

TRANSFORMS ON TIME SCALES

by

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(Under the direction of Thomas Gard)

ABSTRACT

Time scales theory provides a means for unifying and extending real and discrete analysis. Transforms play a crucial role in analysis in part because of transform methods for solving differential equations. Two of the most commonly used transforms are the Laplace and Fourier transforms. We define the Laplace transform for time scales noting that it is an extension of the Laplace transform for real numbers as well as a discrete transform. We give properties of the Laplace transform and discuss instances when results may not be generalized from the real case to time scales. Dynamic equations are solved in examples using the Laplace transform. Next we define the Fourier transform for time scales and discuss how it unifies the different types of Fourier analysis. Finally, we give a discussion on the possibilities for a general transform theory based on time scales analysis.

INDEX WORDS: Time scales, difference equations, q-difference equations, Laplace transform, Fourier transform

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BS, The University of Georgia, 2001

A Thesis Submitted to the Graduate Faculty
of The University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

MASTER OF ARTS

ATHENS, GEORGIA

2003

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ACKNOWLEDGMENTS

I would like to thank my friends and family for not having me committed while I wrote this thesis. I would like to thank the members of my graduate committee, Dr Malcolm Adams and Dr. Elliot Gootman, for agreeing to be on my committee and for the many hours they spent proof reading. Additionally I would like to thank Dr. Joan Hoffacker for introducing me to the subject of time scales. Most importantly, I would like to thank my advisor Dr. Thomas Gard for his invaluable guidance, advice, and endless patience. Without his help I would still be lost in this pursuit.

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CHAPTER 1

INTRODUCTION

The subject of calculus on time scales is a young one being first introduced by Stefan Hilger in 1988 in his Ph.D. dissertation. The purpose of this new calculus was to do away with the discrepancies between continuous and discrete analysis. In particular, the subject allows one to prove results for both differential and difference equations simultaneously. Instead of proving a result for a differential equation defined on the real numbers \mathbb{R} , or a difference equation defined on the integers \mathbb{Z} , one considers a general dynamic equation defined on a time scale, \mathbb{T} , a closed subset of the real line. Thus when proving a result for a time scale, one is not only proving the result for \mathbb{R} and \mathbb{Z} , but for many other possible spaces as well. For example, much recent attention has been given to q -difference equations, that is difference equations defined on the set $q^{\mathbb{N}_0}$ (see page 5). This recurring theme is often described as being one of unification and extension.

With so much of time scale study focused on dynamic equations, it seemed logical to develop a Laplace transform method for solving initial value problems. Early efforts made by Hilger in this area attempted to unify the Laplace transform for \mathbb{R} with the \mathcal{Z} -transform for \mathbb{Z} . However the transform he developed only worked for very special time scales and could be difficult to employ (see page 57). Later, Martin Bohner and Allan Peterson developed a transform that unified the Laplace transform for \mathbb{R} with the $\tilde{\mathcal{Z}}$ -transform (see Donahue [5] for more information on

this transform) for \mathbb{Z} . The transform they developed appears much more natural and lends itself for use on a broader set of time scales.

While time scale theory is still in its early stages of development, Hilger has begun work on Fourier analysis for time scales. His Fourier transform unifies the different kinds of Fourier analysis. Most notably, it provides a closed form for both the Fourier integral and Fourier series represented by a single times scale integral.

In this thesis, we will continue to center most of our attention on the theme of unification and extension with particular emphasis on how this theme presents itself in the theory of transforms. We will present results that unify and extend results from real and discrete analysis pointing out the existing problems and differences when they exist. The reader should observe that while many things may be said about the theory of arbitrary time scales, we can always say more about a specified time scale. Through such observations, it is possible that one will come to an even greater appreciation of calculus.

Proofs in this work shed greater light on the relationship between analysis and dynamic equations. Real analysis is frequently used to prove results in differential equations. However, conventional analysis techniques often fail for time scales. Many times when trying to find an analog to a result from real analysis we will be unable to employ the analogous analysis techniques for time scales but will succeed in proving a weaker result using what we know about dynamic equations. An example of this is the convolution proposition, Proposition 3.16, given on page 61.

Transforms tend to present themselves in two forms. The first is the series form such as seen in the cases of the Fourier series and \mathcal{Z} -transform. The second form is that of an integral transform such as seen in the cases of the Fourier integral and Laplace transform. Most transforms have their own extensively developed theory, however there is no general theory of transforms. In fact, it is not entirely clear

what is meant by a transform. All integral transforms can be expressed as integration against some function and all series transforms can be expressed as summing against some function. The theory of time scales presents itself as a potential way for defining what we mean as a transform. One might be able to unify the definition of all transforms as time scale integration against a function. Coming up with such a definition would be a first step in developing a general theory of transforms. The reader should keep this in mind when examining the sections on the Laplace and Fourier transforms for time scales.

Although we will not give much consideration to applications, the potential of applications motivates continued research in this area. Dynamic equations on time scales may be used to model populations, say of insects, that are continuous over nonoverlapping periods of time separated by periods of inactivity, such as during a dormant or hibernation period. It is also suggested that calculus on time scales might prove useful in numerical analysis for example, in adaptive grid or variable mesh size theory.

Throughout this paper, results are presented from other works. Many of the sources for these results leave proofs to the reader as exercises. When this is the case, we will present the proof here. Details and explanations have been added to most proofs taken from other works. When a new result is presented we will make note of it and give its proof.

In chapter 2, Preliminaries, we will introduce calculus on time scales. Most of the information in this section is taken from Bohner and Peterson [4] and Agarwal and Bohner [1]. However, the material on time scales integration is taken from Bohner and Peterson [3].

Chapter 3 starts by introducing the Laplace transform for time scales and presents some important results. We then proceed to work a few examples, solving dynamic equations using the Laplace transform method. We also develop additional

results needed in the examples as we go. Information on ordinary differential equations used in this section is taken from Edwards and Penney [6]. However, at times, we will use conventions and information taken from Asmar [2] in this section and the one that follows it.

Chapter 4 presents the Fourier transform for time scales, which unifies the different notions of Fourier transforms by presenting them all in the form of one time scales integral. The material on this time scales transform is taken from Hilger [8] and [9]. In the process of writing this section, Folland [7] was frequently consulted.

CHAPTER 2

PRELIMINARIES

2.1 BASIC NOTATION AND DEFINITIONS

Definition 2.1 A *time scale*, denoted \mathbb{T} , is a nonempty closed subset of the real numbers.

There are a few time scales that will be of particular interest. They are

the real numbers \mathbb{R}

the integers \mathbb{Z}

$h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$, where h is a fixed positive real number.

the natural numbers $\mathbb{N} = \{1, 2, \dots\}$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$ is fixed.

In this thesis, we will assume that a given time scale is endowed with the standard relative topology from the real numbers.

Consider the integers, \mathbb{Z} , and select $t \in \mathbb{Z}$. We know that the next greater integer is given by $t + 1$. Next let's consider the real numbers, \mathbb{R} , and let $t \in \mathbb{R}$. In this case,

there is no next greater real number for t . Now consider the time scale $\mathbb{T} = [-1, 0] \cup \mathbb{N}$. If we choose $t \in \mathbb{T}$ such that $t < 0$, then there is no next greater element in \mathbb{T} . However if we choose $t \in \mathbb{T}$ such that $t \geq 0$, then \mathbb{T} has a next greater element given by $t + 1$. The next definition formalizes such statements in a way that makes sense for arbitrary time scales.

Definition 2.2 For $t \in \mathbb{T}$ the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

and the *graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

The convention here is to let $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

The forward jump operator gives the next greater element than t or gives t if there is none. The backward jump operator works similarly for the next lesser element. Finally, the graininess function gives the distance to the next greater element. The definition that follows uses these operators to classify points on time scales.

Definition 2.3 If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\rho(t) < t$ we say that t is *left-scattered*. If a point is both right-scattered and left-scattered, then it is called *isolated*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are both right-dense and left-dense are called *dense*.

The next two definitions set up conditions that are similar but somewhat weaker than continuity.

Definition 2.4 A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* provided that its right-sided limits exist at all right-dense points in \mathbb{T} and its left-sided limits exist at all left-dense points in \mathbb{T} , that is only jump discontinuities are possible.

Example 2.5 To give the reader an idea of what it means for a function to be regulated, we'll give an example of a function that is not regulated on \mathbb{R} . Let $f : \mathbb{R} \rightarrow [-1, 1]$ be defined by

$$f(t) := \begin{cases} \sin(\frac{1}{t}) & t \neq 0 \\ 0 & t = 0. \end{cases}$$

Notice that while f is certainly continuous for all $t \in \mathbb{R} \setminus \{0\}$, neither the left-sided nor the right-sided limit exist at 0 as

$$\lim_{x \rightarrow \infty} \sin(x) \quad \text{does not exist}$$

and

$$\lim_{x \rightarrow -\infty} \sin(x) \quad \text{does not exist.}$$

However, the restriction of f to the time scale \mathbb{N}_0 is regulated because \mathbb{N}_0 does not have any left-dense or right-dense points.

Definition 2.6 A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* (or right-dense continuous) at a point $t_0 \in \mathbb{T}$ if when t_0 is left dense then the left-sided limit of f exists at t_0 and when t_0 is right-dense then f is continuous at t_0 , that is if it is regulated

and continuous from the right. A function that is rd-continuous at all points in \mathbb{T} is called an rd-continuous function.

Example 2.7 Let

$$\mathbb{T} := \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{2\} \cup \left\{2 - \frac{1}{n} : n \in \mathbb{N}\right\}$$

and define $f : \mathbb{T} \rightarrow \{0, 1\}$ by

$$f(t) := \begin{cases} t & t \neq 2 \\ 0 & t = 2 \end{cases}$$

f is obviously continuous at the isolated points of \mathbb{T} , so we'll focus our attention on the right-dense point 0 and the left dense point 2. The right-sided limit of f at 0 exists and equals $f(0)$. So f is continuous at 0. While f is discontinuous at 2, the left-sided limit of f exists at 2. We can see that although f is not continuous, f is rd-continuous.

Theorem 2.8 *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $g : \mathbb{T} \rightarrow \mathbb{T}$. Then*

- (i) *If f is continuous, then f is rd-continuous.*
- (ii) *If f is continuous and g is regulated or rd-continuous, then $f \circ g$ is respectively regulated or rd-continuous.*

2.2 DIFFERENTIATION AND INTEGRATION

Definition 2.9 The set \mathbb{T}^κ is defined as

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

The times scales derivative of a suitable function can't be defined for all points on all times scales. In particular, we can't define it at the finite supremum of a time scale. However, we can define the times scales derivative at all points of \mathbb{T}^κ . As we'll see in the next definition, \mathbb{T}^κ is needed for the times scale derivative to make sense.

Definition 2.10 A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be Δ -differentiable at $t \in \mathbb{T}^\kappa$ provided

$$f^\Delta(t) := \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad s \in \mathbb{T} \setminus \{\sigma(t)\}$$

exists. $f^\Delta(t)$ is called the Δ -derivative of f at t . The function f is called Δ -differentiable on \mathbb{T}^κ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is called the Δ -derivative of f on \mathbb{T}^κ .

Notice that while we have stipulated that s cannot equal $\sigma(t)$, it could happen that $s = t$. When a point t on the time scale is right scattered, the Δ -derivative at t is the slope of the line through the points $(t, f(t))$ and $(\sigma(t), f(\sigma(t)))$. When t is right dense, the Δ -derivative at t is similar the usual definition of the derivative.

Theorem 2.11 Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. If f is Δ -differentiable at t , then:

(i)

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$$

and

(ii) f is continuous at t .

Proof: (i) Assume that f is differentiable at a point $t \in \mathbb{T}$. Observe that when t is right-dense $\mu(t) = 0$ and $\sigma(t) = t$, so we get

$$f(\sigma(t)) = f(t) = f(t) + \mu(t)f^\Delta(t). \quad \checkmark$$

Consider the case when t is right-scattered. Then, because $\sigma(t) \neq t$ and f is continuous at t , we can rewrite the derivative at t as

$$\begin{aligned} f^\Delta(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \end{aligned}$$

So

$$\begin{aligned} f(\sigma(t)) - f(t) &= \mu(t)f^\Delta(t) \\ \implies f(\sigma(t)) &= f(t) + \mu(t)f^\Delta(t). \quad \checkmark \end{aligned}$$

(ii) First observe that, for any $s \in \mathbb{T}$,

$$\sigma(t) - s = (\sigma(t) - t) + (t - s) = \mu(t) + (t - s). \quad (2.1)$$

Let $1 > \epsilon > 0$, and define $\epsilon' = \epsilon[1 + |f^\Delta(t)| + \mu(t)]^{-1}$. Then $1 > \epsilon' > 0$. By the definition of the derivative, given $1 > \epsilon > 0$ there exists $\delta > 0$ such that when $|t - s| < \delta$, $s \neq \sigma(t)$ we have that

$$\begin{aligned} &\left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f^\Delta(t) \right| < \epsilon' \\ \implies &\left| \frac{f(\sigma(t)) - f(s) - (\sigma(t) - s)f^\Delta(t)}{\sigma(t) - s} \right| < \epsilon' \\ \implies &|f(\sigma(t)) - f(s) - (\sigma(t) - s)f^\Delta(t)| < \epsilon' |\sigma(t) - s|. \quad (2.2) \end{aligned}$$

We will proceed to use (2.1) and (2.2) to show that $|f(t) - f(s)| < \epsilon$. Let $|t - s| < \min\{\epsilon', \delta\}$.

$$|f(t) - f(s)| =$$

$$|f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^\Delta(t)(\sigma(t) - s) - f^\Delta(t)(\sigma(t) - s)|$$

Use equation 2.1 to rewrite this as

$$|f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^\Delta(t)[\mu(t) + (t - s)]$$

$$- f^\Delta(t)(\sigma(t) - s)|$$

$$= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^\Delta(t)\mu(t)$$

$$+ f^\Delta(t)(t - s) - f^\Delta(t)(\sigma(t) - s)|$$

$$= | \{ f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \} + f^\Delta(t)(t - s) |$$

because by part (i) we have that $f(\sigma(t)) - f(t) - \mu(t)f^\Delta(t) = 0$. We can estimate the last line by

$$\leq \epsilon' |\sigma(t) - s| + |t - s| |f^\Delta(t)|$$

$$= \epsilon' |\mu(t) + (t - s)| + |t - s| |f^\Delta(t)|$$

$$\leq \epsilon' \mu(t) + \epsilon' |t - s| + |t - s| |f^\Delta(t)|$$

$$< \epsilon' \mu(t) + \epsilon' + \epsilon' |f^\Delta(t)|$$

$$= \epsilon' [1 + |f^\Delta(t)| + \mu(t)] = \epsilon$$

Hence $|f(t) - f(s)| < \epsilon$. \square

Part (ii) of Theorem 2.11 is not too surprising because it is the same in the real case. Part (i) on the other hand is only useful when t is right scattered, otherwise the statement becomes $f(t) = f(t)$.

The next theorem gives some rules for Δ -differentiation. Notice that parts (i) and (ii) below are the same as for the real case while part (iii) is slightly different.

Theorem 2.12 *Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be Δ -differentiable at $t \in \mathbb{T}^\kappa$. Then the following hold:*

(i) *The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For any constant $\alpha \in \mathbb{R}$, $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) *The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

Proof: We will only give the proof of (iii). Let $\epsilon > 0$ and define

$$\epsilon' = \frac{\epsilon}{1 + |f(t)| + |g(\sigma(t))| + |g^\Delta(t)|}.$$

Then there exists $\delta > 0$ such that when $|t - s| < \delta$ we have the following:

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon' |\sigma(t) - s|$$

$$|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \leq \epsilon' |\sigma(t) - s|$$

and by Theorem 2.11

$$|f(t) - f(s)| \leq \epsilon'.$$

So

$$\begin{aligned} & |(fg)(\sigma(t)) - (fg)(s) - [f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t)](\sigma(t) - s)| \\ &= |[f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)]g(\sigma(t)) \\ &\quad + [g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)]f(t)| \end{aligned}$$

$$\begin{aligned}
& +[g(\sigma(t) - g(s) - g^\Delta(t)(\sigma(t) - s))[f(s) - f(t)] \\
& +(\sigma(t) - s)g^\Delta(t)[f(s) - f(t)] \\
& \leq \epsilon' |\sigma(t) - s| |g(\sigma(t))| + \epsilon' |\sigma(t) - s| |f(t)| \\
& +(\epsilon')^2 |\sigma(t) - s| + \epsilon' |\sigma(t) - s| |g^\Delta(t)| \\
& = \epsilon' |\sigma(t) - s| (|g(\sigma(t))| + |f(t)| + \epsilon' + |g^\Delta(t)|) .
\end{aligned}$$

Assuming that we have chosen ϵ small enough that $\epsilon' < 1$, we get that the above is

$$\begin{aligned}
& < \epsilon' |\sigma(t) - s| (|g(\sigma(t))| + |f(t)| + 1 + |g^\Delta(t)|) \\
& = \epsilon' |\sigma(t) - s| (|1 + |f(t)| + g(\sigma(t))| + |g^\Delta(t)|) \\
& = \epsilon |\sigma(t) - s| .
\end{aligned}$$

We have shown that

$$(fg)^\Delta(t) = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t). \quad (2.3)$$

To get the second equality, switch f and g in equation 2.3. \square

It would be prudent now to give some attention to the chain rule. For functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ the chain rule is

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

However, for arbitrary time scales this does not hold as we will show in the following example which is taken from an exercise given in Bohner and Peterson [4].

Example 2.13 Let $\mathbb{T} = \mathbb{Z}$ and let $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(t) = g(t) = t^2 = t \cdot t$.

Using the product rule, Theorem 2.12, we find that

$$f^\Delta(t) = g^\Delta(t) = t + \sigma(t) = t + t + 1 = 2t + 1.$$

Notice that

$$(f \circ g)(t) = (t^2)^2 = t^4 = f(t)g(t).$$

So again using the product rule, we get that

$$\begin{aligned} (f \circ g)^\Delta(t) &= (f(t)g(t))^\Delta \\ &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) \\ &= (2t+1)t^2 + (\sigma(t))^2(2t+1) \\ &= (2t+1)t^2 + (t+1)^2(2t+1) \\ &= 2t^3 + t^2 + (t^2 + 2t + 1)(2t+1) \\ &= 2t^3 + t^2 + 2t^3 + 4t^2 + 2t + t^2 + 2t + 1 \\ &= 4t^3 + 6t^2 + 4t + 1. \end{aligned}$$

Now we compute

$$\begin{aligned} f^\Delta(g(t))g^\Delta(t) &= (2t^2+1)(2t+1) \\ &= 4t^3 + 2t^2 + 2t + 1. \end{aligned}$$

If we assume that $(f \circ g)^\Delta(t) = f^\Delta(g(t))g^\Delta(t)$, then we obtain

$$4t^3 + 6t^2 + 4t + 1 = 4t^3 + 2t^2 + 2t + 1$$

$$\implies 4t^2 + 2t = 0$$

$$\implies t \in \left\{0, -\frac{1}{2}\right\}.$$

Thus $(f \circ g)^\Delta(t) = f^\Delta(g(t))g^\Delta(t)$ holds for only one point in \mathbb{Z} , namely 0.

Despite this situation, there are a few chain rules for time scales, each of which is weaker than that for the real numbers. We will use only one of these chain rules in this paper and so we will only present one. Information on the other chain rules may

be found in Bohner and Peterson [4]. Before presenting the chain rule, Theorem 2.15, we should give some attention to the following question: For which strictly increasing functions γ is $\gamma(\mathbb{T})$ a time scale? We answer this question in the following new proposition.

Proposition 2.14 *Let $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ be a strictly increasing function. Then $\gamma(\mathbb{T})$ is a time scale if and only if*

(i) γ is continuous

and

(ii) γ is bounded above (respectively below) only when \mathbb{T} is bounded above (respectively below).

Proof: We will do the forward direction using a contrapositive proof. Suppose that γ is not continuous and $\gamma(\mathbb{T})$ is a time scale. Then there exists a point $a \in \mathbb{T}$ that is either left-dense, right-dense, or both such that γ is discontinuous at a . With out loss of generality we will assume that a is left-dense but not right-dense. Let $\{t_n\}_{n \in \mathbb{N}}$, $t_n \in \mathbb{T}$, denote a strictly increasing sequence converging to a . Then because γ is strictly increasing we know that $\gamma(a)$ is an upper bound for the sequence $\{\gamma(t_n)\}_{n \in \mathbb{N}}$. So $\{\gamma(t_n)\}_{n \in \mathbb{N}}$ must converge to a finite supremum, moreover

$$\sup\{\gamma(t_n)\}_{n \in \mathbb{N}} < \gamma(a) \quad (2.4)$$

because γ is discontinuous at a . Again using the fact that γ is strictly increasing, we see that

$$\sup\{\gamma(t_n)\}_{n \in \mathbb{N}} \notin \{\gamma(t_n)\}_{n \in \mathbb{N}}.$$

Because $\gamma(\mathbb{T})$ is closed,

$$\sup\{\gamma(t_n)\}_{n \in \mathbb{N}} \in \gamma(\mathbb{T}).$$

Let $b \in \mathbb{T}$ be such that $\gamma(b) = \sup\{\gamma(t_n)\}_{n \in \mathbb{N}}$. Then because γ is increasing $b = \sup\{t_n\}_{n \in \mathbb{N}}$ hence $b \leq a$. From (2.4) we get that $b \neq a$. Thus $b < a$. This contradicts

our assumption that $\{t_n\}_{n \in \mathbb{N}}$ converges to a . Thus by the contrapositive when $\gamma(\mathbb{T})$ is a time scale, γ is continuous.

