

A CASE STUDY OF STUDENTS' KNOWLEDGE OF FUNCTIONS
IN AN ONLINE COLLEGE ALGEBRA COURSE

by

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(Under the Direction of Nicholas Oppong)

ABSTRACT

This study investigated college students' knowledge of the concept of function in an online environment and explored the difficulties they might have encountered with the notion of function in the areas of recognition, interpretation, and translation. Research on students' knowledge and difficulties with the concept of function in an online domain is in its infancy.

This case study focused on four students from two sections of online college algebra courses at a two-year college. Data were collected from a pretest, three reflections, two interviews, a test, a quiz, a comprehensive final examination, and written responses from the interviews. I used a constant comparative method (Glaser & Strauss, 1967) to give meaning to the collected data.

One of the goals of this study was to examine college students' knowledge in recognition of relations as functions and non-functions given in different representations. They were able to recognize relations in numerical and graphical forms more frequently than relations in algebraic form.

Findings revealed most participants were more knowledgeable about interpretation of the graphical representation of functions in which one of the variables was time. Here they were able

to construct answers, as they were able to logically connect to the functional reasoning. On the other hand, most students showed difficulty interpreting (a) functions given in algebraic format, and (b) some global features of the graphical representations of functions

The results indicated that all students achieved a better overall knowledge of functions in the area of translation from numerical to graphical form, than from symbolic to graphical description. The task of translating functions from verbal description to algebraic representation was the most difficult task for the majority of students. Translation of functions from graphical to algebraic was easier for the participants than vice versa.

INDEX WORDS: College Algebra, Mathematics, Student Knowledge, Online Education, Functions, Recognition, Translation, Interpretation

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DEDICATION

To my sister Elham, who is a source of inspiration for me.

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CHAPTER 1

INTRODUCTION

The invention of the World Wide Web in 1992 (Harasim, 2000) has provided universities and colleges with a powerful tool that if used properly, could furnish an effective and efficient teaching tool to bridge the distance between teacher and students, and among students themselves. Web-based instruction is an innovative approach for delivering mathematics instruction to a remote audience. As West (1997) explained, in today's economic situation a worker's frequent career changes make lifelong learning or continuing education necessary, despite the pressures of work, family, and social life.

Distance modes of learning have become popular worldwide over the past three decades, especially in the areas of mathematics and mathematics education (Arnold, Shiu, & Ellerton, 1996). Online learning has become a recognized method of education for traditional and nontraditional students, and a great number of colleges and universities offer online mathematics courses. Online learning offers a solution to meeting educational needs at all levels. It makes college education available to students who are not otherwise able to attend school.

The computer communications revolution of the 21st century brought a "paradigm shift in attitude towards online education," and "our new understanding of the very nature of learning has affected the definition, design, and delivery of education" (Harasim, 2000, p. 42). Computers have increasingly been used as tools for computational purposes and understanding of abstract mathematics. Van Weert (1994) argued that computers will also force mathematics education to change its focus, its organization, and its use of technology. He continued:

The focus will change from teaching to learning, its organization will change from rigid class based learning to flexible team based learning, technology will be integrated into the learning process and will support both this new organization of learning and the learning tasks of the individual student. (p. 621)

During the 1990s, many distance education course developers believed that the main goal of any mathematics education course was for the teacher to enable the student to learn basic knowledge and skills (Arnold et al., 1996). These course developers viewed mathematics as a body of facts, definitions, and theorems that were independent of human reasoning. Most conceded that while mathematics was associated with reasoning, distance mathematics courses should not be very concerned with this aspect of the field. This kind of orientation, as described by Ellerton and Clements (1990), has given rise to the notion that mathematics curriculum for distance courses “should be hierarchical in nature” (p. 719), and that online mathematics teachers should emphasize the importance of basic mathematical facts and skills.

In an online environment, the teacher has to balance the facility of group settings to match instruction of mathematics to different student needs. The following is a summary of some of the desirable features of an online mathematics course.

Course design: As mentioned by Ausubel, “any text used for teaching-learning purposes must be developed in a way to facilitate learning not only by providing information but also by helping the learner to relate newly acquired knowledge to what is already known” (quoted in Holmberg, 1995, p. 88). It is important for the “mathematical descriptions to be precise, necessary, and complete” (Laurillard, 1993, p. 184). Further, as pointed out by Arnold et al. (1996) “When text is itself the learning medium the challenge is to initiate active learning as opposed to passive reading”(p. 725).

Interaction and communication: As per Salomon (1981), education depends upon acts of communication. Not all communications are beneficial to the learner. According to Salomon, educational communication that facilitates learning should be reciprocal (i.e., two-way), consensual (i.e., voluntary), and collaborative (i.e., shared control). Without interaction, teaching becomes simply “passing on content as if it were dogmatic truth,” and the cycle, knowledge acquisition to critical evaluation to knowledge validation, is nonexistent (Shale & Garrison, 1990, p. 29). Chism (1998) argued that distribution lists can be used to share cases, engage students in collaborative problem-solving, and create an online community as students elaborate on discussions and continue to deal with unsolved issues.

Collaborative learning climate: In an environment where students can feel uncertain or isolated, it is important that online instructors make the environment an inviting one. Cooperative and collaborative learning, as put forward by Reeves and Reeves (1997), are instructional methods in which learners work together in pairs or small groups to accomplish a shared goal.

Teacher’s role: In an online environment the teacher is a coach or even a collaborator in the knowledge construction process (Reeves & Reeves, 1997). In this environment, the instructor does not remove himself from educational process. As pointed out by Portela (1999), the teacher’s role is shifted from the deliverer of instruction to being the creator of learning experiences for the students.

Motivation: Multimedia studies have indicated that learners soon tire of the media elements (such as graphics, animation, video, and a user-friendly interface), thus it is important

that motivational aspects be consciously designed into online courses as any other pedagogical dimensions (Reeves & Reeves, 1997).

Distance education in any subject area is inclined to follow a curriculum design and text structure with specific aims and behavioral objectives (Arnold et al., 1996). These courses progress in a linear manner where answers will be found in the text, and interaction with regard to the subject matter, will be principally with teachers and fellow students in the course. Such designs have been common in mathematics courses offered at a distance. As pointed out by Chambers (1995), however, changes in emphasis leading educators away from behaviorism and toward constructivism and reflective learning present strong challenges to objective-driven course designs. The technological developments (Burge, 1995) have enabled educators to offer non-linear course designs through the use of hypertext and multimedia, and have provided students with the opportunity to access coursework through idiosyncratic paths.

Mathematicians and teachers of college-level courses agree that functions play an important role in students' mathematical education. Harel and Dubinsky (1992) argue that the concept of function is an indispensable part of any student's mathematics background. Moreover, a sound understanding of the function concept is needed in order to build additional mathematical concepts (Buck, 1970). Eisenberg and Dreyfus (1994) proposed that having a sense for functions is one of the most important facets of mathematical thinking. It allows students to gain insights into the relationships among variables in problem-solving situations.

The belief that the concept of function should be a central point in mathematics is not a recent phenomenon. During the early twentieth century, the concept of function started to be thought of as an entity by itself and was introduced into the secondary school curriculum (National Council of Teachers of Mathematics [NCTM], 1970). Recommendations during the

1920s suggested that concept of function should be emphasized in every area of high school mathematics, and that it should be the principle to unify the mathematics curriculum (Hedrick, 1922). As put forward by Breslich (1928), since most of mathematics deals with relationships between quantities, “without functional thinking there can be no real understanding and appreciation of mathematics” (p. 42). Over the years the role played by the concept of function in mathematics has been expanded, and today school mathematics curricula are built around it (Fehr, 1966). The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) called for the concept of function to be “an important unifying idea in mathematics” (p. 154).

Technological enhancements can enrich what students know about the function concept in an online environment. This transformation is not automatic, however, and its implementation requires sensitivity to the difficulties students may have with this notion. Many researchers have acknowledged the difficulties students face in learning the concept of function (Tall & Bakar, 1992) in a face-to-face classroom. On the other hand, research studies have done little to examine students’ knowledge of functions and their difficulties with this concept in an online environment. Understanding students’ knowledge and difficulties with this notion in an online environment can contribute significantly to finding ways to guide students’ conceptual learning in a virtual setting.

Rationale

The purpose of this study was to gain insight into college students’ knowledge of functions and explore the difficulties they may have with this notion in an online learning environment. Since the teaching of college algebra in an online environment is relatively new, there is a need for studies concerning this type of instructional mode.

As a young high school student, I was interested in pursuing a diploma in mathematics in my hometown; however, my high school did not offer this option. Determined to pursue this field, at the age of 15, I relocated to a country away from my family. I was still disadvantaged in the new all-female high school where I was pursuing a mathematics diploma, because we did not have high quality mathematics teachers. Good teachers preferred to teach at the all-male high schools where the pay was higher. After completion of high school, I relocated yet to another country on another continent in pursuit of a mathematics degree. These obstacles created many challenges in my life. My educational background and the many difficulties I encountered in this path fueled my interest in online mathematics instruction.

The problem of worldwide scarcity of qualified mathematics teachers in schools and universities can be addressed with the widespread use of the distance mode of teaching (Briggs, 1987). I had to take the courses available at my college, or else I had to apply to another college and relocate. Today, the emergence of online learning at all levels “provides an opportunity to further one’s educational experience without relocating or leaving one’s present job” (Bisciglia & Monk-Turner, 2002, p. 40).

Most of the studies in an online environment focus on the effectiveness of online mathematics courses compared with traditional face-to-face instruction (Hiltz, 1997). Often these studies rely on standardized tests to measure outcomes (Schmidt, Sullivan, & Hardy, 1994). Other studies focus only on learner satisfaction that is most often determined by students’ responses to surveys (Allen, 2001; Anderson, 1999) or self-reports in end-of-course evaluations (Hiltz, 1997).

In recent years, despite the explosive growth of distance education courses at two-year institutions (National Center for Education Statistics [NCES], 2003), research has not

investigated students' knowledge of mathematical concepts in a virtual setting. I began this study with a sound knowledge of the students' struggles to understand and the difficulties they have with the concept of function. I was also aware of what the literature said about the relative complexity of the concept of function for college students in a face-to-face classroom (Seldon & Seldon, 1992). To this end, it was only logical to examine college students' knowledge of functions and their difficulties with the concept of function in an online environment.

Research Questions

The search for greater understanding of students' knowledge of the concept of function and the difficulties they may encounter with this concept in an online environment generated the three research questions of the study.

1. What do students who take an online college algebra course recognize as functions?
What are their specific difficulties in this context?
2. What do students who take an online college algebra course know about interpretation of functions given in equation and graphical form? What are their specific difficulties?
3. What do students who take an online college algebra course know about translation of functions from one representation to another? What are their specific difficulties?

Theoretical Perspective and Definitions

The theoretical framework I used in this study is a modification and combination of the O'Callaghan's (1998) and Markovits, Eylon, and Bruckheimer's (1986) frameworks. O'Callaghan's framework evolved from Kaput's (1989) theory about sources of meaning in mathematics. The components—*interpreting* and *translating*—in O'Callaghan's framework correspond to the sources Kaput categorized as referential extension. The referential category consists of (a) translation between mathematical representation systems, and (b) translation

between mathematical and nonmathematical systems. O’Callaghan studied students’ knowledge of the concept of function by examining the components present in his framework, which he called “component competencies” (p. 24). For Markovits et al., understanding of the concept of function has two levels: (a) passive – the knowledge of identifying and classifying functions and non-functions, and (b) active – the knowledge of transferring information from one context to another in mathematics. The knowledge of transferring information is called *translating* in O’Callaghan’s framework.

In this study, I used the framework that consisted of the elements *recognizing*, *interpreting*, and *translating*. I combined “identifying” and “classifying” from Markovits et al. (1986) framework and used *recognizing* as the first element of my framework. The next two elements *interpreting and translating* contained in my framework were obtained from O’Callaghan’s framework.

For the purpose of this study I used the following definitions:

Recognizing is the ability to identify relations in their various forms as functions and non-functions. This first component of my framework is based on the concept image and concept definition. Every mathematical concept has a verbal definition that accurately explains it. For some concepts we also have a concept image. Concept image, as put forward by Vinner (1983), consists of the mental picture and properties of that concept. For instance, the graph of a quadratic function and the symbols $y = x^2 - 2$ might be considered as someone’s concept image. Students will not necessarily use concept definitions to decide whether or not a given object is an example of a function. In most cases, students decide on the basis of the concept image; that is, the set of all the mental pictures in their minds.

Interpreting, or the interpretation of functions in their different representations, is the second component of my framework. Problems may require students to interpret components of a function, focus on individual points of a graph, or center their attention on global features of a graph.

Translating is the third component of the framework. It is the ability to move from one representation (table, equation, symbols, graph, and verbal) to another. The most commonly used forms of representation, as pointed out by Kaput (1989), are symbols, tables, and graphs. For example, consider how the function $f(x) = |x - 2|$ can be translated into its graphical representation.

Knowledge is “familiarity, awareness, or understanding gained through experience or study” (Pickett et al., 2002, p. 768).

Online Learning is “a formal, structured mode of learning that is interactive, asynchronous and mediated by Internet” (Doherty, 2000, p. 15).

Below are the definitions of a function taken from the traditional textbook and the online college algebra course material.

- A function is a set of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinate. The domain of a function is the set of all the first coordinates of the ordered pairs. The range is the set of all the second coordinates (Aufmann, Barker, & Nation, 2002, p. 148).
- A function is a rule, which maps one set, the domain, onto another set, the range, such that each element of the domain corresponds to one and only one element of the range (online college algebra course material).

CHAPTER 2

REVIEW OF THE LITERATURE

Research on students' knowledge of the concept of function provides an insight into what makes the task of working with functions easy or difficult in a traditional setting. A review of literature on historical perspectives of the concept of function and the teaching of function are provided in this chapter. Chapter 2 also contains a review of literature related to students' knowledge of and difficulties with the concept of function in a face-to-face classroom. Other studies reviewed show the extent of the research in mathematics and online education. Research on students and the concept of function in an online environment was not located.

Historical Perspectives of the Concept of Function

During the early 20th century many mathematics educators believed there was a need for a greater emphasis on functional thinking in school mathematics. In 1921, the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that functional thinking be given a central focus in secondary school mathematics (Hedrick, 1922). Further, the importance of functions during this period was spurred by the fact that functions occurred in real world situations (Hedrick, 1922). Hamley (1934) criticized textbook writers for misinterpreting the recommendations and thinking that "the function concept was synonymous with the graphical representation of functions" (p. 79).

The effects of the recommendations appeared to be mixed. Three mathematics methods textbooks attempted to encourage teachers to emphasize functions as a unifying principle in school mathematics (Butler & Wren, 1941; Schorling, 1936; Young, 1927). On the other hand,

Breslich (1928) examined four secondary textbooks in geometry and algebra and concluded that “[the texts] showed practical disregard of opportunities for training in functions” (p. 43).

In contrast to the uneven treatment of the concept of function during the first half of the century, the recommendations of the 1950s emphasized mathematical structure and defined functions in terms of sets of ordered pairs. Curricular recommendations of the 1950s as stated by the Commission on Mathematics of the College Entrance Examination Board (CEEB), suggested that functions be treated as a topic in its own right, and that a separate course on functions replace the traditional advanced algebra course to provide a unified treatment of elementary functions (CEEB, 1959). Between 1960 and 1967, the School Mathematics Study Group (MSG) produced a series of secondary textbooks to implement the recommendations made by CEEB in 1959, including a high school textbook for a course on function (MSG, 1965). In the program of the Secondary School Mathematics Curriculum Improvement Study, functions were introduced through the use of mapping of set A to set B. The concept of the set of ordered pairs was however introduced in the succeeding chapters. (cited in Thomas, 1969) More recent textbooks place a greater emphasis on graphical representations of functions and on interpreting real-world phenomena using functions (Cooney & Wilson, 1993).

During the late 19th century and early 20th century, while mathematics educators and mathematicians were calling for a greater emphasis on the concept of function, the definition of function was changing (Kleiner, 1989). Dirichlet was one of the early mathematicians during the middle of the 19th century that analyzed Fourier’s work, and illustrated that functions could include arbitrary correspondences (Malik, 1980). The acceptance of a definition of function as an arbitrary correspondence was gradual among the mathematicians (Kleiner, 1989).

During the first part of the 20th century, the definition of function centered on dependence and correspondence with an emphasis placed on the dependence of the second variable on the first, through the use of dependent and independent variables to describe functions (Kleiner, 1989).

The definition of function in the 1950s did not change significantly from that of 1900s. The definition was refined with an emphasis on the concept of set, the acceptance of functions as arbitrary correspondence, and the requirement that each value of the independent variable has a unique image (dependent variable). As pointed out by May and Van Engen (1959), the notion of function as a set of ordered pairs was more general and precise. They argued that although a table, rule, graph, or verbal description might describe a function, it is not logical to consider any of them a function. Instead, it is more precise to refer to a function as a set of ordered pairs defined by a table, rule, graph, or description. For example, the function defined by the formula $y = 2x + 5$ would be referred to as a set of ordered pairs of the form $\{(x, y) / y = 2x + 5\}$. Since the definition involved sets, it was only natural to call the set of all first elements of the function *domain*, and the set of all second elements the *range*. These ideas gave rise to the modern definition of function: A function f from A to B , is defined as any subset of the Cartesian product of A and B , such that for every $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

As Fruedenthal (1983) pointed out, the two essential features of the modern definition of function have evolved: arbitrariness and univalence. The *arbitrary* nature of functions means that the functions do not have to be defined by any specific expression or particular shaped graph. This nature is implicit in the definition of function. The *univalence* requirement is stated explicitly, and means that for each element in the domain there is an element in the range. This feature was not required at the beginning. The development of advanced analysis and the

difficulty to deal with multi-valued symbols created the need to distinguish between dependent and independent variables. It was then that the univalence requirement was added to the definition of function.

The concept of function is one of the most essential topics in mathematics and one for which students have insufficient understanding (Dreyfus & Eisenberg, 1983). Thomas (1969) suggested that students' difficulties with the concept of function seem to center around its complexity and generality. At the same time, the teaching and learning of this concept have been considered most problematic. Thus, as suggested by Leinhardt, Zaslavsky, and Stein (1990), the sequencing of instructional materials is important in the teaching and learning of functions.

Teaching of Functions

Most of the contemporary literature (NCTM, 1989; Vinner, 1983; Vinner & Dreyfus, 1989) emphasize that teaching functions as rules allows students to gain a strong conceptual base in functional thinking before progressing to the more general notion of functions as sets of ordered pairs. During the 1920s, functional thinking was stressed in every area of secondary mathematics. In order to build a real appreciation of mathematics in students, functional thinking must be emphasized (Breslich, 1928). Breslich suggested that ratios, proportions, equations, polynomials, variations, relationships stated in words, graphs, tables, and formulas all provide opportunities for emphasis on functional thinking.

As teachers we think we teach our students the concept of function, but they only seem to be able to put together a few loosely connected procedures and become skillful in applying them (Dreyfus & Eisenberg, 1983). The point is that the students need to get a feel for basic mathematical concepts; in other words, the concepts become part of them (Fischbein, 1976). Kline (1971) stated a similar idea: the mathematics we teach must be tied in with real life

situations where students can see the connection between the concepts and the real life representations.

There are many different approaches to teaching the concept of function. The most common ways are through graphs, tables, word problems, ordered pairs and a combination of these. Lovell (1971) argued in support of teaching functions from an ordered pair perspective. Teaching functions from the standpoint of sets of ordered pairs is a clear and precise way to teach this notion, but many mathematics educators question whether this clarity develops a better understanding of functions for students. Buck (1970), on the other hand, had some reservations about using the ordered pair approach when he stated that this approach “imposes severe limitations upon the student and provides a poor preparation for any further work with functions” (p. 255).

Students bring with them to mathematics classrooms their perceptions of functions and graphing from their past mathematics education and their real-world experiences. The choice of examples is the art of teaching mathematics (Leinhardt et al., 1990). The influence of examples as they are shared with students, whether by the teacher through the instruction, or in the book, is evident in the work of Zaslavsky (1989). The findings of her study suggested that students remember the conceptions and misconceptions presented in the form of examples. If influential examples are incorrect, they too will be remembered and added to the students’ pile of misconceptions.

Several researchers have found that students seem to prefer algebraic representation of functions to its graphical form, and they do not have a full understanding of the information embedded in graphical representation of functions (Clements, 1984). On the other hand, the results of research conducted with a group of teachers (Norman, 1992) contrasted considerably

with the findings of Clements (1984). Norman concluded that the teachers preferred graphical contexts to numerical and algebraic ones, and they all lacked deep understanding of the concept in spite of their years of experience.

Students' Knowledge of and Difficulties with the Concept of Function

The topic of function is a complex notion. This is due to several factors, including the following: (a) It is often associated with other complex mathematical concepts (growth, limit), (b) it pulls together various sub-concepts and fields of mathematics, and (c) it appears in many different representations (Dreyfus & Eisenberg, 1982).

In view of the central importance of the function concept to all of mathematics, it is important to investigate students' knowledge of the function concept and explore their difficulties with this notion. Students who do not have the opportunity to rectify their difficulties may be able to perform in limited circumstances in algebra. At the center of the analysis of students' knowledge of the concept of function is the identification of their difficulties. Concepts learned about function that run counter to earlier learned rules in the algebra curriculum can pose obstacles for students. Educators should be aware of likely difficulties and attempt to minimize them.

In this section students' knowledge and difficulties with the concept of function are categorized under the following headings: (a) recognition, (b) interpretation, and (c) translation.

Recognition

Vinner (1983) found that students demonstrate more patience for patterns other than linear, but had some expectations, such as symmetry and always increasing or always decreasing. In many instances students were able to define a function (a correspondence between two sets that assigns to every element in the first set exactly one element in the second set), but failed to

use it to figure out whether the given graph represented the graph of a function or not. Lovell (1971) reported that students possessed an inaccurate view of what the graph of a function should look like. In addition Vinner and Dreyfus (1989) reported that students do not necessarily use the definition of function when deciding whether a given relation is an example of function. Vinner (1983) suggested that in such cases students' knowledge of the formal definition does not play a role. Instead, what is at work here is students' "concept image" – a concept of what the function is based on, what they have developed through their experience, and examples of functions.

In another study, Barnes (1988) reported that high school juniors and university students did not regard $y = 2$ as a function because it did not have an x in it, but considered $x^2 + y^2 = 4$ a function because it was familiar. Markovits (cited in Dreyfus & Eisenberg, 1983) reported high school students had difficulties in deciding whether a given relation was a function. Vinner (cited in Markovits et al., 1986) investigated the concept of function among a group of 15–17 year-old students after they had studied the formal definition of the concept of function. He found that the students had difficulties with piecewise functions. Students believed that a function must have the same rule of correspondence over the whole domain; otherwise, there are two or more functions involved. Vinner (1983) also reported that students expected a function to be given by one rule. In a related study, Markovits et al. (1986) found that students experienced some difficulty with the piecewise functions and the functions defined by constraints (a few ordered pairs).

Interpretation

Markovits et al. (1986) found that students neglected the issue of domain and range whether they were asked about it implicitly or explicitly. These findings are in agreement with

Tall and Bakar's (1992) finding that students experience difficulty with the concepts of domain and range. Dunham and Osborne (1991), in their study of pre-calculus students, suggested that students tend not to connect numerical values of range and domain with the function's visual characteristics.

Zaslavsky (1997) found that interpretation of graphical information was difficult for high school students. Students considered only the visible part of the graph of the function. For instance, students determined whether a point was or was not on a graph of a function based on what appeared to be "eye measurement."

Students often provide a single point as an answer to interpret a graph although a range of points or an interval is under consideration. Bell and Janvier (1981) found high school students weak in their ability to interpret graphical features so as to extract information about the many every day and scientific situations. Janvier (1981) found that students are better able to interpret graphical representations of functions when one of the variables is time or time-dependent. A finding put forward by Leinhardt et al. (1990) supports the notion that students perform better in such cases because reasoning in a graphical domain is based on real-world situations.

Eisenberg (1992) noted that the topics of *slope* and *y-intercept* do not receive any attention in high schools. In her study, Moschkovich (1990) spent sixteen days getting students to think about the role of slope and the *y*-intercept in the function $y = mx + b$. She found that even with more graphing experience and direct instruction, the role of slope and the *y*-intercept remained problematic for students.

Dunham and Osborne (1991) studied students in a precalculus course and noted that given an ordered pair on a graph, or not on a graph, the students can generally extract the *x*- and

the y -values. In contrast, extracting the corresponding values of x and $f(x)$ from an ordered pair $(x, f(x))$ —hereafter (X, Y) —when working from a graph was difficult for students.

Translation

Functions can be represented in different formats, and it is possible to translate functions from one type of representation to another. Function translation is one topic in which visual reasoning is particularly important. This topic assumes a good understanding of algebraic and graphical representations of functions and the relationship between them. Markovits et al. (1986) noted that students found the translation from graphical to algebraic form more difficult than vice versa. When functions were less familiar—like piecewise or constant—function translation in both directions was found to be hard. Zaslavsky (1997) reported that high school students demonstrated difficulties translating from graphical to algebraic representation of functions.

Eisenberg and Dreyfus (1994) reported that high school students studying upper level mathematics courses had difficulties with the function transformation. In particular, translations in the vertical direction ($f(x) \rightarrow f(x) \pm k$) were easier for students than similar translations in the horizontal direction ($f(x) \rightarrow f(x \pm k)$). In another study, Schwartz and Dreyfus (1995) found that students had difficulty translating among different representations of functions.

