward calculus; they expected me to ask them about calculus, they would likely have a relatively easy time answering the questions, which then ideally made them more comfortable and open to answering further questions. Immediately afterward, I asked them about their course load, background, strengths in different subjects, and scores on standardized tests, which gave me some insight into their ability and achievement from which I could make comparisons while analyzing the data. These questions also gave me a sense of what the school stressed academically. After asking them about their past experiences, it seemed appropriate to ask about their future plans concerning college and career, which also gave me a sense of how much the goals and motivation toward learning the concepts in the course depended on whether they were likely to need the concepts from the course. In an effort to first develop some rapport with the students and ease them into speaking about themselves and their thoughts, I used the background interview portion at the beginning of the study before the students engaged in problem solving.

To further understand each student's solutions and decisions during the problem-solving portion of the session as well as to probe his or her understanding of derivative, I developed a follow-up interview portion protocol (see Appendix B). I began by first asking about their interpretation of each of the problems they had attempted, followed by individualized questions concerning noteworthy events during their solution process. The purpose was to make explicit their strategies and considerations as well as to increase validity concerning my interpretations of what I had seen during the problem-solving session. I then asked what they thought the problem-solving session was all about to see if they had made some connections themselves between the different optimization tasks and their problem-solving processes. Following this, so as not to influence this past response, I asked the students about their understanding of the derivative by probing them concerning their definitions, images, representations, and applications concerning the concept. One of the main reasons that I included these questions was that the students might have had access to knowledge of the concept that they might not have used during the three optimization tasks, and I needed to find any evidence of understanding that I could in as many ways as possible. After the probing questions about their understanding of the derivative, I asked them specifics concerning their use of the derivative anywhere it had occurred in their solutions in an attempt to understand what had evoked its use. I concluded with a “warm-down” question regarding their experience in the study, which also was intended to give me an insight into the student’s problem-solving processes and answers to questions.

Data Collection

To begin the study, I interviewed students using a semi-structured interview style using the interview guides specified above. The interview-guide approach allowed me “to build a conversation within a particular subject area, to word questions spontaneously, and to establish a conversational style but with the focus on a particular subject that has been predetermined” (Patton, 2002, p. 343). Therefore, I collected information regarding the interview questions and pursued other interesting topics as they arose.

After the completion of the background questions portion, I introduced the students to a printed version of the think-aloud protocol, which I read aloud to them (see Appendix B). Taylor and Dionne (2000) mentioned “analyses of … solutions, even when supplemented with observations of the problem solver silently at work, are often inaccurate because they require high levels of inference about covert thinking processes” (p. 413). In order to elicit the thinking processes of the students while solving the task, I used a think-aloud protocol, which provides a more reliable picture than simply asking students to recount their experience afterwards (Patton, 2002). In short, I asked the students to remember to read the upcoming problems aloud, interpret them, and solve them while speaking aloud. While each student worked on the problems, I instructed the student not to erase any written work, but instead to scratch or cross out work if necessary. When the occasional student did not follow this request, I reminded him or her of it and made a note to myself.

After the think-aloud protocol had been introduced and I had addressed any questions the student had about it, I gave him or her the optimization problems one at a time on separate sheets of paper. I made brief notes about the solution process that might not have been caught by the data collection instruments as well as points to ask about later in the follow-up interview portion. This direct observation and interaction during the solution process allowed me to get at the students understanding which "comes … from trying to discern how others think, act, and feel" (Patton, 2002, p. 49). Furthermore, "to get at deeper meanings and preserve context, face-to-face interaction is both necessary and desirable" (p. 49). Additionally, Shapiro (1973) supports that direct observations and interactions with the students provides insight into important differences in students. Also during the problem-solving portion, if the student did not speak or work for 10 to 15 seconds, I used minimal speech prompts, such as *keep talking*, so as to limit interruption of the thought process and to avoid unintentionally guiding the student in a particular direction (Taylor & Dionne, 2000). If the student had such trouble during the task such that he or she made no progress, I probed further responses using Polya's (1962, 2004) problem-solving framework.

I audiotaped the entire session and videotaped for back-up. These technological tools allowed me to make an accurate record of the session in a less obtrusive manner. I could focus primarily on taking notes about key aspects of the solution process and identify questions to ask in the follow-up portion of the interview without being entirely distracting (Patton, 2002). After meeting with the student, I transcribed the entire session using primarily the audio recordings. I used the video recordings to provide insight and detail when the events in the audiotaped recordings were unclear. These transcripts made it easier to revisit various parts of each session and to analyze using my theoretical framework described in the next section. The video recordings specifically allowed me the ability to reexamine parts of the solution process in detail during analysis to ensure proper coding. Additionally, I kept all work the student produced and any notes that I took during the session. After the session, I made some final notes of the session about any unexpected or interesting events that took place during the interview to review when analyzing the data. The only such events were when the videotape or audiotape did not function properly. Luckily, at least one of the two recorded as expected in each interview.

Pilot Study

After interviewing the two students in the pilot study, I made some adjustments to the interview process. One adjustment regarded how much information I was willing to give the participants. During their work on Problem 2, I found that the two students could not recall the distance formula accurately. Because knowing the formula was key to making further progress in the problem and because I was more interested in their problem-solving process than whether the remaining students knew this relatively unimportant fact, I decided to give them that or any other formula if they could not recall or determine it themselves. As justification for my decision concerning the study of problem-solving processes, I note that Schoenfeld (1985) said that "whether or not the students will accurately remember the formula [is] virtually irrelevant" (p. 289).

I made a few other adjustments concerning time spent on the background interview questions, time spent doing the tasks, and questions to ask during the follow-up interview portion. Originally, I planned to spend 10 to 15 minutes on the background interview portion, 20 to 25 minutes on each problem, and 30 to 40 minutes on the follow-up interview portion, totaling between 1 hour 40 minutes and 2 hours 10 minutes. Because the students took 5 to 10 minutes to get warmed up to Problem 1, I decided to spend more time there. Additionally, I realized that I wanted to spend more time in the follow-up interview portions probing their responses and their apparently minimal understanding of derivative. As a result, I spent less time probing information concerning the students' background and instead allowed them for the most part to give me as much information as they would volunteer on their own. In the follow-up portions, I generally asked about their thoughts concerning word problems because that issue tended to come up a lot while they worked the tasks. I felt that their attitudes toward and definitions of word problems might give me some insight into their disposition toward certain types of optimization problems and may, therefore, have had some influence on their problem solving. Due to these changes, I decided to spend about 5 to 10 minutes on the background interview portion, 20 to 25 minutes on each problem, and 45 to 55 minutes on the interview portion, totaling between 1 hour 50 minutes and 2 hours 20 minutes.

Additionally, the three chosen problems, which I discuss in more detail in the next section, seemed adequate because the two pilot students had an easy time understanding them and recognized some necessary tools that could have been helpful even though they still struggled to put together coherent and valid solutions. Because I expected the students to struggle in this manner, as mentioned in Chapter 1, and because I felt their struggles were not due to misunderstanding the problems, I did not make any changes to the chosen problems. The two students expressed a strong desire to know the solutions after the conclusion of the session, which gave me the impression that they had accepted the problems for themselves.

Data Analysis

The final interview portion concerning the student's concept of derivative was analyzed using Zandieh's framework (2000). After reviewing the transcripts and video of the follow-up interview portion from each session, I noted the representations and layers that the student had readily available concerning the concept of derivative and whether they appeared to be pseudo-objects or not by using her method of filling in a chart (see Figure 1) with the three layers as the rows and the four columns as the representations. For example, if a student mentioned that a derivative could be used to find the velocity of a person at a particular point in time, then I would place a circle at the intersection of the limit process-object layer row and paradigmatic physical context column. When the student displayed an object-process as only a pseudo-object, the chart received an open circle. On the other hand, a closed circle denotes that the student provided evidence that he or she had an understanding of the underlying process. If a student talked about the slope of a graph at a point but did not seem to know how that slope was calculated other than by finding the derivative and plugging in a value, then I would place an open circle at the intersection of the limit process-object layer row and graphical context column.

	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1. Outline of the framework for the concept of derivative (Zandieh, 2000, p. 106).

Student performance on the optimization tasks, as observed in the transcripts, student work, and video, was analyzed using the framework developed by Roorda, Vos, and Goedhart (2007) to further inform me about each student's understanding of the concept of derivative. For each problem, sequences of codes and arrows were created to follow the flow of representations and layers during each use of the derivative in each task. For example, if a student received the sequence F1 → F4 → F3, that student noted the function, found the derivative of that function, and then used the derivative function to determine the value at a point. This is an example of transformations within a single type of representation. An example of switching between representations, F1 → G1 represents a student first noting a function and then creating a graph of that function. If a student stopped one strategy to begin another, a new sequence of codes and arrows began.

Once the student performance was analyzed concerning the student's understanding of the concept of derivative, I further analyzed it following Schoenfeld's (1985) framework for analyzing problem-solving sessions, which involves identifying six types of macroscopic episodes, the transitions between them, and then asking particular questions depending on the type. In order to do that, I read through the transcripts looking for details outlined by Schoenfeld to help me identify the episodes, highlighting each with a color: reading (red), analyzing (orange), exploration (yellow), planning (green), implementation (blue), planning-implementation (blue-green), verifying (purple), and transitions (brown). In Schoenfeld's discussion of his framework, he mentions that he used three undergraduates, in consensus, to parse the protocols, which matched his parsings reliably well. In my study, a mathematics education professor independently coded the first problem for each person. Afterwards, we discussed our coding results. When necessary, I made adjustments to mine. I then alone coded the second and third problem, which we reviewed together and reached a consensus. Schoenfeld's questions designed specifically for each type of episode also helped me code identified episodes.

And generally, I looked for evidence as to how the student interpreted and understood the problem, as well as how the student approached solving optimization problems. For each student, I searched for similarities among how he or she solved each task. Also, I looked for similarities among students across the tasks. The follow-up interview portion was used to further inform this analysis.

Lastly, I looked for patterns and connections between the students' understanding of derivative and their performance on the optimization tasks individually and as a group. Therefore, I noted any patterns or connections between the chart, sequences, and problem-solving activity. Additionally, I used information from the background questions to determine potential reasons behind the connections between students.

Limitations

This was the first time that I worked with all three frameworks, which may have yielded some error. I made attempts to limit error by coding data from several research articles in order to calibrate my coding methods with experts and continued to do so until I gave nearly identical codes as the researchers themselves. As mentioned above, I also came to a consensus with another mathematics education professor to limit error. Our individual parsings on Problem 1 agreed 72.4%. I measured this by counting the number of words coded the same as in my parsing and divided by the total number of words that were coded. We then only had to come to a consensus on the remaining 27.6%. From strongest to weakest, each episode type agreed as such: verification 100.0%, reading 99.8%, exploration 85.0%, implementation 75.4%, planning-implementation 65.1%, planning 60.7%, and analysis 40.7%. Schoenfeld's *exploration* category has some similarities to the *analysis* category as well as the *planning* category, which makes determining the process the student was using somewhat subjective, which Schoenfeld admits. He mentions that behaviors characterizing them are not necessarily mutually exclusive. On the other hand, Schoenfeld emphasizes that behaviors regarding reading, being stuck, and verifying are mutually exclusive. These behaviors seem to explain the high agreement in reading, exploration, and verification episodes. At times, I provided some direction to students that seemed to have no idea how to proceed, which may have influenced some of the patterns in their problem-solving activity.

CHAPTER 4

RESULTS

The results of this research study are first organized by question. Concerning Question 1, the results are organized by problem and then by participant within each problem. Regarding Question 2, the students did not give me enough data to answer the question because only one student used the derivative to solve the problems. I supply a bit more detail concerning this in that section. I provided a background of each participant in Chapter 3. I gathered data using audio and video observations, field notes, and the students' work on their problem-solving and the interview portions. The time spent in the background (B), problem-solving (P1, P2, P3), and follow-up (F) portions for each participant are displayed in Figure 2.

Alice:	**Brenda:**	**Chris:**	**Donna:**	**Ethan:**
B: 10min 10sec P1: 22min 47sec P2: 20min 44sec P3: 23min 9sec F: 46min 1sec Total: 2hrs 2min 51sec	B: 8min 50sec P1: 22min 44sec P2: 22min 38sec P3: 25min 13sec F: 50min 58sec Total: 2hrs 10min 23sec	B: 7min 15sec P1: 21min 41sec P2: 25min 42sec P3: 20min 23sec F: 52min 2sec Total: 2hrs 7min 3sec	B: 6min 6sec P1: 22min 44sec P2: 23min 16sec P3: 13min 21sec F: 31min 36sec Total: 1hr 37min 3sec	B: 6min 5sec P1: 35min 56sec P2: 23min 41sec P3: 24min 49sec F: 53min 45sec Total: 2hrs 24min 26sec

Figure 2. Time spent in the interview portions by each participant.

Question 1: AP Calculus Students' Solutions to Optimization Problems

In order to study how students solve optimization problems, I first observed them solving three optimization problems, which are included in Appendix A, and the discussion of the choice of the problems can be found in Chapter 3. Using the Follow-up Questions portion of the Interview Guide (Appendix B), I asked them about their solutions and various moments occurring during the solution process. I analyzed the solution attempts data using Schoenfeld's (1985) Framework for the Macroscopic Analysis of Problem-Solving Protocols. For each participant, organized by problem, I provided a brief summary of my analysis of the solution attempt using the framework, a summary of his or her solution attempt, and a summary of the follow-up interview portion. With each problem-solving framework summary, I include a figure to help visualize the solution attempt in terms of the macroscopic episodes.

> Problem 1. A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs $2 per foot, while the fence for the other three sides costs $1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

Alice

Problem-Solving Framework. In Alice's solution attempt, her episodes flowed nearly in the order presented by Schoenfeld (1985) from reading to verification (see Figure 3). Each type occurred once, except for planning, which was absent. She spent most of her time in analysis and implementation. The entire attempt lasted close to 22 minutes 47 seconds.

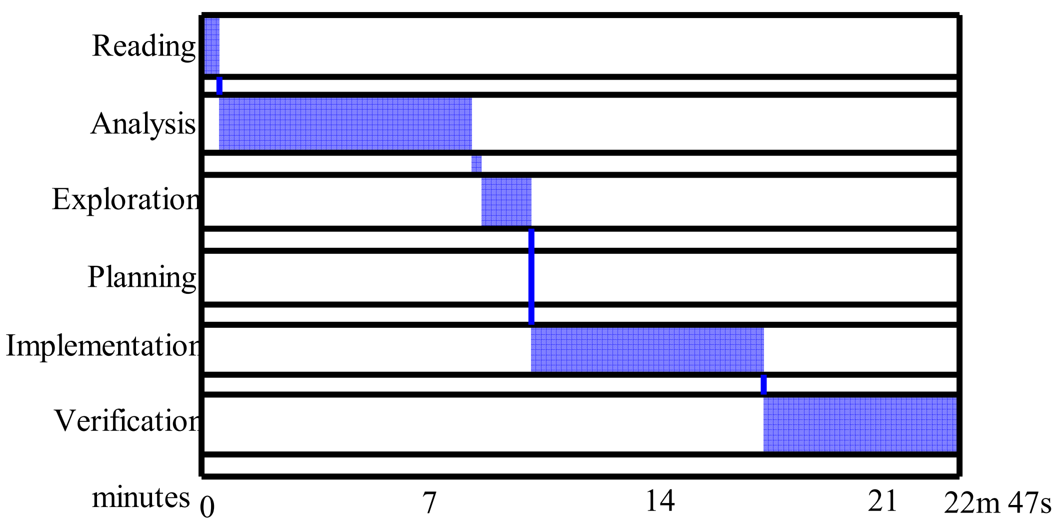

Figure 3. Summary of Alice's problem-solving episodes for Problem 1.

Solution attempt. Alice began by reading and orienting herself to the problem, failing initially to note the goal state. She first drew a rectangle and labeled the sides *a* and *b*. Not knowing what she could do with the variables *a* and *b*, she tried to bring the monetary values into the problem by setting $b = 2x$ and $a = x$. (see Figure 4). During this time, she concentrated on the area and explored any ideas she thought might be related to area such as integration. Alice also considered at this point whether the graphing calculator might be helpful. Realizing that the problem simply involved the base and height of a rectangle, she scratched the integration idea in favor of the geometric formula for area, which is base times height. Using this formula, Alice set up the equation $60000 = (2x)(x)$, solved for *x*, which is approximately 173.21, doubled the value to determine the other side 346.42, and then rounded them up to 174 and 347. Afterward, Alice reread a portion of the question to determine what remained and began to calculate the cost of the fence using the dimensions recently discovered (erroneously). Alice made a number of errors concerning the calculation of total cost, adding two sides costing $174 and one side costing $347, for a total of $695. She then seemed confused about the values she had found. Part of this confusion may have been because she was not exactly sure what the values represented, as evidenced by her responses in the follow-up interview. As a result, she paused to think about whether she was answering the question correctly and reread part of it. When asked about checking her result, she simply verified that the area worked, which led her to make minor adjustments to her rounded values so that the area was closer to 60,000. I then asked her whether other side lengths could give the same area, to which she responded that she did not think so. During the attempt, she never tried to create an objective function concerning cost.

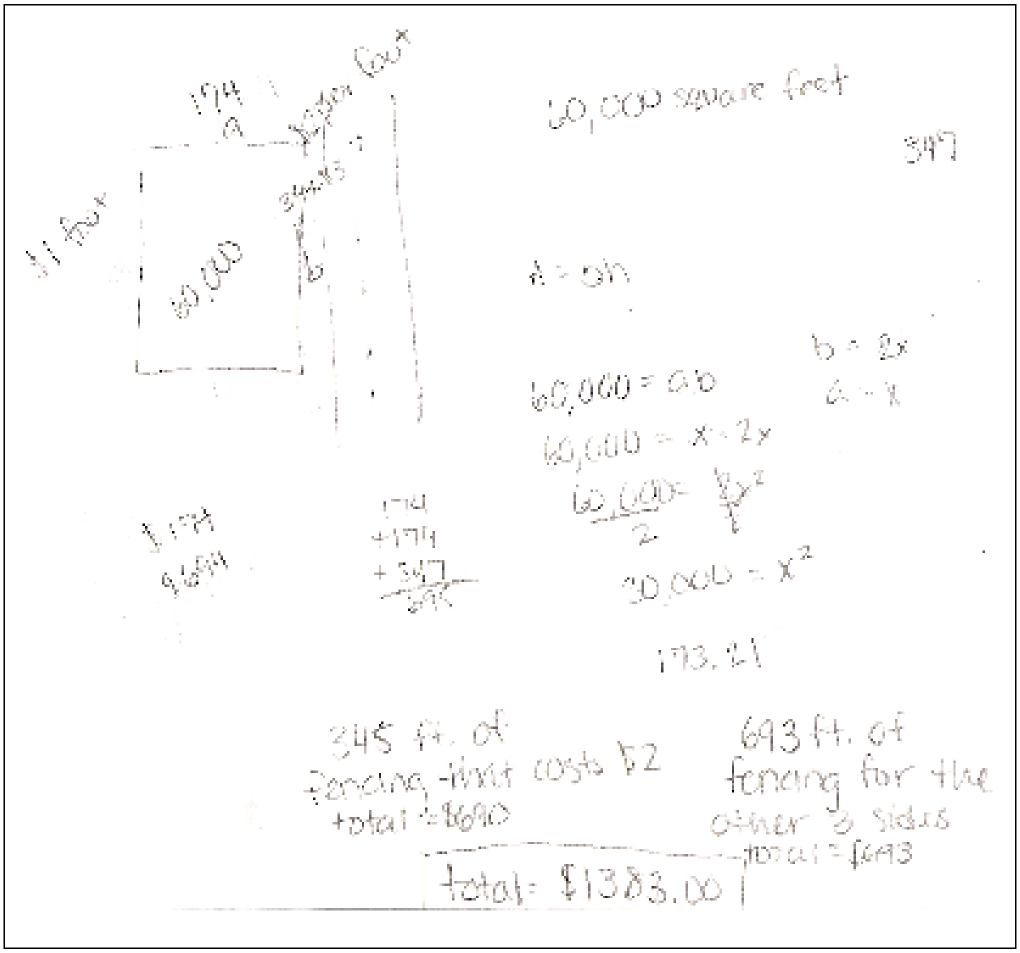

Figure 4. Student work by Alice during Problem 1.

Alice: Um, I was thinking that anything you really do on the calculator really doesn't do much. Because this is basic algebra, you can pretty much do it on your own. So, this is pretty much all this is, and a little geometry mixed in with it, I guess, because of the sides. But the calculator to me is more about graphing than anything. That is the main thing that I use it for, and that is why it was very useful for Problem 2, I think it was.

Interviewer: And, there is no graphing in this problem?

Alice: Mmm mmm.

Interviewer: Why not?

Alice: I don't think so, because you really can't—. There is really no *x* and *y* variables; there is no…. It is pretty much set out straightforward, so I don't think there is any need to graph in this problem.

Alice appeared to believe that graphs have no use in basic algebra and geometry problems and that one must have an *x* and *y* in order to create a graph. Afterward, I asked about her use of the integral and why she discarded it. She mentioned that the integral is used for area under a curve and not for solving problems involving base and height. Following this response, I questioned her about rounding up her answers. She responded that rounding to whole numbers was much easier and that you would rather have too much area than too little. I concluded this part of the interview by asking her if she felt she had everything she needed to solve the problem, whether it was clear, and what she found most difficult. Alice responded that everything was laid out for her in the problem, that she understood it easily, and that the most difficult part was that you had to "undo area" to find the sides.