Now suppose that $\sup \mathbb{T} = \infty$ and $\gamma(\mathbb{T}) < M < \infty$ for some $M \in \mathbb{R}$. Then we can find an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \in \mathbb{T}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$. Once again we use the fact that γ is strictly increasing to determine that $\{\gamma(t_n)\}_{n \in \mathbb{N}}$ converges to a finite supremum with

$$\sup\{\gamma(t_n)\}_{n \in \mathbb{N}} \notin \{\gamma(t_n)\}_{n \in \mathbb{N}}.$$

So

$$\sup\{\gamma(t_n)\}_{n \in \mathbb{N}} \notin \gamma(\mathbb{T})$$

and hence $\gamma(\mathbb{T})$ is once again not closed. By the contrapositive, when $\gamma(\mathbb{T})$ is a time scale, we get (ii).

Suppose that (i) and (ii) hold, and let a be a limit point of $\gamma(\mathbb{T})$. Then without loss of generality we can assume there exists an increasing sequence $\{a_n\}_{n \in \mathbb{N}}$, $a_n \in \gamma(\mathbb{T})$, converging to a . Let $\{t_n\}_{n \in \mathbb{N}}$, $t_n \in \mathbb{T}$, be a sequence such that $\gamma(t_n) = a_n$. Then $\{t_n\}_{n \in \mathbb{N}}$ is also an increasing sequence and $\lim_{n \rightarrow \infty} \gamma(t_n) = a$. Suppose $\sup \mathbb{T} = \infty$, then by (ii), $\sup \gamma(\mathbb{T}) = \infty$. Thus $\lim_{n \rightarrow \infty} t_n < \infty$, otherwise $a = \infty$. Let $t_0 = \lim_{n \rightarrow \infty} t_n$, then the closure of \mathbb{T} gives us that $t_0 \in \mathbb{T}$. Finally, the continuity of γ implies that $\gamma(t_0) = a$, hence $a \in \gamma(\mathbb{T})$. \square

Theorem 2.15 *Chain Rule*

Let $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}} := \gamma(\mathbb{T})$ is a time scale. Let $\omega : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$ and let $\omega^{\tilde{\Delta}}$ denote the derivative of ω on $\tilde{\mathbb{T}}$. If $\gamma^{\Delta}(t)$ and $\omega^{\tilde{\Delta}}(\gamma(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then

$$(\omega \circ \gamma)^{\Delta} = (\omega^{\tilde{\Delta}} \circ \gamma) \gamma^{\Delta}.$$

Proof: Let $1 > \epsilon > 0$ and define $\epsilon' := \epsilon \left[1 + |\gamma^\Delta(t)| + |\omega^{\tilde{\Delta}}(\gamma(t))| \right]^{-1}$. Then $1 > \epsilon' > 0$. The Δ -differentiability of $\gamma(t)$ implies that there exists $\delta_1 > 0$ such that for $t, s \in \mathbb{T}$, when $|t - s| < \delta_1$, we have

$$|\gamma(\sigma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^\Delta(t)| \leq \epsilon' |\sigma(t) - s|.$$

Similarly, the Δ -differentiability of $\omega(t)$ implies that there exists $\delta_2 > 0$ such that for $r, \gamma(t) \in \tilde{\mathbb{T}}$, when $|\gamma(t) - r| < \delta_2$ we have

$$|\omega(\tilde{\sigma}(\gamma(t))) - \omega(r) - (\tilde{\sigma}(\gamma(t)) - r)\omega^{\tilde{\Delta}}(t)(\gamma(t))| \leq \epsilon' |\tilde{\sigma}(\gamma(t)) - r|$$

where $\tilde{\sigma}(t)$ denotes the forward jump operator on $\tilde{\mathbb{T}}$. Let

$$\delta := \min \{ \delta_1, t - \gamma^{-1}(\gamma(t) - \delta_2), \gamma^{-1}(\gamma(t) + \delta_2) - t \}.$$

Notice that because γ is strictly increasing

$$\gamma(t) > \gamma(t) - \delta_2 \implies t > \gamma^{-1}(\gamma(t) - \delta_2) \implies t - \gamma^{-1}(\gamma(t) - \delta_2) > 0.$$

Similarly $\gamma^{-1}(\gamma(t) + \delta_2) - t > 0$. Then for $s \in \mathbb{T}$ such that $|t - s| < \delta$, we also have that $|t - s| < \delta_1$. For such an s we also have

$$\begin{aligned} |t - s| &< t - \gamma^{-1}(\gamma(t) - \delta_2) \\ t - s &< t - \gamma^{-1}(\gamma(t) - \delta_2) \\ -s &< -\gamma^{-1}(\gamma(t) - \delta_2) \\ s &> \gamma^{-1}(\gamma(t) - \delta_2) \\ \gamma(s) &> \gamma(t) - \delta_2 \\ \gamma(s) - \gamma(t) &> -\delta_2 \\ \gamma(t) - \gamma(s) &< \delta_2 \end{aligned}$$

Similarly we can use $|t - s| < \gamma^{-1}(\gamma(t) + \delta_2) - t$ to show that $\delta_2 < \gamma(t) - \gamma(s)$. So $|t - s| < \delta$ implies $|\gamma(t) - \gamma(s)| < \delta_2$. Thus

$$\left| \omega(\gamma(\sigma(t))) - \omega(\gamma(s)) - (\sigma(t) - s)[\omega^{\tilde{\Delta}}(\gamma(t))\gamma^\Delta(t)] \right|$$

$$\begin{aligned}
&= \left| \omega(\gamma(\sigma(t))) - \omega(\gamma(s)) - (\tilde{\sigma}(\gamma(t)) - \gamma(s))\omega^{\tilde{\Delta}}(\gamma(t)) \right. \\
&\quad \left. + [\tilde{\sigma}(\gamma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^{\Delta}(t)]\omega^{\tilde{\Delta}}(\gamma(t)) \right| \\
&\leq \left| \omega(\tilde{\sigma}(\gamma(t))) - \omega(\gamma(s)) - (\tilde{\sigma}(\gamma(t)) - \gamma(s))\omega^{\tilde{\Delta}}(\gamma(t)) \right| \\
&\quad + \left| [\tilde{\sigma}(\gamma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^{\Delta}(t)]\omega^{\tilde{\Delta}}(\gamma(t)) \right| \\
&< \epsilon' |\tilde{\sigma}(\gamma(t)) - \gamma(s)| + \epsilon' |\sigma(t) - s| \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \\
&= \epsilon' |\tilde{\sigma}(\gamma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^{\Delta}(t) + (\sigma(t) - s)\gamma^{\Delta}(t)| \\
&\quad + \epsilon' |\sigma(t) - s| \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \\
&\leq \epsilon' \left\{ |\tilde{\sigma}(\gamma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^{\Delta}(t)| + |(\sigma(t) - s)| |\gamma^{\Delta}(t)| \right. \\
&\quad \left. + |\sigma(t) - s| \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \right\}
\end{aligned}$$

Again because γ is strictly increasing, we get that $\tilde{\sigma}(\gamma(t)) = \gamma(\sigma(t))$. So we can rewrite this last line as

$$\begin{aligned}
&= \epsilon' \left\{ |\gamma(\sigma(t)) - \gamma(s) - (\sigma(t) - s)\gamma^{\Delta}(t)| + |(\sigma(t) - s)| |\gamma^{\Delta}(t)| \right. \\
&\quad \left. + |\sigma(t) - s| \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \right\} \\
&< \epsilon' \left\{ \epsilon' |\sigma(t) - s| + |(\sigma(t) - s)| |\gamma^{\Delta}(t)| + |\sigma(t) - s| \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \right\} \\
&= \epsilon' \left\{ \epsilon' + |\gamma^{\Delta}(t)| + \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \right\} |\sigma(t) - s| \\
&< \epsilon' \left\{ 1 + |\gamma^{\Delta}(t)| + \left| \omega^{\tilde{\Delta}}(\gamma(t)) \right| \right\} |\sigma(t) - s| \\
&= \epsilon |\sigma(t) - s|. \quad \square
\end{aligned}$$

We will now revisit Example 2.13 and confirm that

$$(f \circ g)^{\Delta}(t) = (f^{\hat{\Delta}} \circ g)(t)g^{\Delta}(t).$$

First note that $g(\mathbb{Z}) = \{t^2 : t \in \mathbb{Z}\}$. So

$$\begin{aligned} (f^{\hat{\Delta}} \circ g)(t) &= \frac{f(\hat{\sigma}(g(t))) - f(g(t))}{\hat{\sigma}(g(t)) - g(t)} = \frac{f(\hat{\sigma}(t^2)) - f(t^2)}{\hat{\sigma}(t^2) - t^2} = \frac{f((t+1)^2) - f(t^2)}{(t+1)^2 - t^2} \\ &= \frac{(t+1)^4 - t^4}{(t+1)^2 - t^2} = \frac{(t^4 + 4t^3 + 6t^2 + 4t + 1) - t^4}{(t^2 + 2t + 1) - t^2} = \frac{4t^3 + 6t^2 + 4t + 1}{2t + 1}. \end{aligned}$$

Recalling that $g^{\Delta}(t) = 2t + 1$, we get that

$$(f^{\hat{\Delta}} \circ g)(t)g^{\Delta}(t) = 4t^3 + 6t^2 + 4t + 1$$

which is what we have previously found $(f \circ g)^{\Delta}(t)$ to be from the product rule.

We now set the stage for integration on time scales.

Definition 2.16 Let \mathbb{T} be a time scale, and let $a, b \in \mathbb{T}$ such that $a < b$. A *partition* of $[a, b]$ (where $[a, b]$ denotes the interval on the time scale) is any finite ordered subset

$$P := \{t_0, t_1, \dots, t_n\} \quad \text{where} \quad a = t_0 < t_1 < \dots < t_n = b, \quad t_i \in \mathbb{T}.$$

More precisely P separates the interval $[a, b] \cap \mathbb{T}$ into a collection of subsets:

$$[t_0, t_1) \cap \mathbb{T}, [t_1, t_2) \cap \mathbb{T}, \dots, [t_{n-2}, t_{n-1}) \cap \mathbb{T}, [t_{n-1}, t_n] \cap \mathbb{T}.$$

We denote the collection of all such partitions as $\mathcal{P}(a, b)$.

We will now define what it means to be a δ -partition.

Definition 2.17 Let $\delta > 0$. A partition $P \in \mathcal{P}(a, b)$ given by $a = t_0 < t_1 < \dots < t_n = b$ is called a δ -*partition* if for each $i \in \{1, 2, \dots, n\}$

$$t_i - t_{i-1} \leq \delta$$

whenever $(t_{i-1}, t_i) \cap \mathbb{T} \neq \emptyset$. We denote the collection of all such partitions for a given δ by $\mathcal{P}_{\delta}(a, b)$.

Notice that it may occur that $t_i - t_{i-1} > \delta$ but only if $\rho(t_i) = t_{i-1}$. We illustrate this in the next example.

Example 2.18 Consider the time scale $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\} \cup \{0\}$. Suppose $a = 0$ and $b = 32$. Let P_α be the partition of $[0, 32]$ on \mathbb{T} given by $\{0, 1, 2, 4, 8, 16, 32\}$ and similarly let P_β be given by $\{0, 1, 8, 16, 32\}$. P_α is a δ -partition of $[0, 32]$ for all $\delta > 0$. This occurs because $(t_{i-1}, t_i) \cap \mathbb{T} = \emptyset$ for all $i \in \{1, 2, \dots, n\}$. However, this is not the case for P_β . While $(t_{i-1}, t_i) \cap \mathbb{T} = \emptyset$ for all $i \neq 2$, when $i = 2$ we have that $(t_1, t_2) \cap \mathbb{T} = (1, 8) \cap \mathbb{T} = \{2, 4\} \neq \emptyset$. So P_β is a δ -partition of $[0, 32]$ only when $\delta \geq 8 - 1 = 7$.

Next we have the definition of the Riemann Δ -integral which is very similar the the usual Riemann integral.

Definition 2.19 Let $f : [a, b] \cap \mathbb{T} \rightarrow \mathbb{C}$ be a bounded function and let $P \in \mathcal{P}(a, b)$. For each pair t_{i-1} and t_i in P , choose a point $\tau_i \in \mathbb{T}$ such that $t_{i-1} \leq \tau_i < t_i$. We call the sum

$$S := \sum_{i=1}^n f(\tau_i)(t_i - t_{i-1})$$

a *Riemann Δ -sum* corresponding to P . We say that f is *Riemann Δ -integrable* on $[a, b]$ if there exists a number $I \in \mathbb{C}$ with the following property: Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S - I| < \varepsilon$$

for every *Riemann Δ -sum* of f corresponding to any $P \in \mathcal{P}_\delta(a, b)$. The complex number I is called the *Riemann Δ -integral* (or simply the *Δ -integral*) of f on $[a, b]$ and is denoted

$$\int_a^b f(t) \Delta t.$$

Now we will compute some Δ -integrals directly from the definition to give the reader an idea of what it is for time scales other than \mathbb{R} in which case it is just the usual Riemann integral.

Example 2.20 Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be defined by $f(t) = t^2$. Suppose that

$$\mathbb{T} = \{2^n : n \in \mathbb{N}_0\} \cup \{0\}$$

and consider the integral

$$\int_0^{32} f(t) \Delta t. \quad (2.5)$$

We have seen in this previous example that the partition P_α given by $\{0, 1, 2, 4, 8, 16, 32\}$ is in $\mathcal{P}_\delta(0, 32)$ for all $\delta > 0$. Because we must choose our $\tau_i \in \mathbb{T}$ such that $t_{i-1} \leq \tau_i < t_i$, our only choice is to let

$$\tau_1 = 0, \quad \tau_i = 2^{i-2}, \quad 2 \leq i \leq 6.$$

Thus

$$\begin{aligned} \int_0^{32} f(t) \Delta t &= f(0) + \sum_{i=2}^6 f(2^{i-2})(2^{i-1} - 2^{i-2}) = \sum_{i=1}^5 f(2^{i-1})(2^i - 2^{i-1}) \\ &= \sum_{i=1}^5 2^{2i-2}(2^i - 2^{i-1}) = \sum_{i=1}^5 (2^{3i-2} - 2^{3i-3}) = \sum_{i=1}^5 2^{3i-3} \\ &= 1 + 8 + 64 + 512 + 4096 = 4681. \end{aligned}$$

Compare this with the real case

$$\int_0^{32} t^2 dt = \frac{1}{3}(32^3 - 0) \approx 10,922.66.$$

Now suppose that

$$\mathbb{T} = \left\{ \frac{32}{n} : n \in \mathbb{N} \right\} \cup \{0\}$$

and let P_n denote the partition of $[0, 32]$ given by

$$\left\{0, \frac{32}{n}, \frac{32}{n-1}, \dots, 32\right\}.$$

Notice that for this time scale $P_n \in \mathcal{P}_\delta(0, 32)$ so long as $\delta \geq \frac{32}{n} - 0 = \frac{32}{n}$. We must choose our as

$$\tau_1 \in [0, \frac{32}{n}), \quad \tau_i = \frac{32}{n-i+2}, \quad 2 \leq i \leq n.$$

If we let S_n denote the Riemann Δ -sum corresponding to P_n , then we get that

$$\begin{aligned} S_n &= f(\tau_1) \frac{32}{n} + \sum_{i=2}^n f\left(\frac{32}{n-i+2}\right) \left(\frac{32}{n-i+1} - \frac{32}{n-i+2}\right) \\ &= f(\tau_1) \frac{32}{n} + \sum_{i=0}^{n-2} f\left(\frac{32}{n-i}\right) \left(\frac{32}{n-i-1} - \frac{32}{n-i}\right) \\ &= \tau_1^2 \frac{32}{n} + \sum_{i=0}^{n-2} \frac{32768}{(n-i)^2} \left(\frac{1}{(n-i-1)(n-i)}\right) \\ &= \tau_1^2 \frac{32}{n} + \sum_{i=0}^{n-2} \frac{32768}{(n-i)^3(n-i-1)} \end{aligned}$$

Note that

$$0 < \tau_1^2 \frac{32}{n} < \left(\frac{32}{n}\right)^3$$

so

$$\lim_{n \rightarrow \infty} \tau_1^2 \frac{32}{n} = 0.$$

We can evaluate the second term of S_n as $n \rightarrow \infty$ numerically, in this case we used Maple ¹ to evaluate

$$\sum_{i=0}^{n-2} \frac{32768}{(n-i)^3(n-i-1)}$$

for $n = 10,000$. We get that

$$\int_0^{32} f(t) \Delta t \approx 5013.79989477$$

with error bounded by $\frac{32^3}{10,000^3} < 10^{-7}$.

¹Maple 8, Waterloo Maple Inc.

It is worth noting that for each of our two time scales in the previous example, the Δ -integral was less than that in the real case. In fact, for any increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ and time scale \mathbb{T} ,

$$\int_a^b g(t) \Delta t \leq \int_a^b f(t) dt$$

where $g(t)$ is the restriction of f to \mathbb{T} . Notice also that the Δ -integral on a discrete interval of a time scale is just a weighted sum with the weight of $t \in \mathbb{T}$ being given by $\mu(t)$. If there is a single right-dense point in our interval, it is weighted by a factor of zero. The following proposition states this formally.

Proposition 2.21 *Let $f : \mathbb{T} \rightarrow \mathbb{R}$, then*

$$\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t) f(t). \quad (2.6)$$

Proof: Suppose $\sigma(t) = t$. Then

$$\int_t^t f(\tau) \Delta \tau = 0 = \mu(t) f(t).$$

Now suppose that $\sigma(t) > t$. Then $\{t, \sigma(t)\}$ is a δ -partition for every $\delta > 0$. Thus $\tau_1 = t$ and

$$\int_t^{\sigma(t)} f(\tau) \Delta \tau = f(\tau_1)(\sigma(t) - t) = \mu(t) f(t). \quad \square$$

Theorem 2.22 *Every rd-continuous function f on the interval $[a, b]$ is Δ -integrable on $[a, b]$.*

Proof: First note that the backward jump operator ρ may be canonically extended to the real interval $[\inf \mathbb{T}, \sup \mathbb{T}]$ by

$$\rho(x) := \sup\{s \in \mathbb{T} : s < x, x \in \mathbb{R}\}.$$

Let g be an extension of f to the real line defined by

$$g(x) = \begin{cases} f(x) & x \in \mathbb{T} \\ f(\rho(x)) & x \in \mathbb{R} \setminus \mathbb{T}. \end{cases}$$

Then

$$\int_a^b f(t) \Delta t = \int_a^b g(x) dx$$

provided one of these integrals exists. Because f is rd-continuous, we know that the set of discontinuities of g is at most countable. So g is piecewise continuous on $[a, b]$ (here we mean the interval on the real line) and thus integrable on the same interval. Therefore, f must also be integrable on $[a, b] \cap \mathbb{T}$. \square

The following theorem establishes the linearity of the Δ -integral. The linearity of time scales transforms, which will be introduced later, follows directly from this result.

Theorem 2.23 *Let f and g be Δ -integrable functions on $[a, b]$, and let $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g$ is Δ -integrable and*

$$\int_a^b (\alpha f + \beta g)(t) \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t.$$

Theorem 2.22 along with Theorem 2.11, tells us that a differentiable function is also integrable, a fact which is used in the Fundamental Theorem of Calculus.

Theorem 2.24 *Fundamental Theorem of Calculus*

a) *Let g be a function defined on $[a, b] \cap \mathbb{T}$ such that g is Δ -differentiable on $[a, b]$.*

If g^Δ is Δ -integrable on $[a, b)$, then

$$\int_a^b g^\Delta(t) \Delta t = g(b) - g(a) \quad (2.7)$$

b) Let f be Δ -integrable on $[a, b]$. For $t \in [a, b] \cap \mathbb{T}$ define

$$F(t) := \int_a^t f(\tau) \Delta \tau.$$

Then $F(t)$ is continuous on $[a, b)$. If $t_0 \in [a, b)$ and if f is continuous at t_0 when t_0 is right-dense, then F is Δ -differentiable at t_0 and

$$F^\Delta(t_0) = f(t_0).$$

Theorem 2.25 (*Change of Variable*)

Let $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}} = \gamma(\mathbb{T})$ is a time scale. Let $\tilde{\Delta}$ denote the Δ -derivative on $\tilde{\mathbb{T}}$. Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -integrable on each finite interval of \mathbb{T} . Suppose also that γ is Δ -differentiable and γ^Δ is Δ -integrable on each finite interval of \mathbb{T} . Then if $f\gamma^\Delta$ is Δ -integrable, we have that

$$\int_a^b f(t) \gamma^\Delta(t) \Delta t = \int_{\gamma(a)}^{\gamma(b)} (f \circ \gamma^{-1})(s) \tilde{\Delta} s.$$

for $a, b \in \mathbb{T}$.