An outline of the difficulties given here provides a glimpse at the learning problems students often encounter in face-to-face teaching. Although much work has been dedicated to finding students' learning difficulties with the concept of function in a face-to-face setting, very little attention has been given to this issue in an online environment.

Mathematics and Online Education

Online education is time- and place-independent, and largely text-based. Harasim (2000) suggested that the critical difference between distance education and online education is that

“online education is fundamentally a group communication phenomenon,” and “it is closer to face-to-face seminar-type courses” (p. 49).

Online education influences the educational process and the individual learners in different ways. As more and more colleges and universities embark on offering mathematics courses to geographically scattered populations of students, it is best to know if this method of course delivery is a successful and appropriate method of teaching and learning mathematics. The discipline of mathematics is often considered to be culture free (Ellerton & Clements, 1989) in the sense that, online education materials in mathematics prepared in one part of the world could be used elsewhere with minor modifications. In this section, I will explore the research specifically related to mathematics education.

Schmidt, Sullivan, and Hardy (1994) reported on the evaluation of a pilot program to teach algebra to Texas migrant students through audio-conferencing. They found the medium to be effective in terms of both increased access and educational outcomes. They reported that all the students in this course, most of whom had previously failed algebra, passed, and the final class average was 88%.

Anderson (1999) investigated the impact of the three principals—dialogue, structure, and learner autonomy—on mathematics distance education programs, specifically calculus. He analyzed the difference among distance learning experiences at different institutions, as each program used different technologies and platforms. The results of his surveys indicated that most students were satisfied with the Internet-based calculus and thought it was as good as campus-based courses.

Hiltz (1997) examined the impacts of online college-level courses at the New Jersey Institute of Technology. Among the courses offered were mathematics and statistics. The

quantitative analysis of the results of this study showed that students in these courses performed equal to or better than those taught with traditional modes of course delivery. Furthermore, a majority of the online students reported they had better access to their teacher and would take another online course.

Allen (2001) reported the survey results of an online Calculus I course at Texas A&M University. His results showed that students adapt to online learning without difficulty and perform well on traditional examinations and in successive calculus courses.

Research on the impact of online education in the field of mathematics is limited. For instance, review of literature provides no input into students' knowledge and difficulties with the concept of function in an online environment.

Who Is the Online Student?

Online education is a special form of distance education that uses the Web as the primary environment for course discussion and interaction. The review in this section is broadened to include distance education.

Online education provides students the opportunity to complete a degree without relocating or even attending school full time. Students enroll in online education courses for many of the same reasons as the students in face-to-face classes. They want to gain experience and degrees in order to be more competitive in the labor market as well as for general self-improvement (Peruniak, 1983). May (1994) asserted that distance education courses are particularly important for women because the courses allow them to juggle responsibilities of home and work while attending school and furthering their education. Additionally, such courses provide those who live in rural areas or in communities with no access to local schools, with an avenue for completing their educational and career goals. Kahl and Cropley (1986) indicated that

an individual who is typically enrolled in online education courses would be a married, nontraditional student who is likely to have chosen this particular educational environment for a specific reason.

Distance education provides single parents and economically disadvantaged adults who have to work full time an opportunity to attend school. Working adults who travel too much to attend regular campus-based courses can also benefit greatly from this mode of course delivery. This type of education is particularly feasible for undergraduates who need or want an alternative to on-campus programs for economic, social, and personal reasons (MacDonald, Stodel, Farres, Breithaupt, & Gabriel, 2001).

CHAPTER 3

METHODOLOGY

I have organized this chapter to reflect the course of actions taken as I attempted to answer the research questions I posed earlier:

1. What do students who take an online college algebra course recognize as functions?
What are their specific difficulties in this context?
2. What do students who take an online college algebra course know about interpretation of functions given in equation and graphical form? What are their specific difficulties?
3. What do students who take an online college algebra course know about translation of functions from one representation to another? What are their specific difficulties?

The Nature of the Study

Because this study investigated online college algebra students' knowledge of functions and their difficulties with this concept, a qualitative research approach was used. As an instructor in a two-year college, I have my own perspective on what students know about functions and their difficulties with this concept, rooted in my daily interactions with the students. To go beyond that perspective to their perspective, I went to the students themselves. The case study approach became my tool for attempting to understand the students' perspectives. Goetz and LeCompte (1984) suggest that the case study approach allows the researcher to pursue the participants' perspective. According to Merriam (2001), case studies are particularly useful for studying educational innovations. As Wilson (1979) observed, case study research is a process "which tries to describe and analyze some entity in qualitative, complex and comprehensive terms not infrequently as it unfolds over a period of time" (p. 448).

I needed to provide a detailed description of participants' knowledge and difficulties with the concept of function to enable me to keep track of the participants' perspective. This is what Geertz (1973) described as "thick description." I used methodological triangulation through the use of multiple methods over an extended period of time.

The Online College Algebra Course

This study was conducted at a two-year college in the southeastern part of the United States. During the fall quarter 2003, Ms. Sanders (all names in this study are pseudonyms) taught two sections of online college algebra. I have shared the following account with Ms. Sanders, and she confirmed the accuracy of the course description.

The college algebra course was part of the general education requirement for variety of disciplines. Students in college algebra courses at this college were expected to learn the concepts and know their applications without any dependence on graphing utilities as an aid in translating graphs. Thus, the students were not allowed to use graphing calculators on their tests and final examination. Ms. Sanders' learning goals for her online college algebra students were that they (a) gain a thorough understanding of basic concepts, (b) develop problem-solving skills, (c) develop graphing techniques, and (d) connect the content in this course with that of subsequent courses and their future professions.

The online college algebra course met the same standards as its traditional counterpart; therefore, the main topics of the face-to-face college algebra (preliminary concepts, equations and inequalities, functions and graphs, exponential and logarithmic functions, topics in analytic geometry, and systems of equations) remained the same in the online version.

This online college algebra course used the Blackboard system of course delivery for presentation of material, management of communications, and administration. The various components of this course were as follows:

Announcements: Announcements included timely information critical to success in the course.

Announcements occupied the main frame upon entry to the course.

Course Information: Course information displayed descriptive information about the course.

Course Documents: Course documents included the lecture notes.

Assignments: Assignments comprised homework, tests, and quizzes posted by the instructor.

Staff Information: Staff information provided background and contact information on the course instructor and the technical assistant.

Communication: Course users communicated through the communication center. The communication center allowed users to take part in a distribution list and a virtual classroom. The virtual classroom focused on direct discussion and collaboration through the use of chat features and whiteboards. Thus, it was a means for interaction among the members of a class, rather than a physical space.

Tools: The tools that were used in this course were the digital drop box, the personal information page, a course calendar, and a check grade option.

Ms. Sanders used Microsoft Word, Scientific Notebook (MacKichan software, 1999), and Acrobat Reader for the presentation of the online course material. Students were required to have an online connection with a browser and software (Microsoft Word and Acrobat Reader) for accessing the course material. The students who had no training with these programs were able to acquire the necessary skills at the same time as they learned the mathematics.

The course consisted of five units (regular chapters in a book). Units (Appendix A) were studied through the virtual learning environment website. Every unit consisted of objectives, lecture notes, practice problems and solutions, summary of the unit, assignments, unit quizzes, and unit tests. Each unit was modularized, web-based, and self-paced. In some modules, students were asked to submit solutions for some or all of the practice questions. These problems were not graded. Ms. Sanders only reviewed them to provide feedback.

The unit, “Functions and Graphs,” on mathematical functions was covered in 3 weeks. The main topics in the unit were presented in nine modules.

The online quizzes and tests were developed using the test bank for the college algebra textbook (Aufmann, Barker, & Nation, 2002) assigned for this course. A structured timeline for quizzes and tests, linked with homework completion, was incorporated. Unit quizzes and tests were all taken at home, and the students had 24 hours to complete them. The students submitted their quizzes electronically, and at the completion of every quiz, they were given their score. These scores were registered in the database, which was accessed by Ms. Sanders. Most of the computerized multiple choice questions provided the students with feedback. The unit tests were open-ended questions, and the students submitted their tests via fax or uploaded responses in the drop box. The students received their graded tests in the mail.

At the end of the quarter, the students took a proctored final examination. Students who lived at a distance took the examination with an approved proctor. The grade for the students in this course was a composite of the results of unit tests (50%), online quizzes (10%), and the final examination (40%).

The instructor made herself available through the Virtual Classroom and Distribution List, and via office hours. She made her expectations clear through the announcement page

informing students about any upcoming tests or quizzes and clarifying any key points. Although the students had some autonomy in this course about when to study what section within a chapter, they were required to complete their tests and quizzes within the 24-hour time period specified by the instructor.

Selection of Participants

The participants were chosen from two sections of the online College Algebra course with a combined enrollment of 34 students. The instructor gave me access to her course for the entire quarter, provided me with the class roster, posted a flyer for recruitment of participants on the class website, and made the results of the tests, quizzes, and final examination of the research participants available to me. I applied the principle of purposeful sampling to my data. Merriam (2001) explained, “purposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned” (p. 61).

I explained the focus and the intent of the research to all of the students in two sections of the course, and asked them to complete a form indicating their willingness to participate. As an incentive for students who volunteered their time for this study, I offered a monetary incentive and free online tutoring. I used three selection criteria: (a) a score of 70% or higher on the pretest (Appendix B), (b) willingness and responsiveness to the needs of the study, and (c) lack of scheduling conflicts for interviews. The participants were assured that their participation would be kept confidential and the results would not be released in any individually identifiable form without their prior consent, unless required by law. The participants were also assured that their participation was voluntary, and they could withdraw from the study at any time. In that case, all of the information about them would be removed from the research records or destroyed.

The choice of participants was greatly reduced when all three criteria were applied. I chose four participants—Marsha, Ivy, Nancy, and Andrew—whose profiles are presented below.

Marsha

Marsha was a 32-year-old student who graduated from high school in 1991. Because of her grades she received a scholarship for college but declined to take it and instead joined the Army. After leaving the military, she worked different jobs and later decided to go back to school and earn a degree. Marsha was working on her associate's degree in accounting and hoped to transfer to a four-year college to pursue a degree in finance. She worked full time as an accounts receivable manager. She started taking classes at the two-year college in the summer quarter of 2003, and she had carried a full course load every quarter. Prior to the online college algebra course, she had taken five other online courses.

Ivy

Ivy was a 19-year-old Asian female who graduated from high school in a neighboring county in 2002. Soon after graduating from high school, she started taking classes at the two-year college. During the fall quarter of 2003 she was a full-time student, taking four classes at two different colleges and working part time at a medical facility. She intended to work on her associate's degree in nursing. Apart from the online college algebra course during the fall quarter of 2003, Ivy had taken one other online class. She had taken college algebra in a face-to-face setting once before and had received a D in the course.

Nancy

Nancy was a 27-year-old mother of two girls ages seven and two. She married after graduating from high school in a neighboring county in 1994 and did not work outside the home. She started taking classes at the college in the spring quarter of 2003 and planned to enter the

nursing program. So far she had maintained a grade point average of 4.0. Due to family commitments, Nancy registered for an online course (college algebra) for the first time during the fall quarter of 2003. During this quarter she was a full-time student.

Andrew

Andrew was a 23-year-old student who graduated from an all-male high school on the west coast in 1999. Soon after his high school graduation, he enlisted in the military and had been there ever since. He planned to get his associate's degree from this two-year College and then move to a four-year school and work on his bachelor's degree in a computer related field. He had taken college algebra online twice before. The first time he withdrew and the second time he received a D in class.

Data Collection

The data sources for each participant included a pretest, three reflections, two interviews, and a collection of artifacts. Figure 1 summarizes the data collection schedule. A rationale and description for each data source is provided below.

A pretest was given to all potential participants. This criterion helped me select the four participants and to identify the difficulties students had with the function concept.

The first reflection focused on the recognition aspect of functions. In this reflection, I asked participants if they had seen a mathematical relation they were unable to recognize as either a function or a non-function. I asked the reasons for their difficulty. The second reflection focused on the translation of functions from one representation to another and the difficulties they encountered with this aspect of functions. The focus of the third reflection was interpretation of functions, and the difficulties they encountered doing problems of this nature. (See Appendix C for a listing of the three reflection questions)

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
10/12	10/13	10/14 Nancy Pretest	10/15	10/16 Andrew Pretest	10/17 Marsha Pretest	10/18 Ivy Pretest
10/19 Andrew Reflection1	10/20 Marsha Reflection1	10/21 Nancy Reflection1	10/22 Andrew Reflection2	10/23	10/24 Nancy Reflection2	10/25 Ivy Reflection1
10/26 Nancy Reflection3	10/27 Ivy Reflection2	10/28 Andrew Reflection3	10/29 Marsha Reflection2	10/30	10/31	11/1
11/2	11/3	11/4	11/5	11/6 Marsha Reflection3	11/7	11/8 Ivy & Nancy Interview1
11/9	11/10	11/11	11/12	11/13 Andrew Interview1	11/14	11/15 Marsha Interview1
11/16	11/17	11/18 Nancy Interview2	11/19	11/20 Ivy Interview2	11/21	11/22
11/23	11/24	11/25 Andrew Interview2	11/26	11/27 Break	11/28 Break	11/29
11/30 Ivy Reflection3	12/1	12/2	12/3 Marsha Interview2	12/4 Final Exams	12/5 Final Exams	12/6 Final Exams

Figure 1. Data collection schedule for the fall quarter of 2003.

The first interview was the same for each participant (Appendix D). During the first round of interviews, I provided the students with mathematical tasks in individual sessions and asked them to think aloud as they solved the problems (questions were printed on colored construction paper). The interviews were important as they offered me data on students' knowledge and difficulties with the three main categories of functions identified in this study (recognition, interpretation, and translation). Patton (1990) explained, 'the purpose of interviewing, then, is to allow us to enter into the other person's perspective' (p .196). Students who had difficulty solving a problem were given hints. Interviews were scheduled at times convenient for the participants on weekends at the college's main campus. Each interview lasted

about two hours. One of the participants was from out of state, so I conducted a phone interview with him. He received the interview questions ahead of time and faxed me his responses. During the interview I asked him the questions again and probed further into his conception of functions. The second round of interviews was conducted in order to verify existing information and gather additional data. All interviews were audiotaped and transcribed. During these interviews a mathematics colleague was present, and at a later date, we reflected on those interviews and shared notes.

The artifacts included participants' graded chapter quizzes and tests (on functions and graphs), departmental final examinations, and students' written responses during the interviews. These artifacts provided information about the kinds of difficulties students encountered while studying functions and offered additional insight as to how each student responded to similar questions on the concept of function. In addition, documents related to the online course, such as daily messages, virtual classroom logs, and online lecture notes, were analyzed. This analysis enabled me to determine the kind of difficulties students encountered while studying functions. I reviewed these documents and took notes from them for further analysis.

Data Analysis

I used a constant comparative method (Glaser & Strauss, 1967) to give meaning to the collected data. The purpose of analysis was to identify and categorize students' knowledge and difficulties with the concept of function.

Data analysis occurred in two phases. The first phase involved categorizing the interview transcripts and highlighting any piece of data that seemed significant in terms of the research questions or the theoretical framework. As I examined the interview transcripts, I wrote

categories in the margins and color-coded these highlighted pieces. I did the same for the reflections and artifacts.

As the study progressed, I categorized the data, noting the *convergence* and *divergence* of the color-coded material (Lincoln & Guba, 1985). I grouped statements and events according to their categories and color codes, and those with the same color code were placed in the same folder. For example “algebraic” and “numerical” were color-coded yellow, and they were used as subcategories of “recognition.” I kept a log of these categories in order to compare them and revise them. Hypotheses and constructs that were available from other research helped guide this process and were used as general categories for sorting events. My goal was to establish categories that (a) reflected the purpose of the study, (b) were exhaustive and reflected all relative data, and (c) were mutually exclusive.

During the second phase of analysis, I returned to the data for a theoretical look at the themes that emerged during the first phase. I looked at the first phase of analysis as reporting of the findings. In the second phase, I explained what happened in theoretical terms. At this point I returned to the data to look specifically for incidences in which I could glean information about recognition, interpretation, and translation.

Validity and Reliability

To increase the likelihood that readers will believe and trust the findings of research, researchers traditionally have concerned themselves with the validity and reliability of their research. Likewise, those goals have concerned me.

Validity can be thought of on two levels, internal validity and external validity. Internal validity refers to the extent to which one’s research findings reflect the real situation and that the claims are backed by evidence and there are no good grounds to doubt the results. Lincoln and

Guba (1985) referred to this as the “truth value” of research. To increase the internal validity of my research, I used several strategies. Triangulation, for example, is a method used by qualitative researchers to check and establish validity in their studies: “... triangulation is supposed to support a finding by showing that independent measures of it agree with it or, at least don’t contradict it” (Miles & Huberman, 1984, p. 235). One way to triangulate is to use “several methods in different combinations” (Fontana & Frey, 1994, p. 373). Then, as Denzin (1978) suggested, “the flaws of one method are often the strengths of another, and by combining methods, observers can achieve the best of each while overcoming their unique deficiencies” (p. 302). For triangulation in my study, I used multiple sources of data, namely, pretests, reflections, class tests and quizzes, interviews, written responses from the interviews, and final examination results.

In addition to using triangulation to increase validity, while analyzing my data, I worked with my major professor and committee members who provided peer evaluation of my work. Finally, in a later section of this chapter, I acknowledge the biases I bring to this research.

In contrast to internal validity, which concerns the match between the findings and the reality of the participants, external validity refers to the generalizability of the findings. In qualitative research, “findings cannot be generalized in the statistical sense, that is, from a sample to a population” (Merriam & Simpson, 1995, p. 103). Instead of statistical generalizability, I have sought reader generalizability. Hopefully, the findings of this research study will be applicable to other educational settings by the detailed accounts I have provided of my findings and the context of my study.

Another concept, which is related to the credibility of research, is the reliability of the research. This concept refers to the extent to which research findings can be replicated. Lincoln

and Guba (1985) suggested that qualitative researchers strive for “consistency” of the results, that is, “rather than demanding that outsiders get the same results, a researcher wishes outsiders to concur that, given the data collected, the results make sense—they are consistent and dependable” (Merriam, 2001, p. 206). To this end, triangulation and the acknowledgement of my biases will enhance the reliability of my study. In addition, I have provided an “audit trail” in the methodology section that describes in detail how data was collected, how categories were derived, and how decisions were made throughout the study.

Researcher’s Biases

In qualitative studies, as explained by Merriam (2001), the researcher is the primary instrument for data collection and analyses, thus all findings are filtered through the researcher. Although the researcher may strive for objectivity, the researcher’s perspective on the whole research process cannot be eliminated. As suggested by Jansen and Peshkin (1992), “qualitative researchers, whether in interviewing or in participant observation, or so palpably, inescapable present that they cannot delude themselves that who they are will not make a difference in the outcomes of their study” (p. 720). Therefore, in order to help understand their perspectives, researchers need to reveal the biases they bring to their research. In this way, they will better be able to analyze how their biases shape their research, and it will help their readers to better understand and evaluate their findings.

Peshkin (1988) identified “six I’s” that affected his fieldwork at a multi-ethnic high school in California. Similarly, I will identify my “I’s” that interacted with my research with online college algebra students at a two-year college.

The very first “I” brought to this study was the *Knowledgeable Teacher I*. As a faculty member at a two-year college who has taught mathematics for fourteen years and online

mathematics courses for over seven years, I came to this study with the tendency to believe I have understanding of students' knowledge of the concept of function and their difficulties with this notion. Therefore, I read students' reflections carefully; reviewed their tests, quizzes, and final examinations critically; responded impartially to the questions arising from the reflections; listened openly to each interviewee as they attempted to work through the questions; and did not submerge each participant's knowledge and difficulties with the concept of function into my already existing views. Also, because I have been influenced by my previous readings, I was careful not to project those ideas onto interviewees.

In addition, I brought to this study my *Advisor I*, the part of me that works with students to plan their future goals and assist them with their personal and academic difficulties. As an interviewer, I refrained from giving advice and was instead an active listener. As Merriam (2001) suggested, a good listener not only "hears" what was explicitly stated but also what was implied "as well as noting the silences, whether in interviews, observations, or documents" (p. 23).

I also brought certain assumptions that should be acknowledged. They were as follows: (a) Students' knowledge of mathematical concepts is affected by their active participation and interaction throughout the course, (b) interaction is "vitally important" (Moore, 1989, p. 6) in the design of online education, and that interaction is the key to effective learning (Keegan, 1990), (c) when a mathematics course is designed for online instruction it should provide opportunity for advancement towards the criteria set out for the course as well as reflective learning for the students, (d) participants in my research would reflect on their understanding of the concept of function and sincerely communicate their difficulties during reflections, free online tutoring, and interviews, and (e) understanding of participants' knowledge and possible difficulties help

educators to understand the perspectives of these students, to realize the impact of online instruction, and to serve future online college algebra students more effectively.

Limitations

This study has some limitations. First, because the researcher is the primary instrument of data collection and analysis, the investigator has to “rely on his or her own instincts and abilities throughout most of this research effort” (Merriam, 2001, p. 42). As a researcher I cannot deny my biases but should acknowledge them as I try to stay as true to the collected data as possible. In reporting my research findings, I discuss my biases including my role as a faculty member, so that the readers of this report have a better understanding of my findings.

Secondly, this study is not statistically generalizable to the larger population. This research will provide readers with detailed descriptions of online college algebra students’ knowledge and difficulties with the concept of function. The readers will hopefully take from this research what is applicable to their own situations, “truths” that seem to transfer from the phenomenon studied to their own worlds.

Finally, in qualitative case studies as suggested by Merriam (2001), although “rich, thick description and analysis of a phenomenon may be desired; a researcher may not have time or money to devote to such an undertaking” (p. 42). Therefore, in this case study, the issues arising outside the scope of this study will be opportunities for future research.

CHAPTER 4

CASE STUDIES

The case studies inform the reader of the themes that emerged from the preliminary phase of data analysis. In this chapter, I report what each student knows in the areas of recognition (the question in regard to symbolic representation of functions is omitted from the list of reported findings due to its lack of clarity), interpretation, and translation of functions, and the difficulties he or she encountered throughout these tasks. In addition, I outline my findings and document some episodes from their solution methods. At the end of the chapter, findings from the study are situated in the broader literature on students' knowledge and difficulties with the concept of function.

The Case of Marsha

Marsha scored 75% and 94%, respectively, on the chapter (Functions and Graphs) quiz and test. Her score on the comprehensive final examination, which counted 40% of the final grade, was 72.5%. Marsha finished the online college algebra course with a B (on a scale of A to F).

Recognition

Marsha's responses in the area of recognition fell into three categories: (a) algebraic, (b) numerical, and (c) graphical.

Algebraic. On the pretest, Marsha's response to the question "Do $y = -1$ and $x = 6$ represent functions?" was, "None of the above relations represent a function because there is only one variable." In addition when I asked, "Is $x^2 = y + 4$ a function?" her response was "yes."

During the interview, similar questions reappeared for which she had to determine if $y = 2$ and $x = 5$ represented functions. Her response was “ $y = 2$ is a function because there is only one y for each x -value.” She added, “The second one [$x = 5$] I would say is also a function for the same reason.” I asked for clarification, and Marsha explained, “There is only one variable in each of these, so I am not perfectly clear on whether or not that makes them a function.” I pointed out that she had just written that $y = 2$ and $x = 5$ are functions. She replied, “I did, but now I am thinking over that, and I would say no they are not because there is only one variable, but I am not 100% clear on whether or not that would be a good reason.” (Interview 1, 11/15)

The presence of one variable in the relations of the form $y = k$ and $x = k$, where k is some fixed constant, created uncertainty in Marsha’s mind, and she classified them together. For her, the simple fact that these two relations had only one variable did not qualify them as functions. Marsha had a consistent approach in her responses during the pretest and interview.

Further, when I asked if $x = y^2 + 4$ was a function, Marsha said, “That is not a function because for each value of x , you have two different values for y .” On the other hand, the question “Is $x^2 + y^2 = 25$ a function?” made her pause for a second and say, “I would say yes [it is a function] because even though you can have two values of y , x is squared so that is going to give you...” and after a long pause she said, “I do not know.” I asked her to draw a comparison between the two questions, and after a short pause she added, “So the same principle would apply. There is going to be more than one value of y for each value of x , so then it is not a function for the same reason that the other one was not.” (Interview 1, 11/15)

Algebraic relations in terms of x and y presented another category of difficulty for Marsha. She attempted to verify that algebraic representations were functions by using the procedure “no x -value is associated with two different y -values.” Marsha demonstrated no

difficulty identifying $x = y^2 + 4$ as a non-function. However, $x^2 + y^2 = 25$ was problematic. She knew that in this case there would be two different values of y for each value of x , but the fact that x was squared confused her. Marsha did not recognize $x^2 + y^2 = 25$ as the equation of a circle. Once again on the comprehensive final examination, Marsha recognized $x^2 + y^2 = 9$ as a function. She was consistent in her response to such relations throughout the study. Maybe if the topic of *circle* was not covered in the “Functions and Graphs” chapter, or it was clearly stated in the online notes that circle is not a function, then Marsha would not have had this difficulty.