Brenda

Problem-Solving Framework. Like Alice, Brenda initially moved from reading to verification in an order similar to that presented by Schoenfeld (1985), skipping exploration. She returned, however, to exploration afterward, twice alternating between it and planning-implementation. Most of the time, she remained in the planning-implementation episodes (Figure 5). The entire solution attempt lasted about 22 minutes and 44 seconds.

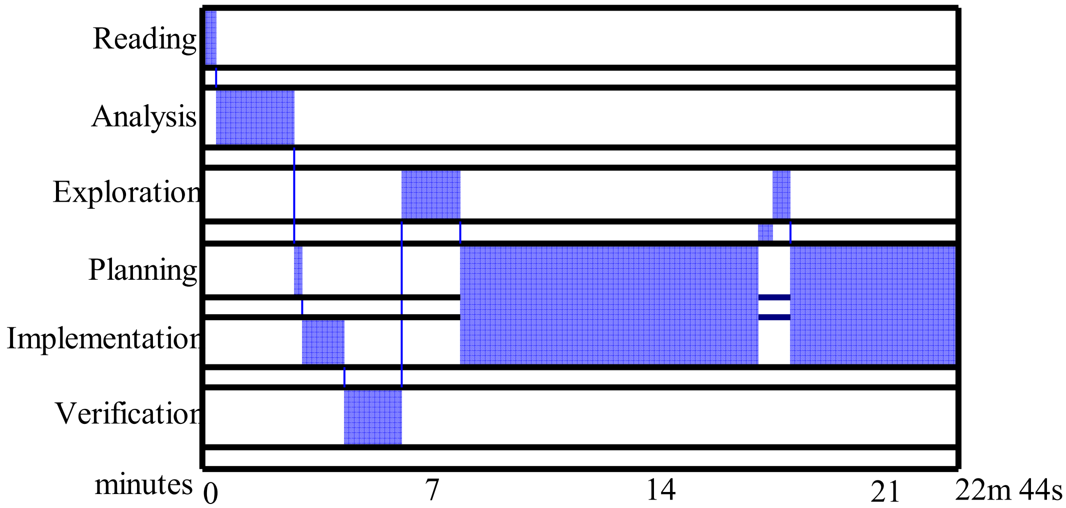

Figure 5. Summary of Brenda's problem-solving episodes for Problem 1.

Solution attempt. Brenda began by reading the problem and then drawing a rectangle along a highway (see Figure 6). She labeled the all the sides with different variables before realizing that the sides across from each other were equal. Soon after, she recalled the area formula, length times width. A source of some confusion later in the solution attempt, she made some incorrect statements, setting each variable equal to its cost per foot. Brenda took note of the minimum condition and realized that the side along the highway should have been the shorter side. Because of this realization, she attempted to make that side as short as possible which in her mind was 2 feet. This yielded a cost of $60006 that she stated was the minimum. Afterwards, however, she asked whether it could have been smaller and might have been somewhat unrealistic. When trying with a 1 foot side, she calculated an incorrect $60003.

Fortunately, this calculation led her to explore some other ideas leading to cheaper costs. During this exploration, the idea that the most expensive side should be the shorter one remained firm. She decided she should try some other values and implemented that approach until she had the side lengths 500 and 120, which yielded $1360. Brenda followed with 100 and 600, which she realized would yield more. When asked when the process would stop, Brenda said that it would have to stop by the time the side lengths parallel to the highway were longer, because they were more expensive. During this implementation, she made a comment that they had learned a better way in class that would combine the variables but that she knew how to do it only when the side lengths were equal in cost.

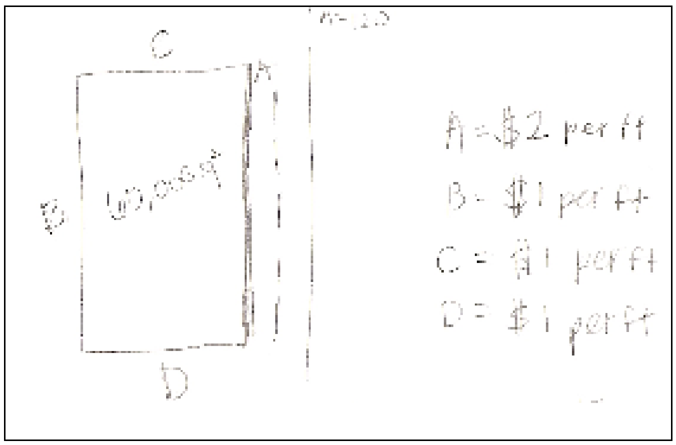

Figure 6. Brenda's drawing and variable definitions.

Because of this comment, I offered to change the problem for her to $1 per side to see what she could do with it. Brenda began with a plan to divide up the side evenly but disliked the solution because the side lengths were not particularly nice. She then considered a ratio of 1 to 2 for the sides. When she made these comments, I believed she was faintly remembering the end results for when the rectangle sides were given equal weight and for when one side was missing. When asked about the process she had used in class, Brenda commented that they had one but that she was never good with word problems, concluding the solution attempt.

Follow-up interview portion. I began this interview portion (14 minutes 50 seconds) by asking about the beginning of her solution attempt where she tried 30000 feet for the less expensive pair of sides. Brenda explained again that she felt that the more expensive sides should have been as short as possible and decided to test that idea by using 2 feet and 1 foot to determine if that would have worked. During her explanation, she mentioned that she was hoping to "correlate" the variables on the sides opposite each other because they had to be equal. I am not sure why she used that term, but it appeared to mean to reconcile them or reduce the number of variables in some way.

When asked if she could follow the same process that she used over and over with variables instead, she found a way to eliminate some of her variables by letting *a* and *b* equal *x* and *c* and *d* equal *y*. Brenda then created the cost formula correctly. Afterward, she tried to use that to determine something about the problem but again could not help using actual values (see Figure 7). That did help her to determine if the cost formula worked in its simplified form. Brenda also determined that *y* equaled 60000 divided by *x* and tested it to make sure she had it written down correctly. When asked whether she could use the area relationship to simplify things, she substituted for *y* but made some type of error by dividing *x* and *y* by 2, thinking that they were a combination of her initial variables.

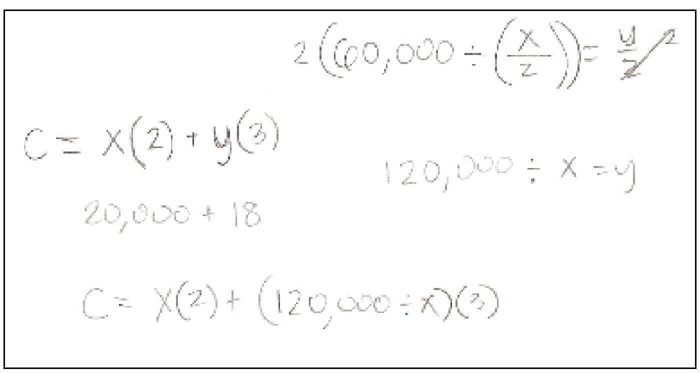

Figure 7. Brenda's work toward creating the objective function in the follow-up interview portion.

I overlooked the error in combination to ask more about what could be done next because she now had the cost formula in terms of *x*. She mentioned then that she could use what she had to find the lowest value but was not sure how to go about it. When asked about how her teacher would have had her find the minimum, she suggested setting the formula equal to 0. When I asked her why, Brenda recalled that her teacher had taught her that setting the derivative equal to 0 gave you either a max or a min value but could not explain why that happened. When asked about the general process she had learned for solving these types of problems, she referred to the problem as a “word problem,” mentioning that the teacher would have had the students note all the important parts of the problem such as the rectangle and costs of the sides. After that, she had no memory of what they had done next but that it involved substituting in some values and finding the minimum in some way.

I concluded the interview portion by asking Brenda about her comment concerning the 1 to 2 ratio that she mentioned when I altered the problem for her. She explained that she figured the sides would be different and that a 1 to 2 ratio was the simplest she could think of to try.

Chris

Problem-Solving Framework. Chris regularly returned to reading and analysis episodes, seeming not to want to make or try out any guesses concerning values and strategies (Figure 8). Much of his working memory may have been taken up by his concern about his ability to solve word problems. This concern led him to spend time in transition between episodes, considering generally his success with this type of problem, and this one in particular. The solution attempt lasted about 21 minutes and 41 seconds.

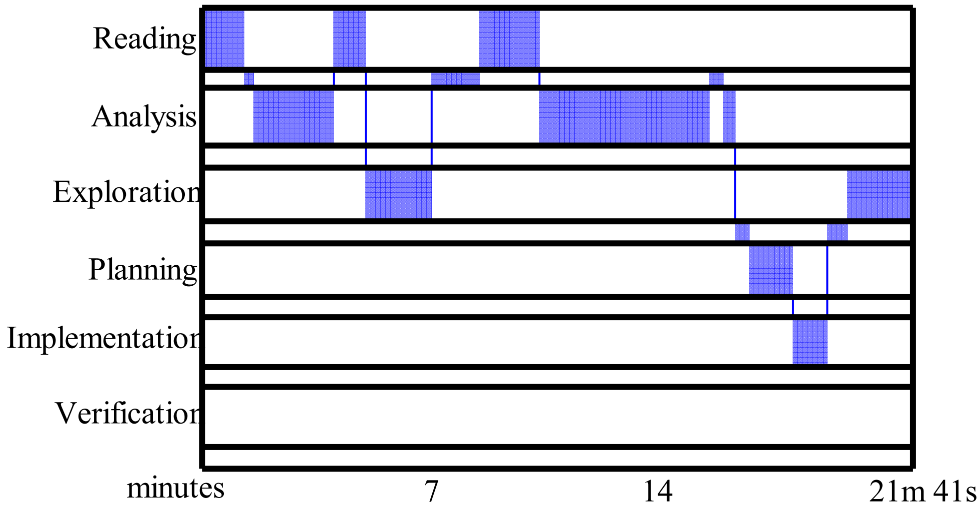

Figure 8. Summary of Chris's problem-solving episodes for Problem 1.

Solution attempt. Chris began by reading the problem and made note of the goal state although not correctly: "to make, to use the max amount of area and the minimum expense." He described the problem as a word problem and then began to analyze it. Chris drew a picture (Figure 9) and mentioned some things to himself about rectangle area. Afterward, he considered where the short side of the rectangle should be, first saying along the highway and then saying otherwise. He then reread the problem to himself again at least once. This time, he began to use some variables for the sides of the rectangle and tried to simplify the number of variables by solving for length in the area formula. Unfortunately, he plugged the expression "area over width" into the area formula for length, resulting in an absolutely true statement, area equals area.

For a moment, Chris stepped away from the problem, mentioning that he probably needed the derivative since this was calculus and that he had never been good at word problems. He then reread the problem after noting that they (those who pose problems) usually did not give you useless information. After rereading, he focused on the prices, which he had not really taken into account, and he struggled to create a formula using them.

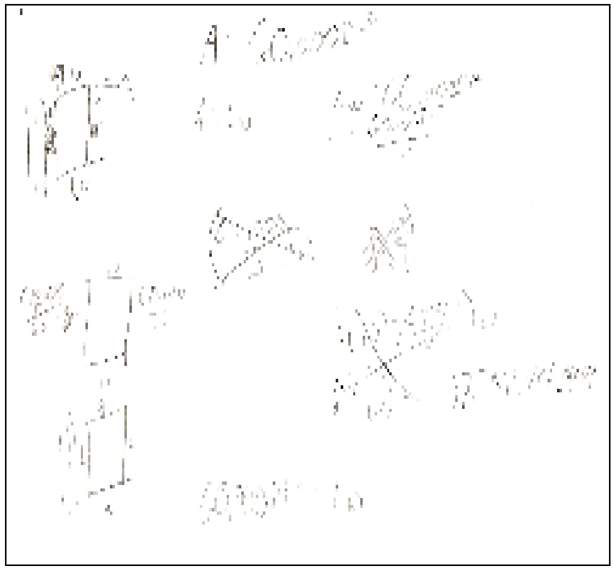

Figure 9. Chris's beginning work concerning Problem 1.

After he mentioned again that he had trouble with word problems, I changed the problem to something more manageable, where the sides cost the same, because he thought that would be easier. He began planning how to find the length and width and decided that he needed to determine the perimeter. For the fourth time, he mentioned that he had trouble with word problems and that he felt no closer. At this point, Chris seemed to recall that a square was best in this situation. The follow-up interview portion showed that he was familiar with this adjusted problem and that the square gave the maximum area for a given rectangle perimeter. Towards the end of the episode, he seemed to realize that perimeter and cost might not necessarily match and was not satisfied with simply finding the square root of the area. After again struggling to determine how to find the appropriate side lengths, he concluded the attempt.

Follow-up interview portion. This interview portion (23 minutes 44 seconds) began with me asking Chris to explain what a word problem was and what about it gave him trouble. He defined it in a couple of ways:

1. Just problems where they don't give you a formula, well, not a formula—. They don't just give you the facts and tell you to solve it. They make you figure it out. They tell you the information, but it is not exactly how you need it.
2. Because instead of … instead of giving you … word problems—.Better, word life applications problems—that was what I was trying to think of—where it gives you real-life instances instead of just giving you numbers and diagrams. It gives you a real-life application.

Note the two qualities of a word problem: not procedural and involving applications. Chris believed that Problems 1 and 3 were definitely word problems because they did not provide a "formula or numbers to plug into a formula, [they give]… a sentence or paragraph structure where I have to figure out all of it." Chris decided that Problem 2 was not a word problem, but he did pause in making this evaluation because that problem did not provide a straightforward method for solving it either.

Afterward, I asked Chris about his use of the area formula and the notation $f(w)$. He mentioned that he was trying to reduce the variables and create a function but that the function was not helpful. Following this, we discussed why he chose the $2 per foot side to be the short end. Chris responded that in order to minimize the cost, the expensive side would have had to be shorter. When he mentioned that he had trouble representing the formula for cost, I gave him two values he mentioned earlier, 100 feet and 600 feet, asking him what the cost would have been.

He easily calculated the cost. Then when given variables, he easily created the cost formula (Figure 10).

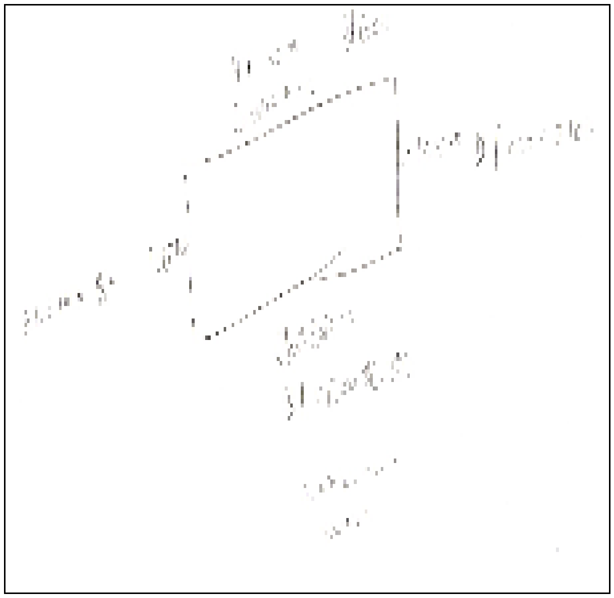

Figure 10. Chris's work toward creating the cost formula.

Unfortunately, even though Chris realized this formula was useful, he did not know how to minimize the cost. He mentioned that we could try values like a 1 foot or less fence length for the side along the highway but then ended up with an extreme cost. He explained that we needed to also keep the perimeter down because the fence was along it. I then asked him why he chose to use a square to represent the fence, to which he replied that he knew the square would give the least perimeter for the given area. When calculating the cost for building a fence using a square, he was momentarily convinced that a square gave the best cost, but then realized that minimizing cost was different from minimizing perimeter in the original problem because we count one of the sides an extra time.

In the meantime, however, I asked him about what he did in class to solve these types of problems. Chris recalled that the students wrote down and defined the variables but did not know what to do after that. This response led him to talk about the specifics of the problem and that the square would give you the smallest area for a given perimeter. When discussing this point, he realized that perimeter and cost were different in this problem so that his intuition concerning the square was incorrect, ending the interview portion concerning this problem.

Donna

Problem-Solving Framework. Even though Donna transitioned from reading and analysis directly into planning-implementation during the first part of the solution attempt, she spent most of her time in a lengthy exploration episode (Figure 11). She never entered into verification, mainly because she never had a potential answer. Additionally, Donna spent some time in transition thinking about potentially useful tools and problems from her past. The solution attempt lasted about 22 minutes 44 seconds.

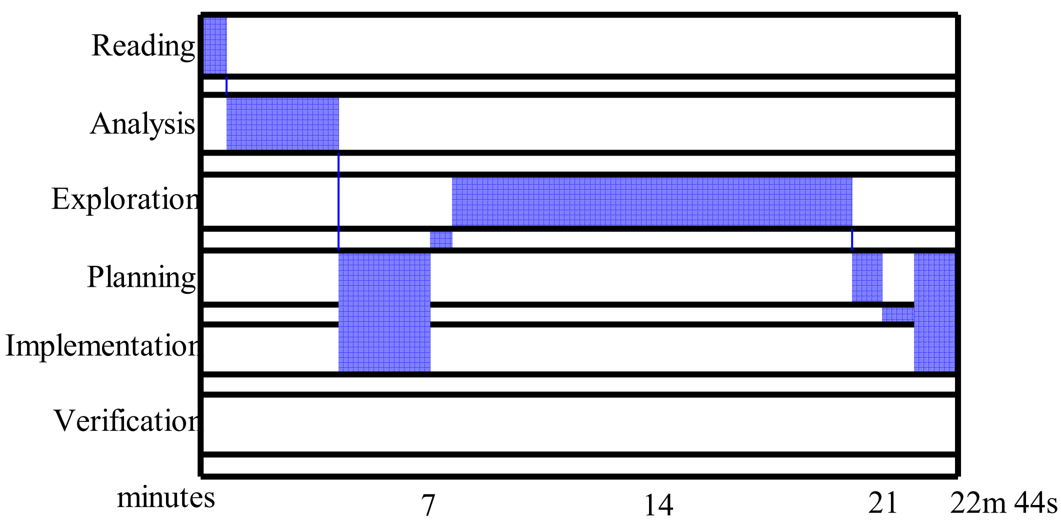

Figure 11. Summary of Donna's problem-solving episodes for Problem 1.

Solution attempt. Donna began by reading the problem and immediately pointing out its main features: the conditions, data, and goal. After noting that it was a word problem, she analyzed it and oriented herself to it, attempting to create equations involving the area and cost. Initially, she assumed that the shape of the plot was a square but quickly fixed her error (see

Figure 12). Assuming the expensive side was short, she attempted to use that piece of intuition and manipulations of the area formula to solve for the width or length of the rectangle. When she solved for the length and then substituted it back into the original equation, she ended up using circular reasoning that resulted in the tautology, 60000 = 60000. Donna then mentioned that she knew from the past that she should solve for one variable and find the derivative but could not determine how to go about it.

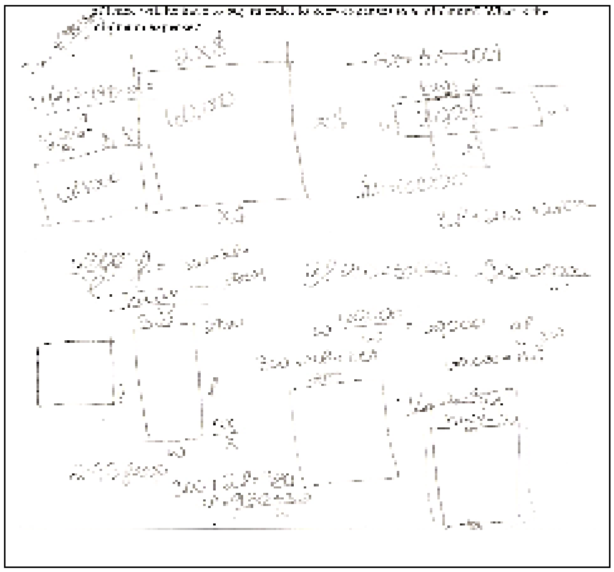

Figure 12. Donna's work concerning Problem 1.

At this point, she began a lengthy exploration session where she tried a square shape and attempted to find a derivative of an equation with no variables, realizing this was problematic. Unfortunately, Donna seemed reluctant to use an equation with three variables, looking instead for one containing the two essential variables, cost of the total fence and the length along the highway. She did state the cost formula out loud along with the perimeter formula but appeared overwhelmed with variables she could not reconcile. When I asked what she was attempting, she discussed a plan that attempted to find the cost and go from there because she could not solve for one of the side lengths. During a transition from this planning attempt, she mentioned that she knew there was more than one possibility concerning values for the side lengths and that she had done max and min problems before, making a few comparisons to this problem. She then outlined a generic strategy that had some of the particulars from this problem and that she typically followed to solve optimization problems, but she did not know how to follow it successfully in this problem, ending the solution attempt.