Proof: Let

$$F(t) := \int_a^t f(t) \gamma^\Delta(t) \Delta t.$$

Then by the previous theorem $F^\Delta = f\gamma^\Delta$ for right-scattered points and those right-dense points at which f is continuous. So

$$\begin{aligned}
 \int_a^b f(t)\gamma^\Delta(t)\Delta t &= \int_a^b F^\Delta(t)\Delta t \\
 &= F(a) - F(b) \\
 &= (F \circ \gamma^{-1})(\gamma(b)) - (F \circ \gamma^{-1})(\gamma(a)) \\
 &= \int_{\gamma(a)}^{\gamma(b)} (F \circ \gamma^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s.
 \end{aligned}$$

Now we employ the chain rule, Theorem 2.15, to get

$$\begin{aligned}
 \int_a^b f(t)\gamma^\Delta(t)\Delta t &= \int_{\gamma(a)}^{\gamma(b)} (F^\Delta \circ \gamma^{-1})(s)(\gamma^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
 &= \int_{\gamma(a)}^{\gamma(b)} ((f\gamma^\Delta) \circ \gamma^{-1})(s)(\gamma^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
 &= \int_{\gamma(a)}^{\gamma(b)} (f \circ \gamma^{-1})(s) \left[(\gamma^\Delta \circ \gamma^{-1})(\gamma^{-1})^{\tilde{\Delta}} \right] (s)\tilde{\Delta}s.
 \end{aligned}$$

Again we use the chain rule,

$$\begin{aligned}
 \int_a^b f(t)\gamma^\Delta(t)\Delta t &= \int_{\gamma(a)}^{\gamma(b)} (f \circ \gamma^{-1})(s)(\gamma \circ \gamma^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
 &= \int_{\gamma(a)}^{\gamma(b)} (f \circ \gamma^{-1})(s)\tilde{\Delta}s. \quad \square
 \end{aligned}$$

Theorem 2.26 (*Integration by Parts*)

Let $a, b \in \mathbb{T}$ and let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be rd-continuous. Then

$$\begin{aligned}
 (i) \quad & \int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t. \\
 (ii) \quad & \int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.
 \end{aligned}$$

Proof:

(i)

$$\begin{aligned} & \int_a^b f(\sigma(t))g^\Delta(t)\Delta t = \\ & \int_a^b f(\sigma(t))g^\Delta(t)\Delta t + \int_a^b f^\Delta(t)g(t)\Delta t - \int_a^b f^\Delta(t)g(t)\Delta t \\ & = \int_a^b [f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t)]\Delta t - \int_a^b f^\Delta(t)g(t)\Delta t \end{aligned}$$

The product rule, Theorem 2.12, tells us that $f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t) = (fg)^\Delta(t)$.

This yields

$$\int_a^b (fg)^\Delta(t)\Delta t - \int_a^b f^\Delta(t)g(t)\Delta t \quad (2.8)$$

So applying the Fundamental Theorem of Calculus, Theorem 2.24, to equation (2.8) gives

$$(fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t.$$

The proof of (ii) proceeds by switching f and g in part (i). \square

2.3 POLYNOMIAL AND EXPONENTIAL FUNCTIONS

While 0 is a derivative of 1 and 1 is a derivative of t , it is not possible in the time scales calculus to figure out what t is the derivative of in a way that provides a closed form for arbitrary time scales. For example, t is a derivative of $t^2/2$ on \mathbb{R} , but for an arbitrary time scale

$$\left(\frac{t^2}{2}\right)^\Delta = \left(\frac{1}{2} \cdot t \cdot t\right)^\Delta$$

which by the product rule (Theorem 2.12) is

$$= \frac{1}{2} (t \cdot t^\Delta + t^\Delta \cdot \sigma(t)) = \frac{t + \sigma(t)}{2}.$$

Furthermore, $\frac{t+\sigma(t)}{2}$ is not necessarily differentiable even if $\sigma(t)$ is continuous. What we would like to do is come up with a set of functions that work like powers of

t under the times scales calculus. We do this by starting with 1 and recursively defining functions by integration. As we'll see, there are at least two ways of doing this.

Definition 2.27 We define the *time scales polynomials*

$g_k, h_k : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ for $k \in \mathbb{N}_0$ as follows:

$$\begin{aligned} g_0(t, s) &= h_0(t, s) \equiv 1 & \forall s, t \in \mathbb{T} \\ g_{k+1}(t, s) &= \int_s^t g_k(\sigma(\tau), s) \Delta\tau & \forall s, t \in \mathbb{T} \\ h_{k+1}(t, s) &= \int_s^t h_k(\tau, s) \Delta\tau & \forall s, t \in \mathbb{T} \end{aligned}$$

Theorem 2.28 Let $s \in \mathbb{T}$ and let $h_k^\Delta(t, s)$ denote the Δ -derivative of $h_k(t, s)$ for fixed s . Then the following hold:

(i)

$$g_1(t, s) = h_1(t, s) = t - s \quad \forall t \in \mathbb{T}$$

(ii)

$$g_2(t, s) = h_2(s, t) \quad \forall t \in \mathbb{T}$$

(iii)

$$h_k^\Delta(t, s) = h_{k-1}(t, s) \quad \forall t \in \mathbb{T}^\kappa, k \in \mathbb{N}$$

(iv)

$$g_k^\Delta(t, s) = g_{k-1}(\sigma(t), s) \quad \forall t \in \mathbb{T}^\kappa, k \in \mathbb{N}$$

The second variable, s , of our polynomials centers them at the point s . Notice that when our time scale is \mathbb{R} , $g_k(t, s) = h_k(t, s) = \frac{1}{k!}(t - s)^k$ for all $k \in \mathbb{N}_0$.

Now we would like to come up with a function that serves the same purpose as the exponential function does in the real case. That is, we would like to find a solution to the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1.$$

Before we can define find such a function we need several concepts in order for the definition to make sense.

Definition 2.29 For $h > 0$ the *Hilger complex plane* is defined as

$$\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -\frac{1}{h}\}.$$

Definition 2.30 The *circle plus* addition, denoted \oplus , on \mathbb{C}_h is defined by

$$z \oplus w := z + w + zwh.$$

The *circle negative* of $z \in \mathbb{C}_h$ is defined by

$$\ominus z := -\frac{z}{1 + zh}.$$

The *circle minus* subtraction is defined by

$$z \ominus w := z \oplus (\ominus w).$$

Theorem 2.31 *The pair (\mathbb{C}_h, \oplus) forms an abelian group.*

Now we will begin considering variable graininess $\mu(t)$ instead of h .

Definition 2.32 A function $f : \mathbb{T} \rightarrow \mathbb{C}$ is called *regressive* if

$$1 + \mu(t)f(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by

$$\mathcal{R}(\mathbb{T}, \mathbb{R}) = \mathcal{R}(\mathbb{R}).$$

Occasionally we will also consider $\mathcal{R}(\mathbb{C})$.

The following proposition is given as an exercise by Bohner and Peterson [A]. For this reason we will give a proof of it here.

Proposition 2.33 $\mathcal{R}(\mathbb{C})$ forms a group under pointwise circle plus addition.

Proof: Let $p, q \in \mathcal{R}(\mathbb{C})$.

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$$

Let $s \in \mathbb{T}^\kappa$ be an right-dense point. Then $\sigma(s) = s \implies \mu(s) = 0$. So

$$(p \oplus q)(s) = p(s) + q(s).$$

p and q are each continuous at s , so $p \oplus q$ is continuous at s . Now let $s \in \mathbb{T}^\kappa$ be a left-dense point. Then

$$\lim_{t \rightarrow s} (p \oplus q)(t) = \lim_{t \rightarrow s} [p(t) + q(t) + \mu(t)p(t)q(t)]$$

Now $\lim_{t \rightarrow s} p(t)$, $\lim_{t \rightarrow s} q(t)$, and $\lim_{t \rightarrow s} \mu(t)$ are each finite because p and q are both rd-continuous and $\mu(t)$ is finite for all $t \in \mathbb{T}$. So the left-sided limit of $p \oplus q$ exists at

s. For fixed $t \in \mathbb{T}^\kappa$, $(p \oplus q)(t)$ is regressive by Theorem 2.31. So $p \oplus q$ is regressive and rd-continuous. Hence $p \oplus q \in \mathcal{R}(\mathbb{C})$.

For fixed $t \in \mathbb{T}^\kappa$, $p(t) \oplus q(t)$ is associative, commutative and $p(t)$ has inverse $\ominus p(t)$ by Theorem 2.31. So if we allow $t \in \mathbb{T}^\kappa$ to vary, $p \oplus q$ is associative, commutative and p has inverse $\ominus p$. Now we need to show that $\ominus p$ is rd-continuous. Again let $s \in \mathbb{T}^\kappa$ be a right-dense point. Then

$$\ominus p(s) = -\frac{p(s)}{1 + \mu(s)p(s)} = -p(s)$$

because $\mu(s) = 0$. So $\ominus p$ is rd-continuous because $-p(s)$ is rd-continuous. Therefore the pair $(\mathcal{R}(\mathbb{C}), \oplus)$ is an abelian group. \square

Definition 2.34 For $h > 0$, define \mathbb{Z}_h to be the strip

$$\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h}\}$$

Definition 2.35 For $h > 0$, the *cylinder transform* $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ is defined by

$$\xi_h(z) := \frac{1}{h} \operatorname{Log}(1 + zh)$$

where Log is the principal branch of the logarithm. For $h = 0$, define

$$\xi_0(z) := z$$

for all $z \in \mathbb{C}$.

For $x \in \mathbb{R}$, $\xi_h(x)$ takes the form

$$\xi_h(x) = \frac{1}{h} \begin{cases} \log(1 + xh) & \text{if } x > -\frac{1}{h} \\ \log |1 + xh| + i\pi & \text{if } x < -\frac{1}{h} \end{cases}$$

where by \log we mean the natural logarithm. We refer to ξ_h as the cylinder transform because if we associate the lines $\operatorname{Im}(z) = \frac{\pi}{h}$ and $\operatorname{Im}(z) = -\frac{\pi}{h}$ together then \mathbb{Z}_h forms a cylinder.

Definition 2.36 For $p \in \mathcal{R}(\mathbb{C})$ we define the *time scales exponential function* to be

$$e_p(t, s) := \exp \left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right) \quad \text{for } s, t \in \mathbb{T}$$

Notice that if didn't have the regressivity, then the definition of the time scales exponential function wouldn't make sense. The next theorem tells us that this is indeed the solution to the initial value problem we gave at the beginning of the section.

Theorem 2.37 *Let $p(t) \in \mathcal{R}(\mathbb{C})$ and fix $t_0 \in \mathbb{T}$. Then the unique solution to the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = y_0 \tag{2.9}$$

is given by

$$y(t) = y_0 \cdot e_p(t, t_0) \tag{2.10}$$

on \mathbb{T} .

We will now use Theorem 2.37 to find the exponential function for the time scale $q^{\mathbb{N}_0}$, $q \in \mathbb{N}$. This example is given as an exercise in Bohner and Peterson [4].

Example 2.38 Let $\mathbb{T} = q^{\mathbb{N}_0}$, let $p(t) \in \mathcal{R}(\mathbb{R})$, and fix $t_0 \in \mathbb{T}$. Then by the preceding theorem, $e_p(t, t_0)$ is the solution to the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1.$$

Because each point of \mathbb{T} is scattered, we can rewrite this dynamic equation as a recurrence relation noting that $\sigma(t) = qt$:

$$\begin{aligned} \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t} &= p(t)y(t) \\ \frac{y(\sigma(t)) - y(t)}{qt - t} &= p(t)y(t) \\ y(\sigma(t)) - y(t) &= (q - 1)tp(t)y(t) \\ y(\sigma(t)) &= [1 + (q - 1)p(t)t]y(t) \end{aligned}$$

Observing that $e_p(t_0, t_0) = 1$ we are able to come up with a closed form for $e_p(t, t_0)$:

$$\begin{aligned} e_p(qt_0, t_0) &= 1 + (q - 1)p(qt_0)qt_0 \\ e_p(q^2t_0, t_0) &= [1 + (q - 1)p(qt_0)qt_0][1 + (q - 1)p(q^2t_0)q^2t_0] \\ &\vdots \\ e_p(t, t_0) &= \prod_{s \in [t_0, t)} (1 + (q - 1)p(s)s), \quad t_0 < t, \quad s \in \mathbb{T} \end{aligned}$$

Theorem 2.39 *If $p, q \in \mathcal{R}(\mathbb{C})$ and $t, s, r \in \mathbb{T}$, then*

- (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $\frac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$;
- (iv) $e_p(t, s) = e_{\ominus p}(s, t)$;
- (v) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (vi) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;

$$(vii) \frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t, s).$$

Part (ii) of Theorem 2.39 will be of particular importance to us in the next section. The ability to rewrite $e_p(\sigma(t), s)$ without $\sigma(t)$ inside the function will allow us to prove many results about the Laplace transform of special functions.

The following Corollary is a consequence of Theorem 2.39 and will be used frequently in the next section to compute the Laplace transform of special functions.

Corollary 2.40 *Let $p \in \mathcal{R}(\mathbb{C})$. Then*

$$e_{\ominus p}(\sigma(t), 0) = \frac{1}{1 + \mu(t)p(t)} e_{\ominus p}(t, 0) = -\frac{(\ominus p)(t)}{p(t)} e_{\ominus p}(t, 0)$$

Proof: Theorem 2.39, gives us

$$\begin{aligned} e_{\ominus p}(\sigma(t), s) &= \frac{e_{\ominus p}(t, 0)}{1 + \mu(t)p(t)} \\ &= (1 + \mu(t)(\ominus p)(t)) e_{\ominus p}(t, 0) \end{aligned}$$

Using the definition of \ominus we get

$$\begin{aligned} e_{\ominus p}(\sigma(t), s) &= \left(1 - \frac{\mu(t)p(t)}{1 + \mu(t)p(t)}\right) e_{\ominus p}(t, 0) \\ &= \frac{1}{1 + \mu(t)p(t)} e_{\ominus p}(t, 0) \\ &= -\frac{(\ominus p)(t)}{p(t)} e_{\ominus p}(t, 0). \quad \square \end{aligned}$$

We will now use the time scales exponential function to define four new functions that work like sine, cosine and their hyperbolic counterparts in the time scales calculus.

Definition 2.41 Let p be an rd-continuous function on \mathbb{T} such that $\mu p^2 \in \mathcal{R}(\mathbb{R})$.

Then we define the *time scales trigonometric functions* \cos_p and \sin_p by

$$\cos_p(t, t_0) = \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2} \quad \text{and} \quad \sin_p(t, t_0) = \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i}$$

where $i = \sqrt{-1}$. Now let p be an rd-continuous function on \mathbb{T} such that $-\mu p^2 \in \mathcal{R}(\mathbb{R})$. Then we define the *time scales hyperbolic trigonometric functions* \cosh_p and \sinh_p by

$$\cosh_p(t, t_0) = \frac{e_p(t, t_0) + e_{-p}(t, t_0)}{2} \quad \text{and} \quad \sinh_p(t, t_0) = \frac{e_p(t, t_0) - e_{-p}(t, t_0)}{2}$$

Notice that μp^2 is regressive iff ip and $-ip$ are both regressive and $-\mu p^2$ is regressive iff p and $-p$ are both regressive, hence the trigonometric and hyperbolic trigonometric functions are well defined. When $\mathbb{T} = \mathbb{R}$, $\cos_1(t, s) = \cos(t - s)$ where in the later half of the equation we mean the usual cosine function. Similarly $\sin_1(t, s) = \sin(t - s)$. Furthermore, for this time scale,

$$\cos_p(t, s) = \cos\left(\int_s^t p(\tau) d\tau\right) \quad \text{and} \quad \sin_p(t, s) = \sin\left(\int_s^t p(\tau) d\tau\right).$$

Proposition 2.42 Let p be an rd-continuous function on \mathbb{T} such that $\mu p^2 \in \mathcal{R}(\mathbb{R})$.

Then

(i)

$$\cos_p^\Delta(t, t_0) = -p(t) \sin_p(t, t_0)$$

and

(ii)

$$\sin_p^\Delta(t, t_0) = p(t) \cos_p(t, t_0).$$

Proof:

$$\begin{aligned}
\cos_p^\Delta(t, t_0) &= \left(\frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2} \right)^\Delta \\
&= \frac{e_{ip}^\Delta(t, t_0) + e_{-ip}^\Delta(t, t_0)}{2} \\
&= \frac{ip(t)e_{ip}(t, t_0) - ip(t)e_{-ip}(t, t_0)}{2} \\
&= \frac{ip(t)(e_{ip}(t, t_0) - e_{-ip}(t, t_0))}{2} \\
&= -p(t) \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i} \\
&= -p(t) \sin_p(t, t_0)
\end{aligned}$$

The proof of (ii) proceeds similarly. \square

We end this subsection with a theorem for time scales functions of two variables which will prove useful in the next subsection.

Theorem 2.43 *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$ and suppose $f : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^\kappa$, $t > a$. Suppose that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Also suppose that $f(\cdot, \tau)$ is Δ -differentiable for each $\tau \in [a, \sigma(t)]$. Denote by f^Δ the derivative of f with respect to the first variable. Then*

$$g(t) := \int_a^t f(t, \tau) \Delta\tau \implies g^\Delta(t) = \int_a^t f^\Delta(t, \tau) \Delta\tau + f(\sigma(t), t). \quad (2.11)$$

Proof: The fact that $f(t, \tau)$ is Δ -differentiable with respect to t implies that there exists $\delta_1 > 0$ such that when $|t - s| < \delta_1$ we have

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \frac{\epsilon}{2(\sigma(t) - a)} |\sigma(t) - s|. \quad (2.12)$$

The continuity of f at (t, t) tells us that there exists $\delta_2 > 0$ such that when $|t - s| < \delta_2$ and $|t - \tau| < \delta_2$ we have

$$|f(s, \tau) - f(t, t)| \leq \frac{\epsilon}{2}. \quad (2.13)$$

Let $\delta := \min\{\delta_1, \delta_2\}$ and suppose that $|t - s| < \delta$. Then

$$\begin{aligned} & \left| g(\sigma(t)) - g(s) - \left(\int_a^t f^\Delta(t, \tau) \Delta\tau + f(\sigma(t), t) \right) (\sigma(t) - s) \right| \\ &= \left| \int_a^{\sigma(t)} f(\sigma(t), \tau) \Delta\tau - \int_a^s f(s, \tau) \Delta\tau \right. \\ & \quad \left. - (\sigma(t) - s)f(\sigma(t), t) - (\sigma(t) - s) \int_a^t f^\Delta(t, \tau) \Delta\tau \right| \end{aligned}$$

by the definition of g . We can now manipulate the integration limits and subtract of the difference to rewrite this as

$$\begin{aligned} & \left| \int_a^{\sigma(t)} [f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)] \Delta\tau \right. \\ & \quad \left. - \int_{\sigma(t)}^s f(s, \tau) \Delta\tau - (\sigma(t) - s)f(\sigma(t), t) \right. \\ & \quad \left. - (\sigma(t) - s) \int_{\sigma(t)}^t f^\Delta(t, \tau) \Delta\tau \right|. \end{aligned}$$

Using Theorem 2.21 we can rewrite the last integral in this expression as $\mu(t)f^\Delta(t, t)$ thus obtaining

$$\begin{aligned} & \left| \int_a^{\sigma(t)} [f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)] \Delta\tau - \right. \\ & \quad \left. \int_{\sigma(t)}^s f(s, \tau) \Delta\tau - (\sigma(t) - s)f(\sigma(t), t) - (\sigma(t) - s)\mu(t)f^\Delta(t, t) \right|. \end{aligned}$$

Theorem 2.11 tells us that $f(t, t) = f(\sigma(t), t) - \mu(t)f^\Delta(t, t)$ yielding

$$\begin{aligned} & \left| \int_a^{\sigma(t)} [f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)] \Delta\tau \right. \\ & \quad \left. - \int_{\sigma(t)}^s f(s, \tau) \Delta\tau - (\sigma(t) - s)f(t, t) \right| \\ &= \left| \int_a^{\sigma(t)} [f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)] \Delta\tau \right. \\ & \quad \left. + \int_s^{\sigma(t)} f(s, \tau) \Delta\tau - (\sigma(t) - s)f(t, t) \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \int_a^{\sigma(t)} [f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)] \Delta\tau \right. \\
&\quad \left. + \int_s^{\sigma(t)} [f(s, \tau) - f(t, t)] \Delta\tau \right| \\
&\leq \int_a^{\sigma(t)} |f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \Delta\tau \\
&\quad + \int_s^{\sigma(t)} |f(s, \tau) - f(t, t)| \Delta\tau
\end{aligned}$$

Now we'll use equations 2.12 and 2.13 to bound this by

$$\begin{aligned}
&\int_a^{\sigma(t)} \frac{\epsilon}{2(\sigma(t) - a)} |\sigma(t) - s| \Delta\tau + \int_s^{\sigma(t)} \frac{\epsilon}{2} \Delta\tau \\
&= \frac{\epsilon}{2} |\sigma(t) - s| + \frac{\epsilon}{2} |\sigma(t) - s| \\
&= \epsilon |\sigma(t) - s|. \quad \square
\end{aligned}$$

2.4 DYNAMIC EQUATIONS

We already seen an example of a dynamic initial value problem in Theorem 2.37. We will now consider the solutions of some more initial value problems that will be of use later.