I asked Marsha whether the following relation was a function:

$$\begin{cases} x \geq 0, y = x + 2 \\ x < 0, y = 1 - 2x \end{cases}$$

She responded, “No.” When asked to expand on her answer, she said, “I plugged in 1 for x in both equations and got 3 and -1 for y .” I inquired if she could substitute the same number for x in both cases. She responded, “Oh, I’m sorry...well, no, I guess you can’t.” Then she substituted 1 and -1 in both equations respectively and explained, “In each of the equations you are going to get a different value for y . I think it is [a function] but I do not know why.” I asked, “For a relation to be a function is it the x or the y -values that should not repeat themselves?” After a pause she said, “I am not good at explaining this. So if you substitute 1 for x , you would only have one value of y for that.” Then she added, “The x cannot be repeated here at all.”

(Interview 1, 11/15)

Marsha also had some difficulty when the relation was defined as piecewise. She did not realize that different rules of correspondence applied to different parts of the domain. The topic of piecewise defined functions was part of the online college algebra curriculum, and students

were expected to graph these types of relations. The course instructor, Ms. Sanders, did not cover this topic.

Numerical. Figure 2 shows Marsha's response to the question, "Which of the following represent a function (a) $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$, or (b) $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$?"

(a) y is a function of x , as for each input y (3, -4, 8, -1) there is only one output x .
 x is not a function of y , as the input $x = 5$ has multiple outputs.

(b) x is a function of y , as for each input x (8, 7, 9, -2) there is only one output y .
 y is not a function of x , as the input $y = -1$ has multiple outputs $x = 8$ and $x = -2$,
 and input $y = 2$ has multiple outputs $x = 7$ and $x = 9$.

Figure 2. Marsha's response to the "recognition" question on the pretest.

I inquired about her response and the fact that she called x and y input and output interchangeably. She said, "I was not clear about the input and output at the time and did not know much about functions." (Interview 2, 12/13)

Next, I asked if the relation $\{(1, 2), (1, 5), (2, 10), (-3, 5)\}$ represented a function. She responded, "I am going to say no because for each $x = 1$ you have the values of 2 and 5 for y ." (Interview 1, 11/15) On the final examination, Marsha was presented with the question "Does the relation $\{(1, 2), (1, 5), (-3, 5), (9, -1)\}$ represent a function?" She responded, "No, for the value $x = 1$ there are two values of y ." Early in the study (pretest), Marsha had no conception of the input and output variables, and used them interchangeably. But later she appeared to have no difficulty distinguishing between numerical representations of functions and non-functions.

Graphical. Relations represented in the form of a graph were another category of questions both in the pretest and the interview. On the pretest Marsha identified the graph of $y = |x|$ as a function. For $x = y^2$ she wrote, “It is not a function as the input y has multiple output x .” As before, she had a flawed view of input and output variables.

On the class quiz, she was asked to “use the vertical line test to determine whether graph of $y = x^2$ represents y as a function of x .” In this case, the graph of $y = x^2$ was provided. Marsha’s response to this multiple-choice question was “ y is not function of x .” I inquired about her response, and she said, “I did not know about the vertical line test and that confused me” (Interview 2, 12/3).

I asked if the graphs presented in Figure 3 represented graphs of functions. After a long pause Marsha said the graph in Figure 3, part (a), did not represent a function.

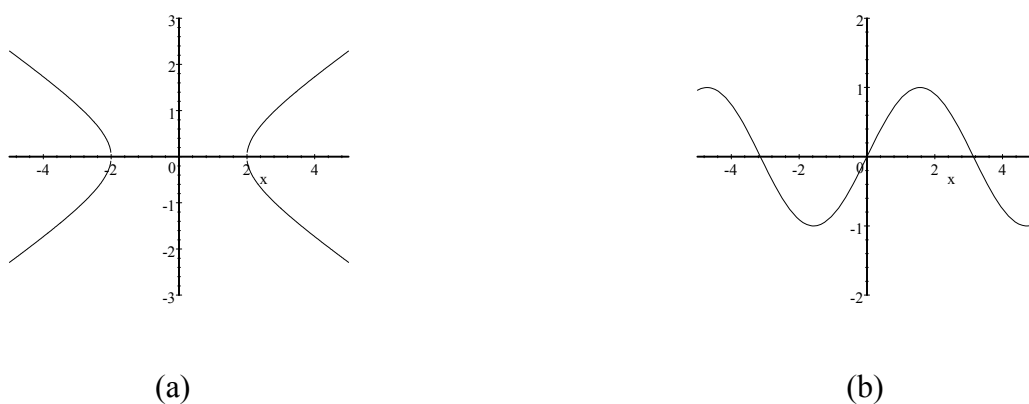


Figure 3. Recognition of graphs.

I inquired, “Why?” She responded, “Just looking at it and for x -value 2, for example, well, yeah, hold on.” Then she explained, “It is a function because the x is not going to be repeated with multiple values of y .” I asked if this works for different values of x . After a short pause, she

picked 3 for the value of x and said, “If x is 3, then you are going to have different y -values so it is not a function.” Marsha continued, “The graph in Figure 3, part (b), is a function because no matter what x -value you have, you will only get one y -value, so it is [a function].” To this end, I asked if she had come across the vertical line test. She commented, “I have, but because I refer to so many different texts it is hard to say where I have seen it. I must have seen it in the online notes, in order for me to look it up in other resources. I cannot say I fully understand it.” When asked whether she was comfortable choosing values for x , she explained, “I have been doing it in class and it seems to be working okay so far...you know, if you have an x -value that has more than one y , then it is not going to be a function.” (Interview 1, 11/15)

Just like the algebraic and numerical representations, Marsha used the procedure “no x -value is associated with two different y -values” in order to verify that graphical representations are functions. She knew the rule well, but applying it to some graphs (Figure 3, part (a)) was problematic for her, especially when the choice of certain values of x would make a difference in whether the graph was a function or not. For instance, in this case $x = 2$ corresponds to $y = 0$, but $x = 3$ maps to $y = \pm 1$. Marsha did not apply the vertical line test to graphs for verifying graphical representations as functions. By her own admittance she did not fully understand the vertical line test, and thought her way was the easiest way—it “seemed to be working okay so far.” The closest mention of the ‘vertical line test’ in this online course was that “Vertical lines do not represent functions.” Yet students were expected to know how to apply the test to different graphs and determine if the given graph was the graph of a function (e.g., on the class quiz).

Interpretation

I have organized Marsha's responses in the area of interpretation into two categories:

(a) algebraic to verbal, and (b) graphical to verbal.

Algebraic to verbal. On the pretest there were questions that involved interpretation of the slope and the intercepts of a linear function. Marsha either skipped these problems or did not complete them. In one instance, I inquired about the reason. She responded, "It was not intentional; when I saw the word *interpret* I thought it required a lot of explanation. I planned to do research and get back to it but never made it back" (Interview 2, 12/3).

Figure 4 shows the response Marsha gave to the following question.

A company's weekly revenue in dollars is given by $R(x) = 2000x - 2x^2$, where x is the number of items produced during a week. (a) For what x is $R(x) > 0$? (b) On what interval is $R(x)$ increasing? Decreasing?

$$2000x - 2x^2 > 0$$

$$x(2000 - 2x) > 0$$

$$x > 0$$

$$2000 - 2x > 0 \Rightarrow -2x > -2000 \Rightarrow x < 1000$$

(a) $R(x) > 0$ for $0 < x < 1000$

(b) $R(x)$ increasing $(-\infty, -1)$

$R(x)$ decreasing $(1, \infty)$

Figure 4. Marsha's response to the "weekly revenue" question on the class test.

I asked how she came up with the intervals on which the function $R(x)$ is increasing or decreasing. Her response was “I really do not know. I guess it just looked right.” I asked if she knew how to do the problem now. She added, “Absolutely no idea, we have not done anything like this in class.” (Interview 2, 12/3) Determining the intervals of increasing and decreasing of functions was part of the online college algebra curriculum, yet it was not covered in class. Interestingly the students were expected to know about it because a question appeared on the test.

I posed the question:

The book value of a machine is \$4,500. It is estimated that after 9 years the value of machine will be \$900. The linear function that determines this relationship is given by

$V = -400t + 4,500$: (a) interpret the meaning of the slope and the y -intercept, (b) identify and interpret the x -intercept.

She identified the slope as -400 and said, “The value of the machine is depreciating at a rate of four hundred dollars per time period.” She continued, “The y -intercept is 4500,” and when I acknowledged her response, she laughed and said, “That was a complete guess because I had already used the 400.” She added, “The y -intercept is the starting point of the depreciating process.” When asked to explain the meaning of the y -intercept in layman’s terms, she said, “It is the value before it starts going down at the interval of \$400.” Subsequently, she moved to part (b) of the problem and said, “Is there a formula [for x -intercept]?” I pointed out that I had not come across one [a formula], but that I knew how to find it. She asked, “Is it 900 maybe?” I responded by asking, “What is the x -intercept of a graph?” She said, “You are looking for the place where it crosses the x -axis.” I continued, “What is the value of y where the graph crosses the x -axis?” She said, “Zero” and added, “If I set this equation equal to zero then I have...,” then

she solved for t . When asked, “How do you interpret the x -intercept?” She commented, “That is going to be the time it takes for the value of machine to equal zero.” (Interview 1, 11/15)

Marsha knew how to identify the slope and interpret its meaning. Although the identification of the y -intercept was an educated guess on Marsha’s part, she demonstrated competence in interpreting its meaning. Identifying and interpreting the x -intercept was troublesome for her as she was looking for a formula to find the x -intercept. When she realized that I did not know a formula, she settled on guessing a value for the x -intercept. Marsha knew how to define the x -intercept and wrote its coordinate. Soon after this exchange, she proceeded and interpreted the x -intercept with no difficulty.

Later, I posed the question:

A manufacturer of graphing calculators determined that 8,000 calculators per week would be sold at a price of \$75. At a price of \$70, it was determined that 10,000 calculators would be sold. The linear function that determines this relationship is given by $y = -400x + 38,000$. (a) Interpret the meaning of the slope of the graph of this linear function, (b) explain the meaning of the y -intercept, and (c) explain the meaning of the x -intercept.

In response to part (a) of the question Marsha said, “It would be the difference in dollar amount of the sales between selling them at \$75 and \$70.” I asked, “Why is the slope negative?” She said that it is negative because if they lower the price they sell more. She continued, “ y -intercept is 38,000 and it is...I am not sure...I am guessing, it would be the sales if they sold the calculators at \$75.” In astonishment I said, “\$75! How do you define the y -intercept?” She replied, “It is where the graph crosses the y -axis.” When asked, “What is the value of x where the graph crosses the y -axis?” she responded: “Zero.” She explained, “The 38,000 is the base sales when

the price is zero. But in an accounting mind that makes no sense.” She concluded the problem when she said, “The x -intercept is 95, and if the price of calculators was \$95 then they sell zero calculators.” (Interview 1, 11/15)

Marsha could sometimes correctly identify and interpret different components of a function and then fail to identify and interpret the same component later. In this case, although she knew exactly how to find the x -intercept and interpret its meaning, interpretation of the slope and y -intercept was problematic for her. I attribute these difficulties to the fact that the linear function was given in a generic form of x and y , and not in terms of variables that corresponded to real-life examples.

Graphical to verbal. On the pretest, I posed the question: “The graph shows how the amount of gas in a tank of a car decreases as the car is driven. Find the slope of the line. Write a statement that states the meaning of the slope.” (Appendix B) Marsha was able to use the graphical representation of the function and interpret the slope. Her response is stated in Figure 5.

$\text{Slope} = \frac{7}{-140} = -0.05$ <p>As more miles are driven, the amount of gas decreases by 0.05 gallons.</p>

Figure 5. Marsha’s solution for the “amount of gas in a tank” question on the pretest.

I posed the question (Appendix D), “Use the graph of $y = h(x)$ to find $h(-2)$.” The four points $(-4, 1)$, $(-2, 4)$, $(3, -1)$, and $(7, 1)$ were marked on the given graph. Marsha said, “It is $(-2, 4)$.” I asked her, “In $f(2) = 3$, can you tell me the value of x and y ?” After a long pause she said, “The x is two, and y is three.” I asked, “Now, in $h(-2)$ what is -2 ?” She said, “It is x .”

When asked to find $h(-2)$, she put her head down and said, “My y -value is -2 .” In surprise, I asked her to consult the graph. She added, “It is one of the points on here”, and then she continued, “When x is -2 , y is 4 . That is what I meant to say.” Finally, she wrote down $(-2, 4)$. I asked why she wrote her response as $(-2, 4)$. She said, “So you want the answer to be the equation!” Another part of this question asked, “What is $h(7)$?” She responded, “ $h(7)$ is $(7, 1)$.” When I said, “ $h(7)$ is not 1 ,” she countered, “But the point is on the graph.” When I posed the question to her again, she said, “ $h(7)$ is 1 . I was saying it wrong.” (Interview 1, 11/15) Marsha had a great deal of difficulty extracting from an ordered pair (X, Y) the corresponding values of x and $h(x)$ when working from a graph.

The question continued and asked, “On what interval is $h(x)$ increasing or decreasing?” Marsha answered, “I feel like I’m on the spot now. All these things I should know, and I’m drawing blanks.” Then, after a long pause, she continued, “But it is increasing at two different intervals.” I pointed out that she needed to concentrate on one interval at a time. She added, “This one is increasing at an interval of two.” When asked to explain this step, she said, “Is it how x is increasing?” After I assured her, she said, “It is $-4 - (-2) = -2$, but you can’t have a negative increase!” She moved to the next interval and said, “This one is increasing at an interval of $7 - 3 = 4$.” (Interview 1, 11/15)

The intervals of increase and decrease of a given graph was troublesome for Marsha. Her knowledge of this concept was flawed because she knew that she had to state how the x was increasing, but she proceeded and subtracted the two x -coordinates between which a segment of the graph was increasing. For the first interval she came across a negative increase that shocked her; but she did not stop, and proceeded to find the second interval. This time around she

reversed the order of the x -values and found a positive number for the “interval.” This topic was not covered in the course.

The latter part of this question (Appendix D) asked Marsha to find the domain and range of the graph of $h(x)$. She responded, “The domain is the x -values, and it would be $[-4, 7]$,” and added, “The range is the y -values, and it would be $[-1, 4]$.” She added, “I understand this lot more now.” She knew how to find the domain and range of functions presented in graphical form.

I posed a question (Appendix D) that presented her with two graphs of the more developed $-f(x)$, and less developed $-g(x)$ regions of the world that were plotted against year. In response to my question “What does the function $f + g$ represent?” she said, “Just from deductive reasoning, I would think it represents the total population.” She read the problem out loud: “Use the graph to estimate $(f + g)(2000)$.” After a short pause she continued, “Just looking at the graph it looks like f is going to be around 1.75 and g is going to be around 5.9.” She continued, “A rough estimate of $(f + g)(2000)$ is 7.65.” (Interview 1, 11/15) Marsha handled this real life example where the graph of time was depicted against the population with no difficulty.

I presented Marsha with a line segment AB that extended from (2, 2) to (6, 5) and asked if the points $(-1, 0)$ and $(5, 4)$ were on that line (Appendix D). She read the problem and said, “Well, (5, 4) is, I can see that $(-1, 0)$ is not.” I asked whether she would arrive at the same conclusion if she extended the line. She said, “I guess if you extended it, [the line] would pass through $(-1, 0)$.” She was deep in thought, so I asked, “What is your next step?” She responded, “I am thinking that the line AB is only from (2, 2) to (6, 5), and point $(-1, 0)$ could not be a part of line AB.” She looked at me and laughingly said, “Don’t try to trick me. I had it right, $(-1, 0)$ is not on the line.” I assured her that I was not trying to trick her. It seemed that extending the line

was out of the question. I asked what she would do if she was given the equation of a line and wanted to determine whether a given point was on that line. After a long pause she said, “You plug it in.” At last she said, “The line is basically infinite, and both points are on that line.” The question that followed consisted of the equation of parabola $y = (x - 4)^2 + 1$ along with its graph. She was asked whether the point $(0, 17)$ was on the parabola. Without hesitation she substituted the point into the equation, and when she saw that it satisfied the equation, she responded, “Yes it is.”(Interview 1, 11/15)

When Marsha was presented with the line segment AB, she determined whether a certain point was or was not on the graph, based on what appeared to the naked eye, although such an approach cannot give a definite answer. On the other hand, Marsha showed no difficulty verifying the point, which did not appear to be on the parabola.

Translation

I have organized Marsha’s responses in this area into five categories: (a) algebraic to graphical, (b) verbal to algebraic, (c) numerical to graphical, (d) symbolic to graphical, and (e) graphical to algebraic.

Algebraic to graphical. On the pretest, Marsha translated $f(x) = x^2 - 2$ to graphical form by plotting points. One graphing problem appeared on the only class quiz she had to do on this chapter. The question stated: “Use the graph of $f(x) = x^2$ to sketch the graph of $g(x) = x^2 - 1$.” The quiz was multiple choice, and she missed that question altogether. The chapter test contained two translations from algebraic to graphical. One was very similar to the question above, and the other asked her to graph $f(x) = -2|x - 2| + 4$. Marsha successfully attempted those by plotting points and received credit for them.

I asked Marsha if she could graph the functions $y = |x + 2|$ and $y = -(x - 2)^3$ without plotting points. Marsha wrote:

Translation is one of the items we are studying in this chapter. To be completely honest, I do not quite get it, yet. From what I understand, it has something to do with the basic parabola, but that's about all that I have got right now. (Reflection 2, 10/29)

When I asked Marsha to graph $f(x) = |x + 2|$, she said, "So...all the y -values are going to be positive. Is there a particular way you want me to do this or is my convoluted version okay?" I asked, "How do you usually graph these functions?" She said, "Plugging in values." I asked if the topic of graphing techniques was covered in the lecture notes. She said it was, but "I just thought my way was easier." In response to my question of whether plotting points takes longer, she said, "Sometimes, but I can generally get an idea of the shape." So I asked her to attempt a few of the plots. She chose four values for x and found the following ordered pairs: (1, 3), (-1, 1), (-4, 2), and (4, 6). After she plotted the points, I asked if she had any idea about the shape of this graph. She said, "It is a smiley face." When she saw my surprise, she said, "Hold on let me do a little sketch, Oh it is like a Chevron, a little V." We moved on, and I asked her to graph $f(x) = x^2 - 2$. For this one, she guessed correctly and said the graph was a smiley face. I asked her if by "smiley face" she meant parabola, and she said, "Yes." The third graph I presented to her was $f(x) = (x - 3)^3$. Like before, she substituted values for x and arrived at the following ordered pairs: (3, 0), (2, -1), (0, -27), and (-1, -64). She plotted the points and said, "It is a frowney face, I am sure it has a name. Is it a hyperbola?" When she worked $f(x) = -\sqrt{x + 2}$ she added, "Oh no, this is one of the things I am having a hard time getting." (Interview 1, 11/15)

Marsha missed two of the graphing questions on the final examination when she tried to plot points. One was $f(x) = (x - 1)^3$ which she missed because she connected the points and plotted a straight line. The second one was an absolute value function $f(x) = |x - 1| - 2$, and she graphed this function in her usual way of plotting points; the graph she plotted was a parabola.

The only way Marsha knew how to translate functions from algebraic to graphical was by plotting points. She was locked into thinking that her way was easier. She did not try to learn the graphing techniques that were discussed in class. She was not always lucky in guessing the correct graphs. The four graphs of $y = x^2$, $y = x^3$, $y = |x|$ and $y = \sqrt{x}$ were the types of functions that were discussed in the online college algebra course. On the class test, Marsha graphed $f(x) = x^2 + 1$ and $f(x) = -2|x - 2| + 4$ successfully by plotting points. However, during the interview when she was asked to graph $f(x) = |x + 2|$, she had difficulty recognizing what the graph would be. Her first choice was a “smiley face,” then a “Chevron, little V.” The square root function ($f(x) = -\sqrt{x + 2}$) was definitely one of the functions she did not know how to translate. For the cube function $f(x) = (x - 3)^3$, she again had difficulty plotting the graph.

On the topic of graphing techniques, the online lecture notes provided limited examples of each type of translation. The only function that had several examples of both vertical and horizontal translations was the square function. This might be the reason Marsha thought translation had “something to do with the basic parabola” (Interview 1, 11/15). On the other hand, the square root and absolute value functions were only introduced in class notes with no mention of any examples. This may have contributed to Marsha’s reluctance to attempt such problems. As for the cube function, it was only demonstrated with vertical translations. While Marsha was plotting points, she knew how the graph of a square function must turn out. Since

the square function was covered more extensively than other types of functions in class, she was rehearsing this type of function elaborately. As a consequence, she was more likely to remember it during the interview. Throughout the course of the study, Marsha showed no interest or desire to learn the graphing techniques. If she had difficulties with a topic discussed in class she used her own resources. I inquired about this practice. Marsha said,

When I am online I do not know what kind of instructor I am dealing with. I do not know if it is someone who is going to take my questions seriously and actually be able to help me, or if it is someone who does not know the answer, or just cannot explain it the way it can help. Unfortunately I have encountered this in a classroom situation. Instead, I prefer to look for people I know and resources that I know can help me. (Interview 1, 11/15)

Verbal to algebraic. I asked Marsha to find the linear function that determined the relationship between the cost to build a house and the number of square feet of floor space. It was estimated that the cost to build a new home was \$25,000 plus \$80 for each square foot of floor space. Marsha wrote the equation: $C = 25,000 + 80n$. (Pretest, 10/17) During the class test on functions and graphs, Marsha answered a similar question and demonstrated no difficulty.

During the first interview I posed a question that required Marsha to translate from verbal to algebraic by finding the function that described the following situation:

A police department believes that the number of serious crimes, which occur each month, is dependent upon the number of police officers for preventive purposes. If 140 police officers are assigned for preventive patrol, there will be no crime. However, it is expected that assignment of no police officers would result in 1,500 serious crimes per month. Find a linear function demonstrating that relationship.

After a short pause Marsha said, “The number of crimes is dependent on the number of officers on patrol.” She summarized the information into two ordered pairs (140, 0) and (0, 1500).

Marsha looked at me in despair and put her hands up. I reworded the problem and mentioned that she needed to find an equation that related the number of crimes to the number of police officers. She continued, “So I need to find the slope.” I provided the formula for the slope because she was not able to recall the formula. She proceeded and utilized the given y -intercept in the slope-intercept formula, writing: $C = -10.71x + 1500$. (Interview 1, 11/15)

I posed another question that described how the profit of a doll company is dependent on the number of dolls sold:

A doll company had a profit of \$50,000 per year when it had sold 1,600 dolls. When the sales of dolls increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P , is a linear function of the number of dolls sold. Find the profit function.

Marsha wrote the two ordered pairs (1600, 50 000) and (1800, 60 000), and used them to find the slope. She added, “So $P = 50x + 50,000$?” When asked if 50,000 was the y -intercept, she responded, “No it is not.” I inquired, “When the y -intercept is not given, what is the remedy?” She said, “I am guessing from your tone that there is [a remedy], so [do] I get a hint?” I asked how she would find the equation of the line if she was given a point and a slope. She explained, “I can see that sheet of my notes in my head, and I know what you are talking about. What is the formula?” I gave her the point-slope formula, and she found $y - 30,000 = 50(x - 1600)$. When I asked her to write the profit function, she wrote $P = 50x - 30,000$. (Interview 1, 11/15) Marsha skipped a similar problem that appeared on the final examination.

In this study the task of translating from verbal to algebraic dealt with only linear functions. Despite Marsha’s lack of confidence in herself to do word problems, her command of

language helped her when it came to reading the verbal problems and translating them into algebraic format. She however, faced difficulty remembering the slope formula. After the formula was provided, she used $y = mx + b$ to find the algebraic function. In another instance, Marsha stumbled and could not recall what she was supposed to do when the y-intercept was not among one of the ordered pairs. Marsha's response indicated that knowledge of the ways in which an equation is used would have helped her remember it.

Numerical to graphical. On the pretest Marsha was asked to graph the function represented by:

x	4	1	0
y	-1	2	3

She plotted the points and obtained a straight line with negative slope. A similar question was asked during the interview, and she graphed the linear function. Translating from numerical to graphical representation was an easy task for Marsha.

Symbolic to graphical. I asked Marsha to graph the function represented by $f(2) = -2$, $f(0) = -3$, and $f(4) = -1$. Marsha plotted the points $(2, -2)$, $(0, -3)$ and $(4, -1)$ and obtained a straight line with positive slope. (Interview 1, 11/15)

Marsha had no difficulty translating from symbolic to graphical, and she seemed to be a quick learner. Early in the study she did not know the meaning of $f(2) = -2$, but once she figured it out, she was able to complete the task.

Graphical to algebraic. I presented Marsha with an absolute value function (Appendix D) that was both shifted two units to the left and reflected along the x-axis. I asked her to find the algebraic form of the function that was translated. She said, "You flipped it; that would make it

negative.” She added, “You moved it back two, so that would be negative two.” Then she wrote, “ $g(x) = -|x| - 2$.” I asked how the function would look if I moved the graph down. She replied, “You would have it [the constant] inside the absolute value.” In astonishment I said, “Inside the absolute value?” Marsha thought for awhile and finally said, “Oh, Okay moving back [horizontal translation] goes inside the absolute value. So it is $g(x) = -|x + 2|$.” (Interview 1, 11/15)

The next problem was a square function (Appendix D) that was translated two units to the right. Marsha looked at the graph and said, “It is going to be $x^2 - 2$.” I reminded her that this problem was similar to the one we just did. Then said, “It is just $x^2 + 2$ ” I invited her to write her response and stressed that this was very similar to the problem she did earlier. When I asked her if the shift was horizontal or vertical, she said, “So it is $(x + 2)^2$.” In response to my question of “Is it plus?” She said, “No it is minus $g(x) = (x - 2)^2$.”