Follow-up interview portion. I began this portion of the interview (8 minutes 58 seconds) by asking Donna about her analysis of the problem. She mentioned that she was a very visual student, and so she started by drawing a picture. She also constructed some formulas to help her remember aspects of the problem, intending also to create a cost formula, but realized she had constructed a perimeter formula instead. Additionally, she admitted that she confused cost and perimeter. Then, I asked Donna about her attempt to solve for width using the area formula, which had resulted in a tautology. She mentioned that she realized this was a mistake because a number of different lengths and widths could have given the same area.

Next, I questioned her about mentioning her strategy to use the derivative after solving for a single variable. She mentioned that the derivative was used to find the max or min in a problem. Donna did not know how the derivative did this, but she did know that once you had some values, you plugged them into a table and plotted them on a number line. When questioned further, she mentioned that you had to solve for zeroes, choose numbers higher and lower than them, plug them into a table, and then plot the result on a number line. Even though she seemed to know the process, she did not understand what it meant when the derivative was zero.

Afterward, I asked Donna about trying a square shape. She knew that she could solve for some side lengths and try them out. This observation led her to think about the perimeter, but she noted that she had made an error. The perimeter was not exactly the same as the cost because it was as if there were three widths. When asked about attempting to find the derivative of an equation with no variables, she commented that it was a mistake, because without a variable, there was nothing that changed, resulting in "a constant thing," and that you could not find the derivative of a constant. I then asked her about previous problems similar to this that she had solved. Donna mentioned that those problems had only one equation with one variable to work with and that she did not know how to consolidate multiple variables and multiple equations. She mentioned that given more time, she might have tried manipulating the equations more and taking derivatives, which would eliminate half the variables (I assumed she meant when there were two variables and you were substituting zero in for the new *dy/dx* variable). She did not know how the derivative did it but that it could somehow give the minimum expense, concluding the follow-up interview portion concerning this problem.

Ethan

Problem-Solving Framework. Ethan's overall progress appeared to follow the order of episodes laid out by Schoenfeld (1985) while backtracking periodically to former types of episodes (see Figure 13). He also regularly spent time in transition between episodes, assessing his knowledge and progress up to that point. By the time we hit the 20-minute mark, I realized that Ethan had begun something that might very well lead to the answer, and I allowed him more time. His solution attempt lasted about 35 minutes 56 seconds.

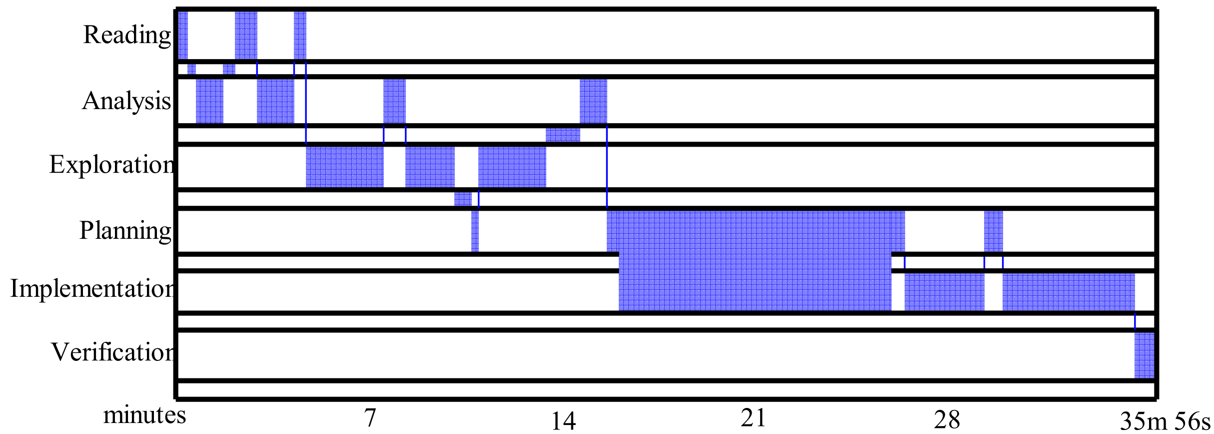

Figure 13. Summary of Ethan's problem-solving episodes for Problem 1.

Solution attempt. Ethan began by reading the problem aloud. While mentioning a couple of times that he had not "done one of these in a while," he drew the situation, labeled the rectangle's sides, and wrote out the area formula using the side labels (Figure 14). Then, he asked whether to find the derivative of the $2 part and began to reread the problem. After rereading and focusing on the goal state, he mentioned that the expensive side needed to be kept to a minimum and then again reread the problem. At this point, he realized that he needed something like perimeter and attempted to solve for a particular side length but noted that it could be any number of values. After he paused, I asked him what he was trying to do. He analyzed the problem and mentioned that he was looking for the minimum expense, which would be easy once he had the side lengths. He admitted, however, that he was not sure how to find those. Again, he explored the area and perimeter functions, considering other variables he could assign without making any progress.

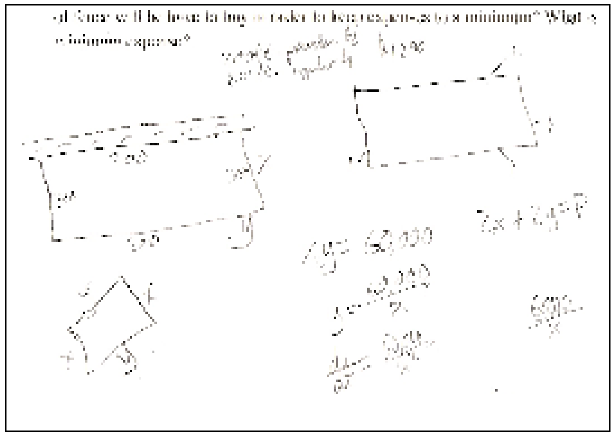

Figure 14. Ethan's early progress concerning Problem 1.

Next, Ethan reviewed what processes he might have wanted to use, noting that he would have needed to find the derivative of one of the equations at some point in order to determine the minimum expense. When I asked why the derivative was useful, he mentioned that my question had started a train of thought, after which he tried taking the derivative using the area equation. When he could not find the graph on his graphing calculator, he abandoned this attempt.

After considering some previous problems similar to this, Ethan decided to attempt a numerical approach that he called an analytic one. He tried several values and compared the cost of the results. He also mentioned that he had noticed that these problems often had convenient results, leading him to try those first. After I asked when he knew to stop, he said that he would stop once the cost began to increase rather than decrease and that he would then narrow the minimum from there. Ethan continued onward with this plan and graphed his result as well until I asked him again about when he would know to stop. He mentioned that he would know that by when the graph curved back up and that it would have been faster if he had been able to create a "numerical" graph in the calculator instead. After finding the point where it curved up, he would look at smaller and smaller increments near the bottom of the graph. When I asked how he would know what the actual answer was, he said he would stare at it and hope it came to him.

Continuing onward with his plan, he discovered that 200 ft along the highway resulted in the lowest cost because it was the last value that he tried before it ended up increasing again at 210. He did attempt to verify this result by choosing a closer value, 205, and then concluded that 200 was best, with the resulting minimum cost being $1200 (Figure 15). When asked how he knew, he mentioned that it was because of the process he just used. After I asked for more details, he simply said that if you go either direction that the cost would increase, using his table of attempted values and graph as a reference, concluding the solution attempt.

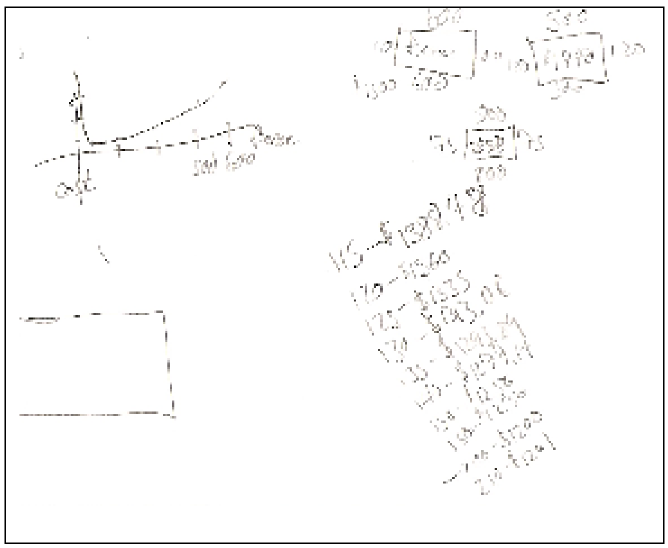

Figure 15. Ethan's numerical approach to solving Problem 1.

Follow-up interview portion. I began this interview portion (19 minutes 42 seconds) by asking Ethan why he had mentioned using the derivative at the beginning. He replied that if he had found the necessary equation, he could have used the zeroes of the derivative to accurately determine the minimum. When I asked how the zeroes of the derivative do that, he mentioned that it gives critical points where the graph peaks or troughs. He further explained that the derivative gives the velocity of the graph, which relates to the direction the graph is traveling. And if you were looking for a max or min point, you were looking for zero velocity because that was when the graph changes direction. In response to how this applied to the problem, he mentioned that he could have found much quicker the precise point where the graph had a zero velocity and then changed direction.

At this point, I asked him how this idea was related to his chart of values. Ethan explained that at the beginning as the side length increased, the cost dropped and that this was negative velocity, which he represented on the graph as a downward change. Using this change helped him to determine the minimum because he was looking for when the negative velocity changed to a positive one. After I asked him about his choice and change of intervals of the length, he mentioned that he had tried a few and then decided to jump ahead to get a better idea of when the graph would change direction. When asked about the rate of change in cost values, he connected it to the graph of the derivative and said that it was steeply increasing from the negative side at first, which matched the sharp decline in the original graph. Then it crossed the x-axis at the minimum value, after which the original graph increased. He further explained that the point where the graph crossed the x-axis was a minimum because that was when it changed from a negative to a positive velocity. I then asked him about finding the derivative of $y = 60000/x$, to which he replied that it was an equation he had and that he had thought to attempt to see if he could do anything with it.

At this point, I asked Ethan about his comment about not solving the problem mathematically. He explained that he had hoped to find an equation for cost that he could differentiate and find the minimum of. After I drew attention to the fact that he had followed some process over and over when creating his chart, he realized that he could represent it generally using x and discovered the cost formula with respect to length by also substituting a portion of the area formula (Figure 16). Immediately, he graphed the equation and used the calculator to find the minimum, which matched his minimum cost of \$1200, where $x = 200$. I also asked him about his use of variables since he had two *y*'s representing different things. Ethan had no trouble explaining to me the meaning of each.

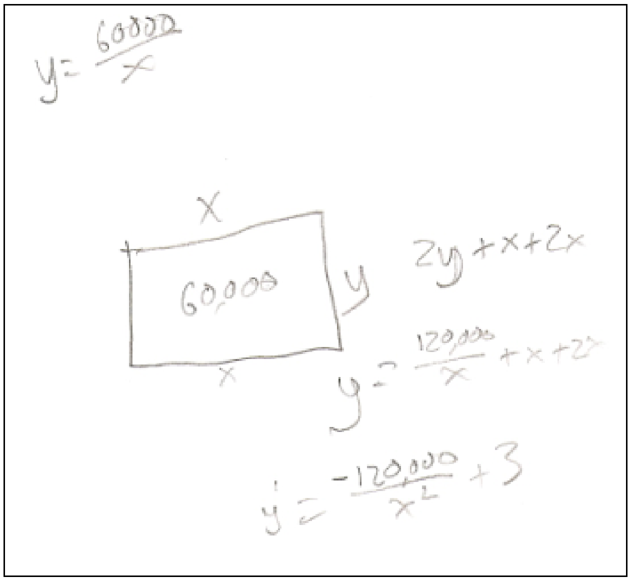

Figure 16. Ethan's development of the objective function.

When I asked Ethan how the derivative came into play, he calculated the derivative and graphed it. After fixing an error with signs that I pointed out, he determined that the minimum occurred at $x = 200$ using the *calculate zero* button. I then questioned him about how he knew this point was indeed a minimum on the graph. Ethan explained that he found the function that gave the change in velocity and that this graph changed from negative to positive at 200. That meant that the original graph was changing from decreasing to increasing, concluding this interview portion.

Problem 2. Find the point *P* on the parabola, $y = x^2$ closest to the point (1, 0). (Rogawski, 2008, p. 225)

Alice

Problem-Solving Framework. Although Alice entered into each type of episode, she spent a great deal of time in exploration and verification (Figure 17). These episodes apparently resulted in part from her incomplete understanding of how to determine distance between two points. The solution attempt lasted about 20 minutes and 44 seconds. It might be important to note that when the other three of the four students convinced me that they had some idea of how to effectively use the distance formula relatively early in their solution attempts and could not generate it themselves, I provided it for them. Having this formula could have sent Alice in a more productive direction as well and influenced her time spent in the various types of episodes.

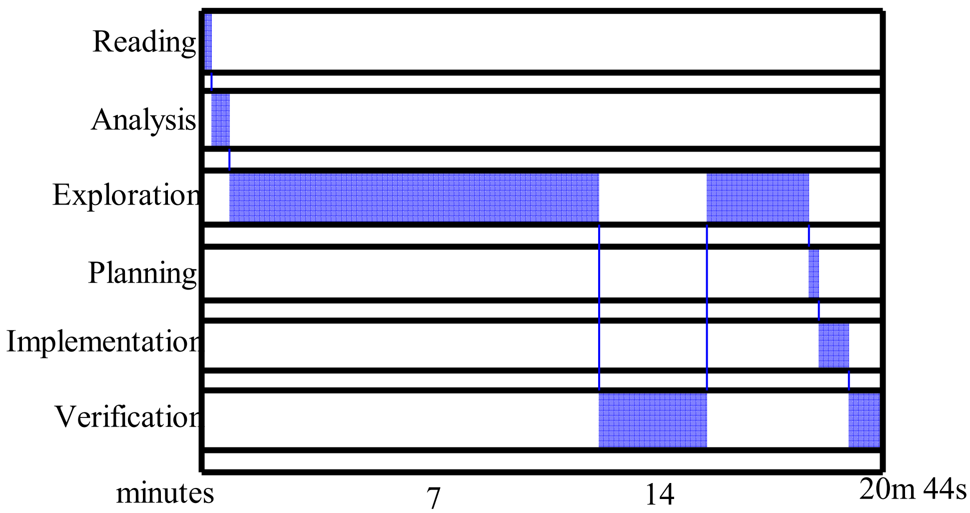

Figure 17. Summary of Alice's problem-solving episodes for Problem 2.

Solution attempt. Alice began by reading the problem, partially aloud. Immediately afterward, she graphed $y = x^2$ in the calculator, which told her that it was a parabola (Figure 18). She then attempted to find the closest point by using some calculator functions. First, she tried the *intersect* function but had nothing to intersect with the parabola. Second, she tried the *calculate* function at 1, which gave her (1, 1), and for a moment she believed this was the closest point. Then, she used the *calculate* function at 0 to get (0, 0). She considered using a table but then used the graph again, examining points like .9 and .99 while zooming in. At this point, she brought in the term *asymptote,* which appeared to come to mind because she was getting closer and closer to a boundary, 1, as evidenced in the follow-up interview portion and hinted at in the solution attempt. She tried to graph the boundary line so that she could intersect the two, but the calculator would not allow her to graph both. Instead, she decided to zoom in closer and considered $x = .5$ but returned to the point (.9, .81), claiming that this was the closest. When asked to explain, she said that you could actually get closer by adding more 9s such as .9999, but that .9 put it in the broader spectrum. At this point, she considered using two dimensions to make points around the middle closer and looked for a calculator function to help.

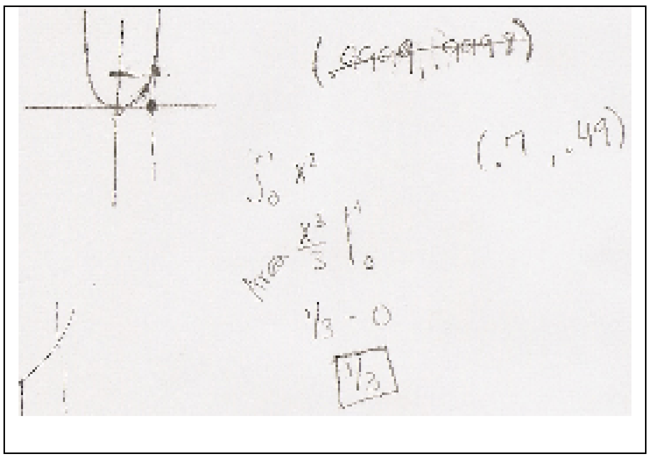

Figure 18. Alice's written progress during Problem 2.

When I asked how she would tell how far apart two points were, she mentioned finding distance. After being asked how to do that, she talked about subtracting them and incorrectly went into how the integral might be useful for finding the space between. When asked about this, she realized that integrals gave area and not distance, calling for the distance formula. Afterward, she began to eyeball some points, such as at $x = .5$ and .7, claiming then that .7 appeared to be the closest. Her justification was that choosing other values, which she seemed to mean were closer horizontally, would be further away vertically. When she could not determine the actual distance nor remember the distance formula, the solution attempt concluded.

Follow-up interview portion. I began this part of the interview (15 minutes 55 seconds) by asking about the intersection attempt Alice made between $y = x^2$ and other curves. She mentioned she was trying to find the intersection between the line $x = 1$—which she typed as $y = 1$ in the calculator and also called an asymptote—and the parabola. When asked why she believed this was an asymptote, she mentioned that it was a boundary that you did not want to cross. Interestingly, she seemed to have no concern about a graph intersecting its asymptote or that the intersection happened at a boundary.

When I asked what she would have liked to know concerning this problem, Alice understandably mentioned the distance formula. She further explained that it would have helped her determine if (0, 0) or (1, 1) were closer, points from which she should have known the distance to (1, 0). When asked how knowing which of those two was closer would help, she added that it would have given her a starting place for trying other values, either down near the origin or up near the point (1, 1). Looking at the point (.7, .49) that she chose to be closest, she explained that the points above it were closer to other points such as (0, 1) and (1, 1) rather than (1, 0). At this point, I asked about the points below (.7, .49) along the parabola. To this question, she responded that .5 might have worked and that the distance formula would have given some direction concerning that. Given time, she said she would have tried other points around the closer point such as .8 and .6 if .7 had turned out to be the best. When asked if she would have continued further with her process and when to stop, she admitted that you would not stop but instead try points to find the closest that you could, concluding this portion of the interview.

Brenda

Problem-Solving Framework. Brenda initially followed the order outlined by Schoenfeld (1985) until the planning-implementation episode (see Figure 19). Instead, she spent a good deal of time in transition afterward, determining whether she needed to or could derive it. After receiving it, she used much of her time planning and exploring. The solution attempt lasted about 22 minutes and 38 seconds.

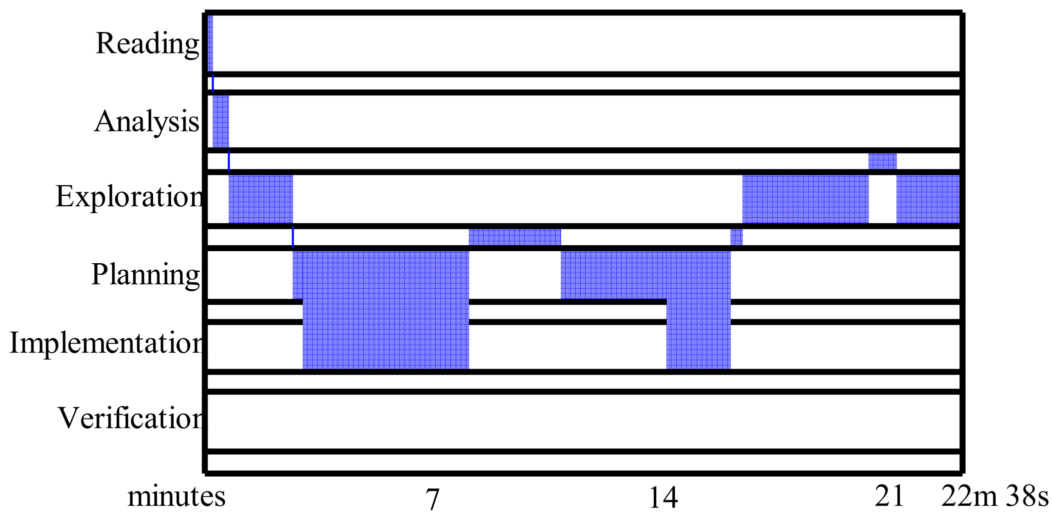

Figure 19. Summary of Brenda's problem-solving episodes for Problem 2.

Solution attempt. Brenda began by reading the problem and then drawing the parabola (Figure 20). Afterward, she determined the derivative of the parabola at (1, 1), began to think about how slope might be informative, and mentioned initially that (0, 0) must be closer without giving reasons. She then decided to create an *x,y* chart of points and proceeded to do so for $x = 1$, 1/2, 1/4, and 2/3 (Figure 20). At the same time, she determined the horizontal and vertical distance to the point (1, 0), making an error or two that she soon fixed. Brenda compared the values but was not satisfied, because she wanted to compare one decimal answer for each, which meant she needed the distance formula, something she claimed she had not used since the 10th grade.