Theorem 2.44 *Variation of Constants I*

Suppose $p \in \mathcal{R}(\mathbb{R})$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem

$$x^\Delta(t) = -p(t)x(\sigma(t)) + f(t), \quad x(t_0) = x_0 \quad (2.14)$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau. \quad (2.15)$$

Proof: We start by showing that $x(t)$ given in equation 2.15 satisfies the initial value problem given in equation 2.14.

$$\begin{aligned} x^\Delta(t) &= \left(e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \right)^\Delta \\ &= \ominus p(t)e_{\ominus p}(t, t_0)x_0 + \left(\int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \right)^\Delta \end{aligned}$$

By Theorem 2.43 we get that

$$\begin{aligned} x^\Delta(t) &= \ominus p(t)e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t (e_{\ominus p}(t, \tau)f(\tau))^\Delta \Delta\tau \\ &\quad + e_{\ominus p}(\sigma(t), t)f(t) \\ &= \ominus p(t)e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t \ominus p(t)e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &\quad + e_{\ominus p}(\sigma(t), t)f(t). \end{aligned}$$

By Corollary 2.40, we can rewrite this last term to obtain

$$\begin{aligned} x^\Delta(t) &= \ominus p(t)e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t \ominus p(t)e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &\quad + \frac{1}{1 + \mu(t)p(t)}e_{\ominus p}(t, t)f(t) \\ &= \ominus p(t)e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t \ominus p(t)e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &\quad + \frac{1}{1 + \mu(t)p(t)}f(t) \\ &= \frac{-p(t)}{1 + \mu(t)p(t)}e_{\ominus p}(t, t_0)x_0 \\ &\quad + \frac{-p(t)}{1 + \mu(t)p(t)} \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &\quad + \frac{1}{1 + \mu(t)p(t)}f(t). \end{aligned}$$

Multiplying both sides by $(1 + \mu(t)p(t))$ gives

$$\begin{aligned} (1 + \mu(t)p(t))x^\Delta(t) &= -p(t)e_{\ominus p}(t, t_0)x_0 \\ &\quad - p(t) \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau + f(t) \\ &= -p(t)x(t) + f(t). \end{aligned}$$

Thus we get

$$\begin{aligned}
x^\Delta(t) &= -\mu(t)p(t)x^\Delta(t) - p(t)x(t) + f(t) \\
&= -p(t)[x(t) + \mu(t)x^\Delta(t)] + f(t) \\
&= -p(t)x(\sigma(t)) + f(t)
\end{aligned}$$

by Theorem 2.11. It is easy to see that x satisfies the initial conditions because

$$e_{\ominus p}(t_0, t_0) = 1 \text{ and } \int_{t_0}^{t_0} e_{\ominus p}(t, \tau) f(\tau) \Delta\tau = 0.$$

Now we proceed to showing the uniqueness of the solution. Suppose that $x(t)$ is a solution of (2.15). Then we can solve the dynamic equation for $f(t)$ to get

$$f(t) = x^\Delta(t) + p(t)x(\sigma(t)) \quad (2.16)$$

$$e_p(t, t_0)f(t) = e_p(t, t_0)[x^\Delta(t) + p(t)x(\sigma(t))] \quad (2.17)$$

$$= e_p(t, t_0)x^\Delta(t) + p(t)e_p(t, t_0)x(\sigma(t)) \quad (2.18)$$

$$= [e_p(t, t_0)x(t)]^\Delta. \quad (2.19)$$

Equation (2.19) follows from equation (2.18) by an application of Theorem 2.12.

Integrating both sides yields

$$\begin{aligned}
\int_{t_0}^t e_p(\tau, t_0)f(\tau)\Delta\tau &= \int_{t_0}^t [e_p(\tau, t_0)x(\tau)]^\Delta\Delta\tau \\
&= e_p(t, t_0)x(t) - e_p(t_0, t_0)x(t_0) \\
&= e_p(t, t_0)x(t) - x_0 \\
e_p(t, t_0)x(t) &= x_0 + \int_{t_0}^t e_p(\tau, t_0)f(\tau)\Delta\tau.
\end{aligned}$$

Solving for $x(t)$ we get

$$\begin{aligned}
x(t) &= e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, t_0)e_p(\tau, t_0)f(\tau)\Delta\tau \\
&= e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, t_0)e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau \\
&= e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau
\end{aligned}$$

by parts (iii), (iv), and (v) of Theorem 2.39. \square

Corollary 2.45 *Variation of Constants II*

Suppose $p \in \mathcal{R}(\mathbb{R})$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem

$$x^\Delta(t) = p(t)x(t) + f(t), \quad x(t_0) = x_0 \quad (2.20)$$

is given by

$$x(t) = e_p(t, t_0)x_0 + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Proof: By Theorem 2.11 we can write $x(t) = x(\sigma(t)) - \mu(t)x^\Delta(t)$. Using this, we rewrite (2.20) as

$$\begin{aligned} x^\Delta(t) &= p(t)[x(\sigma(t)) - \mu(t)x^\Delta(t)] + f(t) \\ &= p(t)x(\sigma(t)) - p(t)\mu(t)x^\Delta(t) + f(t). \end{aligned}$$

Thus we have

$$\begin{aligned} x^\Delta(t) + p(t)\mu(t)x^\Delta(t) &= p(t)x(\sigma(t)) + f(t) \\ (1 + \mu(t)p(t))x^\Delta(t) &= p(t)x(\sigma(t)) + f(t) \\ x^\Delta(t) &= \frac{p(t)}{1 + \mu(t)p(t)}x(\sigma(t)) + \frac{1}{1 + \mu(t)p(t)}f(t) \\ &= -(\ominus p)(t)x(\sigma(t)) + \frac{1}{1 + \mu(t)p(t)}f(t) \end{aligned} \quad (2.21)$$

Notice that $\ominus(\ominus p) = p(t)$ and apply this along with Theorem 2.44 to equation (2.21) to get that

$$x(t) = x_0 e_p(t, t_0) + \int_{t_0}^t e_p(t, \tau) \frac{f(\tau)}{1 + \mu(\tau)p(\tau)} \Delta\tau.$$

Theorem 2.39 tells us that

$$1 + \mu(\tau)p(\tau) = (1 + \mu(\tau)p(\tau))e_p(\tau, \tau) = e_p(\sigma(\tau), \tau).$$

Thus

$$\begin{aligned}
x(t) &= x_0 e_p(t, t_0) + \int_{t_0}^t \frac{e_p(t, \tau)}{e_p(\sigma(\tau), \tau)} f(\tau) \Delta \tau \\
&= x_0 e_p(t, t_0) + \int_{t_0}^t e_p(t, \tau) e_{\ominus p}(\sigma(\tau), \tau) f(\tau) \Delta \tau \\
&= x_0 e_p(t, t_0) + \int_{t_0}^t e_p(t, \tau) e_p(\tau, \sigma(\tau)) f(\tau) \Delta \tau \\
&= x_0 e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau)) f(\tau) \Delta \tau
\end{aligned}$$

as desired. \square

Theorem 2.46 *Variation of Constants III*

Let g be an rd-continuous function. Then the solution to the initial value problem

$$x^{\Delta^{k+1}}(t) = g(t), \quad x^{\Delta^i}(0) = 0 \text{ for all } 0 \leq i \leq k \quad (2.22)$$

is given by

$$x(t) = \int_0^t h_k(t, \sigma(s)) g(s) \Delta s. \quad (2.23)$$

Proof: We proceed by induction on k . Suppose $k = 0$. Corollary 2.20 gives us that the solution to

$$x^\Delta(t) = p(t)x(t) + g(t), \quad x(t_0) = x_0$$

is

$$x(t) = e_p(t, t_0)x_0 + \int_{t_0}^t e_p(t, \sigma(s))g(s)\Delta s.$$

So for $p(t) \equiv 0$, we get that the solution to

$$x^\Delta(t) = g(t), \quad x(0) = 0$$

is

$$x(t) = \int_0^t e_0(t, \sigma(s))g(s)\Delta s = \int_0^t g(s)\Delta s.$$

Recalling that $h_0(t, s) \equiv 1$, we get

$$x(t) = \int_0^t h_0(t, \sigma(s))g(s)\Delta s.$$

So our hypothesis holds for $k = 0$. Now suppose our hypothesis holds for $k = n$ and consider

$$\begin{aligned} x(t) &= \int_0^t h_{n+1}(t, \sigma(s))g(s)\Delta s. \\ x^{\Delta^{n+2}}(t) &= \left(\int_0^t h_{n+1}(t, \sigma(s))g(s)\Delta s \right)^{\Delta^{n+2}} \end{aligned}$$

By Theorem 2.43, the right side of this is

$$\left(\int_0^t h_{n+1}^{\Delta}(t, \sigma(s))g(s)\Delta s + h_{n+1}(\sigma(t), \sigma(t))g(t) \right)^{\Delta^{n+1}}$$

Recalling that $h_k(a, a) = 0$ and $h_k^{\Delta}(t, s) = h_{k-1}(t, s)$ we get

$$x^{\Delta^{n+2}}(t) = \left(\int_0^t h_n(t, \sigma(s))g(s)\Delta s \right)^{\Delta^{n+1}}$$

By our induction hypothesis

$$\int_0^t h_n(t, \sigma(s))g(s)\Delta s$$

is the solution to the initial value problem

$$y^{\Delta^{n+1}} = g(t), \quad y^{\Delta^i}(0) = 0 \text{ for all } 0 \leq i \leq n.$$

So

$$x^{\Delta^{n+2}}(t) = g(t).$$

Furthermore,

$$x^{\Delta^i}(0) = y^{\Delta^{i-1}}(0) = 0 \text{ for all } 1 \leq i \leq n+1.$$

The case $i = 0$ is trivial as $x(0) = 0$ directly from the definition. \square

CHAPTER 3

LAPLACE TRANSFORM

3.1 DEFINITION AND EXAMPLES

We start this section by introducing the Laplace transform. The goal is to exploit some of the properties of the transform in solving dynamic equations.

Definition 3.1 Let \mathbb{T}_+ be a time scale such that $0 \in \mathbb{T}_+$ and $\sup \mathbb{T}_+ = \infty$. Let $f : \mathbb{T}_+ \rightarrow \mathbb{C}$ be a regulated function. Then the *Time Scale Laplace transform* is defined by

$$\mathcal{L}\{f\}(z) := \int_0^\infty f(t) e_{\ominus z}(\sigma(t), 0) \Delta t \quad (3.1)$$

for $z \in \mathcal{D}\{f\}$ where $\mathcal{D}\{f\}$ is the set of $z \in \mathbb{C}$ such that the integral exists and $1 + \mu(t)z \neq 0$, $\forall t \in \mathbb{T}_+$.

We will see why it is important to have $\sigma(t)$ instead of t in the exponential function when we do Example 3.12. Before presenting the properties of the Laplace transform, we will compute the Laplace transform of a function directly from the definition.

Example 3.2 Let $\mathbb{T}_+ = 2^{\mathbb{N}_0} \cup \{0\}$ and let $\chi_{[2^2, 2^5]}$ be the characteristic (indicator) function of $\mathbb{T}_+ \cap [2^2, 2^5]$. Then

$$\begin{aligned} \mathcal{L}\{\chi_{[2^2, 2^5]}\}(z) &= \int_0^\infty \chi_{[2^2, 2^5]} e_{\ominus z}(\sigma(t), 0) \Delta t \\ &= \int_{2^2}^{2^5} e_{\ominus z}(\sigma(t), 0) \Delta t \end{aligned}$$

The exponential function for $2^{\mathbb{N}_0}$ (see page 33) is given by

$$e_p(t, 1) = \prod_{s \in [1, t)} (1 + p(s)s).$$

Observe that for $2^{\mathbb{N}_0} \cup \{0\}$ we can write

$$e_p(t, 0) = e_p(t, 1)e_p(1, 0) = [1 + p(0)] \prod_{s \in [1, t)} (1 + p(s)s)$$

where we have calculated $e_p(1, 0)$ directly from the definition. So

$$\begin{aligned} \mathcal{L}\{\chi_{[2^2, 2^5]}\}(z) &= \int_{2^2}^{2^5} [1 + \ominus z(0)] \prod_{s \in [1, \sigma(t))} (1 + \ominus z(s)s) \Delta t \\ &= [1 + \ominus z(0)] \sum_{n=2}^4 \left[\prod_{i=0}^n (1 + \ominus z(2^i)2^i) \right] (2^{n+1} - 2^n) \\ &= [1 + \ominus z(0)] \sum_{n=2}^4 \left[\prod_{i=0}^n (1 + \ominus z(2^i)2^i) \right] 2^n \\ &= \left(1 + \frac{-z}{1+z}\right) \sum_{n=2}^4 \left[\prod_{i=0}^n \left(1 + \frac{-z2^i}{1+z2^i}\right) \right] 2^n \\ &= \frac{1}{1+z} \sum_{n=2}^4 \left[\prod_{i=0}^n \frac{1}{1+z2^i} \right] 2^n \\ &= \left(\frac{1}{1+z}\right)^2 \left(\frac{1}{1+2z}\right) \left(\frac{1}{1+4z}\right) 2^2 \\ &\quad + \left(\frac{1}{1+z}\right)^2 \left(\frac{1}{1+2z}\right) \left(\frac{1}{1+4z}\right) \left(\frac{1}{1+8z}\right) 2^3 \\ &\quad + \left(\frac{1}{1+z}\right)^2 \left(\frac{1}{1+2z}\right) \left(\frac{1}{1+4z}\right) \left(\frac{1}{1+8z}\right) \left(\frac{1}{1+16z}\right) 2^4 \end{aligned}$$

Notice that the Laplace transform maps functions defined on time scales to functions defined on some subset of the complex numbers. The region of convergence of the transform, $\mathcal{D}\{f\}$, varies not only with the function f but also with the time

scale. We will illustrate this with a couple of examples.

Example 3.3 Suppose that $\mathbb{T} = h\mathbb{Z}$. Then for any regressive $\alpha \in \mathbb{C}$,

$$\begin{aligned} e_\alpha(t, 0) &= \exp \left(\int_0^t \frac{1}{h} \text{Log}(1 + \alpha h) \Delta t \right) = \exp \left(\frac{t}{h} \text{Log}(1 + \alpha h) \right) \\ &= \exp \left(\text{Log}(1 + \alpha h)^{\frac{t}{h}} \right) = (1 + \alpha h)^{\frac{t}{h}}. \end{aligned}$$

So in particular,

$$e_{\ominus z}(t, 0) = (1 + (\ominus z)h)^{\frac{t}{h}} = \left(1 - \frac{zh}{1 + zh} \right)^{\frac{t}{h}} = \left(\frac{1}{1 + zh} \right)^{\frac{t}{h}}.$$

First we'll find $\mathcal{D}\{1\}$.

$$\begin{aligned} \mathcal{L}\{1\}(z) &= \int_0^\infty e_{\ominus z}(\sigma(t), 0) \Delta t = \int_0^\infty \left(\frac{1}{1 + zh} \right)^{\frac{t+1}{h}} \Delta t \\ &= \sum_{t=0}^\infty \left(\frac{1}{1 + zh} \right)^{\frac{t+1}{h}} = \frac{1}{1 + zh} \sum_{t=0}^\infty \left(\frac{1}{1 + zh} \right)^{\frac{t}{h}}. \end{aligned}$$

Recognizing this as a geometric series, we conclude that the $\mathcal{L}\{1\}(z)$ converges if and only if

$$\begin{aligned} & \left| \frac{1}{1 + zh} \right|^{\frac{1}{h}} < 1 \\ \iff & \left| \frac{1}{1 + zh} \right| < 1 \\ \iff & |1 + zh| > 1 \\ \iff & \left| \frac{1}{h} + z \right| > \frac{1}{h} \\ \iff & \left| z - \left(-\frac{1}{h} \right) \right| > \frac{1}{h}. \end{aligned}$$

If we let $D(a, r) \subset \mathbb{C}$ denote the closed ball of radius r about the point a , then the region of convergence is

$$\mathcal{D}\{1\} = \mathbb{C} \setminus D\left(\frac{-1}{h}, \frac{1}{h}\right).$$

Now we will find $\mathcal{D}\{e_\alpha(t, 0)\}$ for $\alpha \in \mathbb{R}$ regressive.

$$\begin{aligned}
\mathcal{L}\{e_\alpha(t, 0)\}(z) &= \int_0^\infty e_\alpha(t, 0) e_{\ominus z}(\sigma(t), 0) \Delta t \\
&= \int_0^\infty (1 + \alpha h)^{\frac{t}{h}} \left(\frac{1}{1 + zh} \right)^{\frac{t+1}{h}} \Delta t \\
&= \left(\frac{1}{1 + zh} \right)^{\frac{1}{h}} \int_0^\infty (1 + \alpha h)^{\frac{t}{h}} \left(\frac{1}{1 + zh} \right)^{\frac{t}{h}} \Delta t \\
&= \left(\frac{1}{1 + zh} \right)^{\frac{1}{h}} \int_0^\infty \left(\frac{1 + \alpha h}{1 + zh} \right)^{\frac{t}{h}} \Delta t \\
&= \left(\frac{1}{1 + zh} \right)^{\frac{1}{h}} \sum_{t=0}^\infty \left(\frac{1 + \alpha h}{1 + zh} \right)^{\frac{t}{h}}
\end{aligned}$$

Again, we recognize this as a geometric series which converges if and only if

$$\begin{aligned}
&\left| \frac{1 + \alpha h}{1 + zh} \right|^{\frac{1}{h}} < 1 \\
\iff &|1 + zh| > |1 + \alpha h| \\
\iff &\left| z - \left(-\frac{1}{h} \right) \right| > \left| \frac{1}{h} + \alpha \right|.
\end{aligned}$$

So our region of convergence is

$$\mathcal{D}\{e_\alpha(t, 0)\} = \mathbb{C} \setminus D\left(\frac{-1}{h}, \frac{1}{h} + \alpha\right).$$

Before presenting the operational properties of the Laplace transform, there is an issue that deserves some attention as it will present itself frequently. When $\mathbb{T}_+ = \mathbb{R}$, the convergence of

$$\lim_{t \rightarrow \infty} f(t) e_{\ominus z}(\sigma(t), 0) = \lim_{t \rightarrow \infty} f(t) e^{-zt} = 0$$

implies that

$$\lim_{t \rightarrow \infty} f(t) e_{\ominus z}(t, 0) = 0.$$

However for arbitrary time scales it is not clear when this is true because the exponential function could possibly switch sign. For example, when $\mathbb{T} = \mathbb{Z}$, we have from the previous example that

$$e_{\ominus z}(t, 0) = \left(\frac{1}{1+z} \right)^t.$$

For $z < -1$, this changes sign for all $t \in \mathbb{Z}$. What we can do is give a sufficient condition on \mathbb{T}_+ for $\lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) = 0$. The next proposition is a new result for time scales.

Proposition 3.4 *Suppose \mathbb{T}_+ is such that $\mu(t) < M$ for some $M \in \mathbb{R}$ and all $t \in \mathbb{T}_+$. Also suppose that $f : \mathbb{T}_+ \rightarrow \mathbb{R}$ and*

$$\lim_{t \rightarrow \infty} f(t)e_{\ominus z}(\sigma(t), 0) = 0.$$

Then

$$\lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) = 0.$$

Proof:

$$\lim_{t \rightarrow \infty} f(t)e_{\ominus z}(\sigma(t), 0) = 0.$$

From this Theorem 2.40 gives

$$\begin{aligned} \lim_{t \rightarrow \infty} -\frac{(\ominus z)(t)}{z} f(t)e_{\ominus z}(t, 0) &= 0 \\ \implies \lim_{t \rightarrow \infty} \frac{z}{z(1 + \mu(t)z)} f(t)e_{\ominus z}(t, 0) &= 0 \\ \implies \lim_{t \rightarrow \infty} \frac{1}{1 + \mu(t)z} f(t)e_{\ominus z}(t, 0) &= 0. \end{aligned}$$

Because $\mu(t) < M$, we find that

$$\lim_{t \rightarrow \infty} \left| \frac{1}{1 + \mu(t)z} f(t)e_{\ominus z}(t, 0) \right| > \left| \frac{1}{1 + Mz} \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) \right| > 0.$$

Thus

$$\begin{aligned} \frac{1}{1+Mz} \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) &= 0 \\ \Rightarrow \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) &= 0. \quad \square \end{aligned}$$

3.2 PROPERTIES OF THE LAPLACE TRANSFORM

Theorem 3.5 (*Linearity*). *Let f and g be regulated functions on \mathbb{T}_+ and let $\alpha, \beta \in \mathbb{R}$ be constants. Then*

$$\mathcal{L}\{\alpha f + \beta g\}(z) = \alpha \mathcal{L}\{f\}(z) + \beta \mathcal{L}\{g\}(z)$$

for those $z \in \mathcal{D}\{f\} \cap \mathcal{D}\{g\}$.

Proof: The proof of this theorem follows directly from the linearity of the Δ -integral, Theorem 2.23.

Theorem 3.6 *Let $f : \mathbb{T}_+ \rightarrow \mathbb{R}$ be such that f^Δ is regulated. Then*

$$\mathcal{L}\{f^\Delta\}(z) = z\mathcal{L}\{f\}(z) - f(0) + \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0)$$

for those regressive $z \in \mathbb{R}$ such that the limit exists.