Subsequently, I presented her the graphs presented in Figure 6, and explained that the middle graph is the graph of $y = x^2$ and asked her to find the equations of the inner and outer graphs.

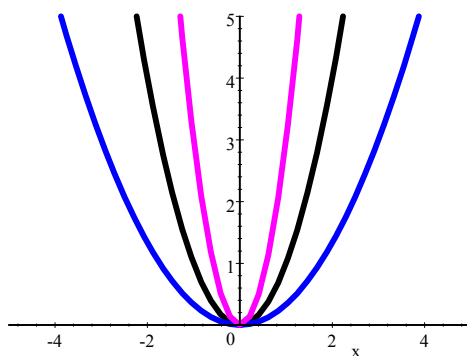


Figure 6. $y = x^2$.

Marsha did not know how the shrinking and stretching property worked because when asked if those words rang a bell, she responded “No.” (Interview 1, 11/15) The shrinking and stretching aspects of graphing techniques were not discussed in class.

The next two questions involved vertical (upward) translation of a cube function and horizontal (left) translation of the square root function. When Marsha saw the first graph she said, “So we get to the mistake I was making,” and she wrote: $g(x) = x^3 + 2$. In doing the next translation she said, “It is going to be $g(x) = \sqrt{x+2}$.” Marsha had no experience with the square root function, however, when presented with a square root function translated 2 units to the left, she was able to write the algebraic function of the graph. In addition, I presented Marsha with graphs of $x = y^2$ and $y = x^3$, and asked her to sketch the graphs of $y = f(-x)$. Marsha reflected the given graphs along the y -axis. (Interview 1, 11/15) She had no difficulty with the reflection tasks along y -axis.

Translating from graphical to algebraic seemed to be an easier task for Marsha. Earlier she had mentioned that she “does not quite get it.” This in fact turned out to be a wonderful learning experience for Marsha. She had difficulty with the horizontal translation, while vertical translation was easy for her. As we were working through these translations, she realized her mistakes.

Summary

Marsha clearly demonstrated that she was an independent learner. She supplemented her mathematical knowledge of functions by exploring resources other than the course instructor (friends, college algebra textbooks, and Algebra For Dummies). This may have contributed to some of the problems Marsha faced in remembering and retrieving her previously learned mathematics concepts. Having tapped so many resources in search of answers to her problems, at

times created cloudy or incomplete mental notes. She was not comfortable communicating with her teacher about questions she might have had in class. Therefore, in order to make sense of mathematical concepts she filled in the missing details for herself – sometimes accurately and sometimes not. She was locked into her own “way of doing things.” Further, Marsha did not demonstrate confidence in doing these tasks. She commented, “I do not have a lot of confidence in what I know. Obviously, I was able to work through all the problems with a little bit of prompting, but now I know more than I felt I knew coming here.” (Interview 1, 11/15) Overall, Marsha demonstrated signs of improvement in the areas of recognition and interpretation of functions. Translation of functions was the one area that Marsha continued to have difficulties with.

The Case of Ivy

Ivy scored 75% and 98%, respectively, on the chapter (functions and graphs) quiz and test. She scored 69% on the comprehensive final examination for online college algebra course, which counted 40% of the grade. Ivy had taken this course face-to-face previously and received a D in it. She finished the online college algebra course with a B (on a scale of A to F).

Recognition

I have organized Ivy’s responses in the area of recognition into three categories:

(a) algebraic, (b) numerical, and (c) graphical.

Algebraic. On the pretest, I asked, “Is $x^2 = y + 4$ a function?” Ivy’s response was, “Yes.” In addition when I asked if $y = -1$ and $x = 6$ represented functions, she responded, “Both are functions” (Pretest, 10/18). I inquired about her response and asked, “Why is $x = 6$ a function?” She said, “This equation is not a function. When you graph the equation, it is a

vertical line and according to the vertical line test, it should cross a point once for it to become a function.” (Interview 2, 11/20)

During the interview, similar questions reappeared for which she had to determine if $y = 2$ and $x = 5$ represented functions. She commented, “I am going to graph. I am kind of nervous, I am sorry.” The following excerpt from my first interview with Ivy outlines our conversation.

Interviewer: There is nothing to be nervous about. You need to calm down.

Ivy: I cannot remember how to graph this. Let’s say x is equal to zero. But there is no x in $y = 2$, so it would be zero and you graph point 2 on the y -axis. It is just a straight line.

Interviewer: Which way?

Ivy: Vertical not horizontal.

Interviewer: You want to graph that?

Ivy: Okay. But there is no x -coordinate in order for me to graph.

Interviewer: Do you need to have both x and y to be able to graph?

Ivy: No, I can just plot the point.

Interviewer: How do you graph $y = 2$?

Ivy: It is just a point on the y -axis. Am I doing it wrong?

Interviewer: I claim that graph of $y = 2$ is a straight line, but you have plotted only a point for me.

Ivy: Okay it is a vertical line.

From point 2 on the y -axis she graphed a vertical line on that axis. I asked if $y = 2$ was a function. Her response was, “No.” I continued, “How do you test if a given relation is a

function?” She replied, “I do [a] vertical line test.” After a long pause she continued, “ $x = 5$ is a horizontal line.” Our conversation continued.

Interviewer: How do you classify a line as horizontal or vertical?

Ivy: On the Cartesian plane, I guess, the x -axis is the horizontal line and the y -axis is the vertical line. In here, $x = 5$ is a horizontal line because the point $x = 5$ is on the x -axis.

Interviewer: How would you classify $x = 5$?

Ivy: It is a function because when I do [a] vertical line test, it only goes through one point one time.

Ivy was confused during the interview. Later I inquired about her response and the fact that she graphed $x = 5$ as a horizontal line and $y = 2$ as a vertical line. She said, “After the interview I checked my notes and found my mistake. I know now that [the] graph of $x = 5$ is the equation of a vertical line and $y = 2$ is the equation of a horizontal line” (Interview 2, 11/20).

Relations of the form $y = k$ and $x = k$, where k is some fixed constant, were difficult for Ivy. She knew how to apply the vertical line test but did not know how to graph these relations. Ivy did not know that a straight line was represented by a set of ordered pairs. She graphed $y = k$ as a vertical line and $x = k$ as a horizontal line because she could not identify ordered pairs that were on these lines. Early in the study, Ivy identified $x = 6$ as a non-function and classified it as a vertical line. However, later, during the interview, she identified a similar relation as a horizontal line. Ivy seemed very dependent on her notes.

Further, when I asked, “Is $x = y^2 + 4$ a function?” Ivy replied, “I need to graph this to see if it is a function.” I inquired if she knew of other methods, she noted, “I do, but graphing is easier for me.” After a short pause she said, “I put this equation into standard form.” She solved

for y^2 then took the square root of both sides, and wrote: " $y = \sqrt{x-4}$." I asked, "When you take the square root of both sides, would you get only one answer?" She said, "Plus or minus?" and wrote: " $y = \pm\sqrt{x-4}$." Then Ivy put her head down and quietly said, "I do not know what to do now." I asked if she had graphed equations of this nature, and she explained, "I have, but cannot remember how to do them now. Can I use my graphing calculator?" I allowed the use of the calculator. She proceeded and plotted the equation $y = \sqrt{x-4}$ and concluded that it is a function. I drew her attention to the \pm sign in front of the radical. She said, "For $y = -\sqrt{x-4}$ the graph will be below the x -axis." Ivy completed the graph, used the vertical line test, and concluded that the given relation was not a function because it failed the vertical line test.

The question: "Is $x^2 + y^2 = 25$ a function?" led Ivy to pause for a second. She answered, "I am going to put that [equation] into a standard form and take the square root of both sides." She wrote: $y^2 = -x^2 + 25$ and in amazement said, "You cannot take the square root of a negative." She was exhibiting a great deal of nervousness, and her hands were shaking. I pointed out that she could rearrange the right-hand side of the equation if she wanted. She completed the arrangement, then took the square root of both sides and wrote: $y = \sqrt{25 - x^2}$. I asked if she recognized this shape. Ivy said, "I do not remember, can I graph it now?" Like the previous problem, she only graphed the positive half of the equation and said, "It is half a circle." In response to my question "Is your graph complete?" Ivy realized, "I forgot the negative in the equation." She graphed $y = -\sqrt{25 - x^2}$, and said, "It is a circle and not a function."

To verify that algebraic relations given in terms of x and y were functions, Ivy solved the equations for the variable y and graphed the equation. She was so driven by her procedural knowledge that it hindered her work and the knowledge of the content matter. For instance,

whenever the given equation was in terms of y^2 , Ivy ignored the “ \pm ” signs. She was only able to recognize $x^2 + y^2 = 25$ as the equation of a circle after she graphed the equation. The topic of circle was discussed in the “Functions and Graphs” chapter of this online college algebra course.

On the comprehensive final examination, Ivy had to determine whether the equations $x^2 + y^2 = 9$ and $x^2 + y = 4$ represented functions. She answered that both were functions. For Ivy to determine if an algebraic relation was a function she had to graph the relation. The use of graphing calculators was not allowed on the final examination; therefore, her answer of “yes” could have been a guess on her part.

During the interview I asked Ivy whether the following relation represented a function:

$$\begin{cases} x \geq 0, y = x + 2 \\ x < 0, y = 1 - 2x \end{cases}$$

She responded, “I put 1 for x in the first one [$y = x + 2$], and 1 for x in the second one [$y = 1 - 2x$].” I inquired if she could substitute the same number for x in both cases. She responded, “Oh, it says $x < 0$, I put 1 in the first equation and -1 in the second one and y is equal to 3 in both cases. The graph of it is a vertical line.” I asked, “How would you graph a straight line? Do you use only one point for graphing a line?” Ivy answered, “No. I can substitute another point for x and put that on the graph.” She proceeded and found the ordered pairs $(1, 3)$, $(-1, 3)$, $(2, 4)$, and $(-2, 5)$. Our conversation continued:

Ivy: All these points are going to be on the y -axis.

Interviewer: Do you only plot the y -coordinates?

Ivy: Yes, I do for these equations.

Interviewer: Okay, let me see how you plot them.

Ivy: It would be a straight line going through the y -axis. It is not a function.

Like Marsha, Ivy did not realize that different rules of correspondence applied to different parts of the domain. After she figured out the rules, she was able to find a set of ordered pairs. To determine whether the given relation was a function, as before, she proceeded to graph the ordered pairs. Ivy demonstrated extreme confusion with regard to graphing. She seemed not to know that plotting both coordinates of a point was necessary.

Numerical. In response to the question: “Which of the following represent a function (a) $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$, or (b) $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$?” Ivy replied that part (a) “was not a function because of $(5, -4)$ and $(5, -1)$. In there, both have the same x -coordinate.” She said part (b) was a function. (Pretest, 10/18)

I asked if the relation $\{(1, 2), (1, 5), (2, 10), (-3, 5)\}$ represented a function. She immediately responded: “It is not because two of the points have the same x -coordinate.” As we continued, I asked if the relation $\{(-1, 0), ((2, 4), (6, 1), (4, 1)\}$ was a function. Ivy said, “It is, [because] none of the x -values are the same.” (Interview1, 11/8)

On the final examination, Ivy was presented with the question: “Does the relation $\{(1, 2), (1, 5), (-3, 5), (9, 1)\}$ represent a function?” She answered, “Not a function because two of the ordered pairs have the same x -value.”

Ivy demonstrated knowledge of functions presented in numerical form. She used the procedure that “no x -value is associated with two y -values” in order to verify that numerical representations were functions.

Graphical. Relations represented in the form of a graph were another category of questions both in pretest and the interview. On the pretest Ivy identified the graph of $y = |x|$ as a function and $x = y^2$ as a non-function. For $x = y^2$, she noted it was not a function because it failed the vertical line test.

On the class quiz she was asked to "use the vertical line test to determine whether the graph of $y = x^2$ represented y as a function of x ." Ivy's response to this multiple-choice question was " y is a function of x ."

I asked if the graphs presented in Figure 7 represented graphs of functions. Without hesitation, Ivy said the graph in Figure 7, part (a) was not a function and the one in part (b) was a function. (Interview 1, 11/8)

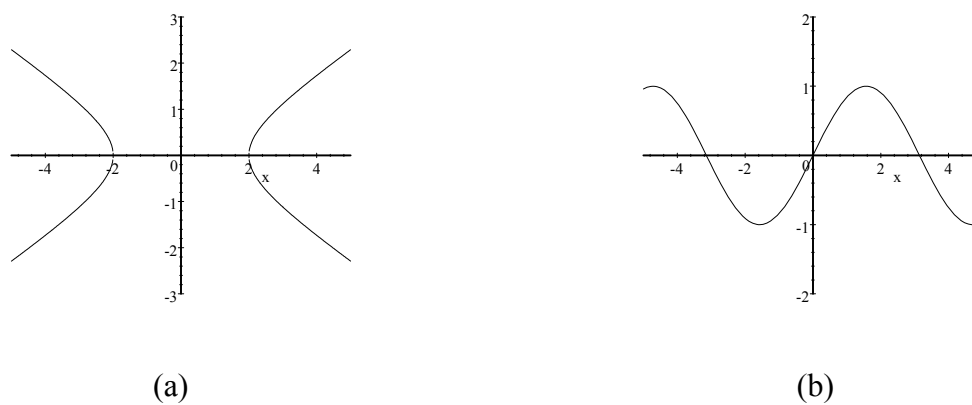


Figure 7. Recognition of graphs.

Ivy showed knowledge of the graphical representations of functions and non-functions. As demonstrated before, Ivy seemed to enjoy the graphical representations.

Interpretation

Ivy's responses in the area of interpretation fell into two categories: (a) algebraic to verbal and (b) graphical to verbal.

Algebraic to verbal. On the pretest, I asked a question that involved interpretation of the slope and the intercepts. Ivy's response to the following question is stated in Figure 8.

An appliance store estimated that 40 transistor radios would be sold per day during an upcoming season if the new radios were priced at \$100 each. At the price of \$150, the

store would sell 25 units. The linear function that models this relationship is given by

$D = -0.30p + 70$. (a) Interpret the meaning of the slope and the y -intercept. (b) Identify and interpret the x -intercept.

In order to find the x - and the y -intercepts, she felt the need to rewrite the equation in terms of x and y . She identified the slope as $-0.30p$, successfully found the x - and the y -intercepts, and skipped the interpretation aspect of this question. In fact, Ivy skipped all of the interpretation questions on the pretest. I asked why she skipped them. She responded, “I am not good in this. I can only do the algebra part” (Interview 2, 11/20).

$$D = -0.30p + 70 \Rightarrow y = -0.30x + 70$$

Slope is $-0.30p$

y -intercept is $(0, 70)$

x -intercept is $(233.33, 0)$

$$y = -0.30x + 70$$

$$y + 0.30x = 70$$

$$0 + 0.30x = 70$$

$$\frac{0.30x}{0.30} = \frac{70}{0.30}$$

$$x = 233.33$$

Figure 8. Ivy’s solution for the “transistor radio” question on the pretest.

Ivy was convinced that she was not good at interpreting functions, and that she was good at the procedural skills. Ivy's lack of self-confidence did not allow her to attempt the interpretation aspect of the questions.

Figure 9 shows Ivy's solution for the following question that appeared on the class test.

A company's weekly revenue in dollars is given by $R(x) = 2000x - 2x^2$, where x is the number of items produced during a week. (a) For what x is $R(x) > 0$? (b) On what interval is $R(x)$ increasing? Decreasing?

Part (a)	Part (b)
$R(x) = 2000x - 2x^2$	Increasing at $(-\infty, 500]$
$0 = -2x^2 + 2000x$	Decreasing at $[500, \infty)$
$x = 0$ or $x = 1000$	
$0 < x < 1000$	

Figure 9. Ivy's response to the "weekly revenue" question on the class test.

I inquired about her response and asked her how she came up with the solution. She stated, "For part (a) I used the method [critical value method] discussed in class, and for part (b) I just used my graphing calculator and figured that part out" (Interview 2, 11/20). Determining the intervals of increasing and decreasing was not covered in her class, but she answered the question correctly.

I posed the question:

The book value of a machine is \$4,500. It is estimated that after 9 years the value of the machine will be \$900. If the linear function that determines this relationship is given by

$V = -400t + 4500$: (a) interpret the meaning of the slope and the y -intercept, (b) identify and interpret the x -intercept.

Ivy identified the slope as -400 and said, “I am not sure how to interpret it [slope]”. She added, “It [slope] is how steep the line is.” Ivy continued, “The y -intercept is $(0, 4500)$ and it is the value of machine.” I asked for clarification: “The value of machine when?” Ivy noted, “When it [machine] is new.” In part (b) of the question Ivy commented, “I cannot remember how to find it [x-intercept]. I am sorry.” I asked, “How do you find the y -intercept?” She said, “I would change the equation into standard form $[y = mx + b]$ and the b in this equation is going to be the y -intercept.” (Interview 1, 11/8) Ivy only knew how to find the y -intercept of a linear function when it was put in a slope-intercept form.

During the interview she identified the slope and the y -intercept; however, Ivy had some difficulty interpreting what they meant. She interpreted the slope as “how steep a line is.” Her interpretation of the y -intercept was accurate. Although Ivy had been able to find the x -intercept on the pretest, she was completely helpless during the interview. During the pretest she had access to her notes; during the interview she did not. Although Ivy could identify the y -intercept as $(0, 4500)$, she was not able to make a connection between the y -intercept found by looking at the slope-intercept equation and the ordered pair $(0, y)$.

Subsequently, I posed the question:

A manufacturer of graphing calculators determined that 8,000 calculators per week would be sold at a price of \$75. At a price of \$70, it was determined that 10,000 calculators would be sold. The linear function that determines this relationship is given by

$y = -400x + 38,000$. (a) Interpret the meaning of the slope of the graph of this linear

function. (b) Explain the meaning of the y -intercept. (c) Explain the meaning of the x -intercept.

Ivy read the problem aloud and laughingly said, “It is the same as the other problem.” She continued, “Slope is -400 and it is the change?” I applauded her response and asked, “It is the change in what?” She explained, “It is the change in value of the calculator when it is sold at $\$75$ and $\$70$.” I pointed out that this problem was about sales and price and requested that she discuss the slope in terms of those features. She added, “When price goes down, more calculators would be sold.” Although the slope was given, Ivy proceeded to calculate it. She incorrectly labeled the independent and dependent variables, and found the slope to be $-1/400$. With a concerned voice she said, “It is not right.” Soon, she tried to fix her problem and realized that the dependent variable was the number of calculators sold. I asked if she could interpret the slope now that she knew what the different variables stood for. She beamed at me with a smile and said, “I still cannot do it.” I said, “If I put the slope over 1 would it help?” She commented, “Is it [slope] a ratio?” I answered affirmatively, and she continued, “As the sales go up by 400 the price goes down by 1.” Ivy read part (b) of the question aloud and said, “38,000 is y -intercept.” I asked, “How would you interpret it?” After a second she added, “It is the number of calculators sold.” I continued, “It is the number of calculators sold at what price?” She paused for a long time, then looked at me in dismay and said, “When price went down?” In part (c) of the problem Ivy said, “The x -intercept is the price.” (Interview 1, 11/8)

Ivy had difficulty identifying the independent and dependent variables when the linear function was given in the generic form of x and y and not in terms of variables that corresponded to real-life examples. She thought of slope as change. When I asked if writing the slope over 1 might help, she realized, “So slope is a ratio.” This exchange sparked a light in her mind and

enabled her to interpret the slope. In this question the slope was given, yet she attempted to find it again. She managed to interpret the y -intercept earlier, but this time she had difficulty doing so. The concept of x -intercept was still an issue for Ivy, and she had difficulty identifying and interpreting it. This implied that Ivy did not possess a correct knowledge of the x - and the y -intercepts and their significance.

Graphical to verbal. On the pretest, I posed the question (Appendix B), “The graph shows how the amount of gas in a tank of a car decreases as the car is driven. Find the slope of the line. Write a statement that states the meaning of the slope.” Her response is stated in Figure 10.

$$\begin{aligned}
 &(40, 13), (180, 6) \\
 &m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 13}{180 - 40} = \frac{-7}{140} = -\frac{1}{20} \\
 &y - y_1 = m(x - x_1) \\
 &y - 13 = -\frac{1}{20}(x - 40) \\
 &y - 13 = -\frac{1}{20}x + 2 \\
 &y = -\frac{1}{20}x + 15
 \end{aligned}$$

Figure 10. Ivy’s solution for the “amount of gas in a tank” question on the pretest.

After she found the slope, she continued to find the equation of the line that connected the two points (40, 13) and (180, 6) even though this was not called for. Like before, she was so driven

by her procedural skills that, instead of addressing the interpretation aspect of the question, she proceeded to find the linear function that passed through those points.

I posed the question (Appendix D), “Use the graph of $y = h(x)$ to find $h(-2)$.” The four points $(-4, 1)$, $(-2, 4)$, $(3, -1)$, and $(7, 1)$ were marked on the given graph. After a short pause, Ivy said, “It is $(-2, 4)$.” I repeated the question, and she added, “ $h(-2) = 4$.” Her original response was similar to what Marsha had said. But when I repeated the question, she was able to extract from the ordered pair $(-2, 4)$ the corresponding value of $h(-2)$ when working from a graph.

The question continued and asked, “On what interval is $h(x)$ increasing or decreasing?” Ivy answered, “It is increasing from $(-4, 1)$ to $(-2, 4)$ and from $(3, -1)$ to $(7, 1)$. It is decreasing from $(-2, 4)$ to $(3, -1)$.” I asked, “Do you remember how we write domain and range in interval notation?” She explained, “Do I give you $x > -4 > -2$?” I explained that this inequality was incorrect. She paused for a long time and said, “I do not know how to correct it.” (Interview 1, 11/8) I inquired about her response and the fact that the inequality $x > -4 > -2$ was not correct. She commented, “I meant to write $-4 < x < -2$. I was very nervous during the interview.” (Interview 2, 11/20)

Ivy’s immediate response for the intervals of increase and decrease were the two points between which the line segment was increasing or decreasing. When I reminded her how to write the domain and range, she realized what she had written was insufficient. She then proceeded and wrote an inequality that supposedly signified the first interval of increase. That inequality was still incorrect, but considering her nervousness, it was impossible for her to proceed. Compared to Marsha, Ivy had some knowledge of intervals although what she had was incorrect.

The latter part of the question (Appendix D) asked Ivy to write the domain and range of the graph of $h(x)$. She responded, “The domain is going to be $[-4, 7]$, and the range is $[-1, 4]$.” Ivy knew how to find the domain and range of functions presented in graphical form.

I posed a question (Appendix D) that presented Ivy with two graphs of the more developed – $f(x)$, and less developed – $g(x)$ regions of the world, that were plotted against year. In response to my question: “What does the function $f + g$ represent?” She said, “You add the less developed regions in year x and more developed regions in year x .” I continued, “What do you get?” After a pause Ivy asked, “Do I get $h(x)$?” I gave affirmation and asked her to continue. She read the second part of the problem aloud: “Use the graph to estimate $(f + g)(2000)$.” After a short pause, she said, “ $h(2000) = 5.5$ billions.” (Interview 1, 11/8) I inquired about her response to the latter part of this question and asked her to consult the graph again to see if 5.5 billions was $h(2000)$. After a few minutes of thinking Ivy said 5.5 billions was $g(2000)$. To find $h(2000)$ I have to add $f(2000) = 1.5$ to $g(2000) = 5.5$ and get 7.0 billions.” (Interview 2, 11/20) Ivy exhibited some difficulty with this real life example in which the graph of time was depicted against the population.

Subsequently, I presented Ivy with a line segment AB that extended from A(2, 2) to B(6, 5) and asked if the points $(-1, 0)$ and $(5, 4)$ were on that line (Appendix D). She read the problem aloud and said, “Well, $(5, 4)$ is on the line but $(-1, 0)$ is not.” The following excerpt from the interview illustrates Ivy’s thinking.

Interviewer: Why is $(5, 4)$ on the line and $(-1, 0)$ is not?

Ivy: Because the line passes $(5, 4)$ and not $(-1, 0)$. The point $(-1, 0)$ is in a different quadrant.

Interviewer: What happens if I extend the line?

Ivy: It [The line] is infinite.

Interviewer: So how do I determine if $(-1, 0)$ is on the line?

Ivy: Use the equation $y - y_1 = m(x - x_1)$.

Interviewer: How do you use the equation?

Ivy: First, find the slope of the line using the points $(2, 2)$ and $(6, 5)$. Then I use [The] point $(-1, 0)$ to see if it is on that line.

Interviewer: What line?

Ivy: I find the equation using [The] points A, B, and slope $\frac{3}{4}$.

Interviewer: What do you do next?

Ivy: I substitute $(-1, 0)$ into that equation.

Interviewer: If $(-1, 0)$ is on the line what do you expect to see?

Ivy: The numbers on both sides of the equation will be equal [get a true statement].

Ivy was able to use an algebraic method to determine if certain points were on the graph of a function. The question that followed consisted of the equation of the parabola $y = (x - 4)^2 + 1$ along with its graph. She was asked whether the point $(0, 17)$ was on the parabola. Without hesitation, she substituted the point into the equation. When she saw that the point satisfied the equation, she commented that the point was on the parabola (Interview 1, 11/8).

When Ivy was presented a line segment AB, like Marsha she only noticed the point that was on the line. But soon she found a way to determine whether a given point was on the graph of a function. She was able to make a connection between the visual representation and the algebraic description of the function.