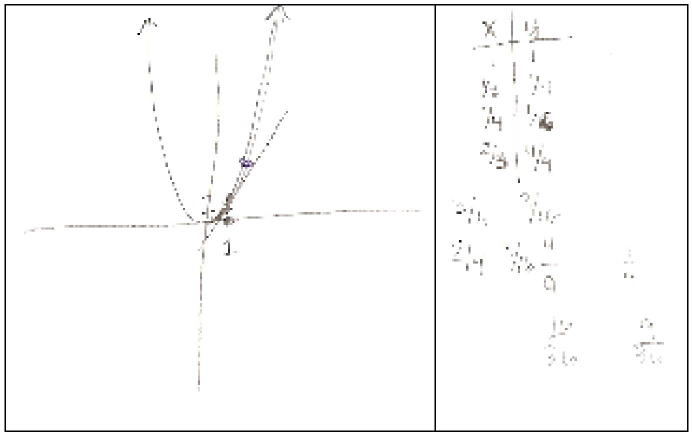

Figure 20. Brenda's initial work concerning Problem 2.

After Brenda explained how she planned to use the distance formula, I gave it to her. When she started to use the distance formula, she realized that she would just be randomly guessing, not an optimal strategy for determining which value was closest because the answer might have a long decimal expression. After finding the distance to the point (1/4, 1/16) using the calculator to be .559, Brenda then considered whether she could determine a shorter distance such as .5 and then solve for the point. She was not, however, confident in this method though, because it required solving an equation that involved square roots. Next, I asked whether she could represent points generally on the parabola, which led her to try the point (x, y), but she still felt the need to pick a distance such as .1 (see Figure 21). She also decided to substitute x^2 for y to reduce the number of variables. Unfortunately, she was concerned about the square root and how to deal with it, so she was not able to move forward. The solution attempt concluded when she mentioned she did not remember ever having to deal with distance but that she could have done anything with area under a curve "lickety split."

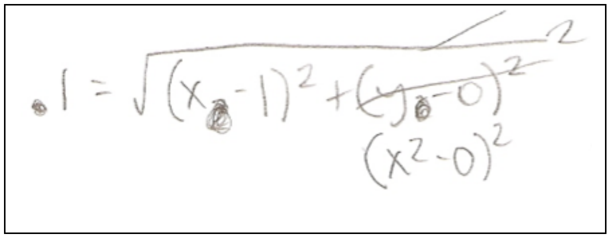

Figure 21. Brenda's development of the distance objective function.

Follow-up interview portion. I began this interview portion (4 minutes 9 seconds) by asking Brenda about her use of the derivative. She mentioned that she had hoped to calculate the derivative at (1, 1) and at the vertex of the parabola (0, 0), using those as a guide in some way to determine points that might have been closer. When I questioned her about details, her description did not give much insight into what she was thinking. I realized that she might have been simply trying out some ideas without having a general sense what she was doing. I then questioned her about what the derivative was giving her, and she mentioned the tangent line. I then asked what tangent line was, to which she replied that the teacher had drawn some type of picture and that the derivative gave the slope of the tangent line at a point. When I questioned her about what the slope of the tangent line at (1, 1) was, she simply gave me the generic derivative, $2x$, again.

I concluded by asking Brenda about how she determined that (1, 1) was closer than what she had said was the closest point, (0, 0). She realized then that they were both 1 away but that (1, 1) was directly above it. She also corrected this error, mentioning that they were points on the parabola and that you could try other points by plugging in values.

Chris

Problem-Solving Framework. Chris, too, followed the flow of episodes in the order presented by Schoenfeld (1985), returning to analysis and exploration toward the end (Figure 22). He also never entered into any verification episodes. At times, he stepped outside his solution attempt during transitions to consider whether he had the resources he needed to continue. The solution attempt lasted about 25 minutes and 42 seconds.

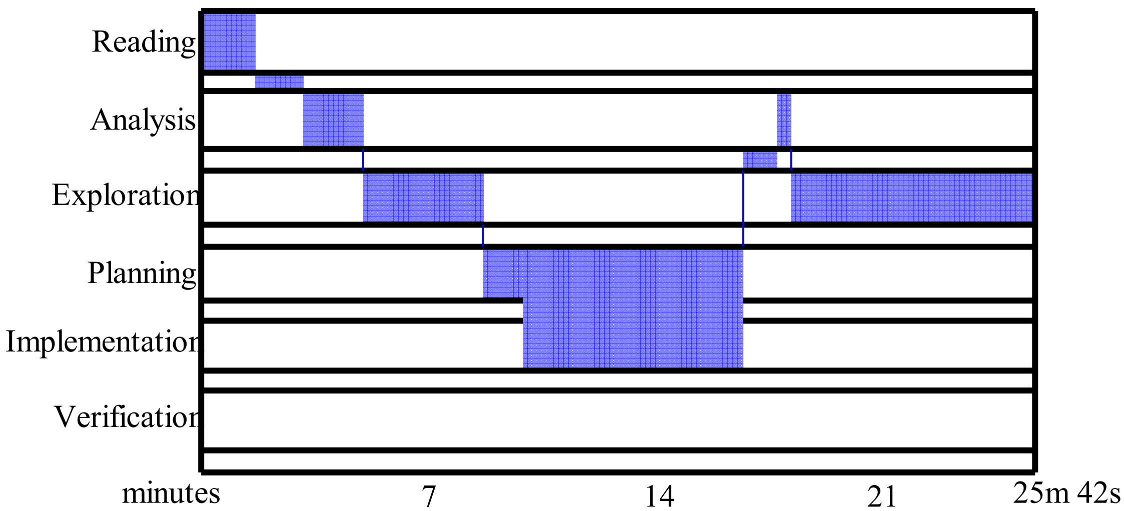

Figure 22. Summary of Chris's problem-solving episodes for Problem 2.

Solution attempt. Chris began by reading the problem through three times with pauses. Afterward, he considered problems in the past related to this, claiming he had solved problems exactly like this in class. Then after examining some relationships between the parabola and the point, he determined the derivative of the quadratic function. He mentioned that he had done this because his class used a lot of derivatives at the beginning of the year when they did this type of problem, and so he expected it to appear somewhere. At the same time, he felt there must have been some type of property or formula they had used for this and realized it might have been the distance formula, which he did not remember. Convinced it would help him, Chris received the distance formula that he almost fully remembered. Once he had the formula, he easily substituted a generic (x, y) into the formula along with $(1, 0)$ and began to simplify the equation (see Figure 23).

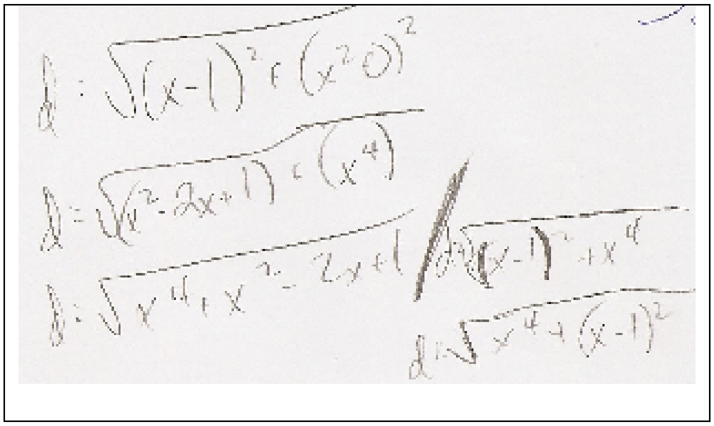

Figure 23. Chris's development of the distance objective function.

Unfortunately, he became worried about the radicand and whether it could be simplified which might have allowed him to correctly use the derivative. Instead, he abandoned this idea and searched his memory for other techniques or tools they used to solve these types of problems. Other than the distance formula, he could not remember much. After thinking about connecting a point on the parabola to (1, 0) with a segment, he hypothesized that he could use similar triangles and trigonometry in some way. As a result, he connected (1, 0) to (0, 1) and found the distance between the two, hoping that would somehow guide him toward finding the closest point (see Figure 24). Because he had trouble doing this and because we had passed the 20-minute mark, the solution attempt concluded.

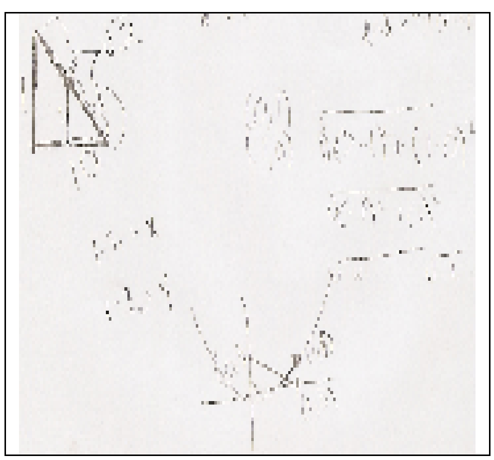

Figure 24. Chris's attempt to use triangles to find the closest point.

Follow-up interview portion. I began this interview portion (3 minutes 59 seconds) by asking Chris about his use of the derivative at the beginning. He replied that he simply remembered using the derivative or antiderivative every single time they worked through a calculus problem and felt it had to be useful somewhere. I then asked him about his work with the distance formula when he substituted a general point and (1, 0). Instead of answering this question, he mentioned that he drew a line from (1, 0) to (0, 1) because he believed it would intersect the parabola at the closest point. He also mentioned that the triangles were supposed to help him do that. When I asked why this method gave you the closest point, Chris backed up a little bit and said that he was not sure that it really would give the closest because he could not remember why. After asking what else he might have wanted to know to work through the problem easier, Chris said that he wished he knew where to use the derivative, assuming the derivative was useful in solving this problem. I mentioned it would be but that there were other techniques as well, which concluded this interview portion.

Donna

Problem-Solving Framework. Donna followed Schoenfeld's order of episodes fairly cleanly until the very end, spending a good deal of time in an exploration episode (see Figure 25). She never entered into verification because she could not overcome algebraic difficulties, despite knowing a reasonable strategy to follow. The solution attempt lasted about 23 minutes and 16 seconds.

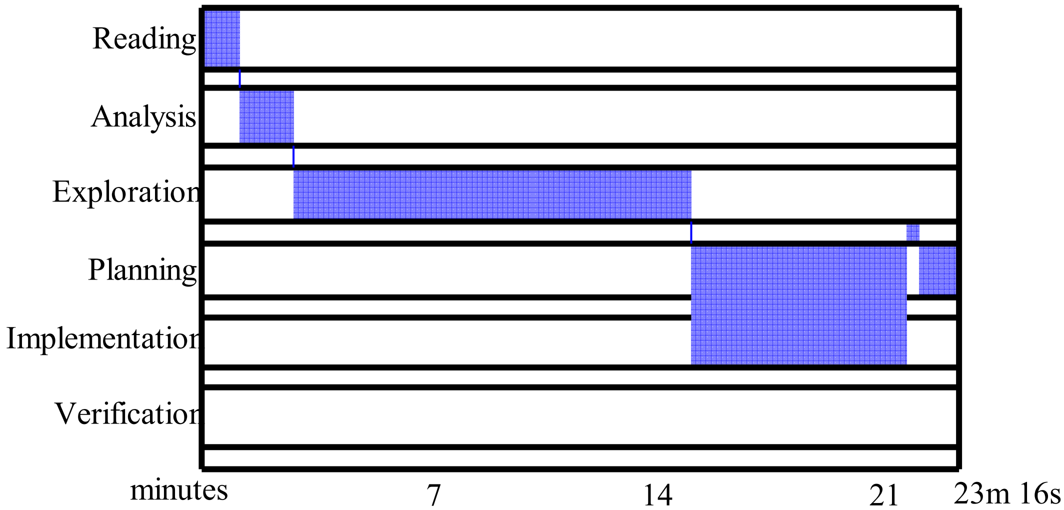

Figure 25. Summary of Donna's problem-solving episodes for Problem 2.

Solution attempt. Donna began by reading the problem and then determined the derivative of the parabola. Afterward, she plugged in 1, the *x*-value of the point. This led her to the point (2, 0) that she was not sure about (see Figure 26). When I asked her about what she was thinking, she mentioned she was not sure how to approach the problem but that she could try plugging values into the derivative, which could have taken forever. She went on to explain that the derivative was useful because it gave the low and high points but still appeared unsure about this. When I asked her to solve a related problem concerning two particular points, she recalled that the distance formula might be useful. Unfortunately, she could not recall it accurately, so I gave it to her.

Once she had the distance formula, she began to think about how to use it, noting she could try some points but that there had to exist a more direct way to find the correct one. I asked if she could represent a point generally to which the answer was yes. Then, she began to substitute (*x, y*) and (1, 0) into the formula. Next, she attempted to simplify the formula and noted that she would have wanted to take the derivative of it but that there were two variables which she would need to reduce to one (Figure 26). Because she did not know how to do that, she tried to determine the derivative anyway. At this point, she felt it would take too long and might not be the correct path. When I asked what she would do given enough time, Donna mentioned that she would have solved for 0 and then plugged in values around the solutions on a number line to determine where a minimum exists, ending the solution attempt.

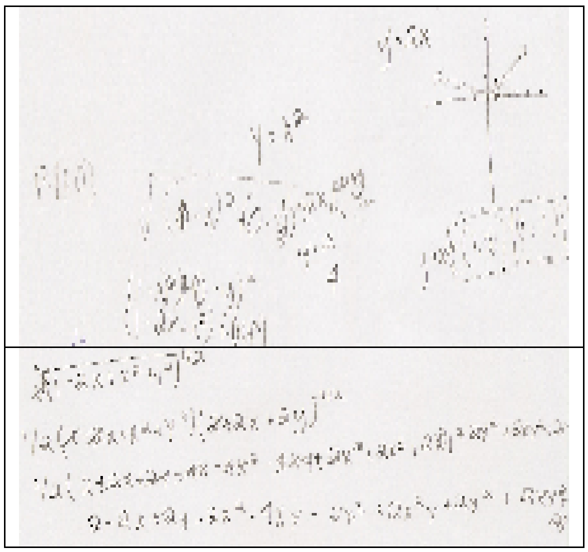

Figure 26. Donna's work concerning the distance objective function.

Follow-up interview portion. I began this interview portion (9 minutes 42 seconds) by asking why she started with a drawing. Donna mentioned that she felt that she was a visual person and needed the picture to help her. When I asked why she took the derivative next, she mentioned that the students in her class used derivatives a lot in calculus and thought it might have been helpful. I then asked her what $y' = 2x$ meant to which she did not give a straight answer. Donna included that if you used it you could see that the minimum of the graph was at 0. I asked if there was another way to determine this and she replied that you could just look at the graph as well. Following this, I asked what the derivative provided. She mentioned that it could be used to tell you what was going on in the graph. This could be done by solving for 0 and then plugging in values to the left and right of the solutions to determine where minimums and maximums occur. I then asked what plugging in one told her at the start of the problem. She mentioned that it told her that the derivative was two at one but that she did not know what that meant.

At the point, I asked her why she was not sure at first whether (1, 0) was a point. Donna mentioned that she thought it might have been interval notation. I then returned to asking how the derivative was used to find minimums and maximums. She mentioned that setting the derivative equal to zero helped you determine the critical points around which the graph was either increasing or decreasing. It can also be used for determining concavity.

Next, I asked her if she had seen a problem like this before. She said that she had not and that her class had only gone so far into calculus up to Chapter 4, which had involved word problems like Problem 1 but not as difficult. When I asked what a word problem was, she described it as follows:

> "Um, to me, a word problem is kind of easier to figure out because you are applying it to something that is actually there and I can visualize it better than just writing out a bunch of numbers."

When I asked what a non-word problem would look like, she mentioned something like "find the derivative of dot dot dot." After asking her why this description did not define a word problem, she then essentially tried again to define a word problem in the following way:

> "Um, I don't know the technical definition but to me a word problem is trying to do something that is pertaining to real life things. And I don't know, it seems like more, I am trying to think of the word, but like word problems, things that aren't word problems are just straightforward. And word problems you kind of have to figure them out instead of just using one simple thing."

Overall, a word problem for her appeared to have a real life application or connection.

Following this discussion, I asked why she chose to use a derivative in the first place. Donna responded that she had hoped it would spark something because calculus was all about derivatives. Given more time, she mentioned that she might have worked with the distance formula further and potentially tried out some points. After asking her if the problem was clear other than the notation of the point, she said yes. I then asked if there was anything she would have found helpful to which she responded how to solve the problem, ending the follow-up interview portion.

Ethan

Problem-Solving Framework. Even though Ethan followed the given episode order provided in the framework, he returned to previous types of episodes on occasion as well (see Figure 27 below). He also spent time in brief transitions, considering his thoughts and direction. His verification episode at the end involved a number of key assessments of his constructed solution. The entire solution attempt lasted about 23 minutes and 41 seconds.

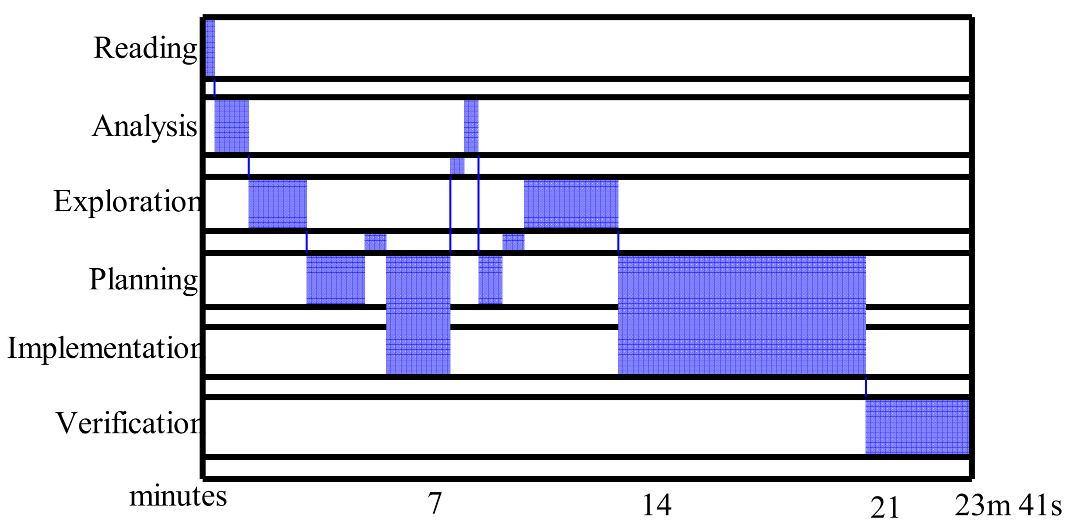

Figure 27. Summary of Ethan's problem-solving episodes for Problem 2.

Solution attempt. Ethan first quietly read the problem to himself. Next, he drew the parabola, the point (1, 0), and a generic point P on the parabola (see Figure 28). He recognized that he needed distance because of the goal to find the closest point as mentioned in the follow-up interview portion. Fortunately, even though he did not know the distance formula offhand, he impressively derived it using two generic points, a triangle, and the Pythagorean Theorem. Afterward, he substituted (x, y) and (1, 0) into the formula and attempted to isolate the variables. He considered using the square root to eliminate the squares. After trying out a pair of numbers, Ethan realized this was not a valid operation. For a short time, he seemed confused about how to proceed commenting that he did not want to have to try out values over and over again because these would have been less convenient as they existed between 0 and 1.

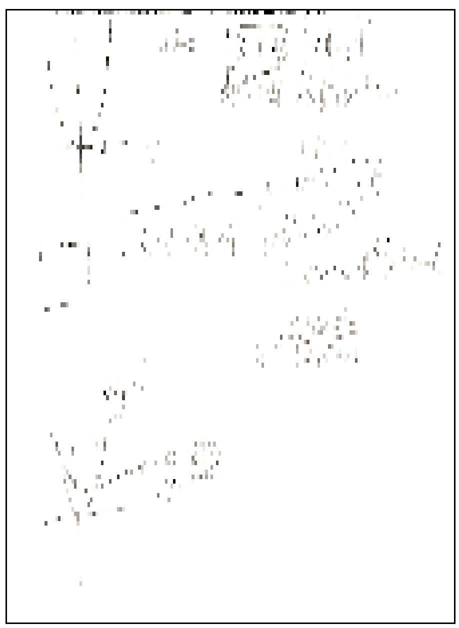

Figure 28. Ethan's solution to Problem 2.

Desiring a more direct approach, he looked for a way to consolidate the variables. Not long afterward, he remembered that $y = x^2$ which he substituted into the formula. After simplifying the formula as much as possible, he determined its derivative (Figure 28). When this did not lend well to solving for zero, he tried graphing the distance formula and used the calculator's minimizing feature leading to the approximate *x*-value .58975. As a verification, he graphed the derivative and used the *solve for zero* feature, which resulted in the same *x*-value. He also determined the approximate *y*-value of the desired point. When asked about further verification, he mentioned he could try out plugging in that point. Realizing this would not tell him anything, he mentioned trying out a bunch of points. Additionally, Ethan recognized that it might have been helpful to try using points that he knew had a certain distance to ensure the formula was working as it should such as (1, 1), which was 1 away, ending the successful solution attempt.