Proof:

$$\mathcal{L}\{f^\Delta\}(z) = \int_0^\infty f^\Delta(t)e_{\ominus z}(\sigma(t), 0)\Delta t.$$

Using integration by parts, Theorem 2.26, and the derivative of $e_{\ominus z}(\sigma(t), 0)$ (see Theorem 2.37) we get

$$\mathcal{L}\{f^\Delta\}(z) = \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) - f(0)e_{\ominus z}(0, 0) - \int_0^\infty f(t)(\ominus z)(t)e_{\ominus z}(t, 0)\Delta t.$$

Recognizing that $e_{\ominus z}(0, 0) = \exp(0) = 1$ and that $-(\ominus z)(t)e_{\ominus z}(t, 0) = ze_{\ominus z}(\sigma(t), 0)$ (see Corollary 2.40), we can simplify this to

$$\mathcal{L}\{f^\Delta\}(z) = \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) - f(0) - z \int_0^\infty f(t)e_{\ominus z}(\sigma(t), 0)\Delta t.$$

The definition of the Laplace transform now gives

$$\mathcal{L}\{f^\Delta\}(z) = \lim_{t \rightarrow \infty} f(t)e_{\ominus z}(t, 0) - f(0) + z\mathcal{L}\{f\}(z).$$

Rearranging the order of the terms gives the desired result. \square

Theorem 3.6 exhibits one of the most important properties of the Laplace transform. This property allows us to remove the Δ -derivatives from dynamic equations when we apply this transform to them.

Corollary 3.7 *Let $f : \mathbb{T}_+ \rightarrow \mathbb{R}$ be such that f^{Δ^n} is regulated. Then*

$$\mathcal{L}\{f^{\Delta^n}\}(z) = z^n \mathcal{L}\{f\}(z) - \sum_{i=0}^{n-1} z^i f^{\Delta^i}(0)$$

for those regressive $z \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} f^{\Delta^i}(t)e_{\ominus z}(t, 0) = 0, \quad 0 \leq i \leq n-1.$$

Proof: By Theorem 2.11, if a function is differentiable at a point $t \in \mathbb{T}$, then it is continuous at t . By Theorem 2.8, the continuity of a function implies that it is regulated. So f^{Δ^n} regulated implies that f^{Δ^k} is regulated for all $0 \leq k \leq n$. We will now proceed by induction on n . For $n = 1$, Theorem 3.6 gives us that

$$\mathcal{L}\{f^\Delta\}(z) = z\mathcal{L}\{f\}(z) - f(0).$$

Suppose that

$$\mathcal{L}\{f^{\Delta^n}\}(z) = z^n \mathcal{L}\{f\}(z) - \sum_{i=0}^{n-1} z^i f^{\Delta^i}(0).$$

Then

$$\begin{aligned}
\mathcal{L}\{f^{\Delta^{n+1}}\}(z) &= z\mathcal{L}\{f^{\Delta^n}\}(z) - f^{\Delta^n}(0) \\
&= z\left(z^n\mathcal{L}\{f\}(z) - \sum_{i=0}^{n-1} z^i f^{\Delta^i}(0)\right) - f^{\Delta^n}(0) \\
&= z^{n+1}\mathcal{L}\{f\}(z) - \sum_{i=0}^n z^i f^{\Delta^i}(0). \quad \square
\end{aligned}$$

Theorem 3.8 *Let $f : \mathbb{T}_+ \rightarrow \mathbb{R}$ be regulated and*

$$F(t) := \int_0^t f(\tau) \Delta\tau$$

for $t \in \mathbb{T}_+$, then

$$\mathcal{L}\{F\}(z) = \frac{1}{z}[\mathcal{L}\{f\}(z) - \lim_{t \rightarrow \infty} F(t)e_{\ominus z}(t, 0)]$$

for regressive $z \in \mathbb{R}, z \neq 0$.

Proof:

$$\mathcal{L}\{F\}(z) = \int_0^\infty F(t)e_{\ominus z}(\sigma(t), 0)\Delta t$$

Corollary 2.40, gives

$$\mathcal{L}\{F\}(z) = -\frac{1}{z} \int_0^\infty F(t)(\ominus z)(t)e_{\ominus z}(t, 0)\Delta t.$$

Integration by parts yields

$$\mathcal{L}\{F\}(z) = -\frac{1}{z} \left[- \int_0^\infty F^\Delta(t)(\ominus z)(t)e_{\ominus z}(t, 0)\Delta t - F(0) + \lim_{t \rightarrow \infty} F(t)e_{\ominus z}(t, 0) \right].$$

Using the fact that $F(0) = 0$ and the Fundamental Theorem of Calculus, Theorem 2.24, we find that

$$\begin{aligned}
\mathcal{L}\{F\}(z) &= -\frac{1}{z}[-\mathcal{L}\{f\}(z) + \lim_{t \rightarrow \infty} F(t)e_{\ominus z}(t, 0)] \\
&= \frac{1}{z}[\mathcal{L}\{f\}(z) - \lim_{t \rightarrow \infty} F(t)e_{\ominus z}(t, 0)]. \quad \square
\end{aligned}$$

Example 3.9

$$\mathcal{L}\{1\} = \frac{1}{z} [1 - \lim_{t \rightarrow \infty} e_{\ominus z}(t, 0)]$$

for all regressive $z \in \mathbb{R}, z \neq 0$.

Proof:

$$\mathcal{L}\{1\}(z) = \int_0^\infty 1 \cdot e_{\ominus z}(\sigma(t), 0) \Delta t.$$

Corollary 2.40 gives

$$\mathcal{L}\{1\}(z) = -\frac{1}{z} \int_0^\infty (\ominus z)(t) e_{\ominus z}(t, 0) \Delta t.$$

Applying the Fundamental Theorem of Calculus yields

$$\begin{aligned} \mathcal{L}\{1\}(z) &= -\frac{1}{z} [e_{\ominus z}(t, 0)]_{t=0}^{t \rightarrow \infty} \\ &= \frac{1}{z} [1 - \lim_{t \rightarrow \infty} e_{\ominus z}(t, 0)]. \quad \square \end{aligned}$$

Theorem 3.10 *Let $k \in \mathbb{N}_0$, then*

$$\mathcal{L}\{h_k(t, 0)\}(z) = \frac{1}{z^{k+1}} - \lim_{t \rightarrow \infty} \sum_{i=0}^k \frac{h_i(t, 0) e_{\ominus z}(t, 0)}{z^{k-i+1}}$$

for those regressive $z \in \mathbb{R}, z \neq 0$.

Proof: We will proceed by induction on k . Because $h_0(t, 0) = 1$, the proceeding example shows that our hypothesis holds for $k = 0$. Now assume that our hypothesis holds for fixed $k \in \mathbb{N}$.

$$\mathcal{L}\{h_{k+1}(t, 0)\}(z) = \mathcal{L}\left\{\int_0^t h_k(\tau, 0) \Delta \tau\right\}(z)$$

by the definition of the time scales polynomials. Theorem 3.8 gives

$$\begin{aligned}
\mathcal{L}\{h_{k+1}(t, 0)\}(z) &= \frac{1}{z}[\mathcal{L}\{h_k(t, 0)\}(z) - \lim_{t \rightarrow \infty} h_{k+1}(t, 0)e_{\ominus z}(t, 0)] \\
&= \frac{1}{z} \left(\frac{1}{z^{k+1}} - \lim_{t \rightarrow \infty} \sum_{i=0}^k \frac{h_i(t, 0)e_{\ominus z}(t, 0)}{z^{k-i+1}} - \lim_{t \rightarrow \infty} h_{k+1}(t, 0)e_{\ominus z}(t, 0) \right) \\
&= \frac{1}{z^{k+2}} - \frac{1}{z} \lim_{t \rightarrow \infty} \sum_{i=0}^k \frac{h_i(t, 0)e_{\ominus z}(t, 0)}{z^{k-i+1}} - \lim_{t \rightarrow \infty} \frac{h_{k+1}(t, 0)e_{\ominus z}(t, 0)}{z} \\
&= \frac{1}{z^{k+2}} - \lim_{t \rightarrow \infty} \sum_{i=0}^k \frac{h_i(t, 0)e_{\ominus z}(t, 0)}{z^{k-i+2}} - \lim_{t \rightarrow \infty} \frac{h_{k+1}(t, 0)e_{\ominus z}(t, 0)}{z} \\
&= \frac{1}{z^{k+2}} - \lim_{t \rightarrow \infty} \sum_{i=0}^{k+1} \frac{h_i(t, 0)e_{\ominus z}(t, 0)}{z^{k-i+2}}.
\end{aligned}$$

We have shown that when our hypothesis holds for k then it also holds for $k+1$. So our claim holds for all $k \in \mathbb{N}$ by the first principle of mathematical induction. \square

Theorem 3.11 *Let $\alpha \in \mathbb{R}$ be regressive. Then*

(i)

$$\mathcal{L}\{e_{\alpha}(t, 0)\}(z) = \frac{1}{z - \alpha}$$

provided that $\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, 0) = 0$

(ii)

$$\mathcal{L}\{\cos_{\alpha}(t, 0)\}(z) = \frac{z}{z^2 + \alpha^2}$$

provided that $\lim_{t \rightarrow \infty} e_{i\alpha \ominus z}(t, 0) = \lim_{t \rightarrow \infty} e_{-i\alpha \ominus z}(t, 0) = 0$

(iii)

$$\mathcal{L}\{\sin_{\alpha}(t, 0)\}(z) = \frac{\alpha}{z^2 + \alpha^2}$$

provided that $\lim_{t \rightarrow \infty} e_{i\alpha \ominus z}(t, 0) = \lim_{t \rightarrow \infty} e_{-i\alpha \ominus z}(t, 0) = 0$.

Proof:

(i)

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \int_0^\infty e_\alpha(t, 0)e_{\ominus z}(\sigma(t), 0)\Delta t$$

Again, we use Corollary 2.40:

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \int_0^\infty \frac{1}{1 + \mu(t)z} e_\alpha(t, 0)e_{\ominus z}(t, 0)\Delta t.$$

Now we can employ Theorem 2.39 to combine the two exponential functions.

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \int_0^\infty \frac{1}{1 + \mu(t)z} e_{\alpha \ominus z}(t, 0)\Delta t.$$

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \frac{1}{\alpha - z} \int_0^\infty \frac{\alpha - z}{1 + \mu(t)z} e_{\alpha \ominus z}(t, 0)\Delta t.$$

In the proceeding line, we have simply multiplied the right side by a appropriate choice of 1, ie $\frac{\alpha - z}{\alpha - z}$. Recognizing $\frac{\alpha - z}{1 + \mu(t)z}$ as $(\alpha \ominus z)(t)$ brings us to the next line.

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \frac{1}{\alpha - z} \int_0^\infty (\alpha \ominus z)(t) e_{\alpha \ominus z}(t, 0)\Delta t.$$

Noticing that $(\alpha \ominus z)(t) e_{\alpha \ominus z}(t, 0)$ is the derivative of $e_{\alpha \ominus z}(t, 0)$, allows us to use the Fundamental Theorem of calculus to get

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \lim_{t \rightarrow \infty} \left[\frac{1}{\alpha - z} e_{\alpha \ominus z}(t, 0) \right] - \frac{1}{\alpha - z} e_{\alpha \ominus z}(0, 0).$$

So

$$\mathcal{L}\{e_\alpha(t, 0)\}(z) = \frac{1}{z - \alpha}$$

provided that $\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, 0) = 0$. Because the time scales trigonometric functions are defined in terms of the exponential function, the proofs of (ii) and (iii) follow directly from the definitions of $\cos_\alpha(t, 0)$ and $\sin_\alpha(t, 0)$, the linearity of the Laplace transform, and part (i). \square .

Example 3.12 Consider the dynamic equation

$$x^{\Delta\Delta} + kx = \sin_\alpha(t, 0) \tag{3.2}$$

for $\alpha, k \in \mathbb{R}$, $k > 0$, $k \neq \alpha^2$. When $\mathbb{T} = \mathbb{R}$ this becomes the differential equation

$$x'' + kx = \sin \alpha t$$

which describes the oscillations of a mass spring system with spring constant k . The solution to this equation with initial value conditions $x(0) = x'(0) = 0$ is

$$x(t) = \frac{1}{(k - \alpha^2)} \sin(\alpha t) + \frac{\alpha}{(\alpha^2 - k)\sqrt{k}} \sin(t\sqrt{k}) \quad (3.3)$$

Now we will use \mathcal{L} to solve the initial value problem

$$x^{\Delta\Delta} + kx = \sin_\alpha(t, 0), \quad x^\Delta(0) = x(0) = 0 \quad (3.4)$$

for all \mathbb{T}_+ under the assumption that $\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, 0) = 0$, $\lim_{t \rightarrow \infty} e_{\sqrt{k} \ominus z}(t, 0) = 0$, $\lim_{t \rightarrow \infty} x(t)e_{\ominus z}(t, 0) = 0$, and $\lim_{t \rightarrow \infty} x^\Delta(t)e_{\ominus z}(t, 0) = 0$. We start by applying \mathcal{L} to both sides of equation (3.4).

$$\mathcal{L}\{x^{\Delta\Delta} + kx\} = \mathcal{L}\{\sin_\alpha(t, 0)\}$$

$$\mathcal{L}\{x^{\Delta\Delta}\} + k\mathcal{L}\{x\} = \mathcal{L}\{\sin_\alpha(t, 0)\}$$

Notice that

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z^2 \mathcal{L}\{x\}(z)$$

by Corollary 3.7 along with our initial conditions and our assumption that $\lim_{t \rightarrow \infty} x(t)e_{\ominus z}(t, 0) = 0$, and $\lim_{t \rightarrow \infty} x^\Delta(t)e_{\ominus z}(t, 0) = 0$. Using this along with the transform of sine, part (iii) of Theorem 3.11, under the assumption that $\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, 0) = 0$ we get

$$z^2 \mathcal{L}\{x\}(z) + k\mathcal{L}\{x\}(z) = \frac{\alpha}{z^2 + \alpha^2}$$

$$(z^2 + k)\mathcal{L}\{x\}(z) = \frac{\alpha}{z^2 + \alpha^2}$$

$$\mathcal{L}\{x\}(z) = \frac{\alpha}{(z^2 + \alpha^2)(z^2 + k)}.$$

Decomposing this by partial fractions yields

$$\mathcal{L}\{x\}(z) = \frac{1}{(k - \alpha^2)} \left(\frac{\alpha}{z^2 + \alpha^2} \right) + \frac{\alpha}{(\alpha^2 - k)\sqrt{k}} \left(\frac{\sqrt{k}}{z^2 + k} \right).$$

Now we can use part (iii) of Theorem 3.11 again to invert the terms, that is:

$$\begin{aligned} \mathcal{L}\{x\}(z) &= \frac{1}{(k - \alpha^2)} \mathcal{L}\{\sin_\alpha(t, 0)\} + \frac{\alpha}{(\alpha^2 - k)\sqrt{k}} \mathcal{L}\{\sin_{\sqrt{k}}(t, 0)\} \\ &= \mathcal{L}\left\{ \frac{1}{(k - \alpha^2)} \sin_\alpha(t, 0) + \frac{\alpha}{(\alpha^2 - k)\sqrt{k}} \sin_{\sqrt{k}}(t, 0) \right\} \end{aligned}$$

Notice that we have used $\lim_{t \rightarrow \infty} e_{\sqrt{k} \ominus z}(t, 0) = 0$ in the inversion process. So the solution to (3.4) is

$$x(t) = \frac{1}{(k - \alpha^2)} \sin_\alpha(t, 0) + \frac{\alpha}{(\alpha^2 - k)\sqrt{k}} \sin_{\sqrt{k}}(t, 0). \quad (3.5)$$

Observe that using \mathcal{L} to solve the initial value problem for all time scales \mathbb{T}_+ was just as easy as restricting ourselves to \mathbb{R} .

In the process of solving this initial value problem, we have made the assumption that \mathcal{L} is injective on the space of functions in its domain. However, there is currently no result for time scales regarding this injectivity. This creates a problem when we try to find $x(t)$ from $\mathcal{L}\{x\}(z)$. There may exist another function, say $y(t)$, such that $\mathcal{L}\{x\}(z) = \mathcal{L}\{y\}(z)$. In order to ensure that the function found is indeed the solution to a given initial value problem, it would be wise to verify it directly. On a historical note, the techniques for using the Laplace transform to solve differential equations were first employed by the engineer Oliver Heaviside before they were proven mathematically.

We will now verify directly that $x(t)$ given in (3.5) is indeed the solution to the initial value problem given in (3.4). Employing Proposition 2.42 we find that:

$$\begin{aligned} x^\Delta(t) &= \frac{\alpha}{(k - \alpha^2)} \cos_\alpha(t, 0) + \frac{\alpha\sqrt{k}}{(\alpha^2 - k)\sqrt{k}} \cos_{\sqrt{k}}(t, 0) \\ x^{\Delta\Delta}(t) &= \frac{-\alpha^2}{(k - \alpha^2)} \sin_\alpha(t, 0) + \frac{-k\alpha}{(\alpha^2 - k)\sqrt{k}} \sin_{\sqrt{k}}(t, 0) \end{aligned}$$

Now computing the left side of our dynamic equation, we have that

$$\begin{aligned}
 x^{\Delta\Delta}(t) + kx(t) &= \left(\frac{-\alpha^2}{k - \alpha^2} + \frac{k}{(k - \alpha^2)} \right) \sin_\alpha(t, 0) \\
 &\quad + \left(\frac{-\alpha k}{(\alpha^2 - k)\sqrt{k}} + \frac{k\alpha}{(\alpha^2 - k)\sqrt{k}} \right) \sin_{\sqrt{k}}(t, 0) \\
 &= \left(\frac{k - \alpha^2}{k - \alpha^2} \right) \sin_\alpha(t, 0) \\
 &= \sin_\alpha(t, 0)
 \end{aligned}$$

as desired. Now we will check the initial conditions. First notice from their definitions that $\sin_p(t, t) = 0$ and $\cos_p(t, t) = 1$ as one would expect. Thus it is clear that $x(0) = 0$. Now consider

$$x^\Delta(0) = \frac{\alpha}{(k - \alpha^2)} + \frac{\alpha\sqrt{k}}{(\alpha^2 - k)\sqrt{k}} = \frac{\alpha}{(k - \alpha^2)} + \frac{-\alpha}{(k - \alpha^2)} = 0. \quad \checkmark$$

There is another Laplace transform defined for time scales with constant graininess by

$$\mathcal{L}_h\{f\}(z) := \int_0^\infty f(t)e_{\ominus z}(t, 0)\Delta t.$$

When $\mathbb{T} = \mathbb{Z}$, $\mathcal{L}_h\{f\}(z - 1) = \mathcal{Z}\{f\}(z)$ the \mathcal{Z} -transform defined by

$$\mathcal{Z}\{f\}(z) := \sum_{t=0}^\infty \frac{f(t)}{z^t}.$$

We'll use properties of the \mathcal{Z} -transform to solve the initial value problem 3.4 with \mathcal{L}_h when $\mathbb{T} = \mathbb{Z}$. In this case the problem presents itself in the form of a difference equation:

$$\Delta\Delta x + kx = \sin_\alpha(t, 0) \tag{3.6}$$

In solving this problem, the well known properties

$$\mathcal{Z}\{\Delta f\}(z) = (z - 1)\mathcal{Z}\{f\}(z) - z \cdot f(0)$$

$$\mathcal{Z}\{a^t\}(z) = \frac{z}{z - a}$$

of the \mathcal{Z} -transform along with the fact that $\mathcal{L}_h\{f\}(z) = \mathcal{Z}\{f\}(z+1)$ will be used. As with the previous two examples, this one is started by applying the transform in question to both sides.

$$\mathcal{L}_h\{\Delta\Delta x + kx\} = \mathcal{L}_h\{\sin_\alpha(t, 0)\}$$

$$\mathcal{L}_h\{\Delta\Delta x\} + k\mathcal{L}_h\{x\} = \mathcal{L}_h\{\sin_\alpha(t, 0)\} \quad (3.7)$$

Write $\sin_\alpha(t, 0)$ in terms of the exponential function.

$$\sin_\alpha(t, 0) = \frac{e_{i\alpha}(t, 0) - e_{-i\alpha}(t, 0)}{2i}.$$

We know for this time scale that $e_a(t, 0) = (1 + a)^t$, so we have

$$\sin_\alpha(t, 0) = \frac{(1 + i\alpha)^t - (1 - i\alpha)^t}{2i}.$$

Thus

$$\begin{aligned} \mathcal{Z}\{\sin_\alpha(t, 0)\}(z) &= \frac{\mathcal{Z}\{(1 + i\alpha)^t\} - \mathcal{Z}\{(1 - i\alpha)^t\}}{2i} \\ &= \frac{1}{2i} \left(\frac{z}{z - 1 - i\alpha} - \frac{z}{z - 1 + i\alpha} \right) \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}_h\{\sin_\alpha(t, 0)\}(z) &= \mathcal{Z}\{\sin_\alpha(t, 0)\}(z+1) \\ &= \frac{1}{2i} \left(\frac{z+1}{z-i\alpha} - \frac{z+1}{z+i\alpha} \right) \\ &= \frac{2i\alpha}{2i} \left(\frac{z+1}{z^2 - \alpha^2} \right) \\ &= \alpha \left(\frac{z+1}{z^2 - \alpha^2} \right) \end{aligned}$$

Now we find $\mathcal{L}_h\{\Delta\Delta x\}$.

$$\begin{aligned} \mathcal{L}_h\{\Delta\Delta x\} &= \mathcal{Z}\{\Delta\Delta x\}(z+1) = z \cdot \mathcal{Z}\{\Delta x\}(z+1) - (z+1) \cdot \Delta x(0) \\ &= z \cdot [z \cdot \mathcal{Z}\{x\}(z+1) - (z+1) \cdot x(0)] - (z+1) \cdot \Delta x(0) \end{aligned}$$

The initial conditions give

$$\mathcal{L}_h\{\Delta\Delta x\} = z^2\mathcal{Z}\{x\}(z+1) = (z-1)^2\mathcal{L}_h\{x\}(z)$$

Putting this and the transform of the sine into equation 3.7 yields

$$\begin{aligned} (z-1)^2\mathcal{L}_h\{x\}(z) + k\mathcal{L}_h\{x\}(z) &= \alpha \left(\frac{z+1}{z^2-\alpha^2} \right) \\ [(z-1)^2 + k]\mathcal{L}_h\{x\}(z) &= \alpha \left(\frac{z+1}{z^2-\alpha^2} \right) \\ \mathcal{L}_h\{x\}(z) &= \frac{\alpha(z+1)}{[z^2-\alpha^2][(z-1)^2+k]} \end{aligned} \tag{3.8}$$

We see at this point that the transformed equation is quite ugly, and trying to separate the right side of (3.8) only makes it worse. We will not proceed beyond this point in solving (3.6) but we will make a few observations about \mathcal{L}_h . Solving the same problem with \mathcal{L} was not nearly as involved as using \mathcal{L}_h . Furthermore, the transform \mathcal{L} is defined for all time scales \mathbb{T}_+ such that $0 \in \mathbb{T}_+$ and $\sup\mathbb{T}_+ = \infty$. \mathcal{L}_h is only defined for $\mathbb{T} = h\mathbb{Z}$ and \mathbb{R} . While we could probably extended \mathcal{L}_h to a greater class of time scales, we can't be sure it will work for all time scales \mathbb{T}_+ . It is for this reason that throughout the rest of this thesis, we'll focus on \mathcal{L} .