Translation

I have organized Ivy's responses into five categories: (a) algebraic to graphical, (b) verbal to algebraic, (c) numerical to graphical, (d) symbolic to graphical, and (e) graphical to algebraic.

Algebraic to graphical. On the pretest, Ivy translated $f(x) = x^2 - 2$ to a graphical form by plotting the points: $(1, -1)$, $(2, 2)$, $(0, -2)$, $(-1, 1)$, and $(-2, 2)$. She connected the points and arrived at a straight line with a positive slope (Pretest, 10/18).

One graphing problem appeared on class quiz for this chapter. The question asked, "Use the graph of $f(x) = x^2$ to sketch the graph of $g(x) = x^2 - 1$." The quiz was multiple-choice, and Ivy decided that none of the provided choices were correct.

A few days prior to the chapter test on functions and graphs, I posed the question, "have you come across any function that you were not able to translate from algebraic to graphical, verbal to algebraic, numerical to graphical, etcetera?" Ivy explained, "I have a little bit of difficulty graphing equations. What I always do is, to give x certain number and try to work that out and solve for y to get a point. This takes too much time to do. What other strategies can I use to graph an equation?" I discussed the different graphing techniques with her.

(Reflection 2, 10/27)

The chapter test for this online college algebra course contained two translations from algebraic to graphical. One involved a parabola that was shifted one unit vertically upward. The second translation asked students to graph $f(x) = -2|x - 2| + 4$. Ivy successfully attempted those by plotting points. The absolute value function was introduced without any examples in this online course. Yet students were expected to know how to perform translations on these graphs.

I asked Ivy to graph the function $f(x) = |x + 2|$. She said, "Okay, so that is $y = |x + 2|$." She paused for a long time, then looked at me, and said, "I can only graph this using the graphing

calculator.” I asked if she knew what the shape of this function should look like. She shrugged her shoulders and said, “I have no idea.” (Interview 1, 11/8) Ivy was reluctant to attempt any of the other questions that involved translating functions to graphical representations.

There were two graphing questions of this nature on the final examination, and Ivy attempted to answer them by plotting points. One equation, $y = (x - 1)^3$ was given; and Ivy sketched the graph of a cube function shifted one unit to the right. The second was an absolute value function $y = |x - 1| - 2$. Ivy plotted this function in her usual way of plotting points. She made arithmetic errors and graphed an absolute value function that passed through the origin.

Ivy knew how to translate functions from algebraic to graphical by plotting points or using the graphing calculator. By her admission plotting points “takes too much time,” and she never tried to learn the graphing techniques. Ivy did not know how to translate functions from algebraic to graphical representations using the graphing techniques.

Verbal to algebraic. I asked Ivy to find the linear function that determined the relationship between the cost to build a house and the number of square feet of floor space. It was estimated that the cost to build a new home was \$25,000 plus \$80 for each square foot of floor space. Ivy’s response was: “ $f(x) = 80x + 25000$.” (Pretest, 10/18) Ivy had no difficulty responding to a similar question on the class test. During the interview, however, she did not know how to translate functions from verbal to algebraic form.

On the final examination, one of the questions stated:

A manufacturer of graphing calculators had a profit of \$50,000 per year when it had sold 1,600 calculators. When the sales of calculators increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P , is a linear function of the number of calculators sold. Find the profit function.

Figure 11 shows the response Ivy gave for this question. In here the word *profit* caught Ivy's attention and she responded with memory of the formula for the profit function.

$P(x) = R(x) - C(x)$ $60,000 = 1,800 - C(x)$
--

Figure 11. Ivy's response to the "graphing calculator" question on the final examination.

Ivy had difficulties with the word problems that involved translations to algebraic representation. If the problem involved single step translations, like those on the pretest and the class test, she was okay; otherwise, she did not know how to handle them. Even though she was able to find the equation of a linear function, she was not able to use the implicit information related to the linear equation when it was not directly focused on linear equations.

Numerical to graphical. On the pretest Ivy was asked to graph the function represented by:

x	4	1	0
y	-1	2	3

She plotted the points and obtained a straight line with negative slope. A similar question was asked of her during the interview, and she graphed the linear function. Ivy had no difficulty translating from numerical to graphical representations.

Symbolic to graphical. I asked Ivy to graph the function represented by $f(2) = -2$, $f(0) = -3$ and $f(4) = -1$. Ivy read the problem aloud and said, "I do not know." I asked her "how do you find $f(2)$ given $f(x) = x - 4$?" She said, "Plug in 2 for x in the

function $f(x) = x - 4$.” I continued, “The result $[-2]$ is the value of what variable?” She responded, “ y ,” and added, “So your x is $[2, 0, 4]$ and your y is $[-2, -3, -1]$.” She plotted the points incorrectly and obtained a V-shaped graph. (Interview 1, 11/8)

Like Marsha, Ivy did not know the meaning of $f(2) = -2$. When this was cleared for her, she proceeded with the problem and attempted to translate the function defined by the constraints to its graphical representation; along the way she made a mistake in plotting one of the points and obtained an incorrect graph.

Graphical to algebraic. I presented Ivy with an absolute value function (Appendix D) that was both shifted two units to the left and reflected along the x -axis. She said, “Would it be $g(x) = -|x + 2|$?” I gave affirmation and we moved to a square function (Appendix D) that was translated two units to the right. She looked at the graph and said, “It is going to be $x^2 - 2$?” In response to my question, “This graph shifted which way?” she remarked, “Right side, so it would be $g(x) = x^2 + 2$.” She continued, “When you shift to the right it is plus and to the left it is minus.” I asked, “How would that function have looked if the graph was shifted up?” She said, “I do not know.” (Interview 1, 11/8)

Subsequently, I presented the graphs in Figure 12 to Ivy and explained that the middle graph was the graph of $y = x^2$. I asked her to find the equations of the inner and outer graphs. Ivy used her hands to show me that the inner graph was narrower and the outer graph was wider than the original graph. I asked, “How do you write the function for the narrower graph?” She replied, “You have a value in front of the x^2 .” I applauded her and asked for that value. She said, “A positive number, say two.” She wrote, “ $g(x) = 2x^2$.” I asked for the function that represented the wider graph. She added, “Is the coefficient of x^2 negative?” In astonishment I asked, “Would it

be negative?” After a short pause she asked, “Would it be a fraction?” I replied with affirmation.

She added, “It is $h(x) = \frac{1}{2}x^2$.”

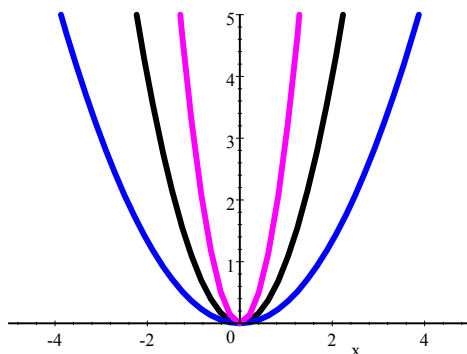


Figure 12. $y = x^2$.

The next two questions involved vertical (upward) translation of a cube function and horizontal (left) translation of a square root function. When Ivy saw the first question she wrote, “ $g(x) = x^3 + 2$.” In doing the next translation she said, “You are shifting to the left.” After a long pause she added, “I do not know.” I encouraged her to write down her thought on this function. She said, “Is it $g(x) = \sqrt{x} - 2$?” I asked her again, “How would the function have looked if the graph was shifted down?” She replied, “So this is for going down. I do not know how to write the function for the left and right shifts.”

Finally, I presented Ivy with the graphs of $x = y^2$ and $y = x^3$ and asked her to sketch the graph of $y = f(-x)$. She thought for a short while then said, “The graphs would open the other way.” I asked which way. She replied, “Over the y -axis.” She reflected the two graphs along the y -axis. (Interview 1, 11/8)

Translating from graphical to algebraic was easier for Ivy than vice versa. Her knowledge of the shrinking and stretching techniques and reflections was exceptionally good. On the other

hand, vertical and horizontal translations posed some difficulty for her, as she was not clear about the translation rules.

Summary

Ivy had previously taken college algebra in a face-to-face setting, yet she seemed to have some conceptual misunderstandings for the concept of functions. Like Marsha, she had developed her own way of doing things and seemed to be comfortable using her methods. She was not as explicit about “her way of doing things,” but her actions proved it to be true. She acknowledged that doing certain problems “her way” took “too much time,” but she never attempted to try or learn the new method. She had an overwhelming procedural disposition that slowed her learning of mathematics.

In doing problems, Ivy did not pay much attention to the directions. She had a tendency to find components (slope) of a linear function when it was provided, or to find the linear function when it was not called for—hoping that it might be useful. Her knowledge of verbal representation of functions seemed particularly blurred. Recognition of functions is the one area in which she did not demonstrate any improvement. She was very comfortable finding the equation of a line. Ivy exhibited gaps, sometimes disturbing ones, in her conceptualization of functions. She preferred to think of functions graphically rather than algebraically.

The Case of Nancy

Nancy scored 83% and 89%, respectively, on the chapter (functions and graphs) quiz and test. She scored 96.5% on the comprehensive final examination for the online college algebra course, which counted 40% of the grade. Nancy finished the online college algebra course with an A (on a scale of A to F).

Recognition

I have organized Nancy's responses in the area of recognition into three categories:

(a) algebraic, (b) numerical, and (c) graphical.

Algebraic. On the pretest, when I asked if $x^2 = y + 4$ was a function, Nancy's response was, "No, because y will have two values of 2 and -2 ." I asked Nancy the reason behind her decision, and she said, "On this one I was simply thinking y^2 instead of x^2 . I just mixed them up in my brain" (Reflection 1, 10/21). During the interview when I asked if $x = y^2 + 4$ was a function Nancy said, "It is not a function because it has y^2 and that would give it two possible y -values." On the other hand, the question: "Is $x^2 + y^2 = 25$ a function?" made her pause for awhile. I asked if she knew what the graph of $x^2 + y^2 = 25$ looked like. Nancy's immediate response was, "[A] circle, and it is not a function." After a long pause she continued, "But then a circle is [a function]; the way a circle goes each x -value only has one y -value. So that confuses me." I asked if she wanted to draw a circle and see it visually. She drew a circle and pointed to the points on the circumference of the circle and said, "In here, each x -value is going to have one y -value. That is what makes sense in my head, so I guess it is a function." I reminded her about the "no" response she had given me originally. She added, "I am confused on that." I asked if she would get the same result if she chose a point inside the circle. She proceeded and picked a point inside the circle and said, "I get a positive and a negative y , so it is not a function." On the comprehensive final examination, Nancy had to determine if $x^2 + y^2 = 9$ and $x^2 + y = 4$ represented functions. In response to the first part she wrote it is not a function as " y^2 can be either y or $-y$." For the second one she just wrote "yes." Unlike Marsha and Ivy, Nancy was able to recognize $x^2 + y^2 = 25$ as the equation of a circle. Her first instinct was that it was not a

function, but like Marsha she wanted the relation to be a function. With a little guidance she figured it out.

In addition, when I asked if $y = -1$ and $x = 6$ represented functions, Nancy's response was " $y = -1$ is a function and $x = 6$ is not a function because vertical lines are not functions" (Pretest, 10/14). During the interview in response to the question "do $y = 2$ and $x = 5$ represent functions," she said, " $y = 2$ is a horizontal line, [a] horizontal line is a function. And $x = 5$ is a vertical line, and [a] vertical line is not a function." Nancy had knowledge of relations of the form $y = k$ and $x = k$ where k is some fixed constant. I asked Nancy whether the following relation represented a function:

$$\begin{cases} x \geq 0, y = x + 2 \\ x < 0, y = 1 - 2x \end{cases}$$

At first she said, "I am going to take this as two separate functions." I pointed out that the existence of brace indicated that this was one function. After a long pause she replied, "I have moved on and this has thrown me off a little bit." Then after a while she laughingly concluded, "I am confused, and I am going to say no." Nancy only had difficulty with relations that were defined as piecewise.

Numerical. In response to the question: "Which of the following represent a function (a) $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$, or (b) $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$?" Nancy wrote that both were functions (Pretest, 10/14). I inquired about her response and asked why she thought $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$ was a function. She said:

I was not clear on this one if you meant, all the points as one within the brace, or if you meant each individual point within the brace. Each individual point would be a function because with each x -value there is only one y -value. If you meant all within the brace, I

suppose it could go either way. Each x -value still has one y -value, but is it meaning to say that there should only be one y -value, the same y -value for all points? I guess this one is a little fuzzy to me because I am not clear on how it should look. (Reflection 1, 10/21)

During the interview I asked if the relation $\{(1, 2), (1, 5), (2, 10), (-3, 5)\}$ represented a function. Her immediate response was, "It is not, because there is more than one y -value." She explained further, "For it to be a function all y -values should be 5." As we continued, I asked if the relation $\{(-1, 0), (2, 4), (6, 1), (4, 1)\}$ was a function. Nancy said, "I would say no if I stick with my theory." (Interview1, 11/8)

Nancy demonstrated a great deal of difficulty working with relations presented in numerical forms. She was not clear whether all points in the set were to be treated as one or as individual ordered pairs. This issue was cleared up for her during one of the reflections. In addition, she had a flawed theory that for a relation to be a function, all the y -values should be the same. She asked if her approach to solving the problem was correct. I promised to discuss this and other concerns she might have after the interview. She mentioned, that in one of the practice problems in class, it was noted that if we had "three children and one mother, then each child had one mother," that would be a function. But if we had "one mother with three children, that would not be a function because there would be three different y -values." (Interview 1, 11/8)

Nancy felt strongly about the validity of her theory. In the interview she had sensed that her theory might not be right and wanted to know how to do problems. She offered an example from the class notes that seemed to be the instigating factor. In the online college algebra class, the introduction to functions was independent of the study of ordered pairs and relations as a set of ordered pairs. The language and examples used were illustrative of the concept of a function as mapping of set A to set B. Given a numerical relation, Nancy looked only at the y -values to

decide whether a relation was a function. By the final examination, Nancy was able to attempt similar problems with no difficulty.

Graphical. Relations represented in the form of a graph were another category of questions both in pretest and the interview. On the pretest Nancy missed the two graphical questions that asked to identify the graph of $y = |x|$ and $x = y^2$ as a function or non-function. I asked Nancy why she did not answer the two graphical questions. She admitted that she overlooked the two graphs and had no problem explaining whether the given graphs were functions or not. She added, “The graph of $y = |x|$ is a function because there is only one y -value for the x -values. The graph of $x = y^2$ is not a function because each x -value has two y -values.” (Interview 2, 11/18)

On the class quiz she was asked to “use the vertical line test to determine whether graph of $y = x^2$ represented y as a function of x .” Nancy’s chosen answer for this question was “ y is a function of x .”

I asked if the graphs presented in Figure 13 represented graphs of functions.

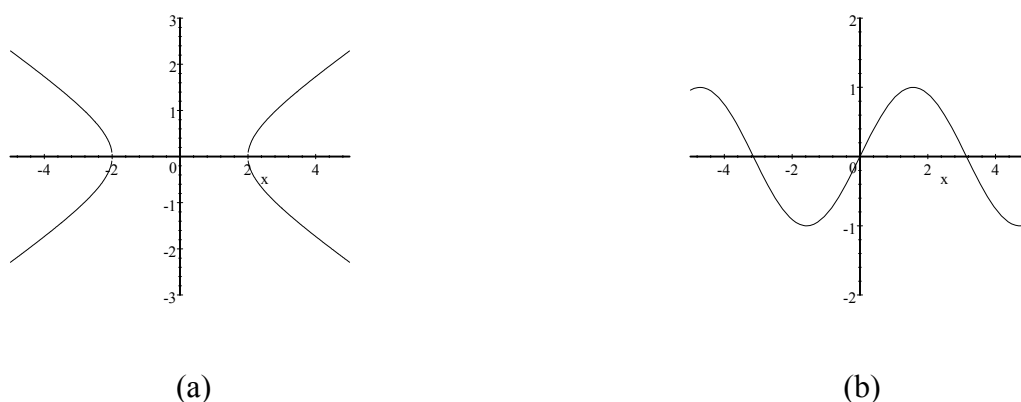


Figure 13. Recognition of graphs.

Without hesitation Nancy said, “My theory threw me off, which is telling me it might be very wrong. I am going with ‘no’ for part (a); in here each x has two y -values.” She continued, “I say ‘yes’ for part (b).”

Nancy was a quick and sharp learner. In her work with the graphical questions, she realized that her earlier theory was wrong. To verify that graphical representations were functions, she used the procedure “no x -value is associated with two different y -values.” However, she did not use the vertical line test. Nancy knew how to recognize functions in their graphical form.

Interpretation

Nancy’s responses in the area of interpretation fell into two categories: (a) algebraic to verbal and (b) graphical to verbal.

Algebraic to verbal. Before starting the unit on “functions and graphs,” Nancy had a limited knowledge of how to interpret algebraic representations of graphs. On the pretest, I asked a question (Appendix B) that involved interpretation of the slope and the intercepts. Nancy’s response to the following question is stated in Figure 14.

An appliance store estimated that 40 transistor radios would be sold per day during an upcoming season if the new radios were priced at \$100 each. At the price of \$150, the store will sell 25 units. The linear function that models this relationship is given by

$D = -0.30p + 70$. (a) Interpret the meaning of the slope and the y -intercept. (b) Identify and interpret the x -intercept.

I asked Nancy about her response and the fact that she (a) did not finish the interpretation of the y -intercept, (b) did not interpret the x -intercept, and (c) dropped the zero from the y -intercept in the function $D = -0.30p + 70$ when calculating the x -intercept.

$y\text{-intercept} = \text{number of radios to be sold} = 70$ $\text{Slope} = -\frac{3}{10} = \text{Change in product sold per price increase}$ $x\text{-intercept} = -\frac{7}{0.3} = -\frac{3}{10}x + 7 = 0$

Figure 14. Nancy's solution for the "transistor radio" question on the pretest.

She said, "At the time I did not think hard enough about how to interpret. As for the calculation of the x -intercept, my fault, I just dropped the zero unintentionally. Sorry." (Interview 2, 11/18)

On the class test Nancy was asked:

A company's weekly revenue in dollars is given by $R(x) = 2000x - 2x^2$, where x is the number of items produced during a week. (a) For what x is $R(x) > 0$? (b) On what interval is $R(x)$ increasing? Decreasing?

Nancy did not answer this question. I asked why she left it blank. She said, "I just did not know how to do it. We did not do anything like it in class" (Interview 2, 11/18). Determining the intervals of increasing and decreasing of functions was part of the online college algebra curriculum, and it was not covered in class. Yet students were expected to know about it since a question appeared on the test. In Nancy's case, her knowledge of the topics on functions and graphs were limited to what was discussed in class notes. Unlike Marsha, she would not tap into other resources in order to complement her existing knowledge of functions or to find solutions to test materials. Students in this course were given 24 hours to complete their tests. So I wondered why Nancy did not see the need to search beyond the course materials.

I posed the question:

The book value of a machine is \$4,500. It is estimated that after 9 years the value of machine will be \$900. If the linear function that determines this relationship is given by $V = -400t + 4,500$: (a) interpret the meaning of the slope and the y -intercept, and (b) identify and interpret the x -intercept.

Nancy's response was, "The slope is -400 and it is the devaluation over time." She continued, "The y -intercept is $4,500$ and it is the value of the machine initially." Subsequently she said, "The x -intercept is going to be... I am trying to remember. I just looked at it the other day. I am thinking. The x -intercept is where the graph crosses the x -axis, right?" After I affirmed her response she said, "If one of the values is zero, it [x -intercept] would have to be zero." I was unsure about her response, so I asked, "What has to be zero?" and she added, "The x -intercept has to be zero. I do not know. I am just guessing out of my head. I cannot remember how to do it." To draw a parallel between the x - and the y -intercept I asked, "How would you define the y -intercept?" She said, "The y -intercept is where it [the graph] crosses the y -axis." Then I reminded Nancy about how she had interpreted the y -intercept earlier. This hint did not help, and she insisted that the x -intercept was zero. Consequently, as she was not able to identify the x -intercept, the interpretation did not take place, and we moved on. (Interview 1, 11/8)

I posed the question:

A manufacturer of graphing calculators determined that 8,000 calculators per week would be sold at a price of \$75. At a price of \$70, it was determined that 10,000 calculators would be sold. The linear function that determines this relationship is given by $y = -400x + 38,000$. (a) Interpret the meaning of the slope of the graph of this linear

function. (b) Explain the meaning of the y -intercept. (c) Explain the meaning of the x -intercept.

Nancy responded, “Slope is -400 and it is the difference in [the] number of calculators sold per week because of the price change. The slope is negative because the sale goes down as the price goes up.” She said, “I am trying to think how to figure out where this $38,000$ came from because I am dumbstruck. Sometimes I know how to work out a problem, but it does not mean I always know exactly why it is what it is. I can plug in anything into a formula and figure it out, but it does not mean [that] I know why I did it.” She added, “ y -intercept is where the graph crosses the y -axis, and the value of x is zero at that point [y -intercept]. So the y -intercept is $38,000$.” Nancy continued, “At a price of $\$0.00$ the company will sell $38,000$ calculators.” To respond to the last part of the question she paused for a while then smiled and explained, “It [x -intercept] is a particular price at which zero calculators would be sold.” I asked, “What is that particular price?” She laughed and said, “I do not know what I am supposed to do.” I pointed out her earlier statement of “zero calculators would be sold” and asked what variable was zero? She said, “Zero is the y -value.” After a long pause she laughed and added, “That is it. I like to make things harder than they are. The x -intercept is $\$95$. If the price is $\$95$, then zero calculators would be sold.”

(Interview 1, 11/8) In the previous problem Nancy was not able to find and interpret the x -intercept, and that bothered her. She did not want to leave the problem. Once we worked a similar question, Nancy succeeded in identifying and interpreting the x -intercept.

Overall Nancy had knowledge of verbal interpretation of functions presented in algebraic form. If she was allowed ample time to work a problem, she would figure it out. She enjoyed the problems and took pride in finding ways to solving them.

Graphical to verbal. On the pretest, I posed the question (Appendix B), “The graph shows how the amount of gas in a tank of a car decreases as the car is driven. Find the slope of the line. Write a statement that states the meaning of the slope.” Nancy’s response is stated in Figure 15. She knew how to find the slope and interpret it accordingly.

$$m = \frac{6-13}{180-40} = -\frac{7}{140} = -\frac{1}{20}$$

The gas decreases $\frac{1}{20}$ of a gallon for every mile driven.

Figure 15. Nancy’s solution for the “amount of gas in a tank” question on the pretest.

I posed the question (Appendix D), “Use the graph of $y = h(x)$ to find $h(-2)$.” The four points $(-4, 1)$, $(-2, 4)$, $(3, -1)$, and $(7, 1)$ were marked on the given graph. Nancy said, “ $h(-2)$ is at point $(-2, 4)$.” I asked what variable she would substitute -2 for to find $h(-2)$.” After a pause, she wrote $(-2, 4)$. In astonishment I asked, “In $h(-2)$ what is -2 ?” Nancy added, “So y would be -2 .” I pointed out that the question asked for $h(-2)$. After a very long pause, Nancy figured it out: “That $[-2]$ is going to be your x -value.” She added, “I am looking for the point on the graph. It is four.” (Interview 1, 11/8) Like Marsha and Ivy, Nancy also had difficulty extracting from an ordered pair (X, Y) the corresponding values of x and $h(x)$ when working from a graph.

The question continued and asked, “On what interval is $h(x)$ increasing or decreasing?” Nancy answered, “From point $(-4, 1)$ to point $(-2, 4)$ it is increasing.” In response to my question about whether what she had written for me was an interval, she concluded, “I have not been asked like this, so I do not know.” (Interview 1, 11/8) Intervals of increase and decrease were troublesome for Nancy, as she did not know how to write those intervals. This topic was

part of the college algebra curriculum, but was not covered in online college algebra course.

Students were expected to know the topic because a question was asked of them in this regard.

The latter part of the question (Appendix D) asked Nancy to write the domain and range of the graph of $h(x)$. She responded, “The domain is going to be $[-4, 1]$ and the range is $[-1, 4]$.” She showed knowledge of domain and range of functions presented in graphical form.

I posed a question (Appendix D) that presented her with two graphs of the more developed – $f(x)$ and less developed – $g(x)$ regions of the world, that were plotted against year. In response to my question: “What does the function $f + g$ represent?” she said, “It represents the population of the world’s most developed regions and the world’s less developed regions in year x .” She read the problem aloud: “Use the graph to estimate $(f + g)(2000)$.” Nancy explained, “I will find $f(2000)$ and $g(2000)$ then I add the two amounts together.” After a short pause she continued, “For f , I get 1.6 million, and for g , I get 5.5 millions. And $(f + g)(2000)$ is 7.1 million.” (Interview 1, 11/8) Like Marsha, Nancy knew how to operate effectively in situations that required her to interpret real-world examples when graphs of time were depicted against population.