Follow-up interview portion. I began this interview portion (5 minutes 23 seconds) by asking him about the triangle he drew to find the distance formula. He mentioned that he knew it would help him determine the distance formula with the help of the Pythagorean theorem because it was also the length of the line between them. I then asked why he immediately jumped to using the distance formula. He responded that he had to find the closest point, which implied using the distance formula to evaluate the closeness and then to study change in the distance. When I asked about substituting (*x, y*) into it, he mentioned that the only variable he had to work with was a point on the parabola. I then asked if he had seen a problem similar to this. Ethan responded that he had seen something like this before but did not remember it exactly.

At this point, I asked about why he chose to use the derivative in this problem. He mentioned that he could have just graphed it and found the minimum but that would have been cheating. The derivative, however, provided the general mechanics of the graph. When asked about how it provided the general mechanics, he focused more on that he needed them to gain a complete picture of the situation at hand. Afterwards, I asked about his use of *y* in multiple places and whether they were the same. He mentioned that they represented different things, the parabola in one case and the distance in another. The reason he chose to use y in multiple places was in part due to convenience and familiarity. Then, I asked him why he chose not to follow up his idea of finding the derivative of $y = x^2$ to which he responded it had no merit because he not only already knew the minimum of the parabola he also knew the result would be a line that had no purpose for determining the answer to the problem, concluding this portion of the follow-up interview portion.

Problem Three. Find the maximum length of a pole that can be carried around a corner joining corridors of widths 8 ft and 4 ft (Rogawski, 2008, p. 229).

Alice

Problem-Solving Framework. Alice strictly followed the order of episodes listed in the framework, except to break up some lengthy explorations with some analysis (see Figure 29 below). She spent over half the episode in exploration episodes. The entire solution attempt lasted 23 minutes and 9 seconds.

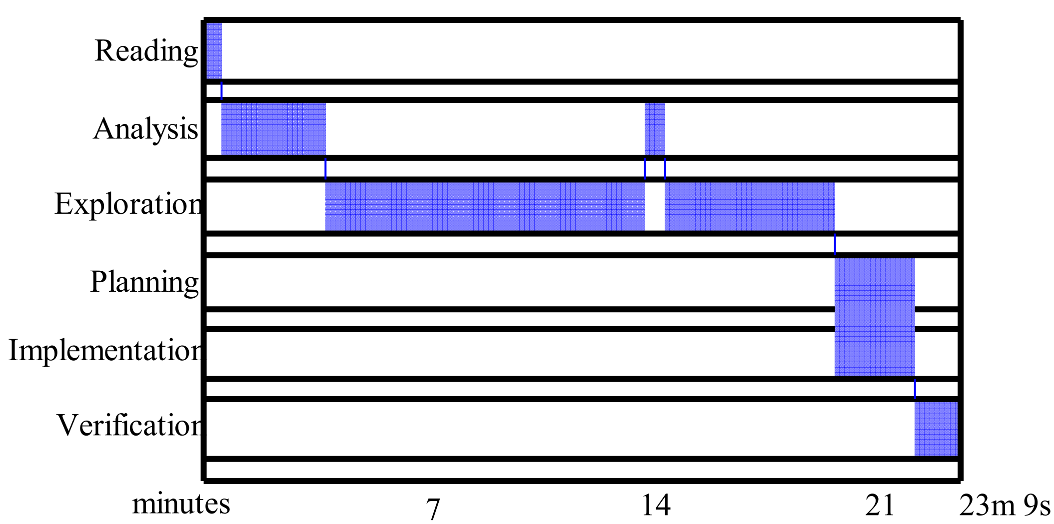

Figure 29. Summary of Alice's problem-solving episodes for Problem 3.

Solution attempt. After reading the problem aloud, Alice drew a picture of the situation and tried to determine where pieces of information fit concerning the different widths of the corridors as well as how the pole might fit through the corner (see Figure 30). She noted that she really had no idea how to do this one at all but trudged forward. Also, she mentioned that it would have been helpful to know the length of the hallways and seemed rather convinced throughout the attempt that this was important missing information even if she did not know how to use it.

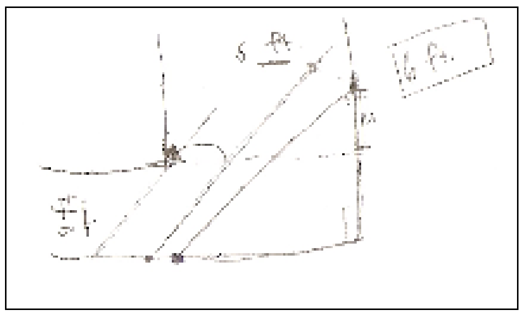

Figure 30. Alice's initial drawings and measurements concerning Problem 3.

At this point, she explored some ideas. First, she figured that the 4 ft corridor would be the most problematic since the 8 ft one was bigger and could fit a longer pole. She then judged a point where the pole would get stuck along the outside wall of the 4 ft corridor. Alice also noted that this formed a right triangle between the pole and the outside walls. When I asked her to draw the picture she had in her mind, she did so and guessed that the length of the pole was 6 ft because it was the median of 4 and 8, despite that it was the hypotenuse of a triangle that appeared to have legs of length at least 4 ft and 8 ft. I then asked if other lengths could have worked to which she mentioned that shorter lengths would but a 7 ft pole would certainly not have fit.

Next, I focused on the point she marked where the pole would hit the outer wall of the 4 ft corridor and asked her how she knew the pole would hit there specifically. She gave a number of explanations that were not entirely rigorous or true. I then asked more generally if the point was before, equal to, or after the inner part of the corner. She mentioned that it was equal to that corner. This was interesting because she then could have calculated the hypotenuse if it worked the same for the 8 ft corridor as well. Also, if it touched the inner corner to give it maximum length then it was not touching the outer wall of the 8 ft corridor because it was parallel to it, which meant the pole could be as long as desired. She then made a number of estimates concerning the range of distances the pole was from the corner.

When she did not zero in on any particular length, I offered to the change the problem to corridors of equal length, 8 ft. She redrew the picture with the new information. Next, she figured the longest the pole could possibly be was 8 ft or a little less. Her reasoning was that if the pole was longer there would have been problems fitting it down the corridor. She also referred to a realistic situation she had faced at home with a mattress but mentioned it could bend whereas a pole could not. With that in mind, she returned to her guess that 6 ft would have been longest. I then asked her about the triangle she drew which led her to note that it was a right triangle with legs each of 8 ft, due to her belief that the point where the pole became stuck was equal to the distance from the inner corner to the outer sides. She mentioned again that she needed to know the length of the hallway.

When I asked her to pick a number that would have been helpful to her, she chose 6 ft. Following this, she used the Pythagorean theorem and the lengths 7.5 for the pole and 6 along the wall to help find the length along the other wall, which was 4.5. When I asked why one side was smaller than the other, she noted that they were probably equal and tried again with legs of length 6 ft, determining the pole to be about 8.4 ft (see Figure 31). When I asked if the pole could be longer, she said no but that perhaps 9 ft could have worked with further consideration.

Nonetheless, she still felt 6 or 7 ft was the longest possible for the original situation because the 4 ft corridor was so much thinner than the 8 ft corridor. Alice also maintained that knowing the lengths of the corridors would have proven helpful, ending the solution attempt.

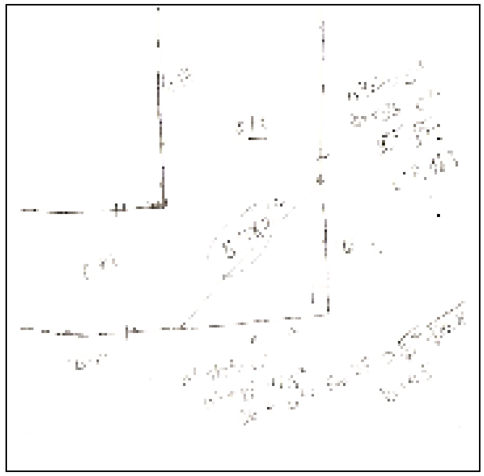

Figure 31. Alice's final work concerning Problem 3.

Follow-up interview portion. During the follow-up interview portion (1 minute 40 seconds), I asked what she might have found helpful to know while solving the problem. Alice again mentioned that the length of the hallways would have been helpful. So, I asked if knowing each hallway was 20 ft would be helpful, she responded in affirmative because it would have helped you determine the lengths of the triangles. When I asked for details, Alice mentioned that she was not sure how it would have worked. I then questioned whether the problem was clear, to which she responded that it was, ending the follow-up interview portion.

Brenda

Problem-Solving Framework. Brenda followed the framework's order of episodes, spending most of the time exploring and verifying (see Figure 32). Over half the episode involved a single exploration episode, mostly regarding the recall and use of trigonometric ratios. The solution attempt lasts 25 minutes and 13 seconds.

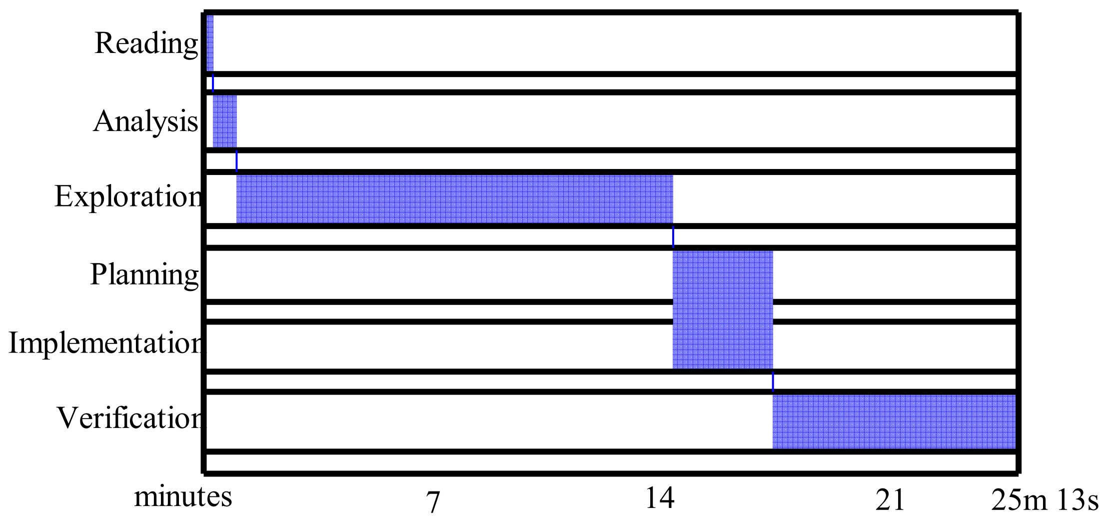

Figure 32. Summary of Brenda's problem-solving episodes for Problem 3.

Solution attempt. After reading the problem, Brenda drew the corner. She also drew a rectangle which she called a square originally, using the corners of the hallway as vertices (see Figure 33). Initially, she drew the pole along the diagonal attaching the two other points along the outer walls, mentioning that this was the maximum pole length. Realizing that she had a right triangle, she began to label her angles, 90, 45, and 45, although the two legs of the triangle were not equal. Brenda also began to recall the trigonometric ratios that she could use to determine the pole length.

Suddenly, she realized that the pole could be much longer if she drew it so it touched the inner corner (see Figure 33). When she redrew her pole, she created right triangles. Brenda then tried to use sine to determine the length of one section of the pole, making an error. She recalled the acronym SOHCAHTOA on her own but failed to set up the calculation correctly despite knowing what each letter meant. She simply calculated the sine of 45 degrees which was approximately 0.85, noting that it was not larger than 4. Next, Brenda added 4 and 0.85 but thought this was not correct.

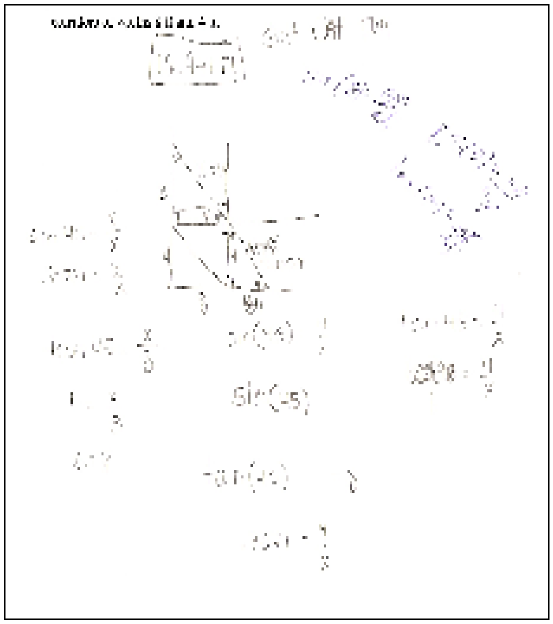

Figure 33. Brenda’s work concerning Problem 3.

After she tried a few things out unsuccessfully, I gave her the actual formulas similar to what students receive on the graduation test now and in the Georgia Performance Standards Mathematics 2 course where they first learned the ratios. Other than receiving help to set her calculator from radians to degrees, Brenda quickly calculated some values, sometimes by hand using the 45-degree angle she chose, and determined the length of the pole at that angle (see Figure 33). She mentioned that she had no idea if that was right.

At this point, I asked her how we determined if it was right or not. She made some complex arguments about how the pole could not move if it was further along the wall and that the 45-degree angles were balanced as opposed to a triangle that was 30-60-90. At my probing, she drew some other possibilities and noted that they were shorter even though that was not true (see Figure 34). I then asked her what about it being shorter was significant to which she gave me an explanation that you want it to be as big as possible to be maximum but that it must be short enough to drop down around the corner, ending the solution attempt.

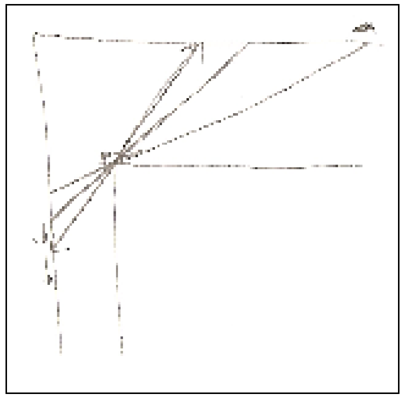

Figure 34. Display of possible, maximal poles around the corner.

Follow-up interview portion. In this interview portion (2 minutes 25 seconds), I first asked about Brenda's first pole drawn along the diagonal of the rectangle in the corner. She mentioned that she was thinking the longest distance inside a rectangle rather than around the corner. Brenda also added that the angles were 45 degrees but she could not explain why. I also asked if there were other ways to find the side of a triangle given the other two sides, something she was struggling with for half the time. She mentioned that she could have used $a^2 + b^2 = c^2$. When I ask how she could have used that, she calculated the length of the first section of the pole and found that she got the same value she did when she used trigonometry. When I asked whether this was good, Brenda mentioned that she felt that she got something correct, ending the follow-up interview portion.

Chris

Problem-Solving Framework. Chris began with a reading episode, followed by a short analysis episode and an extremely long exploration episode (see Figure 35). He spent 18 minutes and 21 seconds of his 20 minute and 23 second solution attempt exploring different ways to combine variables to gain insight into the problem.

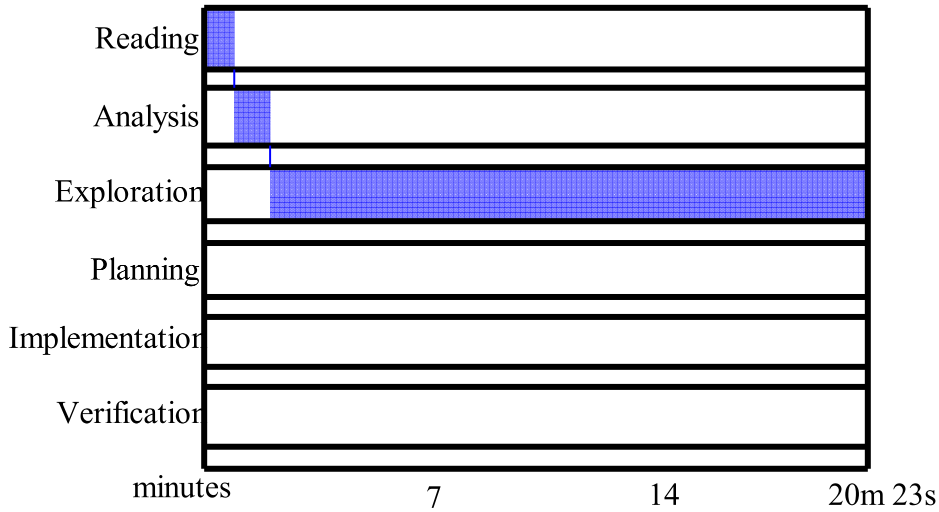

Figure 35. Summary of Chris's problem-solving episodes for Problem 3.

Solution attempt. After reading the problem twice, Chris began analyzing the problem, drew the corner and corridors, and thought about a possible place to draw the pole. He decided to draw a rectangle with the inner and outer corridor corners as vertices. He then labeled the sides of the rectangle, 4 and 8. Afterward, he added and labeled sides of the small triangles in the picture (see Figure 36).

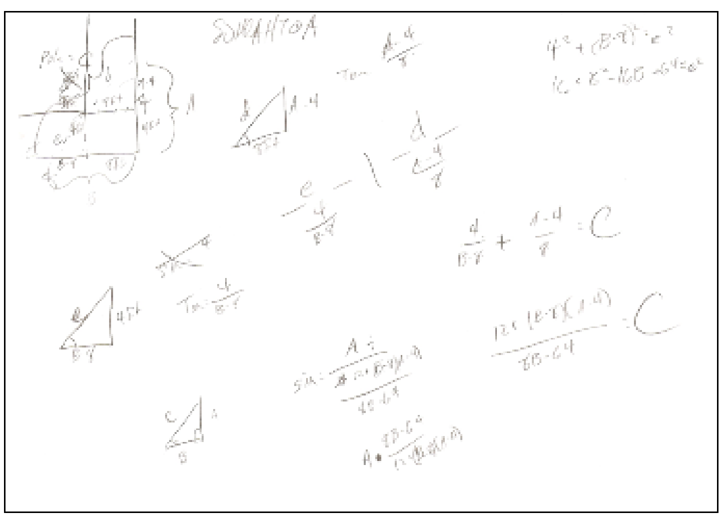

Figure 36. Chris's explorations concerning Problem 3.

At this point, he used trigonometry to set up a number of ratios. At the same time, Chris looked for relationships between his variables using the Pythagorean theorem. The rest of the episode, he spent time searching for ways to determine relationships between the variables and reduce them (see Figure 36). Eventually, he recalled that the derivative should play a role, mainly because this was calculus as mentioned in the follow-up interview portion. Chris mentioned that it could be used to eliminate a variable as well but could not reduce the number of variables enough to use the derivative. When he ran out of manipulations and ideas to try, he ended the solution attempt.

Follow-up interview portion. I began this interview portion (5 minutes 58 seconds) by asking why he jumped to using “SOHCAHTOA.” Chris replied that he figured the best way he knew to find its length was to use the triangle and therefore trigonometry ratios. Next, I asked about the disappearance of some of the trigonometric functions in his manipulations, which he explained relatively well. We realized that I misunderstood one of his substitutions and misread one of his symbols due to his small handwriting.

When I asked about why he brought up the derivative during the solution attempt, Chris mentioned that he always thought of the derivative when he was doing calculus. At this point, I questioned him about his comment concerning using the derivative to eliminate a variable. He recalled that they did something like this in class possibly with the 2^{nd} derivative where they substituted the solutions of the one remaining variable in afterward to solve for the other one. Lastly, I asked him whether other pole lengths were possible. He seemed a bit confused mentioning that there could only be one maximum pole length. While explaining this, he drew other possible lengths which answered my question concerning whether he thought the variables in his drawing had particular unknown variables or could be a variety of non-particular ones. Interestingly, he briefly mentioned that you needed to look for the shortest pole length around the corner to make sure it fit, indicating some intuition that the problem was in fact a minimization problem. This portion of the interview ended when he answered in the affirmative to my question about whether the problem was clear.

Donna

Problem-Solving Framework. Like Chris, Donna read and analyzed the problem for a short time and then entered into a lengthy exploration episode (see Figure 37). She, however, did not spend as much time exploring, ending the solution attempt early. It only lasted 13 minutes and 21 seconds.

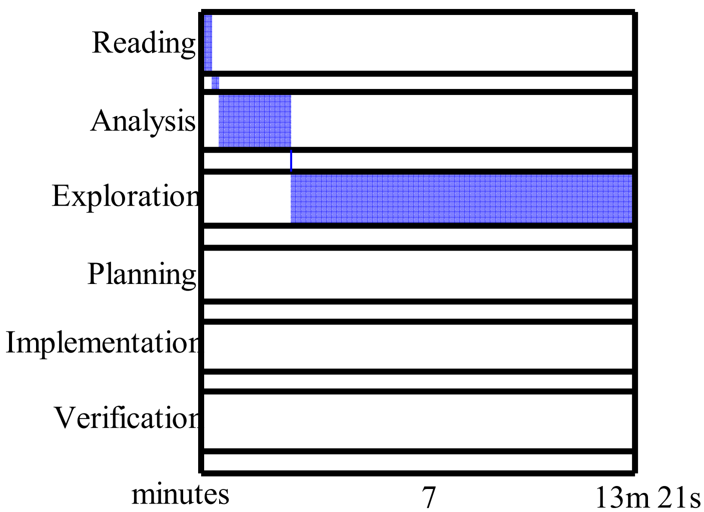

Figure 37. Summary of Donna's problem-solving episodes for Problem 3.