It would now be prudent to examine our Laplace transform in the case that \mathbb{T} is the set of positive integers, but first we have the definition of another discrete transform developed by Donahue in [5].

Definition 3.13 Let $f(t)$ be a function such that $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ and let $z \in \mathbb{R}$. Then the $\tilde{\mathcal{Z}}$ -transform is defined by

$$\tilde{\mathcal{Z}}\{f\}(z) := \sum_{t=0}^{\infty} \frac{f(t)}{(z+1)^{t+1}}$$

provided the series converges.

Proposition 3.14 When $\mathbb{T} = \mathbb{N}_0$, \mathcal{L} is equivalent to the $\tilde{\mathcal{Z}}$ -transform.

Proof: We will start by finding $e_{\ominus z}(t, 0)$ when $\mathbb{T} = \mathbb{Z}$.

$$e_{\ominus z}(t, 0) = \exp \left(\int_0^t \frac{1}{\mu(t)} \text{Log}(1 + \ominus z \mu(t)) \Delta t \right)$$

We know that for this time scale $\mu(t) = 1$ for all $t \in \mathbb{T}$, thus giving

$$e_{\ominus z}(t, 0) = \exp \left(\int_0^t \text{Log}(1 + \ominus z) \Delta t \right).$$

Now we use the definition of \ominus to get

$$e_{\ominus z}(t, 0) = \exp \left(\int_0^t \text{Log} \left(1 - \frac{z}{1+z} \right) \Delta t \right),$$

keeping in mind the value of μ .

$$\begin{aligned} e_{\ominus z}(t, 0) &= \exp \left(\int_0^t \text{Log} \left(\frac{1+z}{1+z} - \frac{z}{1+z} \right) \Delta t \right) \\ &= \exp \left(\int_0^t \text{Log} \left(\frac{1}{1+z} \right) \Delta t \right) \end{aligned}$$

The Δ -integral on this time scale is just the sum of the function values at each point of \mathbb{Z} within the bounds of integration, that is $\int_a^b f(t) \Delta t = \sum_{i=a}^b f(i)$. Thus we obtain

$$\begin{aligned} e_{\ominus z}(t, 0) &= \exp \left(\sum_{i=0}^t \text{Log} \left(\frac{1}{1+z} \right) \right) \\ &= \exp \left(t \text{Log} \left(\frac{1}{1+z} \right) \right) = \exp \left(\text{Log} \left(\frac{1}{1+z} \right)^t \right) \\ &= \left(\frac{1}{1+z} \right)^t. \end{aligned}$$

For this time scale, $\sigma(t) = t + 1$. So

$$e_{\ominus z}(\sigma(t), 0) = e_{\ominus z}(t + 1, 0) = \left(\frac{1}{z + 1} \right)^{t+1}$$

Therefore

$$\mathcal{L}\{f\}(z) = \int_0^\infty f(t) e_{\ominus z}(\sigma(t), 0) \Delta t = \int_0^\infty \frac{f(t)}{(z + 1)^{t+1}} \Delta t$$

$$= \sum_{t=0}^{\infty} \frac{f(t)}{(z+1)^{t+1}} = \tilde{\mathcal{Z}}\{f\}(z) \quad \square.$$

3.3 CONVOLUTION AND SHIFTING PROPERTIES OF SPECIAL FUNCTIONS

The usual convolution of two functions on the real interval $[0, \infty)$ is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds \quad \text{for } t \geq 0.$$

However, this definition does not work for general time scales because $t, s \in \mathbb{T}_+$ does not imply that $t-s \in \mathbb{T}_+$. So the possibility exists that f might not be defined at $t-s$. What we can do is give an alternative definition for the convolution of special functions on a general time scale \mathbb{T}_+ .

Definition 3.15 Assume that f is one of the functions $e_\alpha(t, 0)$, $\cosh_\alpha(t, 0)$, $\sinh_\alpha(t, 0)$, $\cos_\alpha(t, 0)$, $\sin_\alpha(t, 0)$, or $h_k(t, 0)$, $k \in \mathbb{N}_0$. If g is a regulated function on \mathbb{T}_0 , then we define the convolution of f with g by

$$(f * g)(t) := \int_0^t f(t, \sigma(s))g(s)\Delta s \quad \text{for } t \in \mathbb{T}_+.$$

Proposition 3.16 *Convolution Properties*

Assume that $\alpha \in \mathbb{R}$ and f is one of the functions $e_\alpha(\cdot, 0)$, $\cosh_\alpha(\cdot, 0)$, $\sinh_\alpha(\cdot, 0)$, $\cos_\alpha(\cdot, 0)$, $\sin_\alpha(\cdot, 0)$, or $h_k(\cdot, 0)$, $k \in \mathbb{N}_0$. If g is a regulated function on \mathbb{T}_0 such that

$$\lim_{t \rightarrow \infty} e_{\ominus \alpha}(t, 0)(f * g)(t) = 0,$$

then

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z).$$

Proof: First we will prove the case $f(t, 0) = e_\alpha(t, 0)$. Consider the dynamic initial value problem

$$y^\Delta - \alpha y = g(t), \quad y(0) = 0.$$

By the variation of constants formula in Theorem 2.45, the solution of the this problem is given by

$$y(t) = \int_0^t e_\alpha(t, \sigma(s))g(s)\Delta s.$$

But the right side of this is just $(e_\alpha(t, \sigma(s)) * g)(t)$. Now we apply the Laplace transform to both sides of $y^\Delta - \alpha y = g(t)$ to get

$$\begin{aligned} \mathcal{L}\{y^\Delta\}(z) - \alpha \mathcal{L}\{y\}(z) &= \mathcal{L}\{g\}(z). \\ \implies z \mathcal{L}\{y\}(z) - \alpha \mathcal{L}\{y\}(z) &= \mathcal{L}\{g\}(z) \\ \implies \mathcal{L}\{y\}(z) &= \frac{1}{z - \alpha} \mathcal{L}\{g\}(z) \end{aligned}$$

Theorem 3.11 tells us that $\mathcal{L}\{e_\alpha(t, 0)\}(z) = \frac{1}{z - \alpha}$, which leads to

$$\mathcal{L}\{y\}(z) = \mathcal{L}\{e_\alpha(t, 0)\}(z) \mathcal{L}\{g\}(z).$$

So

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \mathcal{L}\{g\}(z).$$

The cases that f is one of the functions $\cosh_\alpha(\cdot, 0)$, $\sinh_\alpha(\cdot, 0)$, $\cos_\alpha(\cdot, 0)$, or $\sin_\alpha(\cdot, 0)$ follows from their respective definitions, the proceeding case and the linearity of the Laplace transform.

We will now prove the proposition in the case $f(t, 0) = h_k(t, 0)$. Consider the dynamic initial value problem

$$y^{\Delta^{k+1}} = g(t), \quad y^{\Delta^i} = 0 \text{ for all } 0 \leq i \leq k.$$

By the variation of constants formula given by Theorem 2.46, the solution to this problem is given by

$$y(t) = \int_0^t h_k(t, \sigma(s))g(s)\Delta s.$$

As before, we recognize the right side of this equation as

$(h_k(t, \sigma(s)) * g)(t)$. Taking the Laplace transform of both sides of $y^{\Delta^{k+1}} = g(t)$, gives

$$\mathcal{L}\{y^{\Delta^{k+1}}\}(z) = \mathcal{L}\{g\}(z).$$

Employing Corollary 3.7 along with the initial conditions tells us that

$$\begin{aligned} z^{k+1} \mathcal{L}\{y\}(z) &= \mathcal{L}\{g\}(z). \\ \implies \mathcal{L}\{y\}(z) &= \frac{1}{z^{k+1}} \mathcal{L}\{g\}(z) \end{aligned}$$

Recall from Theorem 3.10 that $\mathcal{L}\{h_k(t, 0)\} = \frac{1}{z^{k+1}}$; thus

$$\begin{aligned} \mathcal{L}\{y\}(z) &= \mathcal{L}\{h_k(t, 0)\}(z) \mathcal{L}\{g\}(z). \\ \implies \mathcal{L}\{h_k(t, \sigma(s)) * g\}(z) &= \mathcal{L}\{h_k(t, 0)\}(z) \mathcal{L}\{g\}(z) \end{aligned}$$

So we get the desired result

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \mathcal{L}\{g\}(z). \quad \square$$

Corollary 3.17 *Assume f and g are each one of the functions $e_\alpha(t, 0)$, $\cosh_\alpha(t, 0)$, $\sinh_\alpha(t, 0)$, $\cos_\alpha(t, 0)$, $\sin_\alpha(t, 0)$, or $h_k(t, 0)$, $k \in \mathbb{N}_0$, not both $h_k(t, 0)$. Then*

$$f * g = g * f.$$

Proof: First we'll do the case

$$e_\alpha(t, 0) * e_\beta(t, 0) = e_\beta(t, 0) * e_\alpha(t, 0).$$

Let

$$x(t) := e_\alpha(t, 0) * e_\beta(t, 0) \text{ and } y(t) := e_\beta(t, 0) * e_\alpha(t, 0).$$

Notice that $x(0) = 0$ and $y(0) = 0$. Then by the variation of constants formula given in Corollary 2.45, $x(t)$ and $y(t)$ are the solutions to the initial value problems

$$x^\Delta - \alpha x = e_\beta(t, 0), \quad x(0) = 0 \quad (3.9)$$

and

$$y^\Delta - \beta y = e_\alpha(t, 0), \quad y(0) = 0$$

respectively. Then

$$x^\Delta(0) = \alpha x(0) + e_\beta(0, 0) = 1$$

and by the same reasoning $y^\Delta(0) = 1$. We claim that both x and y are solutions to the initial value problem

$$x^{\Delta\Delta} - (\alpha + \beta)x^\Delta + \alpha\beta x = 0, \quad x(0) = 0, \quad x^\Delta(0) = 1.$$

To show this, we rewrite (3.9) as

$$x^\Delta = \alpha x + e_\beta(t, 0), \quad (3.10)$$

and then differentiate both sides to get

$$x^{\Delta\Delta} = \alpha x^\Delta + \beta e_\beta(t, 0). \quad (3.11)$$

Using (3.11) we get that

$$\begin{aligned} & x^{\Delta\Delta} - (\alpha + \beta)x^\Delta + \alpha\beta x \\ &= \alpha x^\Delta + \beta e_\beta(t, 0) - (\alpha + \beta)x^\Delta + \alpha\beta x \\ &= \beta e_\beta(t, 0) - \beta x^\Delta + \alpha\beta x. \end{aligned}$$

Now using (3.10) we rewrite this as

$$\beta e_\beta(t, 0) - \alpha\beta x - \beta e_\beta(t, 0) + \alpha\beta x = 0.$$

One can show y is also a solution to this initial value problem by the same method.

Then because the solution of this problem is unique, we get $x(t) = y(t)$.

The proofs for the trigonometric functions and their hyperbolic counterparts follows the fact that they are defined in terms of the exponential function and from the linearity of the time scales integral. The last thing to show is that

$$e_\alpha(t, 0) * h_k(t, 0) = h_k(t, 0) * e_\alpha(t, 0).$$

Let

$$x(t) := e_\alpha(t, 0) * h_k(t, 0) \text{ and } y(t) := h_k(t, 0) * e_\alpha(t, 0).$$

Because $h_k(0, 0) = 0$, $k > 0$ then $x(0) = 0$ and $y(0) = 0$. (Recall that $h_0(t, s) = 1$ making this case trivial) So by the variation of constants formula given in Corollary 2.45, we have that $x(t)$ is a solution to the initial value problem

$$x^\Delta(t) - \alpha x = h_k(t, 0), \quad x(0) = 0.$$

Differentiating this i times gives

$$x^{\Delta^{i+1}} - \alpha x^{\Delta^i} = h_{k-i}(t, 0)$$

and by a finite induction on i

$$x^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k, \quad x^{\Delta^{k+1}}(0) = 1.$$

Thus $x(t)$ is the solution to the initial value problem

$$x^{\Delta^{k+2}} - \alpha x^{\Delta^{k+1}} = 0, \quad x^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k, \quad x^{\Delta^{k+1}}(0) = 1.$$

Now we will show that y is a solution to the same initial value problem. By Theorem 2.43, the fact that $h_k(0, 0) = 0$, $k > 0$, implies that

$$y^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k.$$

So by the variation of constants formula given in Theorem 2.46, y is the solution of the of the initial value problem

$$y^{\Delta^{k+1}} = e_\alpha(t, 0), \quad y^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k.$$

Thus $y^{\Delta^{k+1}}(0) = 1$ and

$$y^{\Delta^{k+2}} - \alpha y^{\Delta^{k+1}} = 0, \quad y^{\Delta^i}(0) = 0, \quad 0 \leq i \leq k, \quad y^{\Delta^{k+1}}(0) = 1.$$

So x and y are solutions to the same initial value problem and hence must be equal.

□

Now we will focus our attention on shifts. Shifting theorems provide the answers to two important questions:

- 1) What can we do to a function to cause its image under the Laplace transform to be shifted by a factor α ?
- 2) What does shifting a function by a factor α do to its image under the Laplace transform?

For $\mathbb{T} = \mathbb{R}$ the answer to the first question is given by the formula

$$\mathcal{L}\{e^{at}f(t)\}(z) = \mathcal{L}\{f\}(z - a).$$

While there is currently no known way of proving the analog of this for arbitrary functions and time scales, part (ii) of Theorem 2.39 allows us to prove such an analog for the exponential and trigonometric functions.

Proposition 3.18 *Shifting Property I*

If $\alpha, \beta \in \mathcal{R}(\mathbb{R})$ are constants, then

- (i) $\mathcal{L}\{e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(t, 0)\} = \frac{\beta}{(z-\alpha)^2 + \beta^2}$, provided that
 $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(t, 0) = 0$ and $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \left(\sin_{\frac{\beta}{1+\mu\alpha}}(t, 0) \right)^\Delta = 0$;
- (ii) $\mathcal{L}\{e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(t, 0)\} = \frac{z-\alpha}{(z-\alpha)^2 + \beta^2}$, provided that
 $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu\alpha}}(t, 0) = 0$ and $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \left(\cos_{\frac{\beta}{1+\mu\alpha}}(t, 0) \right)^\Delta = 0$;
- (iii) $\mathcal{L}\{e_\alpha(t, 0) \sinh_{\frac{\beta}{1+\mu\alpha}}(t, 0)\} = \frac{\beta}{(z-\alpha)^2 - \beta^2}$, provided that
 $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \sinh_{\frac{\beta}{1+\mu\alpha}}(t, 0) = 0$ and $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \left(\sinh_{\frac{\beta}{1+\mu\alpha}}(t, 0) \right)^\Delta = 0$;

(iv) $\mathcal{L}\{e_\alpha(t, 0) \cosh_{\frac{\beta}{1+\mu\alpha}}(t, 0)\} = \frac{z-\alpha}{(z-\alpha)^2-\beta^2}$, provided that

$$\lim_{t \rightarrow \infty} e_\alpha(t, 0) \cosh_{\frac{\beta}{1+\mu\alpha}}(t, 0) = 0 \text{ and } \lim_{t \rightarrow \infty} e_\alpha(t, 0) \left(\cosh_{\frac{\beta}{1+\mu\alpha}}(t, 0) \right)^\Delta = 0.$$

Proof: (i) Let

$$x(t) := e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0).$$

We start by verifying that $x(t)$ is the solution of the initial value problem

$$x^{\Delta\Delta} - 2\alpha x^\Delta + (\alpha^2 + \beta^2)x = 0, \quad x(0) = 0, \quad x^\Delta(0) = \beta. \quad (3.12)$$

Using the product rule, Theorem 2.12, and Theorem 2.39, we compute x^Δ and $x^{\Delta\Delta}$.

$$\begin{aligned} x^\Delta &= \alpha e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + \frac{\beta}{1+\mu(t)\alpha} e_\alpha(\sigma(t), 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &= \alpha e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + \frac{\beta(1+\mu(t)\alpha)}{1+\mu(t)\alpha} e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &= \alpha e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + \beta e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0). \end{aligned}$$

Now we Δ -differentiate x^Δ to get

$$\begin{aligned} x^{\Delta\Delta} &= \alpha^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + \alpha \frac{\beta}{1+\mu(t)\alpha} e_\alpha(\sigma(t), 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &\quad + \alpha \beta e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) - \beta \frac{\beta}{1+\mu(t)\alpha} e_\alpha(\sigma(t), 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &= \alpha^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + \alpha \frac{\beta(1+\mu(t)\alpha)}{1+\mu(t)\alpha} e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &\quad + \alpha \beta e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) - \beta \frac{\beta(1+\mu(t)\alpha)}{1+\mu(t)\alpha} e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &= \alpha^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) + 2\alpha \beta e_\alpha(t, 0) \cos_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) \\ &\quad - \beta^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0). \end{aligned}$$

So

$$x^{\Delta\Delta} - 2\alpha x^\Delta = -\alpha^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) - \beta^2 e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0)$$

$$= -(\alpha^2 + \beta^2)e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu(t)\alpha}}(t, 0) = -(\alpha^2 + \beta^2)x. \quad \checkmark$$

Noting that $e_\alpha(0, 0) = 1$, $\sin_{\frac{\beta}{1+\mu(t)\alpha}}(0, 0) = 0$, and $\cos_{\frac{\beta}{1+\mu(t)\alpha}}(0, 0) = 1$, it follows that $x(t)$ satisfies the initial conditions. Next we apply the Laplace transform to both sides of the dynamic equation in (3.12) and invoke Corollary 3.7 to obtain

$$z^2 \mathcal{L}\{x\} - zx(0) - x^\Delta(0) - 2\alpha[z\mathcal{L}\{x\} - x(0)] + (\alpha^2 + \beta^2)\mathcal{L}\{x\} = 0.$$

Notice that in the process of using Corollary 3.7, we have used the assumption that $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(t, 0) = 0$ and $\lim_{t \rightarrow \infty} e_\alpha(t, 0) \left(\sin_{\frac{\beta}{1+\mu\alpha}}(t, 0) \right)^\Delta = 0$. Using the initial conditions, we arrive at

$$z^2 \mathcal{L}\{x\} - \beta - 2\alpha z \mathcal{L}\{x\} + (\alpha^2 + \beta^2)\mathcal{L}\{x\} = 0.$$

We now proceed to solve for $\mathcal{L}\{x\}$.

$$\mathcal{L}\{x\}(z^2 - 2\alpha z + \alpha^2 + \beta^2) = \beta$$

$$\mathcal{L}\{x\}(z) = \frac{\beta}{z^2 - 2\alpha z + \alpha^2 + \beta^2}$$

$$\mathcal{L}\{x\}(z) = \frac{\beta}{(z - \alpha)^2 + \beta^2} \quad (3.13)$$

So we get

$$\mathcal{L}\{x\}(z) = \frac{\beta}{(z - \alpha)^2 + \beta^2}.$$

Recalling the definition of $x(t)$ we find that

$$\mathcal{L}\{e_\alpha(t, 0) \sin_{\frac{\beta}{1+\mu\alpha}}(t, 0)\} = \frac{\beta}{(z - \alpha)^2 + \beta^2}$$

as desired. The proofs of parts (ii), (iii), and (iv) proceed in a similar fashion.