Subsequently, I presented Nancy with a line segment AB that extended from $(2, 2)$ to $(6, 5)$ and asked if the points $(-1, 0)$ and $(5, 4)$ were on that line (Appendix D). She responded, “The point $(-1, 0)$ is not on the line but point $(5, 4)$ is on the line.” I asked why $(-1, 0)$ is not on the line, and Nancy said, “Because the line stops at $(2, 2)$.” She explained further that “the line AB goes from $(2, 2)$ to $(5, 6)$.” I asked if the line AB contained only the points $(2, 2)$ and $(5, 6)$ or if it contained all the infinite points along that path. Nancy remarked that if the line was continuous it would have included the point $(-1, 0)$, but “it was not because there were no arrows showing that it was a continuous line.” Nancy’s written response to this question is stated in

Figure 16. Nancy added, “You drew only a segment of a line and that threw me off.” The question that followed consisted of the equation of parabola $y = (x - 4)^2 + 1$ along with its graph. Nancy was asked whether the point $(0, 17)$ was on the parabola. Without hesitation, she substituted the point into the equation, and when she saw that it satisfied the equation, she said, “Yes.” (Interview 1, 11/8)

For ‘segment’ AB point $(-1, 0)$ not on it.
 For ‘line’ AB point $(-1, 0)$ is on it. Point $(5, 4)$ would be on it [AB] either way.

Figure 16. Nancy’s written response from the first interview.

For Nancy the fact that the line segment AB did not have arrows disqualified it as a continuous line; hence, it did not include the point that did not appear to be physically on the line segment. On the other hand, if AB was to be considered as a line then it would contain the point $(-1, 0)$ which was not physically on the line. She determined if a given point was or was not on a line based on “eye measurement.” Like Marsha and Ivy, she had no difficulty verifying that a given point was on a parabola although visually it was not.

Translation

I have organized Nancy’s responses into five categories: (a) algebraic to graphical, (b) verbal to algebraic, (c) numerical to graphical, (d) symbolic to graphical, and (e) graphical to algebraic.

Algebraic to graphical. On the pretest, Nancy translated $f(x) = x^2 - 2$ to its graphical representation by plotting the points: $(1, -1)$, $(2, 2)$, $(3, 7)$, and $(4, 14)$. She connected the points

and arrived at a straight line with a positive slope. I inquired about her response and she said, “It never occurred to me that the graph was anything but a straight line” (Interview 2, 11/18).

One graphing problem appeared on the class quiz for this chapter. The question stated, “Use the graph of $f(x) = x^2$ to sketch the graph of $g(x) = x^2 - 1$.” The quiz was multiple-choice, and Nancy chose the parabola that was shifted one unit down vertically.

A few days prior to the chapter test on functions and graphs, I posed the question, “Have you come across any function that you were not able to translate from algebraic to graphical, verbal to algebraic, numerical to graphical, etcetera?” Nancy explained, “I have a hard time with remembering translations of graphs [algebraic to graphical], when to shift up, left, or right. That confuses me a little bit as does reflections of graphs. I have a hard time remembering what each formula means, something I do not easily comprehend.” (Reflection 2, 10/24)

The chapter test contained two translations from algebraic to graphical. One involved a parabola that was shifted one unit vertically upward. Nancy translated that function without plotting points. The second one asked the students to graph $f(x) = -2|x - 2| + 4$. Nancy was clearly employing the translation techniques, as she was not plotting points at all. She missed this problem altogether as she (a) plotted a parabola instead of the absolute value function, (b) did not reflect the graph, and (c) did not stretch the graph.

I asked Nancy to graph the function $f(x) = |x + 2|$. She proceeded with a parabola and said, “I would graph this at a negative two.” I mentioned that she had moved the graph where it was supposed to be but that it was the wrong shape. In response to my question of how she would graph $f(x) = x^2 - 2$, she said, “I am thrown off because in class we just went with this shape and we just moved it accordingly.” Nancy continued, “I only remember doing [a] parabola.” (Interview 1, 11/8) Nancy was not able to make a connection between the graphical

representation of a parabola and its algebraic form. Further, she was not familiar with graphs other than the parabola.

Nancy was thirsty for knowledge, and when it was given, she learned it eagerly. In the beginning of the study, she did not know about the graphing techniques, she was translating from algebraic to graphical by way of plotting points. Unlike Marsha and Ivy, Nancy learned the graphing techniques and put them into practice. She seemed only to be comfortable with square functions because this type of graph was dealt with in several examples in class. By the end of the course, she demonstrated a greater knowledge of graphing techniques. For instance, on the final examination Nancy applied the graphing techniques and translated the functions $y = (x - 1)^3$ and $y = |x - 1| - 2$ to their graphical representations.

Verbal to algebraic. I asked Nancy to find the linear function that determined the relationship between the cost to build a house and the number of square feet of floor space. It was estimated that the cost to build a new home was \$25,000 plus \$80 for each square foot of floor space. Nancy's response was: " $H = 25,000 + 80(\text{per square feet})$." (Pretest, 10/14) A similar question appeared on the class test and Nancy showed competence in translating it to algebraic form.

In the interview, I posed a question that required Nancy to translate from verbal to algebraic by finding the function that described the situation:

A police department believes that the number of serious crimes, which occur each month, is dependent upon the number of police officers for preventive purposes. If 140 police officers are assigned for preventive patrol, there will be no crime. However, it is expected that assignment of no police officers would result in 1,500 serious crimes per month. Find a linear function demonstrating that relationship.

After a very short pause, Nancy read the problem aloud and summarized the information into two ordered pairs (140, 0) and (0, 1500). She used the ordered pairs to find the slope and then used the point-slope formula to find the linear function. Her solution is stated in Figure 17.

(Interview 1, 11/8)

$$s = \frac{1500 - 0}{0 - 140} = -\frac{1500}{140} = -\frac{75}{7}$$

$$y - 0 = -\frac{75}{7}(x - 140)$$

$$y = -\frac{75}{7}x + \frac{10500}{7}$$

$$y = -\frac{75}{7}x + 1500$$

Figure 17. Nancy's written response to the "police department" question on the interview.

Nancy was able to distinguish between the dependent and the independent variables, and unlike Marsha, she used the point-slope formula to find the linear function that demonstrated the relationship described in the problem.

I posed another question that described how the profit of a doll company was dependent on the number of dolls sold:

A doll company had a profit of \$50,000 per year when it had sold 1,600 dolls. When the sales of dolls increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P , is a linear function of the number of dolls sold. Find the profit function.

Nancy said, "In here the number of dolls is going to be x , and y will be the profit." She then wrote the two ordered pairs as (1600, 50 000) and (1800, 60 000). She explained further "I am

going to find my slope. Next, I will put this slope in point-slope formula to find the function.”

(Interview 1, 11/8) Figure 18 shows Nancy’s solution for this question. Nancy used an approach consistent with her work on other problems to find the linear function. A similar question appeared on the final examination, and Nancy was able to describe the situation in algebraic form.

$$\begin{aligned} \text{slope} &= \frac{60,000 - 50,000}{1800 - 1600} = \frac{10,000}{200} = 50 \\ y - 50000 &= 50(x - 1600) \\ y - 50000 &= 50x - 80000 \\ y &= 50x - 30000 \end{aligned}$$

Figure 18. Nancy’s written response to the “doll company” question on the interview.

Numerical to graphical. On the pretest Nancy was asked to graph the function represented by:

x	4	1	0
y	-1	2	3

She plotted the points and obtained a straight line with negative slope. A similar question was asked of her during the interview, and she graphed the linear function. Nancy knew how to translate from a numerical to a graphical representation of a function and showed no hesitation in working out problems in this area.

Symbolic to graphical. I asked Nancy to graph the function represented by $f(2) = -2$, $f(0) = -3$, and $f(4) = -1$. Nancy plotted the points $(2, -2)$, $(0, -3)$, and $(4, -1)$ and obtained a straight line with positive slope. (Interview 1, 11/8) Nancy demonstrated knowledge of translation of functions from symbolic to graphical.

Graphical to algebraic. I presented Nancy with an absolute value function (Appendix D) that was both shifted two units to the left and reflected along the x -axis. She said, “Let me think.” After a long pause she added, “Might take me a minute. I do not know if it is going to be plus or minus. Take the absolute value of x minus two, and it is going to be $|x - 2|$...think, think. It has to be positive.” After some deep thinking Nancy wrote, “ $g(x) = -|x + 2|$.” (Interview 1, 11/8) Nancy demonstrated difficulty translating an absolute value function from algebraic to graphical representation. Yet translation of a similar graph to algebraic form was comparatively easier for her.

Next, she worked with a square function (Appendix D) that was translated two units to the right. She looked at the graph and said, “ $g(x) = (x^2 - 2)$.” I asked, “What kind of translation is this?” She asserted that the original graph had moved to the right. I continued, “If this was a vertical translation of two units upward, what would the function have looked like?” She responded, “It would have been $x^2 + 2$.” I asked “if it was translated two units down how would the function look?” Nancy added, “ $x^2 - 2$, no parenthesis.” I asked if she thought the parentheses made the difference and she said, “It [the parentheses] did make a difference the way she taught us.” Finally, she said, “Well, it is my understanding that if $x^2 - 2$ is in parentheses, then we are talking about horizontal translation.” It is curious how Nancy struggled. Very small issues that teachers take for granted in a face-to-face classroom can cause a great deal of confusion for an online student. This showed that teaching online courses is more than posting

class notes to the Web. The notes should use simple sentence structure and an organization that contributes to the acquisition and application of knowledge.

Subsequently, I presented Nancy with the graphs in Figure 19 and explained that the middle graph was $y = x^2$ and asked her to find the equations of the inner and outer graphs. Nancy's response was, "I do not know [shrinking and stretching property] this. We need to talk about this after the interview." Nancy had no knowledge of this topic as it was not covered in the course.

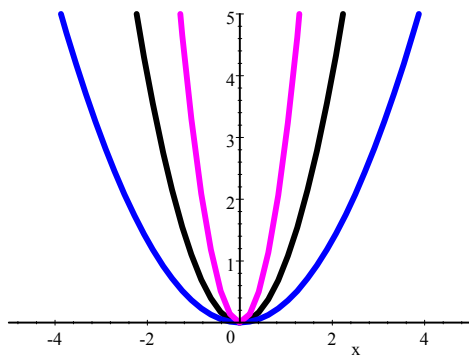


Figure 19. $y = x^2$.

The next two questions involved the vertical (upward) translation of a cube function and the horizontal (left) translation of a square root function. When Nancy saw the first graph, without hesitation, she wrote: $g(x) = x^3 + 2$. This type of problem was covered in class notes, and as a result, she was able to relate to it. In doing the next translation she paused for a while, then wrote: $g(x) = \sqrt{x} + 2$. I asked if 2 was under the radical or not, and she said, "I would say yes because it is talking about the x -value which has been moved. I do not know." I asked if she had come across the square root function in class. Nancy explained, "This [square root function] was in our notes and I went over it very briefly, but we never did anything [translations] on it."

Finally, I presented Nancy with graphs of $x = y^2$ and $y = x^3$ and asked her to sketch the graph of $y = f(-x)$. She thought for a short while then said, “I am translating it over the y -axis.” I asked if she remembered that this process was called *reflection*. She responded, “You do not hear people saying it and even though you read it, you do not necessarily pick up the names and the words.” (Interview 1, 11/8) Nancy demonstrated no difficulty reflecting graphical representation of functions.

Summary

Nancy was dedicated, confident, and a sharp thinker. She was able to recall and perform skills that were covered in the online class; otherwise, she learned them during the interview. She would never give up on a problem. She could sit and think about it for a very long time until a solution was found. She always paused after she had given her responses and did not move to another question until she had exhausted all possibilities. Many times she changed her response as a result of this thinking. If she did not know a problem, she was eager to learn about it. She was the only participant who stayed after the interview to ask questions that had puzzled her during the question and answer session. Nancy valued interaction and used it as a supplemental source for finding answers to her questions. It was interesting to watch her work the problems; she truly enjoyed the process. She was a reflective student, as she was able to assess her own mathematical abilities and search for ways to get help. Throughout the study she showed improvement in all three areas of functions.

The Case of Andrew

Andrew scored 33% and 83%, respectively, on the chapter (functions and graphs) quiz and test. He scored 56% on the comprehensive final examination for the online college algebra course that counted 40% of the grade. He completed the online college algebra course for the

second time and made a C in the course, an improvement from his grade of D the previous quarter.

Recognition

Andrew's responses in the area of recognition fell into three categories: (a) algebraic, (b) numerical, and (c) graphical.

Algebraic. On the pretest Andrew's response to the question, "Do $y = -1$ and $x = 6$ represent functions?" was " $y = -1$ is a horizontal line, and it is a function. Also, $x = 6$ is a vertical line and it is a function." During the interview, similar questions reappeared for which he had to determine if $y = 2$ and $x = 5$ represented functions. His response was " $y = 2$ is a function." I asked, "How do you know it [$y = 2$] is a function?" He replied, " $y = 2$ is a horizontal line and I do a vertical line test." I applauded him for his response, but when I asked him how he would use the vertical line test, he said, "I just referenced my notes and saw the vertical line test." He continued, "I believe for every x -value you should be getting a y -value." Then I asked him to explain to me the reason $y = 2$ passes the vertical line test. He mentioned, "From what I remembered when I was taking the functions, vertical and horizontal lines are functions." When I inquired about $x = 5$ he commented, " $x = 5$ is a vertical line and it passes the vertical line test." (Interview 1, 11/13)

Relations of the form $y = k$ and $x = k$ where k is some fixed constant were troublesome for Andrew. Unlike Ivy, he recognized $y = k$ as a horizontal line and $x = k$ as a vertical line. Although he cited the vertical line test when asked whether $y = k$ represented a function, but he did not know how to use it. Like Marsha, he somehow classified these two types of relations together and had a misconception that both ought to be functions. His responses were consistent. During the pretest and interview, he regarded $x = k$ and $y = k$ as functions.

During the interview, when I asked if $x = y^2 + 4$ was a function, Andrew said it was not a function because “we get two values for y .” His response to the question “Is $x^2 + y^2 = 25$ a function?” was “Yes.” I asked if he recognized this equation. He said, “The sum of squares?” Next I asked, “Why do you think this is a function?” He explained, “I remember it being a function because if you take the square root of it all, you get $x + y = 5$.” I was not sure if I understood him well, so I repeated his response, and he insisted, “It is a function.”

(Interview 1, 11/13)

Algebraic relations in terms of x and y presented another category of difficulty for Andrew. He had no difficulty identifying $x = y^2 + 4$ as a non-function. However, $x^2 + y^2 = 25$ was problematic for him. A similar question appeared on the final examination, and he experienced difficulties with such relations. He was not able to draw a comparison between the two relations. He also demonstrated some difficulties with basic algebraic concepts.

I asked Andrew whether the following relation represented a function:

$$\begin{cases} x \geq 0, y = x + 2 \\ x < 0, y = 1 - 2x \end{cases}$$

He responded, “Yes.” When I asked him to expand on his answer he said, “I put 1 for x in the first equation and y will be 3. Then I put -1 for x in the second equation and y would equal 3.” Then he had two pairs of ordered pairs $(1, 3)$ and $(-1, 3)$ written in a set. (Interview 1, 11/13)

Andrew knew how to recognize an algebraic relation written as piecewise.

Numerical. On the pretest in response to the question: “Which of the following represent a function $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$, or (b) $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$?” Andrew said, “Unsure, I am confused on what to do.”

During the first reflection I asked Andrew “whether he had come across a mathematical relation that he was not able to recognize as a function/non-function.” His response was, “I had trouble with the second question on pretest that asked whether the given relations $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$ and $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$ were functions.” Then he wrote in bold “I do not know how to produce the answers for those questions. I do not know what kind of formula to use.” (Reflection 1, 10/19) Initially Andrew was looking for a formula to help him recognize functions presented in numerical form.

Next, I asked if the relations $\{(1, 2), (1, 5), (2, 10), (-3, 5)\}$ and $\{(-1, 0), (2, 4), (6, 1), (4, 1)\}$ represented functions. In response to the first part he said, “No, since two of the x -values are the same. It does not pass the vertical line test.” He then continued, “The second relation is a function.” (Interview 1, 11/13) A similar question appeared on the final examination, and he demonstrated knowledge of such a question.

In the beginning of the study Andrew had a limited knowledge of the relations presented in numerical form. In fact he was looking for a formula to use. Later, he learned how to distinguish between numerical representations of functions and non-functions. In doing so he used the two methods taught in class: the vertical line test, and the procedure for insuring that none of the x -values were the same.

Graphical. Relations represented in the form of a graph were another category of questions both on pretest and in the interview. On the pretest, Andrew was asked whether the graphs of $y = |x|$ and $x = y^2$ were representations of the graphs of functions. His response was, “Yes” to both of these graphs.

On the class quiz, he was asked to use the vertical line test to determine whether graph of $y = x^2$ represented y as a function of x . Andrew's response to this multiple-choice question was, "y is not a function of x."

I asked if the graphs presented in Figure 20 represented graphs of functions. Andrew's written response for Figures 20, part (a) and (b) were, "Yes it is a function," and "Unsure how to figure out if it is a function" respectively.

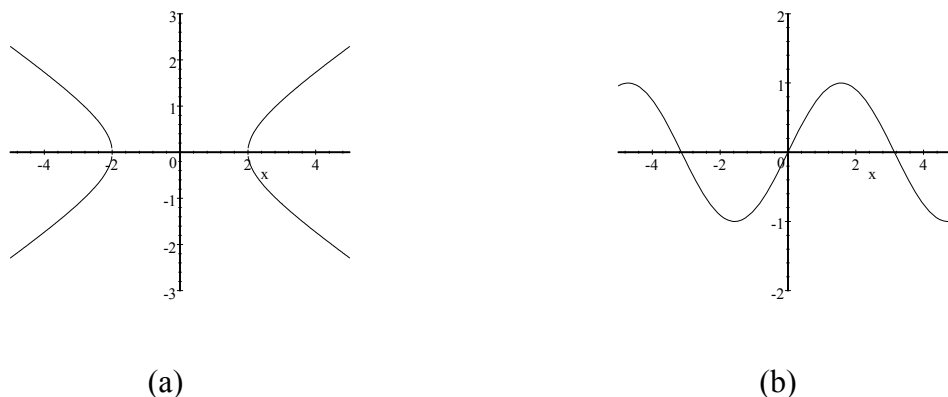


Figure 20. Recognition of graphs.

I asked, "How did you decide on this?" Andrew explained, "Because for every value of x ..." then he paused for a long time. I broke the silence and asked him to choose an x on the graph of the first function (Figure 20, part (a)) and decide which y corresponded to that x -value. He responded, "I am not quite sure what that particular x corresponds to." I asked him to choose 2.5 for the value of x and let me know what y -value(s) correspond to this x . He paused for a while then laughingly said, "I am not sure." I pointed out to him that the $x = 2.5$ corresponded to $y = 0.8$ and $y = -0.8$. This would give two pairs of points $(2.5, 0.8)$ and $(2.5, -0.8)$ in a set. I then asked, "Is the set presented by $\{(2.5, 0.8), (2.5, -0.8)\}$ a function?" Without hesitation he

asserted, “It [the given relation] is a function.” I reminded him of the earlier response he had given me when we were discussing relations presented in numerical form. He said it [the relation] is not a function “since two of the x -values are the same.” This discussion made Andrew quiet, so I asked, “What do you think about this graph? Is it a function?” He added, “Oh, then I think it is not a function.” Andrew’s response for the graph in Figure 20, part (b) was, “It is not a function.” I asked, “Have you come across an easy way of finding whether a given graph is the graph of a function or not?” He said, “No, but I would like to know how.”

(Interview 1, 11/13)

Andrew knew about the vertical line test, but it never occurred to him that he could use it in this case. His knowledge of the two methods—vertical line test and the procedure that “none of the x -values should be the same”—seemed to be very shallow. He had difficulty recognizing functions presented in graphical forms. I inquired about this and he said, “I wish more examples were covered in lectures, when I need to see more examples, or do not understand one, I go on the Internet and search” (Interview 2, 11/25).

Interpretation

I have organized Andrew’s responses in the area of interpretation into two categories: (a) algebraic to verbal and (b) graphical to verbal.

Algebraic to verbal. On the pretest, I asked a question that involved interpretation of the slope and the intercepts. Andrew’s response to the following question is stated in Figure 21.

An appliance store estimated that 40 transistor radios would be sold per day during an upcoming season if the new radios were priced at \$100 each. At the price of \$150, the store will sell 25 units. The linear function that models this relationship is given by

$D = -0.30p + 70$. (a) Interpret the meaning of the slope and the y -intercept. (b) Identify and interpret the x -intercept.

When I inquired about his responses Andrew said, “All I know is that -0.30 reflects a decrease in units, but I am not sure how to expand on that. As to the y -intercept being 70 , the only logical answer I can come up with is, that it is the increase in price.

x -intercept is the price.

y -intercept is 70 which is the increase in price.

slope is -0.30 and it is decrease in units.

Figure 21. Andrew’s solution for the “transistor radio” question on the pretest.

I say this, because in this problem, we have a negative slope, therefore, as the price increases, the unit sold decreases. Also, I do not know how to find the x -intercept.” (Interview 2, 11/25)

Andrew did not know the difference between the y -intercept and the slope.

On the class test Andrew was asked:

A company’s weekly revenue in dollars is given by $R(x) = 2000x - 2x^2$, where x is the number of items produced during a week. (a) For what x is $R(x) > 0$? (b) On what interval is $R(x)$ increasing? Decreasing?

Andrew did not answer that question. I asked him why he omitted it. He said, “I have not seen anything like this before” (Interview 2, 11/25).

I posed the question:

The book value of a machine is \$4,500. It is estimated that after 9 years the value of the machine will be \$900. If the linear function that determines this relationship is given by

- $V = -400t + 4500$: (a) interpret the meaning of the slope and the y -intercept, and
 (b) identify and interpret the x -intercept.

Andrew's written response to this question is stated in Figure 22. During the first interview, I inquired about Andrew's response and drew his attention to the fact that he had interpreted the slope and the y -intercept as basically the same.

Slope is -400 and it is the decrease.
 y -intercept is 4500 it is increase in price.
 x -intercept is the price.

Figure 22. Andrew's solution for the "book value of machine" question on the interview.

He said, "Yes I interpreted them the same." I continued, "If they are the same, then why do they have different names?" He laughed and after a short pause said, "I do not know." He explained, "I think the increase or the decrease should be the slope, and a point within that increase or the decrease is the y -intercept." I asked him to explain to me the statement "the y -intercept of a graph is 2" After a long pause he said, "On the y -axis from the origin you move up two." When I asked him for the coordinate of the y -intercept he said, "I do not know." I talked him through the process of finding the y -intercept and then asked, "How do you find the x -intercept?" Andrew commented, "The x -intercept would be zero?" I pointed out that he needs to find the x -intercept, and asked, "So how could it be zero?" He laughed and said, "I am not sure." Nancy, too, thought the x -intercept was zero. I wondered: Why? Is it because students do not connect the concept image and concept definition of intercepts?

Later, I posed the question:

A manufacturer of graphing calculators determined that 8,000 calculators per week would be sold at a price of \$75. At a price of \$70, it was determined that 10,000 calculators would be sold. The linear function that determines this relationship is given by $y = -400x + 38,000$. (a) Interpret the meaning of the slope of the graph of this linear function. (b) Explain the meaning of the y -intercept. (c) Explain the meaning of the x -intercept.

After Andrew read the problem he paused for a while, so I asked, “In this question what does y stand for?” He responded, “The y is the price?” Surprised I said, “Is y the price?” He asserted, “Yes.” I continued, “What does the x stand for?” He responded, “The number of calculators.” I asked him to address different parts of the question. In response to part (a) of the question, he said, “That is the decrease in price...I mean, the decrease in the total price depending on the price per calculators.” He added, “The y -intercept is 38 000, and it is the increase in price.” He continued the problem saying, “The x -intercept is the price.”

Andrew had a flawed knowledge of the slope and y -intercept. He treated them the same, and thought of slope as “the increase and decrease” and y -intercept as “a point within that increase or decrease.” In addition he had difficulty identifying and interpreting the x -intercept. He could not distinguish between the dependent and independent variables. In all, he did not understand how to interpret algebraic representation of functions.

Graphical to verbal. On the pretest I posed the question (Appendix B), “The graph shows how the amount of gas in a tank of a car decreases as the car is driven. Find the slope of the line. Write a statement that states the meaning of the slope.” Andrew’s response is stated in Figure 23. I inquired about his response. He said, “This is the best I can do. I do not know how

else to describe it” (Interview 2, 11/25). Andrew knew about the relationship between gallons of gas and miles driven, yet he was unable to correctly interpret the slope.

$\text{Slope} = -\frac{7}{140}$ <p>Car uses 7 gallons of gas to reach 140 miles.</p>
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Figure 23. Andrew’s solution for the “amount of gas in a tank” question on the pretest.

I posed the question (Appendix D), “Use the graph of $y = h(x)$ to find $h(-2)$.” The four points $(-4, 1)$, $(-2, 4)$, $(3, -1)$, and $(7, 1)$ were marked on the given graph. Andrew said, “I am confused and do not know where to start.” I prompted him by asking, “To find $f(-2)$ given $f(x) = x + 6$ what would you do?” His response was, “I do not know where to go from there. Do I substitute -2 for x ?” I gave affirmation of his answer and asked him to find $h(-2)$. He asked, “Would y be $h(-2)$?” and when I encouraged him for his response he said, “ $h(-2) = 4$.” Another part of this question asked, “What is $h(7)$?” He said, “One.” I pointed out that he was not confused after all, and he remarked, “No, not when you explained it like this!” Andrew experienced some difficulty extracting the corresponding values of x and $h(x)$ from an ordered pair (X, Y) when working from a graph.