Solution attempt. After reading the problem through once, Donna mentioned she had not seen a problem like this one. She then analyzed the problem. One of the first things she mentioned was that the pole could be as long as it wanted if it was standing up, considering that we put no limit on the height of the ceiling or whether one existed. Next, she drew the situation and the pole by connecting the outer and inner corner (shorter diagonal in Figure 38). Donna then noted that the pole obviously could not be longer than 4 ft, which we later learned in the follow-up interview portion was because she had thought the pole would be carried perpendicular to the corridor walls.

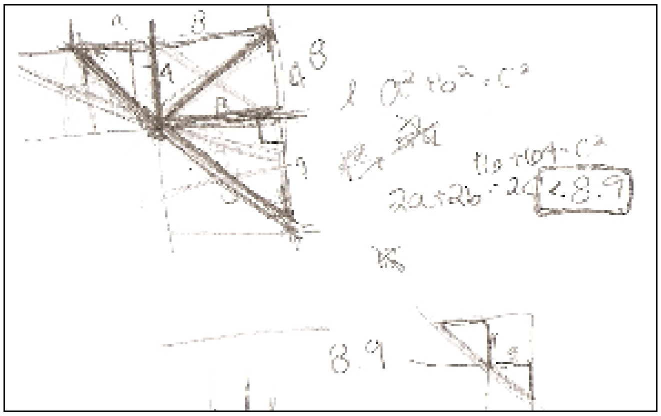

Figure 38. Donna’s drawings and work concerning Problem 3.

At this point, she jumped into a long exploration episode where she tried a number of ideas that were relatively simple in nature, lowering her confidence in her results because she expected the solution to have some challenging aspect. Afterward, Donna tried using the Pythagorean theorem to determine the length of the pole and calculated it to be about 8.9 ft. This again seemed too simple and she claimed she must use the derivative somewhere because that was how they always handled optimization problems. Because of this, she determined the derivative of the Pythagorean theorem but neither included the differentials nor noted with respect to what variable. This gave her 2a + 2b = 2c. Substituting for a and b, she determined that the pole was 12. Because this also seemed too simple, she reconsidered the pole’s position.

Not long after questioning her about her worry concerning the simplicity of her solution attempts, I asked her to use my pen top to illustrate her perception of the motion of the pole through the corridors. This had the desired effect, which was to give her a fresh, more accurate picture of how to fit a much longer pole through the corridors. After redrawing her pole so that it was touching the outer walls and the inner corner, she considered some different ideas for determining the length (longer diagonal in Figure 38). Donna also kept in mind that she would need to use the derivative at some point to find the maximum value. Unfortunately, she could not come up with a reasonable action to take that would have allowed her to make any other reasonable attempts, bringing her efforts to an end.

Follow-up interview portion. I began this portion of the interview (3 minutes 39 seconds) by asking Donna about her comment concerning holding the pole upward. She mentioned that it would be straight up so it would not have had to bend, allowing for any length. Challenging the assumptions embedded in the problem was an interesting, uncommon insight among the students. Donna then mentioned that she assumed next that the pole would be held perpendicular to the walls making it 4 ft in length because that was the skinniest hallway. She confirmed that to find the biggest pole, you had to consider the smallest spot that the pole would have had to pass through.

Next, I questioned her about her use of the Pythagorean theorem and why she said it could not be that simple. To this, Donna responded that when she discovered $c^2 = 80$ to be the max she realized that it could not be true given that you had to use the derivative to find the maximum. Adding further details in response to my questions, she mentioned that this was the only way that she had ever learned to solve for a maximum or minimum.

Afterward, I focused on her drawing of the triangles. Donna explained that you needed to find the hypotenuse of each and add those together to find the length of the pole. When I asked about her assumption that the triangles were similar, she argues that the first had a base of 8 and the second had a base of 4, making one twice the size of the other. I then asked if the problem statement was clear to which she responded yes. Donna also answered no when I asked if there was anything that she wished she knew going into the problem, ending this portion of the interview.

Ethan

Problem-Solving Framework. Ethan followed the order of episodes in the framework relatively close without returning to previous types this time (see Figure 39 below). He also spent a long time exploring, but discovered and began a workable plan. Unfortunately, due to time limitations and because his work seemed familiar in comparison with his solution to Problem 1, I ended the solution attempt after 24 minutes and 49 seconds.

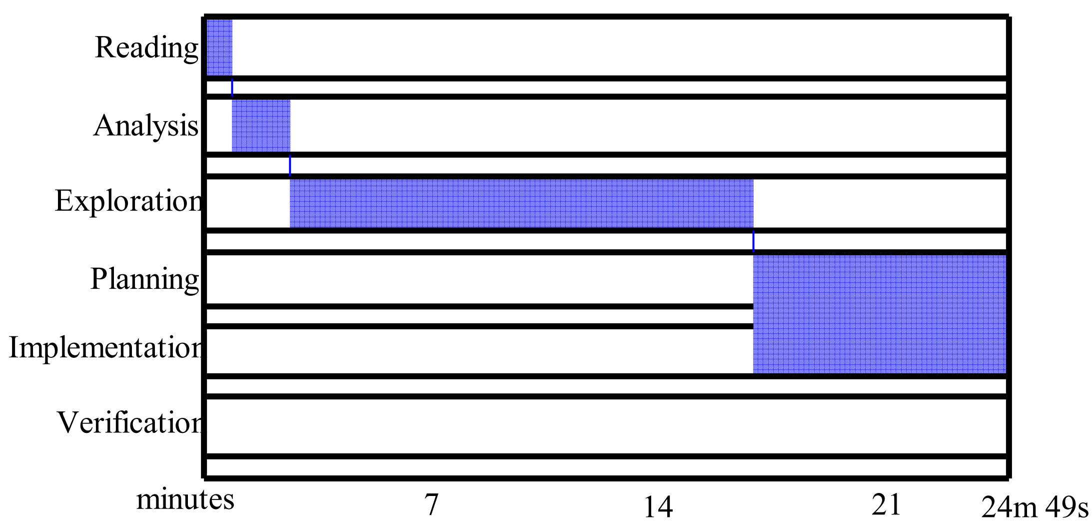

Figure 39. Summary of Ethan's problem-solving episodes for Problem 3.

Solution attempt. After reading the problem, Ethan read it again and examined its different parts. He then attempted to evaluate the assumptions underlying the problem in order to construct a visual representation, which he drew as to scale as possible (see Figure 40). Pausing before beginning to explore different aspects of the problem, he mentioned that he had never solved a problem like this before.

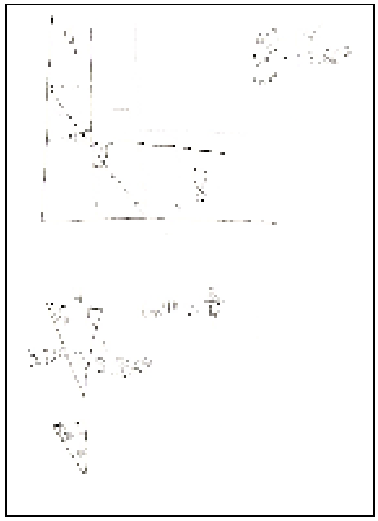

Figure 40. Ethan's written work concerning Problem 3.

Now that he had some understanding of the conditions and goals of the problem, he began to analyze the figure, looking for different shapes that might have been of use. Ethan mentioned that he envisioned a right triangle at the corner with its sides changing while the hypotenuse representing the pole remained constant. Additionally, he searched for a way to make the length of the pole a variable. The angle of the pole against the wall also became another variable of interest. Realizing he had to try some angles, he took his pencil and tried to eyeball the most restricted location of the pole.

Next, he decided to try 45 degrees because the visualization did not really seem to help him. After drawing in the rectangle with the corners at opposite vertices, he noted that he had two right triangles created by the sides of the rectangle, the outer walls, and the pole (see Figure 40). Using the Pythagorean theorem, he determined the length to be about 17. Similarly, he tried 50 degrees between the outer wall of the 8 ft corridor and the pole but had to use trigonometric functions, leading him to a length of 17.6. He noted that the maximum length given these two so far was 17 because that would fit at both locations, leading him to decide to try smaller angles.

Before he began to calculate the length at 40 degrees and because I wanted to make sure I had time to include follow-up questions, I asked him what his process would have been given more time. He answered that he planned to try smaller and smaller angles until the pole length increased. Once he found the minimum length, this would be his answer concerning the longest pole that could fit around the corner because it was the only length that would fit at all angles, concluding the solution attempt.

Follow-up interview portion. I began the follow-up interview portion (3 minutes 49 seconds) by asking how he immediately knew to draw the pole so that it was touching the inner corner. He answered that he envisioned the longest pole physically possible would be touching the corner and scraping along the walls. When I asked if he had seen a problem like this before, Ethan answered no. I then asked what made this question different. He responded that he did not really know what to do with the angles and that he had a hard time determining where to put the variables.

Afterward, I asked if there was a general way to represent his numerical attempts with an equation like he did in Problem 1. Ethan responded that there probably was and that the angle would likely have been the variable in the equation. If he were to graph it and look for a minimum, he would have let x be the angle and y be the pole length (see Figure 41). Lastly, I asked why he was searching for a minimum when the question asked for a maximum. Ethan explained that this was the tricky part of the problem in that to find the maximum pole length, you had to determine the minimum length required of the pole when it was turning or else it would not fit at that location, ending this portion of the follow-up interview portion.

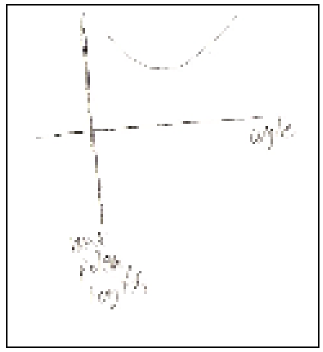

Figure 41. Ethan's display concerning his expectations regarding a length objective function.

Question 2: AP Calculus Students' Understanding of Derivative

In order to study students' understanding of derivative, I first observed them using the concept of derivative while solving the three optimization problems. Then using the Follow-up Interview Portion Questions Guide (Appendix B), I asked them about their use of the derivative and then about their general understanding of the derivative using questions chosen to probe this understanding. Chapter 3 contained an explanation of the choice of questions in the instrumentation section. I analyzed the students' use of the derivative using Roorda, Vos, and Goedhart's (2007) framework developed to study the application of representations of the derivative. Using Zandieh's (2000) framework concerning students' understanding of derivative, I analyzed their use of the derivative and responses to the follow-up questions.

Unfortunately, all but Ethan surprised me with their lack of understanding concerning a concept that students sometimes mentioned appeared throughout the course. Because no one else used the derivative to solve the problem, I could not establish any patterns to answer this research question. With more students like Ethan—who used the derivative and concepts related to it to solve two problems—or an alteration in the structure and methodology of the study, I might have been able to address it as discussed in the Future Research section of Chapter 5.

Of particular note, Ethan displayed an exceptional understanding of the derivative. The other students displayed understanding mainly in the form of pseudo-objects. Two of the other students actually outlined accurate procedures for using the derivative generally, but could not apply them to these problems.

Further Observations

In the follow-up interview portion, I asked the participants which of the three problems they found easiest and which they found hardest. Alice and Chris found the Problem 1 easiest in part because it seemed straightforward, whereas Donna found it to be the hardest because she believed it involved too many variables. Brenda and Donna thought that the Problem 3 was easiest because it was easy to picture, whereas Chris and Ethan thought it was hardest because it was difficult to determine what to look for and do. As for Problem 2, only Ethan found it easiest, saying that it seemed to flow well, but Alice and Brenda thought it was the hardest because they needed the distance formula to solve it.

Afterward, I asked the students about the perceived similarities and differences of all three problems. Noted similarities involved the need to use geometry (Alice & Chris), a requirement to draw a diagram (Brenda), the necessity of formulas (Brenda), a call for testing things (Brenda), a need for measurements (Donna), and a common curve involving some type of minimum (Ethan). Alice thought the problems were all quite different, whereas Ethan and Chris seemed to think the problems had subtle differences. Brenda and Donna believed the differences in the problems stemmed from what they asked for concerning distance, length, and cost.

While working through the problems, all the students except Ethan mentioned that some were word problems. Alice, Brenda, and Chris thought that they typically had difficulties with word problems, whereas Donna thought word problems were relatively easy because they involve applications that could be visualized. Generally, the students identified Problem 1 as a word problem. Some thought Problem 3 was a word problem, but they generally thought that Problem 2 did not fit their definitions.

When I questioned the students about whether they had seen problems like these in their courses, Alice and Brenda replied that they had seen similar problems earlier in the school year. Donna had seen them toward the end of the year. Ethan had seen them earlier that school year and the previous year as well. Chris did not comment on when he had seen similar problems but simply stated that he had encountered one exactly like Problem 2 but nothing as complicated as Problem 1 with no comment on Problem 3.

At the end of each interview, I asked the students to comment on their experience and thoughts of the study. Alice found it beneficial because she was interested in mathematics and wanted to see what the study was like, because it brought back some things she had not used in a while, and because it helped her with her thought process. Brenda thought that the study taught her that she needed to learn more mathematics because she did not think that the first two problems went well and that it was nerve racking overall because of the problem difficulty. Also, she had expected more derivatives, integrals, and areas under curves, which she considered calculus. Similarly, Chris thought the problems were more difficult than expected but believed that difficulty may have been because he did not remember a lot of things he needed. He said he might have had an easier time if the interview had occurred during the school year. Donna liked the study even though it was difficult because it helped her to think about what she was doing and the reasons for it, which she claimed she had never done before. Also, she felt the difficulty might have stemmed from the fact that her class barely got this far in calculus. Humorously, Ethan simply labeled the study “enjoyable.”

Solution Pattern Summary

In this section, I drew out patterns from my analyses of the five students' problem attempts and their follow-up interview portions.

Patterns in Solution Attempts

After looking over the solution attempts of Problem 1 and noting common steps taken by either all or a majority of the students, I pieced together a typical sequence of events. All students began by reading the problem and then drawing a picture of the situation involved, including labels. Afterward, four of the students categorized the problem as a word problem and set up some formulas to describe basic relationships. They then tried to recall techniques from class and, as a result, usually began to try to substitute equations into one another in order to solve for a variable. Lastly, four of them tried out some values in the formulas they had created.

Within the solution attempts of Problem 1, I also noted some less common steps taken by students that could have been beneficial to others. These included considering reasonable possibilities (a realistic feasible set), noting unused information provided by the problem, breaking the problem into parts (data, conditions, goals, etc.), considering available technology (the graphing calculator in this case), recording progress in a chart or graph, considering when a chosen process will terminate, and verifying their result by trying other values.

While working on Problem 2, the majority of the students generated similar steps, but not always in the same order. All of the students naturally began by reading the problem. Soon after, three students decided to determine the minimum of the parabola using the equation, indicating that they at least considered the minimization goal. Then, the students graphed the parabola and considered the utility of various concepts from their previous mathematics classes such as the distance formula, linear equations, and the Pythagorean theorem. This consideration always resulted in the students recognizing the distance formula as a key tool, despite the fact that none of them remembered it. Along with the discovery of the need for the distance formula, the students considered the reasonableness of strategies that they might employ involving the distance formula. Three students then considered if their strategies would ever terminate and attempted to remember more useful techniques from class. Lastly, all but one student used a general point (x, y) on the parabola and defined the distance between that point and the point (1, 0). Unfortunately, many became overwhelmed with the algebra, ending their solution attempts. This seemed to happen because the objective function involved three variables (x, y, distance) and contained a quartic polynomial inside a square root. Among the less common but potentially beneficial steps, I noted students using graphing calculator functions, creating a table or chart to record progress, providing justification for techniques to be embarked upon, calculating points along the parabola to get a sense of the problem space, considering similar problems from class, determining the derivative of the distance function, and verifying their results and formulas.

Even though the students, other than Ethan, did not progress as far on Problem 3, they still followed a number of common steps. As before, all participants began by reading the problem. Soon thereafter, they all drew the situation identified in the problem, making evident some errors in a few students' assumptions about the problem. Next, they noted geometric shapes in their drawings that might prove useful. Three students considered techniques used in class that might have had some utility, such as trigonometric ratios. At times, they reevaluated their basic assumptions and tried out different values, generally resulting in the correction of the initial errors they had made. One or two students benefited from each of the following behaviors: evaluating the reasonableness of their results, considering whether their processes would ever terminate with a realistic answer, and breaking down the problem into parts. Uniquely, one student also envisioned a dynamic picture of varying right triangles with hypotenuses rotating around the inner corner, allowing him to identify key variables to manipulate.

Among these three optimization problems, the students followed some steps consistently in more than one of them. For all three problems, all students began by reading the problem and almost all soon after drew a picture or graph with labels to help orient them to the problem space. The students usually then showed evidence of noting the minimization goal and developed some formulas to describe the problem. Following this activity, they often considered techniques that had been used in class, even though they did not always know how to apply them. Additionally, the students often tried out some values to get a sense of the problem and potential answers. In all three problems of the present study, at least one or student benefited from considering similar past problems, breaking down parts of the problem, using the graphing calculator features, recording their progress in a table or chart, considering when their process would terminate, and verifying their results by trying other values.

Patterns Regarding Episode Order

Looking at the summary charts (Figures 42) of the students' episodes within the solution attempts for each problem, I noted a few patterns. Problem 1 tended to have the largest number of episodes of short duration, the third had much longer episodes that were fewer in number, and the second was somewhere in between. The last two problems have a general downward flow from top left to bottom right in the charts, which may have occurred for a number of reasons. At times, the students seemed to follow a familiar path of looking for something to try and then implementing it without fully considering other ideas. Only Ethan attempted to verify his work in multiple ways, using various representations and strategies. These attempts can be observed in his charts for Problem 1 and 2, where it appears to be a flurry of episodes bobbing up and down. The opposite can be observed in Problem 3 where few students knew how to approach the problem.

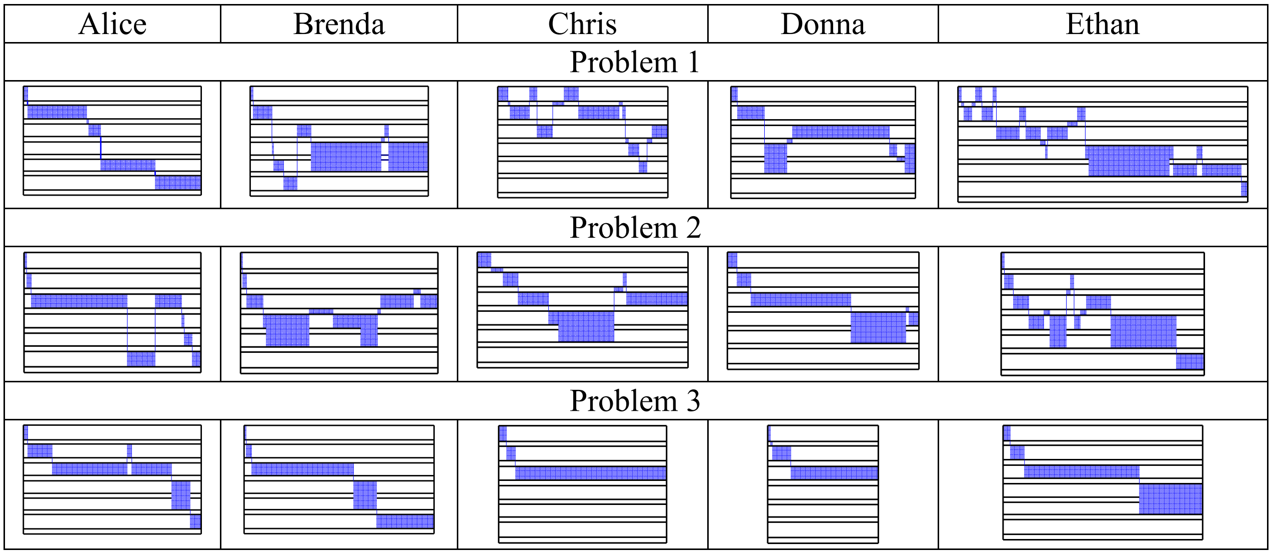

Figure 42. Summary of the problem-solving episode summaries for each problem.

Variable Naming Issues

All five students found themselves unable at some point to define and label variables in a useful way, particularly in the first and last problems, which hindered their use of the analytic approaches they had learned. During Problem 1, all five participants seemed to know that they needed to combine distance and cost in some meaningful way, but rarely managed to do so. Even though they had little trouble defining the area condition in terms of length, they did not easily set up other relationships such as cost in terms of the sides and distance in terms of the points. While working out the objective function in Problem 2, Chris and Donna correctly inserted the generic point (x, y) into the distance equation, but became overwhelmed when the algebraic manipulations became long and they could not find a way to reduce the number of variables. Ethan, on the other hand, who derived the distance formula on his own, used numerical tests to keep himself from performing illegal operations and persevered through long algebraic manipulations. Other than Ethan, these students lacked confidence when working through difficult manipulations and possibly lacked practice in using the calculator to alleviate some of the difficulty.