□

Definition 3.19 Let $a \in \mathbb{T}$, $a > 0$, and define the *step function* u_a by

$$u_a(t) := \begin{cases} 0 & \text{if } t \in \mathbb{T} \cap (-\infty, a) \\ 1 & \text{if } t \in \mathbb{T} \cap [a, \infty). \end{cases}$$

Proposition 3.20 Let $a \in \mathbb{T}_+$, $a > 0$. Then

$$\mathcal{L}\{u_a(t)\} = \frac{e_{\ominus z}(a, 0)}{z}$$

for those $z \in \mathcal{R}(\mathbb{R})$ such that

$$\lim_{t \rightarrow \infty} e_{\ominus z}(t, 0) = 0.$$

Proof:

$$\mathcal{L}\{u_a(t)\}(z) = \int_0^\infty u_a(t) e_{\ominus z}(\sigma(t), 0) \Delta t = \int_a^\infty e_{\ominus z}(\sigma(t), 0) \Delta t$$

By Corollary 2.40 we can re write this as

$$\begin{aligned} \int_a^\infty \left(-\frac{\ominus z}{z} \right) e_{\ominus z}(t, 0) \Delta t &= -\frac{1}{z} [e_{\ominus z}(t, 0)]_0^\infty \\ &= -\frac{1}{z} [0 - e_{\ominus z}(a, 0)] = \frac{e_{\ominus z}(a, 0)}{z}. \quad \square \end{aligned}$$

We will now consider the second type of shift property, the type referred to in question 2 at the beginning of this section. For $\mathbb{T} = \mathbb{R}$ this kind of shift property is expressed by

$$\mathcal{L}\{u_a(t)f(t-a)\}(z) = e^{-az}\mathcal{L}\{f\}(z).$$

Again, though, we have a problem. As mentioned before the convolution proposition, even if both t and a are in our time scale, there is no guarantee that $t - a$ is in our

time scale. Thus $f(t - a)$ may not be defined. Once again, however, we can verify an analogous result for some functions. The following proposition is a new result for time scales.

Proposition 3.21 *Shifting Property II*

Let $a \in \mathbb{T}_+$, $a > 0$. Assume f is one of the functions $e_\alpha(t, a)$, $\cos_\alpha(t, a)$, $\sin_\alpha(t, a)$, $\sinh_\alpha(t, a)$, $\cosh_\alpha(t, a)$. If $z, \alpha \in \mathcal{R}(\mathbb{R})$ are regressive and satisfy

$$\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, a) = \lim_{t \rightarrow \infty} e_{i\alpha \ominus z}(t, a) = \lim_{t \rightarrow \infty} e_{-i\alpha \ominus z}(t, a) = 0,$$

then

$$\mathcal{L}\{u_a(t)f(t, a)\} = e_{\ominus z}(a, 0)\mathcal{L}\{f(t, 0)\}.$$

Proof: First we'll do the case $f(t, a) = e_\alpha(t, a)$. Observe that by using Theorem 2.39,

$$\begin{aligned} e_\alpha(t, a)e_{\ominus z}(\sigma(t), 0) &= \frac{1}{1 + \mu(t)z} e_\alpha(t, a)e_{\ominus z}(t, 0) \\ &= \frac{1}{1 + \mu(t)z} e_{\alpha \ominus z}(t, a)e_{\ominus z}(a, 0) = \frac{1}{\alpha - z} \left(\frac{\alpha - z}{1 + \mu(t)z} \right) e_{\alpha \ominus z}(t, a)e_{\ominus z}(a, 0) \\ &= \frac{1}{\alpha - z} (\alpha \ominus z) e_{\alpha \ominus z}(t, a)e_{\ominus z}(a, 0) = \frac{1}{\alpha - z} e_{\ominus z}(a, 0)(\alpha \ominus z) e_{\alpha \ominus z}(t, a) \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}\{u_a(t)f(t, a)\} &= \int_0^\infty u_a(t)e_\alpha(t, a)e_{\ominus z}(\sigma(t), 0)\Delta t \\ &= \frac{1}{\alpha - z} e_{\ominus z}(a, 0) \int_0^\infty u_a(t)(\alpha \ominus z)(t)e_{\alpha \ominus z}(t, a)\Delta t \\ &= \frac{1}{\alpha - z} e_{\ominus z}(a, 0) \int_a^\infty (\alpha \ominus z)(t)e_{\alpha \ominus z}(t, a)\Delta t \\ &= \frac{1}{\alpha - z} e_{\ominus z}(a, 0)[e_{\alpha \ominus z}(t, a)]_{t=a}^{t \rightarrow \infty} = -\frac{1}{\alpha - z} e_{\ominus z}(a, 0) \\ &= e_{\ominus z}(a, 0)\mathcal{L}\{e_\alpha(t, 0)\} \end{aligned}$$

provided that

$$\lim_{t \rightarrow \infty} e_{\alpha \ominus z}(t, a) = 0 \quad \checkmark.$$

Next consider the case $f(t, a) = \cos_\alpha(t, a)$.

$$\cos_\alpha(t, a) = \frac{e_{i\alpha}(t, a) + e_{-i\alpha}(t, a)}{2}$$

So

$$\begin{aligned} \mathcal{L}\{u_a(t) \cos_\alpha(t, a)\} &= \mathcal{L}\left\{u_a(t) \frac{e_{i\alpha}(t, a) + e_{-i\alpha}(t, a)}{2}\right\} \\ &= \frac{1}{2} \mathcal{L}\{u_a(t) e_{i\alpha}(t, a)\} + \frac{1}{2} \mathcal{L}\{u_a(t) e_{-i\alpha}(t, a)\} \end{aligned}$$

by the linearity of \mathcal{L} . Using the first case of this proof yields

$$\frac{1}{2} e_{\ominus z}(a, 0) \mathcal{L}\{e_{i\alpha}(t, 0)\} + \frac{1}{2} e_{\ominus z}(a, 0) \mathcal{L}\{e_{-i\alpha}(t, 0)\} \quad (3.14)$$

provided that $\lim_{t \rightarrow \infty} e_{i\alpha \ominus z}(t, a) = \lim_{t \rightarrow \infty} e_{-i\alpha \ominus z}(t, a) = 0$. Manipulating (3.14) yields

$$e_{\ominus z}(a, 0) \frac{1}{2} \mathcal{L}\{e_{i\alpha}(t, 0) + e_{-i\alpha}(t, 0)\} = e_{\ominus z}(a, 0) \mathcal{L}\left\{\frac{e_{i\alpha}(t, 0) + e_{-i\alpha}(t, 0)}{2}\right\}$$

This last line is again due to the linearity of \mathcal{L} . Now we use the definition of $\cos_\alpha(t, 0)$ to get

$$= e_{\ominus z}(a, 0) \mathcal{L}\{\cos_\alpha(t, 0)\} \quad \checkmark.$$

The proofs for $\sin_\alpha(t, a)$, $\cosh_\alpha(t, a)$, and $\sinh_\alpha(t, a)$ are the same as that for $\cos_\alpha(t, 0)$ because they are all linear combinations of exponential functions. \square .

We will now proceed to introduce the Dirac delta function. Consider a function $d_{a,\epsilon} : \mathbb{T} \rightarrow \mathbb{R}$ with parameters $\epsilon > 0$ and $a \in \mathbb{T}$, $a + \epsilon \in \mathbb{T}$, given by

$$d_{a,\epsilon}(t) := \begin{cases} \frac{1}{\epsilon} & \text{if } a \leq t < a + \epsilon \\ 0 & \text{otherwise .} \end{cases}$$

Then for $b \in \mathbb{T}$, $b \geq a + \epsilon$, we get that

$$\int_a^b d_{a,\epsilon}(t) \Delta t = \int_a^{a+\epsilon} \frac{1}{\epsilon} \Delta t = 1.$$

We can think of $d_{a,\epsilon}(t)$ as a force acting on a mass over a brief time interval of length ϵ such that the net effect of force is independent of ϵ . An impulse function can be thought of as such a force acting instantaneously (as in the real case) or at least acting over the smallest time interval allowed by a given time scale, that is

$$\delta_a(t) = \lim_{\epsilon \rightarrow \mu(a)} d_{a,\epsilon}(t).$$

Example 3.22 Suppose $\mathbb{T} = \mathbb{R}$, then

$$\delta_a(t) = \lim_{\epsilon \rightarrow 0} d_{a,\epsilon}(t) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose that $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then

$$\delta_a(t) = \lim_{\epsilon \rightarrow h} d_{a,\epsilon}(t) = \begin{cases} \frac{1}{h} & \text{if } t = a \\ 0 & \text{otherwise.} \end{cases}$$

Consider for a continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ the integral

$$\int_a^b f(t) d_{a,\epsilon}(t) \Delta t = \int_a^{a+\epsilon} f(t) d_{a,\epsilon}(t) \Delta t = \int_a^{a+\epsilon} f(t) \frac{1}{\epsilon} \Delta t.$$

As $\epsilon \rightarrow \mu(a)$ this integral approaches $f(a)$. This motivates our next definition.

Definition 3.23 Let $a, b \in \mathbb{T}$ and let $f : \mathbb{T} \rightarrow \mathbb{R}$ be continuous. If $\delta_0(t)$ satisfies the following two conditions:

(i)

$$\int_a^b f(t) \delta_0(t) \Delta t = f(0) \quad \text{if } a \leq 0 < b$$

(ii)

$$\int_a^b f(t)\delta_0(t)\Delta t = 0 \quad \text{if } 0 \text{ not in } [a, b)$$

then $\delta_0(t)$ is called the *Dirac delta function*.

We define the impulse function with parameter $t_0 \in \mathbb{T}$ by

$$\delta_{t_0}(t) = \delta_0(t - t_0).$$

The next Theorem and the following Corollary are new results for time scales.

Theorem 3.24 *Let $a, b \in \mathbb{T}$ and let $f : \mathbb{T} \rightarrow \mathbb{R}$ be continuous. Then*

$$\int_a^b f(t)\delta_{t_0}(t)\Delta t = \begin{cases} f(t_0) & \text{if } a \leq t_0 \leq b \\ 0 & \text{if } t_0 \text{ not in } [a, b] \end{cases}$$

Proof: Let $\gamma(t) = t - t_0$. Then γ is strictly increasing, Δ -differentiable, and $\gamma^\Delta = 1$ is Δ -integrable on each finite interval of \mathbb{T} . Furthermore, because $\gamma(\mathbb{T})$ is just a translation of \mathbb{T} , it is also a time scale. Then by the change of variable theorem (Theorem 2.25)

$$\begin{aligned} \int_a^b f(t)\delta_{t_0}(t)\Delta t &= \int_a^b f(t)\delta_{t_0}(t)\gamma^\Delta(t)\Delta t \\ &= \int_{\gamma(a)}^{\gamma(b)} (f \circ \gamma^{-1})(s)(\delta_{t_0} \circ \gamma^{-1})(s)\tilde{\Delta}s \\ &= \int_{\gamma(a)}^{\gamma(b)} f(s + t_0)\delta_{t_0}(s + t_0)\tilde{\Delta}s \\ &= \int_{\gamma(a)}^{\gamma(b)} f(s + t_0)\delta_0(s + t_0 - t_0)\tilde{\Delta}s \\ &= \int_{\gamma(a)}^{\gamma(b)} f(s + t_0)\delta_0(s)\tilde{\Delta}s. \end{aligned}$$

If $t_0 \in [a, b]$, then $0 \in [\gamma(a), \gamma(b)]$ and so we get

$$\int_{\gamma(a)}^{\gamma(b)} f(s + t_0) \delta_0(s) \tilde{\Delta} s = f(0 + t_0) = f(t_0).$$

If t_0 is not in $[a, b]$, then 0 is not in $[\gamma(a), \gamma(b)]$ and we get

$$\int_{\gamma(a)}^{\gamma(b)} f(s + t_0) \delta_0(s) \tilde{\Delta} s = 0. \quad \square$$

Corollary 3.25 *Let $\alpha \geq 0$ be regressive. Then*

$$\mathcal{L}\{\delta_0(t - \alpha)\}(z) = e_{\ominus} z(\alpha, 0) \quad (3.15)$$

for $z \in \mathcal{R}(\mathbb{R})$.

Proof:

$$\begin{aligned} \mathcal{L}\{\delta_0(t - \alpha)\}(z) &= \int_0^{\infty} \delta_0(t - \alpha) e_{\ominus} z(t, 0) \Delta t \\ &= e_{\ominus} z(\alpha, 0) \quad \text{by Theorem 3.24} \quad \square \end{aligned}$$

Example 3.26 We'll now use \mathcal{L} to solve the following initial value problem:

$$ax^{\Delta\Delta} + bx^{\Delta} + cx = \delta_{\alpha}(t) \quad (3.16)$$

$$x(0) = x^{\Delta}(0) = 0, \quad a, b, c \in \mathbb{R}, \quad a \neq 0, \quad \alpha \geq 0. \quad (3.17)$$

Applying \mathcal{L} to both sides of equation 3.16, we get

$$a\mathcal{L}\{x^{\Delta\Delta}\}(z) + b\mathcal{L}\{x^{\Delta}\}(z) + c\mathcal{L}\{x\}(z) = \mathcal{L}\{\delta_0(t - \alpha)\}(z).$$

Corollary 3.25 implies

$$a\mathcal{L}\{x^{\Delta\Delta}\}(z) + b\mathcal{L}\{x^{\Delta}\}(z) + c\mathcal{L}\{x\}(z) = e_{\ominus} z(\alpha, 0).$$

Using Theorem 3.7 along with the initial value conditions (3.17) yields

$$(az^2 + bz + c)\mathcal{L}\{x\}(z) = e_{\ominus z}(\alpha, 0)$$

provided our solution satisfies $\lim_{t \rightarrow \infty} x(t)e_{\ominus z}(t, 0) = \lim_{t \rightarrow \infty} x^\Delta(t)e_{\ominus z}(t, 0) = 0$. So

$$\mathcal{L}\{x\}(z) = \frac{e_{\ominus z}(\alpha, 0)}{az^2 + bz + c}. \quad (3.18)$$

Suppose $b^2 - 4ac > 0$ and let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

the roots of $az^2 + bz + c$. Then

$$\mathcal{L}\{x\}(z) = \frac{e_{\ominus z}(\alpha, 0)}{a(z - r_1)(z - r_2)}.$$

Decomposing this by partial fractions gives

$$a\mathcal{L}\{x\}(z) = \frac{e_{\ominus z}(\alpha, 0)}{(z - r_2)(r_2 - r_1)} - \frac{e_{\ominus z}(\alpha, 0)}{(z - r_1)(r_2 - r_1)}.$$

By Theorem 3.11

$$a\mathcal{L}\{x\}(z) = \frac{1}{r_2 - r_1}e_{\ominus z}(\alpha, 0)\mathcal{L}\{e_{r_2}(t, 0)\} - \frac{1}{r_2 - r_1}e_{\ominus z}(\alpha, 0)\mathcal{L}\{e_{r_1}(t, 0)\}.$$

Proposition 3.21 gives

$$\mathcal{L}\{x\}(z) = \frac{1}{a(r_2 - r_1)}\mathcal{L}\{u_\alpha(t)e_{r_2}(t, 0)\} - \frac{1}{a(r_2 - r_1)}\mathcal{L}\{u_\alpha(t)e_{r_1}(t, 0)\}$$

So

$$x(t) = \frac{1}{a(r_2 - r_1)}u_\alpha(t)e_{r_2}(t, 0) - \frac{1}{a(r_2 - r_1)}u_\alpha(t)e_{r_1}(t, 0).$$

Now suppose that $b^2 - 4ac = 0$. Then $az^2 + bz + c = a(z + \frac{b}{2a})^2$ and so equation (3.18) becomes

$$\mathcal{L}\{x\}(z) = \frac{e_{\ominus z}(\alpha, 0)}{a(z + \frac{b}{2a})^2} = \frac{1}{a}e_{\ominus z}(\alpha, 0) \left(\mathcal{L}\{e_{-\frac{b}{2a}}(t, 0)\} \right)^2.$$

by Theorem 3.11. Applying Proposition 3.21 yields

$$\mathcal{L}\{x\}(z) = \frac{1}{a} \mathcal{L}\{u_a(t)e_{-\frac{b}{2a}}(t, \alpha)\} \mathcal{L}\{e_{-\frac{b}{2a}}(t, 0)\}.$$

After observing that $u_a(t)e_{-\frac{b}{2a}}(t, \alpha)$ is regulated, we can simplify the right side using Proposition 3.16,

$$\mathcal{L}\{x\}(z) = \frac{1}{a} \mathcal{L}\{e_{-\frac{b}{2a}}(t, \alpha) * (u_a(t)e_{-\frac{b}{2a}}(t, \alpha))\}.$$

So

$$x(t) = \frac{1}{a} e_{-\frac{b}{2a}}(t, \alpha) * (u_a(t)e_{-\frac{b}{2a}}(t, \alpha)). \quad \checkmark$$

CHAPTER 4

FOURIER TRANSFORM

Fourier analysis for locally compact abelian groups may be found in *Fourier Analysis on Groups* by Walter Rudin ¹. The presentation given here by defining the Fourier transform on time scales, however, is more concrete than that which appears in such an abstract environment while still showing the importance of the group structure.

4.1 NOTATION AND DEFINITIONS

Before defining the Fourier transform, we'll need to define some special time scales.

Definition 4.1 The time scale \mathbb{T}_h is defined as

$$\mathbb{T}_h := \begin{cases} 0 & \text{if } h = \infty \\ h\mathbb{Z} & \text{if } h > 0 \\ \mathbb{R} & \text{if } h = 0. \end{cases}$$

Notice that $(\mathbb{T}_h, +)$ forms an abelian group. If $H > 0$ is an integer multiple of h or if $H = \infty$, then \mathbb{T}_H is a proper subgroup of \mathbb{T}_h . This leads us to our next definition.

Definition 4.2 Let $H > h$ such that \mathbb{T}_H is a proper subgroup of \mathbb{T}_h . Then the time

¹W. Rudin. *Fourier Analysis on Groups*. Interscience Publishers, New York, 1962.

scale \mathbb{T}_{hH} is defined as the quotient group

$$\mathbb{T}_{hH} := \mathbb{T}_h / \mathbb{T}_H,$$

and $\hat{\mathbb{T}}_{hH}$ is defined by

$$\hat{\mathbb{T}}_{hH} := \mathbb{T}_{\hat{h}\hat{H}} \quad \text{where } \hat{h} = \lim_{x \rightarrow H} \frac{2\pi}{x} \text{ and } \hat{H} = \lim_{x \rightarrow h} \frac{2\pi}{x}.$$

Notice that if $h < \infty$ then $\hat{h} = \frac{2\pi}{h}$ with the same being true for \hat{H} . Here, h is the graininess of the timescale and H can be thought of as the length. We will illustrate this with a few examples.

Example 4.3

$$\mathbb{T}_{h\infty} = h\mathbb{Z}/\{0\} = h\mathbb{Z}$$

$$\mathbb{T}_{0\infty} = \mathbb{R}/\{0\} = \mathbb{R}$$

Assume that $H < \infty$.

$$\mathbb{T}_{0H} = \mathbb{R}/H\mathbb{Z} = \{x \pmod{H} : x \in \mathbb{R}\}$$

Now also assume that $h > 0$.

$$\mathbb{T}_{hH} = h\mathbb{Z}/H\mathbb{Z} \cong \mathbb{Z}_{H/h}$$

Proposition 4.4 *All time scales that are abelian groups may be written as \mathbb{T}_{hH} for some h and H .*

Proof: Let G be a time scale such that $(G, +)$ is an abelian group. We will show first that G^κ must have constant graininess, that is $\mu(t) \equiv h$, $t \in G^\kappa$, for some $h \in \mathbb{R} \cup \{\infty\}$, $h \geq 0$. Notice that by considering $t \in G^\kappa$ we avoid the case when G

is finite and has $\mu(\sup G) = 0$. Suppose that $a, b \in G^\kappa$. Then clearly $\sigma(a), \sigma(b) \in G$ and also the additive inverses $-a, -b \in G$. So $\mu(a), \mu(b) \in G$ by the closure of G . By the order on \mathbb{R} , we have that $\mu(a) \leq \mu(b)$, $\mu(b) \leq \mu(a)$, or both. With out loss of generality we may assume that $\mu(a) \leq \mu(b)$. Again by the closure of G , we find that $b + \mu(a) \in G$. By the definition of σ , $\sigma(b) \leq b + \mu(a)$. Thus $\sigma(b) - b \leq \mu(a) \implies \mu(b) \leq \mu(a)$. So $\mu(a) = \mu(b)$. Thus if we let h be the graininess of G , then as a set $G \subseteq \mathbb{T}_h$. Let

$$H = \sup_{a, b \in G} |b - a| + h.$$

If $h = 0$ then G is a closed interval with length given by H . If $h > 0$, then H/h is the order of the group G . Thus we get that $G = \mathbb{T}_{hH}$. \square

We now give a couple of definitions to establish relationships between functions defined on \mathbb{T}_h and functions defined on \mathbb{T}_{hH} .