The question continued and asked, “On what interval is $h(x)$ increasing or decreasing?” Andrew answered, “Between $(-4, 1)$, $(-2, 4)$ and $(3, -1)$, $(7, 1)$.” I reminded him what he had said about interval notation, and asked if he knew how to write this in interval notation. He paused for a short while and said, “I really do not know how.” The latter part of the question (Appendix D) asked Andrew to find the domain and range of the graph of $h(x)$, he responded,

“Oh no. The domain...I am not quite sure how to get domain.” Then, I asked him whether domain had to do with the x or the y variable. After a pause he concluded, “I do not know that. I read about it, but when it comes to finding the domain I have difficulty.” (Interview 1, 11/13)

Andrew was shocked when he saw the question about domain. He did not know how to write intervals of increase and decrease and the domain and range of a function.

I posed a question (Appendix D) that presented him with two graphs of the more developed – $f(x)$, and less developed – $g(x)$ regions of the world that were plotted against year. In response to my question “What does the function $f + g$ represent?” He said, “Population of both regions.” He read the problem aloud: “Use the graph to estimate $(f + g)(2000)$.” After a short pause he said, “It would be the world’s population in 2000.” He continued, “It [population of the world in 2000] is ten billion times 2000.” In response to my inquiry of “why do you multiply the numbers?” he said, “Because it says use the graph to estimate $f + g$ times 2000.” (Interview 1, 11/13)

Andrew lacked knowledge of the basic concepts of functions. Instead of thinking of $(f + g)(x)$ as $f(x) + g(x)$, he said $(f + g)(x)$ was the product of $(f + g)$ and x . I was surprised by his response.

Next, I presented Andrew with a line segment AB that extended from (2, 2) to (6, 5) and asked if the points (–1, 0) and (5, 4) were on that line (Appendix D). Andrew’s response was, “(–1, 0) is not and (5, 4) is on the line.” He marked the first point on the x -axis and the second on the graph. The question that followed consisted of the equation of parabola $y = (x - 4)^2 + 1$ along with its graph. He was asked whether the point (0, 17) was on the parabola. Without hesitation he substituted the point into the equation of the parabola, and when he saw that it satisfied the equation, he said, “Yes.” (Interview 1, 11/13)

Like Marsha, Ivy, and Nancy, Andrew, too, only noticed the point that was on the line segment. Like other participants, Andrew showed no difficulty verifying the point that did not appear to be on the parabola.

Translation

I have organized Andrew's responses in this area into five categories: (a) algebraic to graphical, (b) verbal to algebraic, (c) numerical to graphical, (d) symbolic to graphical, and (e) graphical to algebraic.

Algebraic to graphical. On the pretest, Andrew was asked to graph the function $y = x^2 - 2$, but he chose not to answer the question. I asked him the reason he omitted that question on the pretest. His response was, "I did not know how to graph anything but a straight line" (Interview 2, 11/25).

One graphing problem appeared on the only class quiz for the chapter. It stated: "Use the graph of $f(x) = x^2$ to sketch the graph of $g(x) = x^2 - 1$." The quiz was multiple-choice, and Andrew chose for his answer the parabola that was shifted one unit down vertically. The chapter test contained two translations from algebraic to graphical, and Andrew attempted those problems by plotting points. The first one asked him to graph $f(x) = -2|x - 2| + 4$. For this question, he substituted 0, 1, 2, 3, and 4 for x , but due to some arithmetic errors, all of the values of $f(x)$ were positive, and the absolute value function was not reflected. The second problem asked him to graph $f(x) = x^2 + 1$. He substituted 0, 1, 2, and 3 for x and plotted half of a parabola that was shifted one unit vertically upward.

When I asked Andrew to graph $f(x) = |x + 2|$, he said, "I could graph this by plugging in values for x ." I asked if he knew about the graphing techniques. His response was, "Oh, no." I

continued, “Can you tell by looking at a graph what the shape should roughly look like?” He added, “Ah, by looking at the function? Not at all.” I asked, “So how would you graph the given function?” He explained, “I would just make a table and plug in numbers for x . Zero, one, and two.” (Interview 1, 11/13)

On the final examination there were two graphing questions. The first one was $f(x) = (x - 1)^3$, which he missed because he connected the points and plotted a straight line with positive slope. The second one was an absolute value function $f(x) = |x - 1| - 2$, and he graphed this function by plotting points; the graph he plotted was an absolute value function that was translated 1 unit to the right and 2 units vertically down.

Andrew only knew how to translate functions from an algebraic form to a graphical representation by plotting points. But he had difficulty plotting points and arriving at the desired function, because he did not use negative values for x . As a result, he was only finding part of the graph.

Verbal to algebraic. I asked Andrew to find the linear function that determined the relationship between the cost to build a house and the number of square feet of floor space. It was estimated that the cost to build a new home was \$25,000 plus \$80 for each square foot of floor space. Andrew’s response was: “ $y = 25,000 + 80x$ ” (Pretest, 10/23). On the class test, Andrew responded to a similar question and demonstrated no difficulty.

In the interview, I posed a question that required Andrew to translate from verbal to algebraic by finding the function that described the situation:

A police department believes that the number of serious crimes, which occur each month, is dependent upon the number of police officers for preventive purposes. If 140 police officers are assigned for preventive patrol, there will be no crime. However, it is expected

that assignment of no police officers would result in 1,500 serious crimes per month. Find a linear function demonstrating that relationship.

After a long pause Andrew said, “I do not know.” He continued, “The number of crimes depends on the number of police officers, but I am not sure how to write the linear function for that.” He stated that he was not good at word problems. I continued to encourage him to persevere and try harder. He found two pairs of points, (0, 1500), (140, 0) and added, “I would have to find the difference between the two.” “Difference between which two?” I asked. He explained “between the police officers and the crimes.” I asked, “What would that give you? Andrew laughed and concluded, “I do not know.” (Interview 1, 11/13) Andrew was close to finding the slope, but was unable to do so.

We worked the next question that described how the profit of a doll company was dependent on the number of dolls sold. Andrew’s solution is stated in Figure 24.

A doll company had a profit of \$50,000 per year when it had sold 1,600 dolls. When the sales of dolls increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P , is a linear function of the number of dolls sold. Find the profit function.

In doing this problem Andrew did not say, “I do not know,” or “I am not sure.”

<p>(1600, 50000), (1800, 60000)</p> $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{60000 - 50000}{1800 - 1600} = \frac{10000}{200} = 50$ $y - 50000 = 50(x - 1600)$ $y = 50x - 30000$

Figure 24. Andrew’s solution for the “doll company” problem on the interview.

He found the correct ordered pairs, and when I asked about the linear equation he responded it is “ $y = mx + b$.” My next question was, “What is the slope in this problem?” He answered, “The slope would be 200.” He added, “By selling 200 more dolls, it made \$10,000 more.” I asked for the slope formula and he was able to recall it. He proceeded to find the slope, and used the point-slope formula to write the linear function.

Andrew seemed to have low self-confidence in his mathematical abilities to translate verbal representations of functions to algebraic forms. As he worked through the verbal representations of functions, his understanding seemed to improve, and at times, he even surprised himself, as he was not expecting to be able to handle such problems. On the final examination he did not attempt any of the word problems.

Numerical to graphical. On the pretest Andrew was asked to graph the function represented by:

x	4	1	0
y	-1	2	3

He plotted the points and obtained a straight line with negative slope. A similar question was asked of him during the interview and he graphed the linear function. Translating from a numerical to a graphical representation was an easy task for Andrew.

Symbolic to graphical. I asked Andrew to graph the function represented by $f(2) = -2$, $f(0) = -3$, and $f(4) = -1$. He was not sure about this question, so I asked him what the values of x and y were in $f(2) = -2$. He responded, “ $x = -2$ and $y = 2$.” I explained to him that his answer was reversed, and asked him to write $f(2) = -2$ as an ordered pair. He said, “The ordered pair is $(2, -2)$,” and added that there were three points in this case, namely, $(2, -2)$,

$(0, -3)$, and $(4, -1)$. He plotted the points and obtained a straight line. At first he had difficulty writing $f(2) = -2$ as an ordered pair. But later when he figured this out, the task of translating from symbolic to graphical was relatively easy for him.

Graphical to algebraic. I presented Andrew with an absolute value function (Appendix D) that was both shifted two units to the left and reflected along the x -axis. I asked him to find the algebraic form of the function that was translated. After a long pause he said, “ $g(x) = g(-2)$?” I asked, “What kind of translation has taken place in this problem?” He mentioned, “You moved it [$g(x)$] over by -2 .” He questioned himself “would $g(x) = -2$?” and answered, “No that is not right.” He paused and then said, “Not sure.” He asked if I could explain this concept to him after the interview, and we skipped this category of questions. He was not able to deal with functions that were translated vertically and horizontally, and shrunk or stretched. Finally, I presented Andrew with graphs of $x = y^2$ and $y = x^3$ and asked him to sketch the graph of $y = f(-x)$. He translated both graphs over the y -axis. (Interview 1, 11/13)

Andrew did not know how to translate functions from graphical to algebraic representations. The only translation that he had solid knowledge of was the reflection task. He was eager to learn about the graphing techniques, and I promised to discuss it with him.

Summary

Andrew needed a lot of help and encouragement in formulating most of the questions, and lacked self-confidence. He had developed some misunderstandings about a number of functional concepts. He seemed weak in his knowledge of mathematical concepts and the function concept in particular. He understood procedures such as the “vertical line test,” but lacked an understanding of their application. By probing his thinking, I was able to identify how he thought about functions. At times, he surprised himself with his responses. Throughout the

course he showed some improvement in the areas of recognition and interpretation. In contrast, translation of functions remained problematic for Andrew. His understanding of verbal representation of functions was weak.

Situating the Study

The purpose of this section is to situate the present study within the broader empirical literature on students' knowledge of functions and their difficulties with this notion.

Comparisons and contrasts will be drawn between the results of the prior literature and the results of the present study and explanations will be offered for differences.

Prior research in mathematics and online education indicates a gap in the study of students' knowledge of mathematical concepts and their difficulties in an online environment. Allen (2001), Anderson (1999), Hiltz (1997), and Schmidt, Sullivan and Hardy (1994) reported that students who had taken online mathematics courses performed well in their present or future mathematics courses.

Recognition

Barnes (1988) reported that students did not regard $y = 2$ as a function because it did not have an x in it. Among the participants of the study, Marsha was the only one that regarded $y = k$ as a non-function because there was only one variable in that relation. In addition, Ivy also treated $y = k$ as a non-function, but because she believed the relation was a vertical line and therefore failed the vertical line test. I wondered: Could this lack of knowledge be attributed to the fact that the online course material did not explain clearly different aspects of functions? Could it be because students had weak mathematics background? Or could there be other reasons?

Previous research indicated that students considered $x^2 + y^2 = 4$ to be a function because it was familiar (Barnes, 1988). That was the case for Marsha, Nancy, and Andrew. They too wanted the relation $x^2 + y^2 = 25$ to be a function because it was very familiar to them. Nancy knew this was the equation of a circle, and she felt strongly that this should be a function. My findings paralleled the results of Barnes (1988). The analysis of data led me to believe that because the topic of circle was discussed in the chapter called “Functions and Graphs,” students might have thought of the circle as a function. I wondered: Could there have been other reasons?

Further, Vinner and Dreyfus (1989) posited a claim that students do not necessarily use the definition of function when deciding whether a given relation is an example of a function. In most cases they decide on the basis of a concept image. This was the case for Marsha. She treated $x = k$ as a non-function because there was only one variable in that relation. Here the way she defined a function was the result of her experience with the concept. This experience was formed by some examples that did not fit the concept definition (Vinner, 1983). A question was begged by this finding: What kind of examples or previous experiences would lead a student to believe this? Is this experience related to the fact that the course was online?

Markovits et al. (1986) and Vinner (cited in Markovits et al., 1986) indicated that students had difficulties with piecewise functions. This was the case with Marsha. She did not realize that different rules of correspondence applied to different parts of the domain. Both Ivy and Andrew had knowledge of these types of problems. Further, Vinner (1983) reported that students expected a function to be given by one rule. That was the case for Nancy. She thought of a relation given by two disjoint domains as two separate functions. This topic was not discussed in the online class, yet between the four participants the two who had taken the course previously knew about piecewise defined functions. I wondered: Did their previous knowledge

of the concept of function play a role? Did they use their textbook or other resources to supplement their knowledge of the concept of function?

Previous research indicated that students possess inaccurate views of what the graph of a function should look like (Vinner, 1983; Lovell, 1971). That was not the case for Nancy and Ivy. They both demonstrated knowledge of graphical representations of functions. In contrast, Andrew and Marsha exhibited difficulty in recognizing the graphs of functions. A question to address by this finding is: Why do students in an online course find graphical representations of functions difficult? Could it be because the online text of the course was not rich in examples and illustrations?

Interpretation

Moschkovich (1990) indicated that many students had difficulty with the interpretation of the y -intercept. This was the case with most of the participants in this study. My analysis indicated that students may come to this online course with different knowledge bases, therefore, in the absence of a detailed review, students can face difficulty. But could there be other reasons?

Dunham and Osborne (1991) reported that most students had difficulties extracting from an ordered pair the corresponding values of x and $f(x)$. All the participants in this study showed some level of difficulty with this concept. Despite the fact that the topic was covered in the online college algebra course, students struggled. Why? Could it be because the course was online, and the explanations were not explicit enough, and choice of examples not varied?

Previous research indicated that students have difficulty with the concept of domain and range (Dunham & Osborne, 1991; Tall & Bakar, 1992; Markovits et al, 1986). That was the case for one of the participants in the present study. Andrew had absolutely no knowledge of the domain and range. Most of the students in this online course did not have difficulty with the

concept of domain and range. Was this topic well explained or stressed in this online course? My analysis of the data led me to believe that the topic of domain and range received considerable amount of attention in this course. But if that was the case, why did Andrew have so much difficulty? Did students' previous knowledge of this concept also play a role?

Most of the students in this study were more comfortable with graphs of functions in which one of the variables was time or time-dependent. This result was similar to the finding of Janvier (1981). In this case, the medium of instruction did not make a difference.

My findings paralleled the results of Bell and Janvier (1981), who reported that many students had difficulty interpreting some global features of the graphical representation of functions such as intervals of increase and decrease. In addition, Zaslavsky (1997) found that students determined whether certain points were, or were not, on a graph based only on what appeared to be "eye measurement." That was the case for all of the participants in the study. These findings point to students' difficulty to reason visually. A question to ask is: How could online mathematics courses facilitate students' visualization of functions? How could online courses teach students how to make connection between graphical and algebraic representations in order to foster better understanding? Maybe the use of graphical software in this online course could facilitate students' visual understanding of functions.

Translation

Markovits et al. (1986) and Zaslavsky (1997) each indicated that translation of functions from graphical to algebraic was more difficult than vice versa for students. That was not the case for participants in the study. That was likely because in the latter case the function as well as its graph was given, and with minimal knowledge of the graphing techniques, students could make the proper algebraic translation of the function.

Eisenberg and Dreyfus (1994) reported that the translation of functions from algebraic to graphical in vertical direction was easier for students than similar translations in horizontal direction. That was not the case in the present study. Participants demonstrated difficulty translating in both directions. A question was begged by this finding: Why did students in this online course had so much difficulty with translation between algebraic and graphical representations? Analysis of the data led me to believe that the students did not see any reason to learn the graphing techniques, despite the amount of difficulty they faced in the translation of such functions. But why? Maybe if graphical technologies, such as certain dynamic software were used in this online course it would have helped students' learning of such techniques. Could there be other reasons?

My findings also paralleled the results of Markovits et al. (1986), who reported that functions defined by constraints caused difficulties for students. This was the case for all of the participants as they did not know how to write the point $f(2) = 3$ as an ordered pair. Did this suggest that the online class notes on the topic of function were not detailed enough for the students?

Summary

Previous research has reported on students' knowledge and difficulties with the concept of function in a face-to-face setting. The results of the present study provide an overall understanding of students' knowledge and difficulties with the concept of function in the three areas of recognition, interpretation, and translation in an online environment.

Although translations among different representations of functions are discussed in the empirical literature, most tasks focus on translation between algebraic and graphical and vice versa (Lienhardt et al., 1990). This study added some insight into the knowledge and difficulties

of online students with regard to translation of functions in the areas of verbal to algebraic and symbolic to graphical.

CHAPTER 5

SUMMARY AND IMPLICATIONS

Purpose

The purpose of this study was to investigate college students' knowledge of the concept of function in an online environment and explore the difficulties they may have with the notion of function. Several factors motivated me to conduct this study. First, previous studies in online mathematics education focused on learner satisfaction and effectiveness of online mathematics courses compared with traditional face-to-face instruction (Hiltz, 1997; Schmidt, Sullivan, & Hardy, 1994; Allen, 2001; Anderson, 1999) without focusing on any specific mathematical concept. Second, despite the explosive growth of distance education courses at two-year institutions (National Center for Education Statistics [NCES], 2003), research has not addressed students' knowledge of mathematical concepts in an online environment. Third, the concept of function is essential to the algebra curriculum and is one of the most essential topics in mathematics (O'Callaghan, 1998).

Research investigating students' knowledge and difficulties with the concept of function (e.g., see O'Callaghan's, 1998; Dunham & Osborne, 1991; Eisenberg & Dreyfus, 1994) formed the foundation for this study. What captured my interest was the concern that understanding of the concept of function among college students is superficial (Dreyfus & Eisenberg, 1983). To that end, I wanted to learn about students' knowledge of and difficulties with the concept of function in an online environment.

Theoretical Perspective

The theoretical framework I used in this study is a modification and combination of the Markovits et al. (1986) and O’Callaghan’s (1998) frameworks. O’Callaghan’s framework evolved from Kaput’s (1989) sources of meaning in mathematics specifically with regard to the function concept. The components—*interpreting* and *translating*—in O’Callaghan’s framework stem from sources Kaput identified in the referential extension category. Markovits et al. (1986) pointed out that understanding has two stages: (a) passive – the knowledge to identify and classify functions and non-functions, and (b) active – the knowledge to transfer information from one representation to another in mathematics. Stage (b) is similar to O’Callaghan’s *translating* component. In this study, I combined identifying and classifying of functions and non-functions, and used *recognizing* as the first component of my framework.

My theoretical framework consisted of the elements *recognizing*, *interpreting*, and *translating*. *Recognizing* allowed me to portray the knowledge and difficulties that students had with identifying functions and non-functions. *Interpreting* helped me to describe the students’ knowledge and difficulties with interpreting functions in their different representations. *Translating* allowed me to portray students’ knowledge and difficulties moving from one representation to another.

Methodology

Four students from a college algebra online course at a two-year college in the southeastern part of the United States were selected and studied over a ten-week period during the fall quarter of 2003. Each of the four participated in all phases of the study. Data for this study was collected through pretest, three learning reflections, two interviews, and artifacts that included one test, a quiz, and a comprehensive final examination.

I used a constant comparative method (Glaser & Strauss, 1967) throughout the study. The first part of the analysis involved identification of color-coded categories that were significant in terms of the research questions and theoretical framework. The second part of the analysis involved a return to the data for a theoretical look at the themes that emerged during the first phase of analysis.

Findings

The data from four case studies that describe participants' knowledge of and difficulties with the concept of function in the areas of recognition, interpretation, and translation in an online college algebra course were presented in chapter 4 and are summarized in the present chapter. This summary also provides a critical review of the online course.

Research Question Results

Question 1. What do students who take an online college algebra course recognize as functions? What are their specific difficulties in this context?

Relations in numerical and graphical forms were recognized more frequently than relations in an algebraic form. The participants used the vertical line test and a statement similar to “one x -value cannot be paired with two y -values,” to recognize functions. Among the four participants, Ivy was the only one who knew how to use the vertical line test. She graphed all algebraic relations to determine whether they represented functions or not.

Most participants in this study exhibited difficulties with algebraic relations in a piecewise defined form, $x = k$, $y = k$, and $x^2 + y^2 = 25$. Most participants did not recognize $x^2 + y^2 = 25$ as the equation of a circle. The only type of algebraic relation that all of the participants recognized as a non-function was $x = y^2 + 4$.

Nancy outperformed the other participants in recognizing relations in algebraic, numerical, and graphical form. Overall participants showed improvement in their recognition of functions over the course of the study.

Question 2. What do students who take an online college algebra course know about interpretation of functions given in equation and graphical form? What are their specific difficulties?

Interpretation of functions in equation form involved interpretation of the slope, and the y -intercept of a linear function. All participants identified the slope and the intercepts of a linear function prior to starting the interpretation process. Students were more knowledgeable about the interpretation of the y -intercept than the slope. Most participants did not have a correct knowledge of the x -intercept, and that led to their inability to interpret it. Among the four participants, Nancy was the only one who knew how to interpret various components of a linear function.

In the area of graphical representation of functions, all of the participants lacked knowledge of how to extract the corresponding values of x and $f(x)$ from an ordered pair (X, Y) located on the graph of a function. Interpretation of some global features of the graphical representation of functions, such as intervals of increase and decrease, was difficult for most of the participants. All, with the exception of Andrew, had a good grasp of domain and range. Most knew how to interpret graphical representation of functions in which one of the variables was time. Ivy was the only participant who made a connection between the visual representation and the algebraic description of the functions.

Overall, the participants showed signs of improvement in the area of interpretation throughout the study. Students had more difficulty with the interpretation of functions in algebraic form than in graphical representation.

Question 3. What do students who take an online college algebra course know about translation of functions from one representation to another? What are their specific difficulties?

In translating functions from an algebraic representation to a graphical form, most of the participants resorted to plotting points although they admitted that it took longer to graph using their method. They did not attempt to learn other techniques. The difficulties most of the participants showed were as follows: (a) The participants were not able to recognize the shape of the graphs; for instance, for graphing $y = (x - 1)^3$, Marsha and Andrew plotted points and connected them as a straight line, (b) Ivy and Andrew made arithmetic errors in calculating the values of y ; therefore, the resulting graph was not translated properly, and (c) Andrew did not use negative values for x , and the resulting graph of $y = x^2 + 1$ was half of a parabola.

The task of translating functions from a verbal representation to an algebraic description was the most difficult task for all but one participant. The participants were more comfortable with translation of functions from graphical to algebraic than vice versa. The only type of graphical representation of functions, that all of the participants were able to translate without any difficulty, was the reflection task. The translation of functions from a numerical to a graphical representation was easy for all the participants. On the other hand, the translation of functions from a symbolic to a graphical representation was less frequent.

Summary

In all, the participants were more knowledgeable about the recognition of functions than interpretation and translation. Throughout the study the participants demonstrated improvement

in their knowledge of all but one area of functions—translation of functions. Among the four participants, Nancy was the only one that demonstrated improvement in all three areas of recognition, interpretation, and translation of functions.

The findings of this study suggested teachers will challenge students if they are not provided opportunities to explore, and possibly experience, multiple approaches to learning functions. As more and more students take online courses, it is vital that educators give students more than just access to information. Educators must offer educational experiences that empower knowledge construction by unsophisticated learners, and help them make sense of the massive information.

Critical Review of the Course

In an online environment there is no dedicated classroom, and students can study anywhere at anytime. In this environment, the teacher has to balance the facility of group settings to match instruction of the concept of function to different student needs. Here, teachers must give special attention to a wider range of learner management problems.

In this case study, it was discovered that the following practices were missing from the online course design.

- Connection between different representations of functions.
- Examples connecting the function concept to real-life representations.
- Expecting students to interpret graphs and symbols.
- Clear and detailed instructions and explanations.
- Adequate independent practice.

In the area of interaction and communication, this online course did not encourage active participation of the students throughout the course. This created a challenge for the students, as

they were not looked upon as active partners in the acquisition of mathematical knowledge. The other deficiency in this course was that the instructor did not provide students access to an information base (distribution list) that was dynamic.

As for the environment of this online course, it was noted that the students were not engaged in collaborative investigations; rather, in passive learning. Students were not encouraged to think of themselves as guides and directors of their own learning.

In this course, the teacher's role was not one that would develop a mathematical environment in which mathematical understanding was constructed by all participants. Also, the teacher did not organize online activities around students' active explorations of mathematical ideas.

Motivation in an online course must also be considered. This course did not motivate students to engage in academic interaction with the instructor and other students, and did not promote the benefits of such interactions. Also, it did not communicate high expectations and provide a challenge for learners.

Implications for Teaching

Current publications on online education suggest that students find it useful to see questions and responses from other students, as it helps them gauge where the broad standard should be in relation to others (Jones, 1999; McMurray & Dunlop, 2000). Ms. Sanders, the instructor of the online college algebra course, provided some opportunities for interaction (distribution list and the virtual classroom) between herself and the students. However one of the very few, who took part in a meaningful interaction with her teacher, was Nancy. Therefore, as pointed out by Sherry (cited in Curtin, 2002), unless the online participation of students is assessed, on average one third of students will seldom participate in online activities.

Topics such as the translation of algebraic representation of functions to their graphical form, and the vertical line test, were not demonstrated through the use of variety of examples. It is crucial to provide online students with course materials that include different types of examples or ways to access other resources that can complement their knowledge; in this case, that provision was not made.

A brief review of some elementary characteristics of functions at the beginning of the unit on functions might be helpful to students like Ivy, who have difficulty graphing a linear function or a set of ordered pairs, or Nancy, who consider a numerical relation to be a function, when all the y -values are the same.

Implications for Curriculum

The course materials in an online college algebra course should be the embodiment of the curriculum, with more exploratory approaches that are based on interactivity. In this online class, students mostly used course materials that were prepared by the instructor. Therefore, in cases where topics were not covered in the online course materials, students were left with an impression that they are not required. However, it was noticed that certain aspects of functions not discussed in class were present on the class test. It is crucial that when topics are eliminated from the curriculum, students not be assessed on them.