While working through Problem 3, the students faced a number of issues concerning variables. Alice and Brenda used tools at their disposal including the Pythagorean theorem and trigonometric ratios, but relied too heavily on estimates, which kept them from setting up an objective function. Chris and Donna defined a number of helpful variables but could not find meaningful ways to reduce them. Although Ethan struggled to determine the best variable to manipulate, he found one that he could reasonably work with if he had had more time. Rather than creating variables, two students preferred to estimate values to be used in the problems. Two seemed unwilling to rely on any estimates; instead, they chose to create so many variables that they became overwhelmed.

Substitution Issues

Whether or not the students developed the correct equations using reasonable variables, they still tended to struggle with substitution, either to reduce the number of variables or to try specific values. Chris and Donna solved for a variable and then substituted it back into its own equation, rather than into a different one, resulting in a tautology. Alice and Chris also made complicated substitutions that seemed to lack meaning, leading them in an unproductive, unfocused direction. The students other than Alice made reasonable substitutions into the distance formula that could have led to a solution in Problem 2. Brenda, Donna, and Ethan relied on substituting guesses, hoping to head toward a reasonable solution. Only Ethan appeared to fully recognize that varying his guesses could be useful in determining an estimate for the solution.

Minimum/Maximum Issues

Although several of the students demonstrated some intuition about determining the optimal solutions, all but Ethan had some trouble with how to appropriately think about or calculate the maximum or minimum value. Relying on guesswork, Alice and Brenda occasionally forgot that the minimizing or maximizing goal implied that multiple values existed in the feasible set and required evaluation to determine which was best. Other times, a few of students forgot to consider the variable to be minimized and instead found the minimum of a different function. For example, even though he was instructed to find the minimum distance, Chris took the derivative of the parabola given in problem two. Brenda, Donna, and Ethan tried many values to determine an approximation, but the first two abandoned this process once realizing it might continue indefinitely. Only Ethan persevered and found the correct answer using this method. In Problem 3, Brenda, Chris, and Ethan demonstrated an intuition for optimization when they noted that the goal to find the maximum pole length actually involved finding the minimum pole length that fit around the corner among infinitely many maximal possibilities. On the other hand, the students may also have misremembered the goal at times, as Chris did in Problem 1, leading to unfocused explorations. Donna and Ethan seemed particularly able to outline a general algorithm for finding the optimal value using the derivative. Chris, however, was unable to recall these techniques, other than noting that the derivative in combination with some geometry might have some use in the last problem.

Objective Function Issues

All of the students found it difficult to create the necessary objective function in at least one problem. In a couple of instances, the students did not consider creating an objective function because they believed only one value existed in the feasible set: Alice in Problem 1 and

Brenda in Problem 3. Sometimes, these two relied heavily on their own ability to judge the best set of values through estimation and "eyeballing" the situation. In most cases, however, the students all knew they needed some sort of objective function, but did not always know how to create it from the variables they produced. All but Alice created an objective function for at least two problems when including the follow-up interview portion, either by manipulating formulas and variables that they generated or by trying out values to get a sense of the process followed by substituting variables into the process. Despite creating the objective function, three either did not remember how to determine optimal values (Brenda and Chris) or became too overwhelmed with the variables (Chris and Donna). Brenda and Ethan realized that they could alleviate some of these issues by using the graphing calculator's *minimum* and *zero* functions on the graphs of their objective functions.

Intuition about Optimization

Although the students sometimes showed an intuition for optimization, they often relied heavily on guesswork that at times hindered their ability to find the solution. Working through Problem 1, all but Alice, who assumed the rectangle was a square, noted that the shortest side of the rectangle should occur along the highway. Unfortunately, Brenda and Ethan assumed, at least for a brief time, that this meant that the shortest side should be as small as possible, overloading the other sides. Because Problem 2 provided a readily available equation to work with, $y = x^2$, Brenda, Donna, and Ethan minimized values other than the appropriate objective function. The first two students relied confidently on eyeballing lengths and distances even sometimes when they had an objective function to verify their estimates. Alice in Problem 2 and Ethan in Problems 1 and 3 also tried out a range of values, explaining that if they had more time they would continue to narrow their range to reach an approximation. Trying out at least a few values did seem helpful at least for giving the students except Chris a sense of possible solutions. Generally, though, the students tried to avoid relying too heavily on estimations, realizing the benefit of having an objective function with which to judge possible values.

Reasonableness and Verification of Results

Just as the students generally attempted to assess the reasonableness of possible values, they periodically tried to assess the reasonableness of their strategies. Several students reconsidered their basic assumptions underlying their attempts. While working problem two, for example, Alice altered her assumption about the meaning of the term closest. During Problem 3, Alice, Donna, and Ethan kept an inventory of what they had already used and still had not included. Occasionally, the students generated various strategies to try and evaluated them. As an example, Brenda, Donna, and Ethan realized that trying out points in the distance formula for Problem 2 would be cumbersome and likely inefficient. At times, all but Alice considered whether their current plan would lead to the intended goal, preventing them from heading toward a time-consuming dead end. Even though Alice, Brenda, and Ethan evaluated their plans, they did not always make the right assessments concerning whether their strategy would give them a rigorous solution, particularly when those strategies involved some aspect of visual estimation. Toward the end of some attempts, Brenda, Donna, and Ethan sometimes tried to assess their results and conclusions. Although Alice, Brenda, and Ethan assessed one or two results, the first two students did not verify their answers or assumptions unless prompted.

Consideration of Previous Learning

Although the students generally evaluated the reasonableness of their strategies, the participants, during two-thirds of the solution attempts, did not initially make an assessment of the knowledge or strategies used in prior problems, even if they realized they had seen something similar. Occasionally, all but Ethan recalled extremely general tools from their calculus classes or other sources, such as real life, but they did not always know how to apply them. Only Donna and Ethan recalled some useful strategies for optimization problems, with Ethan alone actually capitalizing on them.

Word Problem Difficulties

When the students did consider previous problems that had similarities to the three they were given, they most commonly brought up the concept of word problems, especially while working on Problem 1. The students defined *word problems* as having the following characteristics: They provide background information or a story to sift through (Alice, Brenda, and Chris), contain sentence and paragraph structure from which to determine variables and formulas (Chris), and involve real-life applications that can be visualized (Chris and Donna). These students also listed aspects of problems that they would not label as word problems: they are information based, like the third one (Alice); everything is set concerning the conditions and data (Brenda); there is a set procedure to follow (Brenda and Donna); and they may provide a diagram or equation in lieu of words (Chris and Donna). Unfortunately, the students mentioned they did not have much experience with word problems, having either encountered a few at the beginning of the year or barely worked with them at the end. The students thought that word problems are difficult, with the exception of Donna, for the following reasons: they involve many steps (Alice), they are not common (Brenda), they seem to require more information than given (Chris), and they are advanced (Donna).

Lack of Calculator Use

Even though students could have used the graphing calculator at their disposal to help alleviate some of their struggles, they rarely used of the full functions of the tool. All the students used the graphing calculator for basic arithmetic or determining trigonometric ratios, functions that are common on basic scientific calculators. Although all but Chris occasionally did use the calculator for graphing utilities and functions, more often they merely mentioned it without following through. Those that did use the graphing utilities either simply graphed a particular function (Alice and Ethan), used the *zoom* function (Alice), used the *minimum* function (Ethan), or used the *zero* function (Ethan). Interestingly, Alice thought that a particular equation could not be graphed because it had variables with names other than x and y, which might have kept other students from considering graphs in those problems.

CHAPTER 5

DISCUSSION

Because of my interest in gifted students, I talked at length with a number of AP Calculus teachers to find an interesting problem that might apply to their students. When they explained that their students struggle with optimization problems, I became engrossed in the topic. This fascination was partially due to my own experience as a high school teacher; my students also generally struggled when I gave them application problems, even when those problems involved relatively simple mathematics. I also knew that optimization involves a conceptual understanding of the derivative that allows for its application in nonroutine situations. For that reason, I asked the following questions:

- How do AP Calculus students understand and solve optimization problems?
- How does students' understanding of the concept of derivative affect their approach to solving optimization problems in AP Calculus?

While constructing, conducting, and analyzing the students' interviews to answer these questions, I used three frameworks along with ideas from other authors. During the interviews, I used Polya's (2004) problem-solving suggestions as a guide for directing students when they became unable to move forward during their attempts. To analyze the solution attempts, I used Schoenfeld's (1985) problem-solving framework to identify and examine macroscopic problem-solving episodes produced by the students, which included reading, analysis, exploration, planning, implementation, and verification. Because optimization problems in calculus generally make use of the derivative, I included Zandieh's (2000) framework for students' conceptual understanding of the derivative, which also provided insight into how that understanding played into their solutions. In the follow-up interview portion, I included questions similar to those

Zandieh used to explore students' understanding during her development of the framework. Lastly, I used Roorda, Vos, and Goedhart's (2007) framework to analyze direct uses of the derivative during the solution attempts, looking for patterns that might provide further insight into how students' understanding of the derivative affected their solution attempts. Because only one student actually used the derivative to solve the problem, however, I could not answer the second research question.

After talking to two AP Calculus AB classes at two different schools and an AP Calculus BC class, I recruited seven students for the study. The pilot study allowed me to make necessary adjustments to the interview and problem-solving portions of the study. Using a semi-structured interview approach, I questioned all seven students about their academic backgrounds, interpretations of the problems, their solutions, and their understanding of the concept of the derivative. I analyzed their responses using the three frameworks mentioned above and looked for key similarities, differences, and relationships among their solution structures. I examined their limited use and understanding of the derivative and noted other interesting observations, which did not allow me to draw any conclusions.

Conclusions

As mentioned earlier, the students in this study almost always began their solution attempts by drawing a figure to represent the problem. This may have occured because the College Board (2010) stressed that teachers should use multiple representations, namely graphical, numerical, analytic, and verbal. This emphasis may have been why many of the students also readily created formulas to manipulate and some formed tables from which to compare values. Other researchers have noted that their participants frequently created a graphical representation (Roorda, Vos, & Goedhart, 2010; Selden, Mason, & Selden, 1989;

Selden, Selden, & Mason, 1994; Zandieh, 2000), developed and focused on using formulas (Roorda, Vos, & Goedhart, 2007a), and implemented some form of trial and error (Selden, Mason, & Selden, 1989; Selden, Selden, & Mason, 1994).

Despite representing the problem in different ways, they still generally failed to use their knowledge of calculus to solve the problems. Even when they knew the procedure usually taken by calculus students, the students, except one, did not use it. This may have occurred because the students focused on learning the procedures and not on understanding why they work and how to apply them. Other researchers found that students often failed to apply calculus techniques they learned (Selden, Mason, & Selden, 1989; Selden, Selden, & Mason, 1994).

The one student who solved any problems displayed an overwhelming conceptual understanding. There are a number of possibilities for the connection between the two. For example, his conceptual understanding of the derivative may have allowed him to apply it to new situations. Another possibility can be derived from the idea that he enjoyed solving the problems. As a result of his enjoyment of mathematics, he may have desired to learn to solve problems well and to develop a deep understanding of the concepts in his mathematics classes. Of course, he may have enjoyed the problems and problem solving in general because he first gained conceptual understanding.

Aside their difficulty in solving Problem 1, the students generally found it to be the easiest, which may have stemmed from the fact that they knew at least a few things they could try. This may be because the problem included a number of concepts such as length, area, rectangles, and cost. The regular transfer between episodes of different levels, in this problem may have resulted from having a variety of concepts to work with leading them to check back with the problem and plan and implement different strategies.

Even though two students preferred problem 3 because it was easy to visualize, they did not generate many reasonable strategies that would lead to an answer. Additionally, the students spent a good deal of time in exploration episodes looking for something to try. If they found something, they seemed to try it regardless of its potential. The student who demonstrated the most understanding also tended to do the most verification of his work; the students who struggled seemed less interested in verifying their results. In a discussion of the results from using his framework, Schoenfeld (1985) similarly noted two common types of episode patterns: those involving mostly the reading and exploration typical of problem-solving novices and those that involved the regular assessments and verification typical of problem-solving experts.

The difficulties the students encountered concerning the naming and use of their variables are consistent with findings by Klymchuk, Zverkova, Gruenwald, and Sauerbier (2010), who gave 104 calculus students an application to solve and a questionnaire about their difficulties. Having been the instructors to the calculus course these students took, the authors included some recommendations about how to improve their performance. These involve teaching basic skills with application problems from the beginning of the course, including applications problems commonly throughout, and requiring students to write out their steps in detail on all problems, simple and complicated. Actually writing out the meanings and units of the variables may have helped several of the students in my study who found themselves confused when setting up equations and identifying relationships between them. Even with the recommendations of these authors, the students in both studies did not demonstrate conceptual understanding.

Along with their difficulties involving the variables, the students took some interesting steps involving substitution. As in the cases where students substituted variables back into their original equations and made complicated, meaningless substitions, students appeared to substitute values at times in order to be productive, following some procedures learned in class to reduce the number of variables. Although one of the students recognized that the operations taken involved some sort of circular reasoning, these students did not seem to have a conceptual basis for the operations. Because students often verbally stated different strategies they knew, they may have performed these operations in an attempt to show me what they knew about different operations learned in class rather than what seemed a reasonable strategy at the time. Similarly, students explained to me detailed procedural operations for finding minimums and maximums that they did not follow, indicating potentially that they did not know how to use the procedures or they were displaying their knowledge without intending to actually try it, or both. A further indication that students not only lacked some of the conceptual knowledge to solve the problems and that they may have wanted to show me something, students often mentioned different types of problems they could solve and then could not solve them when I altered the problem space.

When students did not know an analytical approach, they sometimes tried a numerical one, using estimates. Fortunately, generating reasonable estimates seemed to help the students make progress in their solutions as long as they did not become too confident in them. In all, there seemed to be three groups of students among these five: one who avoided making estimates, two who readily accepted their estimations, and two who remained skeptical. Describing problem solvers in a similar way, Polya (1965) noted two types of problem solvers: primitive ones that wait around for a guess to come to them, which they often simply accept, and sophisticated ones that actively make guesses, which they verify and readily change when needed. Only the skeptical students in this study made substantial progress, each generating objective functions and verbalizing a workable strategy for determining optimal values.

The students often made evaluations of their strategies even if it was relatively insubstantial. Occasionally, but rarely, students made global assessments concerning the progress of their strategies in connection to the problem at hand. Instead, they seemed to focus on assessments of the difficulty and time necessary to complete their strategies. These were sometimes warranted, but did not always send the students in a better direction. Students sometimes abandoned a workable strategy in favor of a less efficient one or none at all.

Surprisingly, students rarely recalled workable strategies from similar problems they had solved in the past. Although some did recall similar problems and sometimes similar results, they often did not recall the associated, successful strategies. Considering that the AP Calculus curriculum was intended to be challenging and required them to regularly recall much of their previous learning, I would have expected that more students would have successfully recall strategies that would be beneficial to them. Polya (1965) too noted that problem solvers may make progress by actively considering available knowledge in connection to their current problem, particularly previous problems with similar unknowns or goals, in this case to optimize some variable. Selden, Selden, Hauk, and Mason's (1999) note that many of their participants relied more heavily on algebraic methods than calculus while working to solve nonroutine problems, which may have indicated that the participants lacked some understanding allowing them to transfer their knowledge.

Students often had a difficult time recalling commonly used formulas such as the distance formula and trigonometric ratios. Considering that these formulas show up in prior courses and AP Calculus, the students should have been able to recall them much more accurately and readily. Even though none of the students could quickly recall the distance formula, one student did have the conceptual understanding to derive it, overwhelmingly demonstrated by the connections made between the Pythagorean theorem, the coordinate plane, and generic points. Unfortunately, including the Pilot Study, only one in seven could do this. It would be interesting to know what made the difference, whether understanding, ability, or simply the belief it could be done, as we would like all or most of our students to be able to do derive key formulas when needed.

The students' beliefs that they were poor at solving word problems might have hindered their ability to construct a well thought-out solution. In fact, the only participant who did not apply this label nearly solved all three problems. Researchers also note that students struggle with word problems, do not encounter them as much as they should (Peter-Koop, 2010; Verschaffel & Corte, 1997), and may not perform well because anxiety resulting from previous poor performance uses up some of their working memory (Rayner, Pitsolantis, & Osana, 2009). The students also may have used this labeling as a reason for being unable to solve a given problem, indicating an affective influence. When mentioning that they were bad at word problems, the students may have been indicating that they could have solved other types if provided. For example, one student mentioned that if I had instead given her an area-under-a-curve problem, she could have solved it "lickety split." Schoenfeld (2011) also stated that students may not continue to work on a problem if they believe that mathematics problems should be solved within a time limit, say 5 minutes, even if they might have been able to solve them. Notably, the three students who did not make substantial progress not only labeled many of the problems as word problems, but also mentioned that word problems were difficult. This may indicate that they expected to be unable to solve them and gave an effort reflecting that belief.

Many should have considered using the graphing calculator when the algebraic manipulations became too overwhelming to find a minimum. In most cases, the students might as well have had a scientific calculator in front of them instead, considering the surprisingly unsophisticated operations they performed with it. In agreement with the implications noted here, Lee and McDougall (2010) observed that secondary school students could use the graphing calculator to free themselves from the mundane manipulations in problems and focus on the mathematical concepts involved. Students might have also used the calculator to explore the problem space, especially when they had a formula in front of them to explore.

Even with the graphing calculator and at their disposal, the students had remarkably little success in solving the three nonroutine problems given to them. Three of the five students were unable to even make substantial progress toward the solutions to the given problems, even though four did construct the objective function in Problem 2. Only two students seemed to know what to do with this objective function, but one became overwhelmed with the algebraic operations, which prevented further progress. Of the five, only one student actually pieced together a correct, rigorous response to one problem. These results are supported by previous studies that show students often fail to solve nonroutine problems (Roorda, Vos, & Goedhart, 2007b; Selden, Mason, & Selden, 1989; Selden, Selden, Hauk, and Mason, 1999; Selden, Selden, & Mason, 1994).

Although the students admitted they had seen problems similar to these, they thought they could not deal with the differences. Even after I simplified the problems for two students, they still could not use their experience to progress. In one case, a student outlined a reasonable process for solving optimization problems, but could not follow through. In the same way, the other students may have had the knowledge to solve the problems and could not actually apply it.

Wondering whether or not this was true for their study, Selden, Selden, Hauk, and Mason (1999) discovered that even when students had demonstrated they had the full set of factual knowledge necessary for the nonroutine problems, only 34% of their solutions were substantially or completely correct. On the other hand, the authors of that article cited Schoenfeld's (1985) definition of a problem without establishing that the students accepted the problem for themselves. In the present study, all seven students appeared to accept the problems because they were usually willing to work on them for over 20 minutes and readily desired to know the solutions. Even though the students provided evidence they accepted the problems for themselves, the results of the present study further support that knowing the content is not enough for students to solve nonroutine problems.

Schoenfeld's framework (1985) helped me to focus on the managerial decisions made by students as they solved the problems, even though some alterations may be warranted. Observing their actions and thoughts spoken aloud led me to pinpoint a list of problem points during their solutions attempts and gave me a feel for what they knew but could not actually determine how to use. At the same time, some of the listed questions within the episodes were not always informative concerning the present study. During the reading episodes, the students simply read through the problem and then began to analyze it, at which point they sometimes made note of the goals and conditions. In the same way that Schoenfeld mentions that planning and implementation may coincide, it appears that analysis and exploration might as well. Students may regularly return to analyze the problem while exploring their knowledge space for useful strategies and tools. One of the key aspects of the framework for this present study involved locating and examining assessments made about the students solution progress, new information that they had determined or received, and the reasonableness of their results. The analysis of these particular events existed in Schoenfeld's questions on transitions. Because students' assessments and decisions based on those assessments seemed to influence much of their solution attempts, more detail concerning this aspect of the framework may be needed. Overall, however, the framework proved a useful resource for analyzing the protocols without requiring an overcomplicated and difficult coding procedure.

Implications for Teachers

I did not collect data on how the students were taught calculus concepts; however, the results of this study can provide insight into student struggles teachers could keep in mind while teaching optimization problems. Having these points of difficulty made explicit may help teachers pinpoint particular issues individual students have while solving optimization problems and may help teachers be the catalyst for innovative interventions. Throughout this section, I make suggestions that I have found beneficial during this study, which experienced, talented teachers may already have in their arsenal among other strategies. My hope is that these suggestions, will spark some discussion of beneficial teaching practices to alleviate the struggles encountered by these students.

In the follow-up interview portions, I sometimes asked students to use variables in place of the values they had tried in an attempt to calculate the variable to be minimized or maximized. Sometimes, I asked them first to try values if they did not already. This method worked well several times in the follow-up interview portion for the students in this study. Students were always able then to create the objective function, even if they did not know how to proceed from there.

Towards the end of the interview, three of the students mentioned that participating in it had been beneficial, in part because it helped their thought processes because they had been asked to think about what they were doing and provide reasoning for their actions. Teachers might find it informative to explore the use of teaching experiments periodically throughout the course, either individually or in pairs, where students work through problems followed immediately by an interview where students have to answer questions about their performance and reasoning, in an attempt to nurture their conceptual understanding. Single problem interviews throughout the year might help students through an adjustment of expectations and an acquaintance with such problems.

Future Research

Researchers might consider examining the extent to which the suggestions for teachers, mentioned in the previous section, benefit students. Researchers could also consider exploring which of these implications might extend to teaching other types of applications problems. Perhaps most important, we need to examine the relationship between what students are actually taught during optimization units and what students can or cannot actually do as a result. In particular, researchers might examine why some students seem to have an exceptional grasp of the concept of the derivative and actually use it in their solutions, whereas others do not.