Implications for Research

Teaching mathematics is a complex task, and to be successful requires substantial breadth and depth of knowledge of both mathematics and pedagogical nature, as well as the ability to organize this knowledge in a useful way (Shulman, 1987). This study gave similar findings. These issues were beyond the scope of this research; still, they create avenues for future studies.

Recommending practices for teaching functions in a virtual setting will help teachers to construct new learning opportunities. Teaching mathematics compels us to recognize that the way functions are taught in an online environment is closely linked to teachers' knowledge base, and their orientation towards the concept of function. It is important that teachers have (a) a systematic set of practices that ensure all areas of the function receive equal emphasis, (b) a text that is rich yet simple to understand, and (c) appropriate graphical software for enhancing students' knowledge and understanding of functions. Therefore, it would be beneficial to investigate the content knowledge and pedagogical content knowledge of teachers who teach online mathematics courses.

The concept of interaction is an important factor in the effectiveness of online mathematics courses. Moore (1989) discussed three types of interaction essential in an online learning setting: learner-instructor, learner-content, and learner-learner. All three sets of interactions take place whether the course is in a structured or non-structured environment. Participants in this study valued the interaction they had with the researcher focused on the questions about functions, and they valued the feedback they received. During the interview, the participants pointed to the learning experience as unique because it allowed them to improve their knowledge of the various concepts of function and clear up any difficulties they had. Based on this, it may be very useful for future research to focus on ways to build interaction into an online mathematics course.

Concluding Remarks

Although the focus of this study was on student' knowledge of and difficulties with the concept of function in an online college algebra course, I view it as a part of a larger research agenda in mathematics education.

The analysis of data led me to believe that online instruction benefited students in the areas of recognition and interpretation of functions. In contrast, translation of functions remained problematic, and the online course did not particularly contribute to students' knowledge in this area.

It is my hope that other researchers will continue and extend the investigation of functions in an online environment. In particular the translation component has manifested itself as an appealing area for further analysis and exploration. As more complete understanding of this, and other aspects of functions, is the key to designing ways to help students develop better knowledge of functions.

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APPENDICES

APPENDIX A

Content of Course for Fall Quarter 2003/2004

Taken from:

Aufmann, R. N., Barker, V. C., & Nation, R. D. (2002). College algebra
Boston: Houghton Mifflin.

Unit One

Preliminary Concepts, Equations and Inequalities

Integer & Rational Number Exponents

Linear Equations

Formulas and Applications

Quadratic Equations

Other Types of Equations

Inequalities

Variation and Applications

Unit Two

Functions and Graphs

A Two-Dimensional Coordinate System and Graphs

Introduction to Functions

Linear Functions

Quadratic Functions

Properties of Graphs

The Algebra of Functions

Unit Three

Exponential and Logarithmic Functions

Exponential Functions and their Graphs

Logarithmic Functions and their Graphs

Properties of Logarithms

Exponential and Logarithmic Equations

Unit Four

Topics in Analytic Geometry

Parabolas

Ellipses

Hyperbolas

Unit Five

Systems of Equations

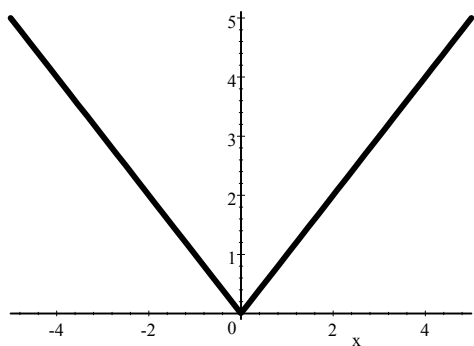
Systems of Equations in Two Variables

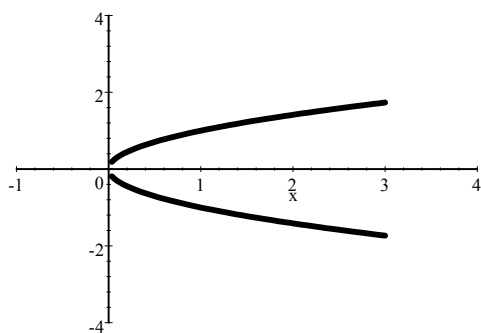
Systems of Equations in More than Two Variables

APPENDIX B

PRETEST

1. A building contractor estimates that the cost to build a new home is \$25,000 plus \$80 for each square foot of floor space. Determine a linear function that will give the cost to build a house that contains a given number of square feet. Use the model to determine the cost to build a house that contains 2,000 square feet.
2. Determine whether each of the following representations present a function? If your answer is “No,” state why?
 - $\{(2, 3), (5, -4), (-3, 8), (5, -1)\}$
 - $\{(8, -1), (7, 2), (9, 2), (-2, -1)\}$
 - $y = -1$
 - $x = 6$
 - $x + y = -3$
 - $x^2 = y + 4$
 -





3. Graph each function:

- $f(x) = 2x + 3$
- $f(x) = 5x$
- $f(x) = x^2 - 2$

4. Graph $f(x) = 2$, then find $f(2)$ and $f(-1)$.

5. Graph the function represented by:

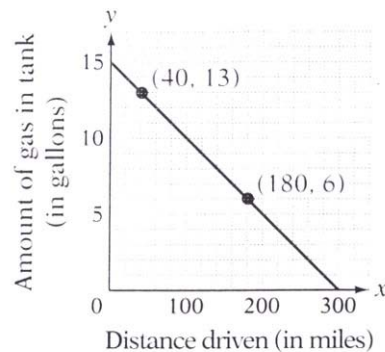
x	4	1	0
y	-1	2	3

6. The monthly cost for receiving messages from a telephone answering service is \$5.00 plus \$0.25 a message. The equation that describes the cost is $C = 0.25n + 5$, where n is the number of messages received. The point $(24, 11)$ is on the graph of this function.

Write a sentence that describes the meaning of this ordered pair.

7. An appliance store estimated that 40 transistor radios would be sold per day during an upcoming holiday season if the new radios were priced at \$100 each. At the price of \$150, the store would sell 25 units. If the linear function that models this relationship is given by $D = -0.30p + 70$:

- Interpret the meaning of the slope and y -intercept.
 - Identify and interpret the x -intercept.
8. The length of a rectangular flowerbed is 3 times the width. Find a linear function for the perimeter of the flowerbed in terms of the width of the flowerbed.
9. Complete the following sentences:
- If a line has a slope of 6, then the value of y _____ by 6, as the value of x increases by 1.
 - If a line has a slope of -3 , then the value of y _____ by 3 as the value of x decreases by 1.
10. The graph below shows how the amount of gas in a tank of a car decreases as the car is driven. Find the slope of the line. Write a sentence that states the meaning of the slope.



Adapted from Aufmann, R. N., Barker, V. C., & Lockwood, J. C. (2003). Intermediate Algebra

11. The temperature of an object taken from a freezer slowly rises and can be modeled by the function $T(x) = 30x - 80$, where $T(x)$ is the Fahrenheit temperature of the object x ($0 \leq x \leq 5$) hours after being removed from the freezer. Find the intercepts of the graph of this function and explain what they mean in the context of this problem.

APPENDIX C

REFLECTIONS ON THE STUDY OF FUNCTIONS

Directions: Reflect on your understanding of the concept of function by addressing the following questions.

1. Have you come across a mathematical relation that you were not able to recognize as a function/non function? What was it? Why do you think you were not able to recognize it?
2. Have you come across any function that you were not able to translate (from equation to graph, from verbal to equation, from table to graph, etc.)? What was it? Why do you think you were not able to translate it?
3. Have you come across any function that you were not able to interpret? What was it? Why do you think you were not able to interpret it?

APPENDIX D

INTERVIEW PROTOCOL

Determine whether each of the following representations present a function? If your answer is “no,” state why.

1. $y = 2$

2. $x = 5$

3. $x = y^2 + 4$

4. $x^2 + y^2 = 25$

5. $x^2 + y = 9$

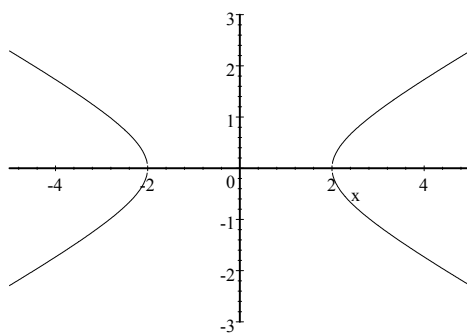
6. $\{(1, 2), (1, 5), (2, 10), (-3, 5)\}$

7. $\{(-1, 0), (2, 4), (6, 1), (4, 1)\}$

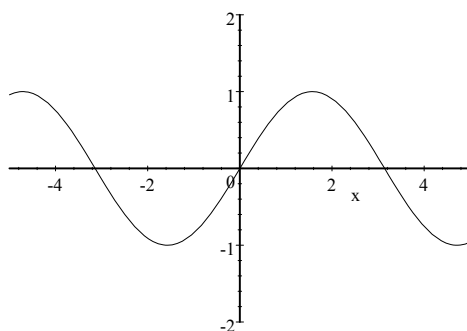
8. $f(2) = 3$ and $f(-5) = 2$

9.
$$\begin{cases} x \geq 0, y = x + 2 \\ x < 0, y = 1 - 2x \end{cases}$$

10. (a)



10. (b)



11. The book value of a machine is \$4,500. It is estimated that after 9 years the value of the machine will be \$900. The linear function that determines this relationship is given by

$$V = -400t + 4500 \quad (0 \leq t \leq 9).$$

a) Interpret the meaning of the slope and the y-intercept.

b) Identify and interpret the x-intercept.

12. Use the following graph of $y = h(x)$ to find

$h(-2)$, $h(7)$, $h(-4)$.

a) What are the zeros of $h(x)$?

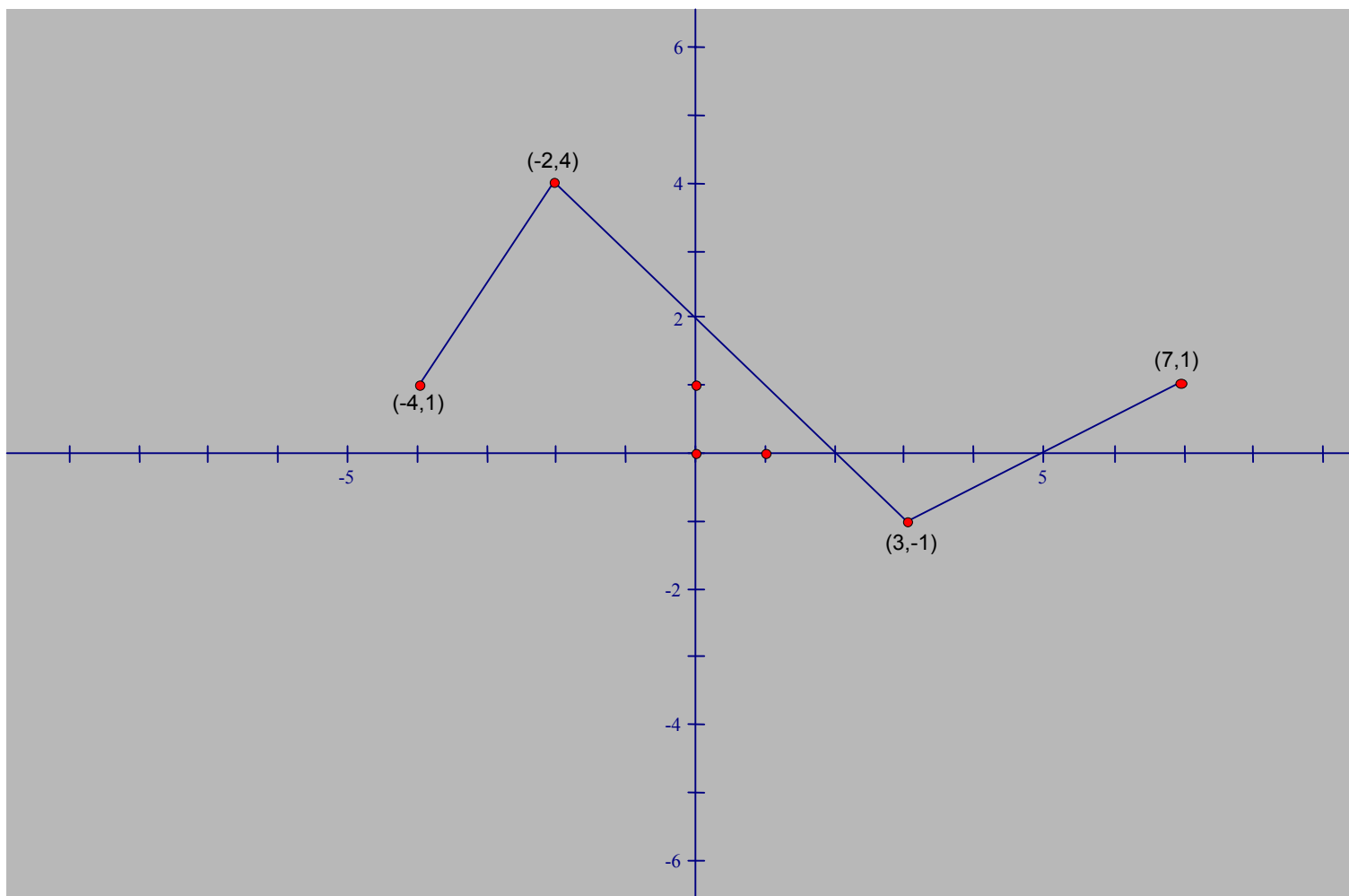
b) On what interval(s) is $h(x)$ increasing?

c) On what interval(s) is $h(x)$ decreasing?

d) What is the domain of $h(x)$?

e) What is the range of $h(x)$?

f) Which is larger $h(-2)$ or $h(4)$?



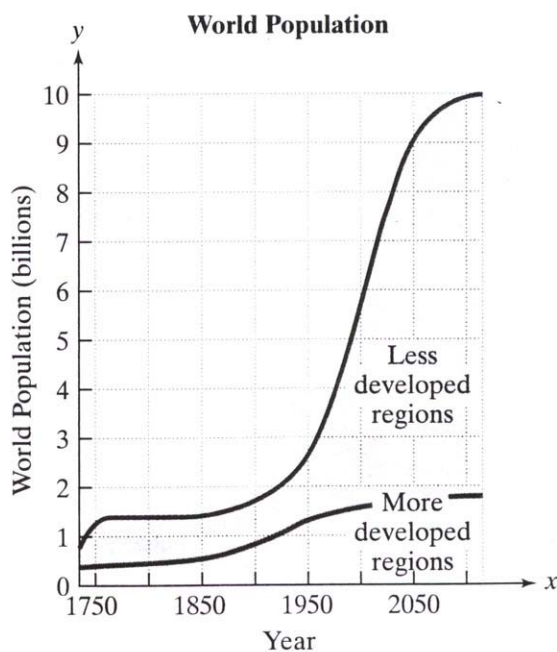
13. Consider the following function:

$f(x)$ = population of the world's more-developed regions in year x .

$g(x)$ = population of the world's less-developed regions in year x .

$h(x)$ = total population in year x .

- What do the function $f + g$ represent?
- What does the function $h - g$ represent?
- Use the graph to estimate $(f + g)(2000)$.
- Use the graph to estimate $(h - g)(2000)$.

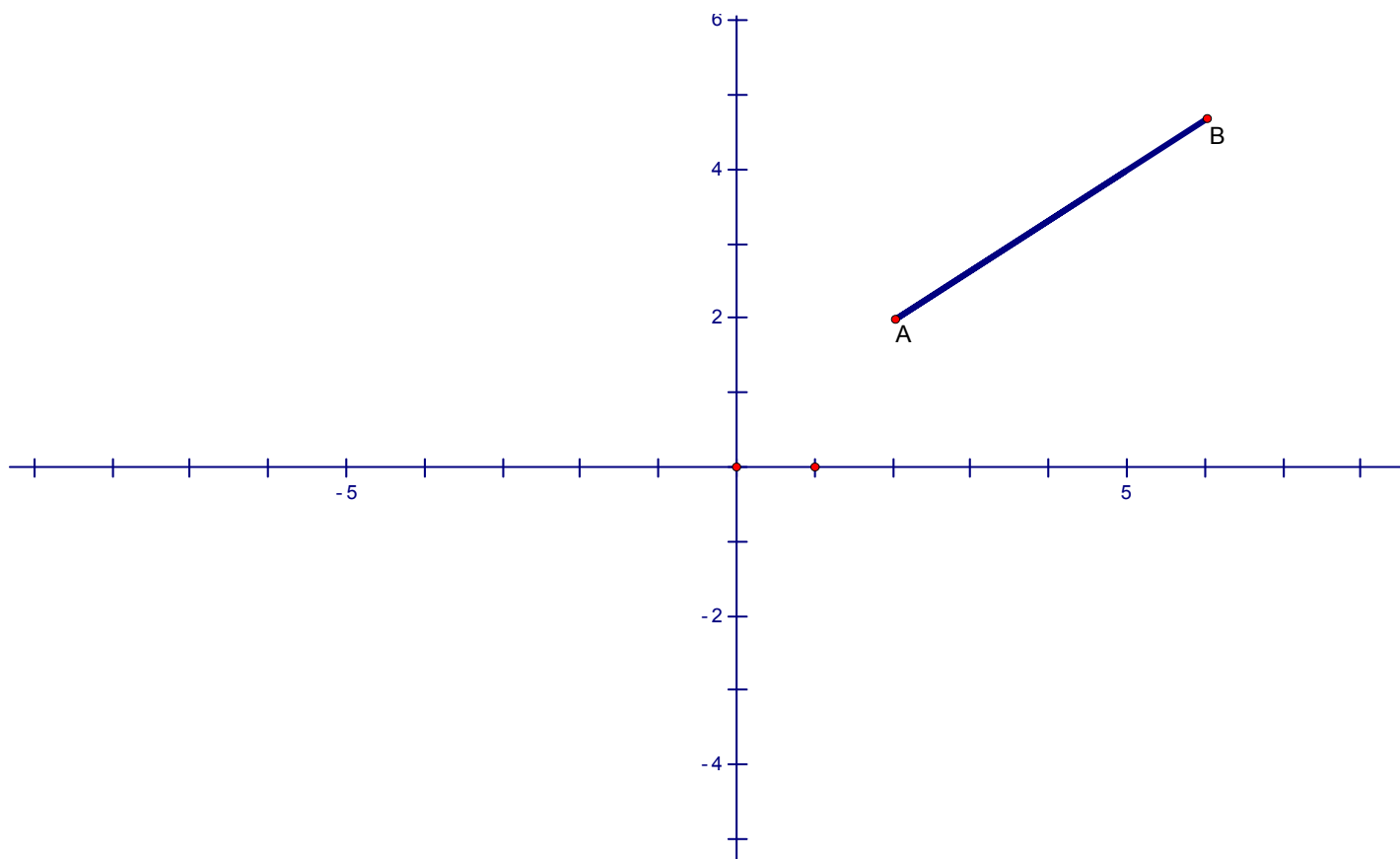


Source: Population Reference Bureau

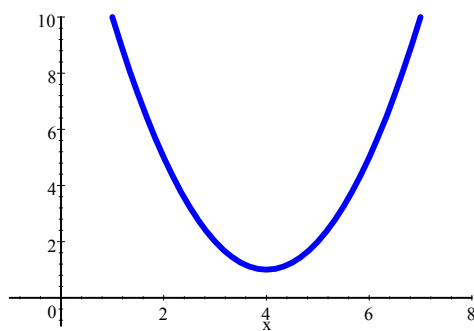
14. A manufacturer of graphing calculators determined that 8,000 calculators per week would be sold at a price of \$75. At a price of \$70, it was determined that 10,000 calculators would be sold. The linear function that determines this relationship is given by $y = -400x + 38,000$.

- Interpret the meaning of the slope of the graph of this linear function.
- Explain the meaning of the y -intercept.
- Explain the meaning of the x -intercept.

15. Are the points $(-1, 0)$ and $(5, 4)$ on line AB?



16. Is the point $(0, 17)$ on the graph of $y = (x - 4)^2 + 1$?



17. Graph each function.

a) $f(x) = |x + 2|$

b) $f(x) = x^2 - 2$

c) $f(x) = (x - 3)^3$

d) $f(x) = -\sqrt{x + 2}$

e) $g(x) = |x - 1| - 2$

18. A police department believes that the number of serious crimes, which occur each month, is dependent upon the number of police officers for preventive purposes. If 140 police officers are assigned for preventive patrol, there will be no crime. However, it is expected that assignment of no police officers would result in 1,500 serious crimes per month. Find a linear function demonstrating this relationship.

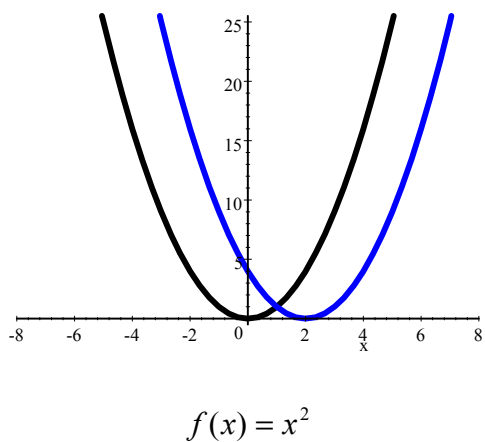
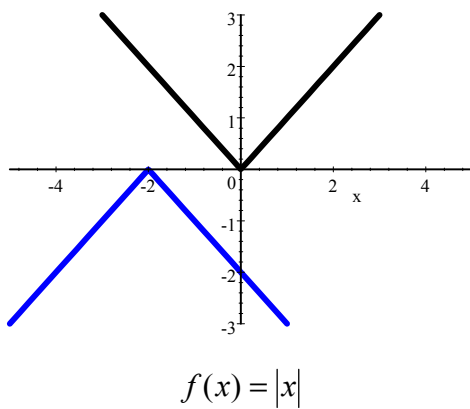
19. A doll company had a profit of \$50,000 per year when it had sold 1,600 dolls. When the sales of dolls increased to 1,800 the company had a profit of \$60,000. Assume that the profit, P , is a linear function of the number of dolls sold. Find the profit function.

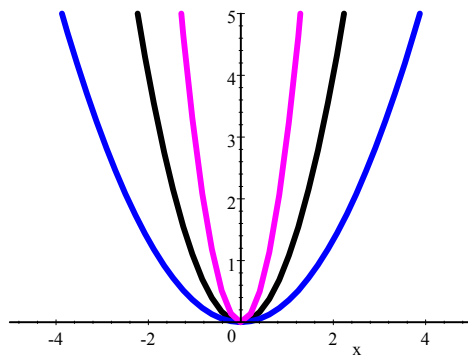
20. Graph the function represented by

x	-2	-1	0	1	2
y	-14	-9	-4	1	6

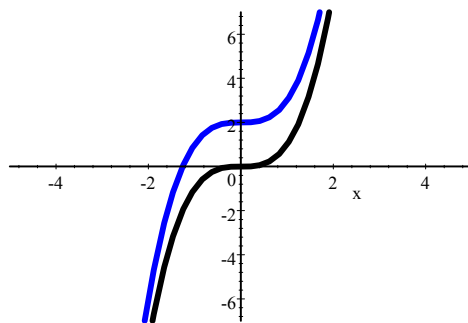
21. Graph the function given by: $f(2) = -2$, $f(0) = -3$, and $f(4) = -1$.

22. In the following exercises the functions of f (black), g (blue), and h (pink) are graphed in the same rectangular coordinate system. If g and h are obtained from f through a sequence of transformations, find the equation of g and h .

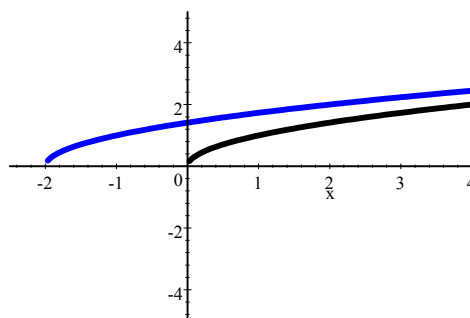




$$f(x) = x^2$$

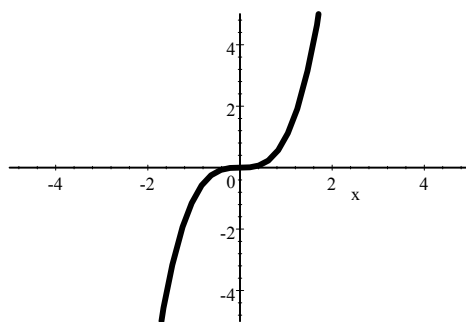
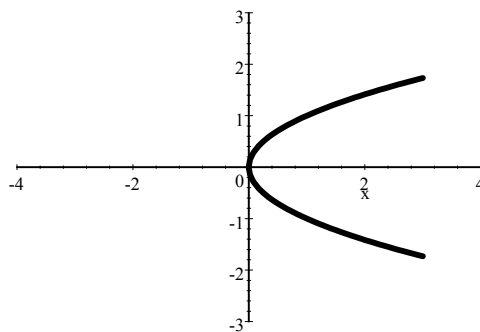


$$f(x) = x^3$$



$$f(x) = \sqrt{x}$$

23. The following figures show graphs of $y = f(x)$. Sketch the graph of $y = f(-x)$ on the same axis.



24. Use the following information to find the equation of the line (red).