Additionally, classroom teachers might consider developing some teaching experiments when addressing the optimization problems unit to explore the effectiveness of these suggestions. For example, Panaoura, Gagatsis, and Demetrioun (2009) demonstrated that "providing students with the opportunity to self-monitor their learning behavior when they encounter obstacles in problem solving, through the use of modeling, is one possible way to enhance students' self-regulation and consequently their mathematical performance" (p. 63). Researchers might structure a study similarly to see if student's work improved in nonroutine problems, optimization problems in particular. Similar to what occurred in the follow-up interview portions where students made additional progress without receiving any direct instructions from me, researchers might also consider allowing students to talk with the interviewer in an attempt to lead to a successful solution attempt, making note of key moments where students struggled and overcame those struggles.

Alterations in the structure of the study could be considered to focus more on particular aspects of how students solve these problems. Limiting the research to a particular problem or two and asking detailed questions of more students may help researchers identify and support patterns common to typical AP Calculus students. It might also be interesting to compare the students' performance immediately after learning optimization problems with what they do towards the end of the year or after the course. First asking teachers to identify students particularly able in the classroom may also provide candidates more likely to successfully solve the optimization problems and thereby provide further information into how successful students solve them. Including data about the actual teaching that occurred in the classroom may also provide insight into why students solve problems the way they do and what knowledge they may have had at their disposal.

Researchers also can investigate some of the student difficulties that were demonstrated throughout this study. Exploring how and why students use the graphing calculator in different problem contexts can help teachers better understand how to direct calculator use in their classrooms. Research has been conducted on the link between calculator use and achievement (Lee & McDougall, 2010), but more research is needed on how students choose to use the calculator. Additionally, research has repeatedly revealed that students struggle to correctly use variables in word problems (Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010), but more research is needed on explicitly why students have these difficulties and what teachers can do to alleviate them.

Helping students to perform well on optimization problems and application problems in general must become a priority, not only because students will encounter problems like this in the field but also because they generally find them motivating. Students may find themselves more able to engage in application problems because they can refer to personal experiences and think about actual physical contexts. These types of problems also provide a natural way to test what students really understand and what knowledge they can transfer to new situations. Focused research on using nonroutine problems regularly in the classroom as teaching and assessment tools may be warranted, desirable, and necessary for teachers to help determine what their students will take with them beyond their courses.

Final Comments

The results of this study show that students, even in the highest mathematical courses, have some very fundamental issues that prevent them from solving optimization problems. At times, students struggled with their use of variables, either creating them or substituting them. Surprisingly, they generally could not remember basic, useful equations such as the distance formula or trigonometric identities on their own. Some students relied too heavily on estimates while others avoided making estimates. Students also used labels such as “word problems” that may have hindered their performance or as a reason for low performance.

Regarding concepts taught in the calculus course, students struggled putting together key formulas and using efficient calculus concepts. Even though they recognized the variable to optimize, they usually could not put together a general equation to calculate it. When they could do this, students still rarely used the derivative to determine optimal values, despite sometimes outlining the general procedures to do this. When asked about the concept of the derivative, they demonstrated a weak understanding of the underlying concepts and representations related to it, which may have hindered their ability to use it.

The present study also pointed out and justified necessary explorations for future researchers regarding these issues. We need to know what differs between successful students in solving these types of problems and those that struggle, even after having taken the same courses. Observations of the classroom and teaching methods might provide information into students' performance. Furthermore, we need to learn why students continue to have fundamental issues while solving these problems, despite reaching the highest level mathematics courses in high school.

Helping students to perform well on optimization problems and application problems in general must become a priority, not only because students will encounter problems like this in the field but also because they generally find them motivating. Students may find themselves more able to engage in application problems because they can refer to personal experiences and think about actual physical contexts. These types of problems also provide a natural way to test what students really understand and what knowledge they can transfer to new situations. Focused research on using nonroutine problems regularly in the classroom as teaching and assessment tools may be warranted, desirable, and necessary for teachers to help determine what their students will take with them beyond their courses.

References

Basson, I. (2002). Physics and mathematics as interrelated fields of thought development using acceleration as an example. *International Journal of Mathematical Education in Science and Technology*, *33*, 679–690.

Black, K. (1997). Putting constraints in optimization for first-year calculus students. *Siam Review*, *39*, 310–312.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, *29*, 41–62.

Brown, S. I., & Walter, M. I. (1990). *The art of problem posing*. Hillsdale, NJ: Erlbaum.

Cai, J. (2003). What research tells us about teaching mathematics through problem solving. In F. K. Lester (Ed.), *Research and issues in teaching mathematics through problem solving*. Reston, VA: National Council of Teachers of Mathematics.

Carvallo, P., Figueiredo, L., Gomez, J., & Velho, L. (2003). *Mathematical optimization in graphics and vision*. Rio de Janeiro, Brazil: Instituto de Matematica Pura e Aplicada, IMPA.

Chieu, V. M., Luengo, V., Vadcard, L, & Tonetti, J. (2010). Student modeling in orthopedic surgery training: Exploiting symbiosis between temporal Bayesian networks and fine-grained didactic analysis. *International Journal of Artificial Intelligence in Education*, *20*, 269–301.

College Board. (2010). *Calculus: Course description*. New York, NY: Author.

Craig, T. (2002). Factors affecting students' perceptions of difficulty in calculus word problems.

Second International Conference on the Teaching of Mathematics, Crete, Greece, July 1-6, 2002. Retrieved April 7, 2012 from http://www.math.uoc.gr/~ictm2/Proceedings/pap411.pdf.

Derigs, U. (2001). *Optimization and operations research*. Köln, Germany: Universität zu Köln.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht, the Netherlands: Kluwer.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, *24*, 8–40.

Etherington, M. B. (2011). Investigative primary science: A problem-based learning approach. *Australian Journal of Teacher Education*, *36*(9), 36–57.

Frederiksen, N. (1984). Implications of cognitive theory for instruction in problem solving. *Review of Educational Research*, *54*, 363–407.

Gundersen, G., & Steihaug, T. (2010). On large-scale unconstrained optimization problems and higher order methods. *Optimization Methods and Software*, *25*, 337–358.

Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, *102*, 395–406.

Hart, D. K. (1991). Building concept images—Supercalculators and students' use of multiple representations in calculus. Dissertation Abstracts International, *52*, 4254A. (UMI No. 9214776)

Hatfield, L. L. (1978). Heuristical emphases in the instruction of mathematical problem solving: Rationales and research. In L. L. Hatfield & D. A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 21–42). Columbus, OH: ERIC.

Hausknecht, A. O., & Kowalczyk, R. E. (2008). Exploring calculus using innovative technology. In Editor's name needed here (Ed.), *Proceedings of the 19th Annual International Conference on Technology in Collegiate Mathematics* (pp.75–79). City, State: Publisher.

Henderson, K. B., & Pingry, R. E. (1953). Problem-solving in mathematics. In H. F. Fehr (Ed.), *The learning of mathematics: Its theory and practice* (pp. 228–270). Washington, DC: National Council of Teachers of Mathematics.

Hiebert, J. (2003). Signposts for teaching mathematics through problem solving. In F. K. Lester Jr. & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten–Grade 6* (pp. 53–61). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., et al. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12–21.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al, (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Jonassen, D., Strobel, J., & Lee, C. B. (2006). Everyday problem solving in engineering: Lessons for engineering educators. *Journal of Engineering Education*, *95*(2), 139–151.

Kallrath, J. (2000). Mixed integer optimization in the chemical process industry: Experience, potential and future perspectives. *Chemical Engineering Research and Design*, *78*(6), 1–19.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for*

Research in Mathematics Education, *8*, 163–180.

Kilpatrick, J. (1978). Variables and methodologies in research on problem solving. In L. L. Hatfield & D. A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 7–20). Columbus, OH: ERIC.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123–147). Hillsdale, NJ: Erlbaum.

Klymchuk, S., Zverkova, T., Gruenwald, N., & Sauerbier, G. (2010). University students' difficulties in solving application problems in calculus: Student perspectives. *Mathematics Education Research Journal*, *22*(1), 81–91.

Kotsopoulos, D. (2007). Unravelling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher*, *63*(2), 19–24.

Krussel, C. E. S. (1995). Visualization and reification of concepts in advanced mathematical thinking. *Dissertation Abstracts International*, *56*(4), 1279. (UMI No. 9517901)

Lambdin, D. (2003). Benefits of teaching through problem solving. In F. K. Lester, Jr. & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten–Grade 6* (pp. 3–14). Reston, VA: National Council of Teachers of Mathematics.

Lee, J. A., & McDougall, D. E. (2010). Secondary school teachers' conceptions and their teaching practices using graphing calculators. *International Journal of Mathematical Education in Science and Technology*, *41*, 857–872.

Lester, F. K. (1978). Mathematical problem solving in the elementary school: Some educational and psychological considerations. In L. L. Hatfield & D. A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 7–20). Columbus, OH: ERIC.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes.* Final report of the National Science Foundation project MDR 85-50346. Washington, D.C.: National Science Foundation.

Machado, J. A., Silva, M. F., Barbosa, R. S., Jesus, I. S., Reis, C. M., Marcos, M. G., et al. (2010). Some applications of fractional calculus in engineering. *Mathematical Problems in Engineering*, *2010*, 1–34.

McIntosh, R., & Jarrett, D. (2000). *Teaching mathematical problem solving: Implementing the vision*. Portland, OR: Northwest Regional Educational Laboratory.

Man, M., Modrak, V., & Grabara, J. K. (2011). Marginal cost of industrial production. *Polish Journal of Management Studies*, *3*, 62–69.

Martin, T. (2000). Calculus students' ability to solve geometric related rates problems. *Mathematics Education Research Journal*, *12*(2), 74–91.

Mebarak, E. (2003). *On the development of an automated design procedure to design optimal robots*. Unpublished master's thesis, Florida International University, Miami.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Panaoura, A., Gagatsis, A., & Demetriou, A. (2009). An intervention to the metacognitive performance: Self-regulation in mathematics and mathematical modeling. *Acta Didactica Universitatis Comenianae Mathematics, 9*, 63–79.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage.

Perels, F., Gürtler, T., & Schmitz, B. (2005). Training of self-regulatory and problem-solving competence. *Learning and Instruction*, *15*, 123–129.

Perkins, P. (2010). *Discrete mathematical modeling: Math 381 course notes.* Unpublished notes, University of Washington, Seattle. Retrieved February 12, 2011, from http://www.math.washington.edu/~perkins/381AWin10/notes/381notes.pdf

Peter-Koop, A. (2004). Fermi problems in primary mathematics classrooms: Pupils' interactive modeling processes. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010*. (Proceedings of The Twenty-Seventh Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 454–461). Sydney, Australia: MERGA.

Polya, G. (1954). *Induction and analogy in mathematics: Mathematics and plausible Reasoning* (Vol. 1**)**. Princeton, NJ: Princeton University Press.

Polya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving*. New York: Wiley.

Polya, G. (2004). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press. (Original publication 1945)

Rayner, V., Pitsolantis, N., & Osana, H. (2009). Mathematics anxiety in preservice teachers: Its relationship to their conceptual and procedural knowledge of fractions. *Mathematics Education Research Journal*, *21*(3), 60–85.

Redish, E. F., & Steinberg, R. N. (1999). Teaching physics: Figuring out what works. *Physics Today*, *52*, 24–30.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*, 175–189.

Rockafellar, R. T. (2007). *Fundamentals of optimization: Lecture notes*. Unpublished notes, University of Washington, Seattle. Retrieved February 12, 2011, from http://www.math.washington.edu/~rtr/fundamentals.pdf.

Rogawski, J. (2008). *Calculus*. New York: W. H. Freeman.

Rogawski, J., & Cannon, R. (2012). *Rogawski's calculus for AP: Early transcendentals*. New York, NY: W. H. Freeman.

Rogers, K. (1999). Research synthesis on gifted provisions. Retrieved March 15, 2012, from http://www.austega.com/gifted/articles/Rogers_researchsynthesis.htm.

Romberg, T. A. (1994). Classroom instruction that fosters mathematical thinking and problem solving. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 287–304). Hillsdale, NJ: Erlbaum.

Roorda, G., Vos, P. & Goedhart, M. (2007a). Derivatives in applications: How to describe students understanding. In G. Kaiser, F. Garcia & B. Sriraman (Eds.), *Proceedings of the WG on 'Mathematical Modeling and Applications' 5th Conference on European Research in Mathematics Education (CERME-5)* (pp. 2160–2169). Nicosia, Cyprus: University of Cyprus.

Roorda, G., Vos, P., & Goedhart, M. (2007b). The concept of derivative in modeling and applications. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling: Education, engineering and economics* (pp. 288 – 293). Chichester, UK: Horwood Publishing.

Roorda, G., Vos, P., & Goedhart, M. J. (2009). Derivatives and applications, development of one student's understanding. In R. Leikin & J. Mamona-Downs (Eds.), *Proceedings of the Working Group on Advanced Mathematical Thinking at the 6th Conference on European Research in Mathematics Education* (pp. 2296–2305). Lyon, France: University of Lyon.

Rossman, A. J., & Chance, B. L. (2002). A data-oriented, active learning, post-calculus introduction to statistical concepts, methods, and theory. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching of Statistics* (pp. 1–6). Voorburg, The Netherlands: International Statistical Institute.

Salas, S. L., Hille, E., & Etgen, G. J. (2003). *Calculus: One and several variables*. New York, NY: Wiley.

Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematics problem solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 345–395) New York, NY: Academic Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H., & Herrmann, D. J. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. *Journal of Experimental Psychology*, *8*, 484–494.

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve nonroutine problems? *Journal of Mathematical Behavior*, *8*, 45–50.

Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). *Do calculus students eventually learn to solve non-routine problems* (No. 1999-5). Cookeville, TN: Tennessee Technological University, Department of Mathematics.

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. *Research Issues in Undergraduate mathematics Learning* (MAA Notes 33), pp. 19–26.

Seltzer, S., Hilbert, S., Maceli, J., Robinson, E., & Schwartz, D. (1996). An active approach to calculus. *New Directions for Teaching and Learning*, *68*, 83–90.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25, pp. 59–84). Washington, DC: Mathematical Association of America.

Shepherd, M. (2007, March 30). *Some Calculus 2 students seem to prefer procedural approaches to exercises over conceptual ones*. Paper presented at the 2007 Missouri Section of the MAA. Retrieved February 12, 2011, from http://www.missouriwestern.edu/orgs/momaa/2007/conceptualproceduralMoMAA.doc.

Shimizu, Y. (2003). Problem solving as a vehicle for teaching mathematics: A Japanese perspective. In F. K. Lester, Jr. & R. I. Charles (Eds.), *Teaching mathematics through problem solving* (pp. 205–214). Reston, VA: National Council of Teachers of Mathematics.

Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *5*, 521–539.

Skemp, R. R. (2006). Relational and instrumental understanding. *Mathematics Teaching in the Middle School*, *12*, 88–95. (Original publication 1976)

Strabala, J. (2003). Motivating students to engage in problem solving. In F. K. Lester Jr. & R. I. Charles (Eds.), *Teaching mathematics through problem-solving: Prekindergarten–Grade 6* (pp. 67–80). Reston, VA: National Council of Teachers of Mathematics.

Tall, D. (1992). Students' difficulties in calculus. *Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7* (pp. 13–28), Quebec, Canada.

Tall, D. (1997). Functions and calculus. In A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education* (pp. 289–325). Dordrecht, the Netherlands: Kluwer.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Taylor, K. L., & Dionne, J. P. (2000). Accessing problem-solving strategy knowledge: The complementary use of concurrent verbal protocols and retrospective debriefing. *Journal of Educational Psychology, 92*, 413–425.

Teuscher, D., & Reys, R. E. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher*, *103*, 519–524.

Van De Walle, J. A. (2003). Designing and selecting problem-based tasks. In F. K. Lester Jr. & R. I. Charles (Eds.), *Teaching mathematics through problem-solving: Prekindergarten–Grade 6* (pp. 67–80). Reston, VA: National Council of Teachers of Mathematics.

Verschaffel, L., & de Corte, E. (1997). Teaching realistic mathematical modeling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, *28*, 577–601.

Villegas, J. L., Castro, E., & Gutierrez, J. (2009). Representations in problem solving: A case study with optimization problems. *Electronic Journal of Research in Educational Psychology, 7*(1), 279–308.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*, 356–366.

Wilson, J. W., Fernandez, M. L. & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57–78). New York, NY: Macmillan.

Wu, H. (1999). Basic skills versus conceptual understanding: A bogus dichotomy in mathematics education. *American Educator*, *23*(3), 14–22.

Zandieh, M. J. (1997). *The evolution of student understanding of the concept of derivative*. Doctoral dissertation, Oregon State University, Corvallis. *Dissertation Abstracts International, 58–8,* 3056.

Zandieh, M. J. (1998). The role of a formal definition in nine students' concept image of derivative. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Klob, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 136–141). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate athematics education. 4: CBMS Issues in Mathematics Education* (pp. 103–127). Providence, RI: American Mathematical Society.

APPENDIX A

OPTIMIZATION TASKS

1. A rectangle is inscribed in the region in the first quadrant bounded by the coordinate axes and the parabola $y = 1 – x^2$. Find of the rectangle that maximize the area (Hollis, 2008).
2. Find the point on the graph of y = 1/x^2, x > 0, that is closest to the origin (Hollis, 2008).
3. A manufacturer sells a certain article to dealers at a rate of $20 each if less than 50 are ordered, and decreases the price by 2 cents per article (for the entire order) if the number ordered is between 50 and 600. What size order will produce the maximum amount of money for the manufacturer?
4. A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs $2 per foot, while the fence for the other three sides costs $1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?
5. Experiments show that if fertilizer made from N pounds of nitrogen and P pounds of phosphate is used on an acre of Kansas farmland, then the yield of corn is B = 8 + 0.3(NP)^.5 bushels per acre. Suppose that nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb. If a farmer intends to spend $30 per acre on fertilizer, which combination of nitrogen and phosphate will produce the highest yield of corn? (Rogawski, 2008, p. 221).
6. Find the point P on the parabola, $y = x^2$ closest to the point (1, 0) (Rogawski, 2008, p. 225).
7. An electrical station is located on one side of a straight river, one kilometer wide. Five kilometers upstream, on the other side of the river, there is a factory. If the owner wishes to lay a cable from the electrical station to the factory, he knows that laying the cable

underground costs [$4.50] per meter, and laying it underwater costs [$7.50] per meter. What would be the most economical route for the cable to be laid? And if the underground cable costs the same as the underwater cable, what would the route be (Villegas, 2009, p. 291)?

8. The cost of running a heavy truck at a constant velocity of v km/h is estimated to be 4 + v^2/200 dollars per hour. Determine at what constant velocity the truck should run at to minimize the total cost of a 100 km trip? Adapted from Klymchuk et al (2010).
9. Find the maximum length of a pole that can be carried around a corner joining corridors of widths 8 ft and 4 ft (Rogawski, 2008, p. 229).
10. Find the angle **θ** that maximizes the area of a trapezoid with a base of length 4 and sides of length 2 (Rogawski, 2008, p. 226).

APPENDIX B

INTERVIEW PROTOCOL

Background Portion Questions Guide

1. Why did you decide to take Advanced Placement Calculus?
2. What do you think of the course? What has challenged you? What do you particularly like about it?
3. Do you plan to take the AP Calculus Exam? Why or why not?
4. Have you taken an honors or AP mathematics course before this one?
5. Are you taking any other AP or honors courses?
6. Have you taken the SAT, ACT, or other similar test? How did you do?
7. What do you plan to study in college? Why?
8. What are your plans for your future career? Why?

Problem-Solving Portion Think Aloud Protocol

1. Read the problem aloud.
2. Restate the problem in your own words.
3. Begin to design and implement a solution while speaking aloud about any thoughts, considerations, strategies, ideas, or procedures as they come to you.
4. Do not worry about making mistakes or saying something that is incorrect.
5. Continue train of thought. Do not return to an old thought for my benefit, such as to restate the problem if you forgot. Do not feel the need to explain or justify what you are doing to me. I will ask you more about it if I find it necessary.
6. You will hear "keep talking" if you have not spoken for 10 to 15 seconds as a reminder to vocalize your thoughts.

7. Your performance will not reflect on you in any way.

Follow-up Interview Portion Questions Guide

1. How did you interpret what Problem #1 was asking? Problem #2? Problem #3?
2. I noticed you did ________, could you talk more about that?
3. Were there any problems you found to be difficult? What in particular did you find difficult about that part of the problem?
4. Is there any information that you could have used that would make this problem easier?
5. Was there anything confusing about the way the problem was worded?
6. What do you think these problems were all about? Did any similarities or differences jump out at you?
7. How would you define what a derivative is?
8. Are there other ways to describe or explain what a derivative is?
9. Are there any other ways to represent a derivative?
10. What else comes to mind concerning derivatives?
11. If someone asked you to explain derivative to them, what might you say to them?
12. What applications can derivatives be used for?
13. What prompted you to use the derivative in this problem?
14. Based on their responses, I might ask them specific questions about their use of different representations of derivative and their understanding of it (i.e. tangent and its creation, instantaneous rate of change, etc.)
15. How was the experience of participating in this study?