Definition 4.5 For each function $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ we define the function $f^\uparrow : \mathbb{T}_h \rightarrow \mathbb{C}$ by

$$f^\uparrow(t) := f(t \bmod H) \chi_{[-\frac{H}{2}, \frac{H}{2}]}(t)$$

where $\chi_{[-\frac{H}{2}, \frac{H}{2}]}$ is the indicator function of the interval $[-\frac{H}{2}, \frac{H}{2}]$.

What we have done here is to periodically extended f to \mathbb{T}_h , then we restricted this continuation to the interval $[-H/2, H/2]$, and finally we halved the function at the end points $\pm H/2$.

Definition 4.6 Let $f : \mathbb{T}_h \rightarrow \mathbb{C}$. We define $f^\downarrow : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ by

$$f^\downarrow(t) := \begin{cases} f(t) & \text{for } |t| < \frac{H}{2} \\ \frac{1}{2} [f(-\frac{H}{2}) + f(+\frac{H}{2})] & \text{for } |t| = \frac{H}{2}. \end{cases}$$

Essentially, what we have done here is to make a periodic function of period H from the restriction of f to the interval $[-\frac{H}{2}, +\frac{H}{2}]$, averaging the endpoints if they are in \mathbb{T}_{hH} . Notice that if $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$, then $(f^\dagger)^\downarrow = f$. These relationships allow us to define integration on \mathbb{T}_{hH} in terms of integration on \mathbb{T}_h .

Definition 4.7 Let $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$. Then we define the integral of f on \mathbb{T}_{hH} by

$$\int_a^b f(t) \Delta t := \int_{-\infty}^{\infty} f^\dagger(s) \chi_{[a,b]} \Delta s \quad t \in \mathbb{T}_{hH}, s \in \mathbb{T}_h$$

where $\chi_{[a,b]}$ is the characteristic (or indicator) function of the interval $[a, b] \subset \mathbb{R}$.

Now we give the definition of the time scale Fourier transform.

Definition 4.8 Let $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$. We define the Fourier transform of f by

$$\mathcal{F}\{f\}(\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \Delta t$$

for those $\omega \in \hat{\mathbb{T}}_{hH}$ such that the integral exists. At times we may also write $\mathcal{F}\{f\}(\omega)$ as $\hat{f}(\omega)$.

For $\mathbb{T}_{0,\infty} = \mathbb{R}$, this becomes

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

the usual Fourier integral. When $h = 0$ and $H < \infty$, we get

$$\mathcal{F}\{f\}(\omega) = \int_{-H/2}^{H/2} f(t) e^{-i\omega t} dt$$

the Fourier transform for H -periodic functions. In the case $\mathbb{T}_{h,\infty} = h\mathbb{Z}$, the transform becomes the Fourier series

$$\mathcal{F}\{f\}(\omega) = h \sum_{t \in h\mathbb{Z}} f(t) e^{-i\omega t} = h \sum_{n=-\infty}^{\infty} f(nh) e^{-i\omega nh}.$$

Similarly in the case, $H/h = N$ we get

$$\mathcal{F}\{f\}(\omega) = h \sum_{t \in \mathbb{T}_{hH}} f(t) e^{-i\omega t} = h \sum_{n=0}^{N-1} f(nh) e^{-i\omega nh}$$

the discrete Fourier transform. So the Fourier transform for time scales incorporates the four classical kinds of Fourier transform for \mathbb{R} .

4.2 PROPERTIES OF THE FOURIER TRANSFORM

Theorem 4.9 (*Linearity*). *Let $f, g : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\mathcal{F}\{\alpha f + \beta g\}(\omega) = \alpha \mathcal{F}\{f\}(\omega) + \beta \mathcal{F}\{g\}(\omega)$$

for those $\omega \in \hat{\mathbb{T}}_{hH}$ such that $\mathcal{F}\{f\}(\omega)$ and $\mathcal{F}\{g\}(\omega)$ converge.

Proof: The proof of this theorem follows directly from the linearity of the Δ -integral, Theorem 2.23.

Theorem 4.10 *Let $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ and $t, \tau \in \mathbb{T}_{hH}$. Then*

(i)

$$\mathcal{F}\{e^{it\nu} f\}(\omega) = \mathcal{F}\{f\}(\omega - \nu)$$

(ii)

$$\mathcal{F}\{f(t + \tau)\}(\omega) = e^{i\omega\tau} \mathcal{F}\{f(t)\}(\omega)$$

(iii)

$$\mathcal{F}\{f(-t)\}(\omega) = \mathcal{F}\{f(t)\}(-\omega)$$

(iv)

$$\mathcal{F}\{\bar{f}\}(\omega) = \overline{\mathcal{F}\{f\}(-\omega)}$$

for those $\nu, \omega \in \hat{\mathbb{T}}_{hH}$ such that the respective integrals exist.

Proof: (i)

$$\begin{aligned}\mathcal{F}\{e^{it\nu}f\}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{it\nu}e^{-i\omega t}\Delta t = \int_{-\infty}^{\infty} f(t)e^{-i\omega(t-\nu)}\Delta t \\ &= \mathcal{F}\{f\}(\omega - \nu) \quad \checkmark\end{aligned}$$

(ii) Here we could employ the change of variable theorem for times scales, but because we have constant graininess on each time scale \mathbb{T}_{hH} the usual change of variable theorem is valid. Thus we have

$$\begin{aligned}\mathcal{F}\{f(t + \tau)\}(\omega) &= \int_{-\infty}^{\infty} f(t + \tau)e^{-i\omega t}\Delta t \\ &= \int_{-\infty}^{\infty} f(t)e^{-i\omega(t-\tau)}\Delta t \\ &= e^{i\omega\tau} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\Delta t.\end{aligned}\tag{4.1}$$

Suppose $H = \infty$, then (4.1) becomes

$$e^{i\omega\tau} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\Delta t = e^{i\omega\tau} \mathcal{F}\{f(t)\}(\omega).$$

Suppose that $H < \infty$, then the definition of \mathbb{T}_{hH} as a quotient group gives us that $f(t + H) = f(t)$ and so (4.1) becomes

$$e^{i\omega\tau} \int_{-H/2}^{H/2} f(t)e^{-i\omega t}\Delta t = e^{i\omega\tau} \mathcal{F}\{f(t)\}(\omega). \quad \checkmark$$

(iii)

$$\begin{aligned}\mathcal{F}\{f(-t)\}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega(-t)}\Delta t = \int_{-\infty}^{\infty} f(t)e^{-i(-\omega)t}\Delta t \\ &= \mathcal{F}\{f(t)\}(-\omega) \quad \checkmark\end{aligned}$$

(iv)

$$\begin{aligned}\mathcal{F}\{\overline{f}\}(\omega) &= \int_{-\infty}^{\infty} \overline{f(t)}e^{-i\omega t}\Delta t = \int_{-\infty}^{\infty} \overline{f(t)}e^{i(-\omega)t}\Delta t \\ &= \overline{\mathcal{F}\{f\}(-\omega)} \quad \square\end{aligned}$$

The following is a new theorem for time scales.

Theorem 4.11

$$\mathcal{F}\{f^{\Delta^k}\}(\omega) = \left(\lim_{\tau \rightarrow h} \frac{e^{i\omega\tau} - 1}{\tau} \right)^k \mathcal{F}\{f\}(\omega) \quad (4.2)$$

for $\omega \in \hat{\mathbb{T}}_{hH}$.

Proof: Observe that when $h = 0$,

$$\lim_{\tau \rightarrow h} \frac{e^{i\omega\tau} - 1}{\tau} = \lim_{\tau \rightarrow 0} \frac{e^{i\omega\tau} - 1}{\tau} = \lim_{\tau \rightarrow 0} \frac{i\omega e^{i\omega\tau}}{1} = i\omega.$$

Thus (4.2) just becomes the well known statement from real analysis. Consider $h > 0$ and $k = 1$. Then

$$\begin{aligned} \mathcal{F}\{f^{\Delta}\}(\omega) &= \int_{-\infty}^{\infty} f^{\Delta}(t) e^{-i\omega t} \Delta t = \int_{-\infty}^{\infty} \frac{f(t+h) - f(t)}{h} e^{-i\omega t} \Delta t \\ &= \frac{1}{h} \int_{-\infty}^{\infty} f(t+h) e^{-i\omega t} \Delta t - \frac{1}{h} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \Delta t. \end{aligned}$$

By part (ii) of Theorem 4.10, we get that this is

$$\begin{aligned} \frac{1}{h} e^{i\omega h} \mathcal{F}\{f\}(\omega) - \frac{1}{h} \mathcal{F}\{f\}(\omega) &= \frac{e^{i\omega h} - 1}{h} \mathcal{F}\{f\}(\omega) \\ &= \lim_{\tau \rightarrow h} \frac{e^{i\omega\tau} - 1}{\tau} \mathcal{F}\{f\}(\omega). \end{aligned}$$

The remainder of the proof proceeds by induction on k . \square

Theorem 4.12 Let $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ and $g : \hat{\mathbb{T}}_{hH} \rightarrow \mathbb{C}$ be functions such that

$$\int_{-\infty}^{\infty} |f(t)| \Delta t < \infty \text{ and } \int_{-\infty}^{\infty} |g(\omega)| \Delta \omega < \infty.$$

Then

$$\int_{-\infty}^{\infty} |f(t) \hat{g}(t)| \Delta t < \infty \text{ and } \int_{-\infty}^{\infty} |\hat{f}(\omega) g(\omega)| \Delta \omega < \infty.$$

Furthermore

$$\int_{-\infty}^{\infty} f(t) \hat{g}(t) \Delta t = \int_{-\infty}^{\infty} \hat{f}(t) g(t) \Delta t.$$

Proof: The proof of this theorem proceeds by examining each of the four cases and using results from real analysis, in particular Fubini's Theorem.

Definition 4.13 For $T \in \mathbb{T}_h$, $T \geq 0$, we define the *rectangle function* χ_T on \mathbb{T}_h by

$$\chi_T(t) := \chi_{[-T, T]}$$

where once again $\chi_{[-T, T]}$ is the characteristic function of the interval $[-T, T]$.

For $T < H/2$, the function χ_T^\downarrow is also a rectangle function on \mathbb{T}_{hH} .

Notice that we have been using $e^{i\omega t}$, the restriction of the usual exponential function to a time scale, and not the time scale exponential function $e_{i\omega}(t, 0)$. So we cannot assume that $(e^{i\omega t})^\Delta = i\omega e^{i\omega t}$. In fact, this does not hold true in general as we will see in the next proposition.

Proposition 4.14 *Let $t \in \mathbb{T}_{hH}$. Then*

$$(e^{i\omega t})^\Delta = i\omega \operatorname{sinc}\left(\frac{\omega h}{2}\right) e^{i\omega(t+\frac{h}{2})}$$

where

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Proof: The case where $h = 0$ is obvious. Consider $h > 0$. Then

$$\begin{aligned} (e^{i\omega t})^\Delta &= \frac{e^{i\omega(t+h)} - e^{i\omega t}}{h} = \frac{e^{i\omega\frac{h}{2}} - e^{-i\omega\frac{h}{2}}}{h} e^{i\omega(t+\frac{h}{2})} = \frac{2i}{h} \sin\left(\frac{\omega h}{2}\right) e^{i\omega(t+\frac{h}{2})} \\ &= i\omega \operatorname{sinc}\left(\frac{\omega h}{2}\right) e^{i\omega(t+h/2)} \quad \square \end{aligned}$$

It follows from Proposition 4.14 that the Δ -derivatives of the usual trigonometric functions are

$$\cos^\Delta(\omega t) = -\omega \operatorname{sinc}\left(\frac{\omega h}{2}\right) \sin\left[\omega\left(t + \frac{h}{2}\right)\right]$$

and

$$\sin^\Delta(\omega t) = \omega \operatorname{sinc}\left(\frac{\omega h}{2}\right) \cos\left[\omega\left(t + \frac{h}{2}\right)\right]. \quad (4.3)$$

Lemma 4.15 *Let $a, b \in \mathbb{T}_h$ and let $f : \mathbb{T}_h \rightarrow \mathbb{C}$. Then*

$$\int_a^b f(t) \Delta t = \int_{-b+h}^{-a+h} f(-t) \Delta t.$$

Proof: For $h = 0$ this is clear. Consider $h > 0$:

$$\begin{aligned} \int_a^b f(t) \Delta t &= h[f(a) + \dots + f(b-h)] \\ &= h[f(-[(-a+h)-h]) + \dots + f(-(-s+h))] \\ &= h[f(-(-s+h)) + \dots + f(-[(-a+h)-h])] \\ &= \int_{-b+h}^{-a+h} f(-t) \Delta t \quad \square \end{aligned}$$

Now we will consider the Fourier transform of the rectangle function $\chi_T^\downarrow(t)$.

Proposition 4.16 *Let $T < H/2$, then*

$$\mathcal{F}\{\chi_T^\downarrow\}(\omega) = 2 \frac{\operatorname{sinc}[\omega(T - \frac{h}{2})]}{\operatorname{sinc}(\frac{\omega h}{2})} \left(T - \frac{h}{2}\right)$$

for $\omega \in \hat{\mathbb{T}}_{hH} \setminus \{0\}$.

Proof: Notice that χ_T^\downarrow is a function on \mathbb{T}_{hH} . Assume $s \in \mathbb{T}_h$. Recalling the definition of the integral on \mathbb{T}_{hH} we get that

$$\begin{aligned}
\mathcal{F}\{\chi_T^\downarrow\}(\omega) &= \int_{-\infty}^{\infty} \chi_T^\downarrow(t) e^{-i\omega t} \Delta t \\
&= \int_{-\infty}^{\infty} (\chi_T^\downarrow(s) e^{-i\omega s})^\uparrow \Delta s \\
&= \int_{-T}^{T+h} e^{-i\omega s} \Delta s \\
&= \int_{-T}^0 e^{-i\omega s} \Delta s + \int_0^h e^{-i\omega s} \Delta s + \int_h^{T+h} e^{-i\omega s} \Delta s \\
&= \int_{-T}^0 e^{-i\omega s} \Delta s + \int_0^{\sigma(0)} e^{-i\omega s} \Delta s + \int_h^{T+h} e^{-i\omega s} \Delta s \\
&= \int_{-T}^0 e^{-i\omega s} \Delta s + \mu(0) e^0 + \int_h^{T+h} e^{-i\omega s} \Delta s \\
&= \int_{-T}^0 e^{-i\omega s} \Delta s + h + \int_h^{T+h} e^{-i\omega s} \Delta s
\end{aligned}$$

By the previous lemma, we can rewrite this as

$$\begin{aligned}
\mathcal{F}\{\chi_T^\downarrow\}(\omega) &= h + \int_h^{T+h} e^{i\omega s} \Delta s + \int_h^{T+h} e^{-i\omega s} \Delta s \\
&= h + 2 \int_h^{T+h} \cos(\omega s) \Delta s.
\end{aligned}$$

Notice that from (4.3) we have that

$$\begin{aligned}
\sin^\Delta(\omega s) &= \omega \operatorname{sinc}\left(\frac{\omega h}{2}\right) \cos\left[\omega\left(s + \frac{h}{2}\right)\right] \\
\frac{\sin^\Delta\left[\omega\left(s - \frac{h}{2}\right)\right]}{\omega \operatorname{sinc}\left(\frac{\omega h}{2}\right)} &= \cos(\omega s).
\end{aligned}$$

Thus we may write

$$\begin{aligned}
h + 2 \int_h^{T+h} \cos(\omega s) \Delta s &= h + 2 \left[\frac{\sin \left[\omega \left(t - \frac{h}{2} \right) \right]}{\omega \operatorname{sinc} \left(\frac{\omega h}{2} \right)} \right]_{s=h}^{s=T+h} \\
&= h + 2 \left[\frac{\sin \left[\omega \left(s - \frac{h}{2} \right) \right] \left(s - \frac{h}{2} \right)}{\operatorname{sinc} \left(\frac{\omega h}{2} \right) \left(s - \frac{h}{2} \right)} \right]_{s=h}^{s=T+h} \\
&= h + 2 \left[\frac{\operatorname{sinc} \left[\omega \left(s - \frac{h}{2} \right) \right]}{\operatorname{sinc} \left(\frac{\omega h}{2} \right)} \left(s - \frac{h}{2} \right) \right]_{s=h}^{s=T+h} \\
&= 2 \frac{\operatorname{sinc} \left[\omega \left(T + \frac{h}{2} \right) \right]}{\operatorname{sinc} \left(\frac{\omega h}{2} \right)} \left(T + \frac{h}{2} \right). \quad \square
\end{aligned}$$

Definition 4.17 We define the trapezoidal function $\phi_T(t)$ for $T \in \mathbb{T}_{\frac{h}{2}}$ by

$$\phi_T := \begin{cases} \frac{1}{2}(\chi_{T-h} + \chi_T), & \text{if } T \in \mathbb{T}_h, \\ \chi_{T-\frac{h}{2}}, & \text{if } T \in \mathbb{T}_h + \frac{h}{2} \end{cases}$$

where by $\mathbb{T}_h + \frac{h}{2}$ we mean the translation $\{t + \frac{h}{2} : t \in \mathbb{T}_h\}$.

It follows from Proposition 4.16 that

$$\mathcal{F}\{\phi_T^\downarrow\}(\omega) = \int_{-T}^{T+h} \phi_T(t) e^{-i\omega t} \Delta t = \begin{cases} 2 \frac{\operatorname{sinc}(\omega T)}{\operatorname{tanc}(\frac{\omega h}{2})} T, & \text{if } T \in \mathbb{T}_h, \\ 2 \frac{\operatorname{sinc}(\omega T)}{\operatorname{sinc}(\frac{\omega h}{2})} T, & \text{if } T \in \mathbb{T}_h + \frac{h}{2}, \end{cases}$$

where $\operatorname{tanc}(x) := \frac{\operatorname{sinc}(x)}{\cos(x)}$.

4.3 FOURIER INVERSION

Definition 4.18 Suppose that $\Omega \in \mathbb{T}_{\frac{h}{2}}$, $0 \leq \Omega \leq \frac{\hat{H}}{2}$, and let $\phi_\Omega(\omega)$ be the trapezoidal

function defined on $\mathbb{T}_{\hat{h}}$. Then we define the *Dirichlet kernel* by

$$D_{\Omega}(t) := \hat{\phi}_{\Omega}^{\downarrow}(T).$$

Lemma 4.19 *Let $f : \mathbb{T}_h \rightarrow \mathbb{C}$ be a continuously differentiable function such that*

$$\int_{-\infty}^{\infty} |f(t)| \Delta t < \infty.$$

Then

$$\lim_{\Omega \rightarrow \frac{H}{2}} \int_{-\infty}^{\infty} f(t) D_{\Omega}(t) \Delta t = 2\pi f(0)$$

Proof: The proof proceeds in manner similar to that used in real analysis.

Theorem 4.20 *Fourier Inversion Theorem*

Let $f : \mathbb{T}_{hH} \rightarrow \mathbb{C}$ be continuously differentiable and assume that

$$\int_{-\infty}^{\infty} |f(t)| \Delta t < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\hat{f}(t)| \Delta t < \infty.$$

Then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \Delta \omega = f(t).$$

Proof:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \Delta \omega = \frac{1}{2\pi} \lim_{\Omega \rightarrow \frac{H}{2}} \int_{-\infty}^{\infty} \phi_{\Omega}^{\downarrow}(\omega) \mathcal{F}\{f(s)\}(\omega) e^{i\omega t} \Delta \omega$$

By Theorem 4.10 we can rewrite this as

$$\frac{1}{2\pi} \lim_{\Omega \rightarrow \frac{H}{2}} \int_{-\infty}^{\infty} \phi_{\Omega}^{\downarrow}(\omega) \mathcal{F}\{f(s+t)\}(\omega) \Delta \omega.$$

Now we employ Theorem 4.12 to obtain

$$\frac{1}{2\pi} \lim_{\Omega \rightarrow \frac{\hat{h}}{2}} \int_{-\infty}^{\infty} \hat{\phi}_{\Omega}^{\downarrow}(s) f(s+t) \Delta s = \frac{1}{2\pi} \lim_{\Omega \rightarrow \frac{\hat{h}}{2}} \int_{-\infty}^{\infty} D_{\Omega}(s) f(s+t) \Delta t = f(t).$$

□

We will now use Theorem 4.20 to solve the heat equation for time scales.

Example 4.21 Let $\mathbb{T}_{h_1 H_1}$ and $\mathbb{T}_{h_2 H_2}$ be two time scales and let

$$u : \mathbb{T}_{h_1 H_1} \times \mathbb{T}_{h_2 H_2} \rightarrow \mathbb{R}.$$

We will denote by u_x the Δ -derivative of $u(x, t)$ with respect to the first variable $x \in \mathbb{T}_{h_1 H_1}$. Similarly we denote by u_t the Δ -derivative of $u(x, t)$ with respect to the second variable $t \in \mathbb{T}_{h_2 H_2}$.

Consider the initial value problem

$$u_t = \kappa u_{xx}, \quad u(x, 0) = f(x) \tag{4.4}$$

where $\kappa \in \mathbb{R}$ is a constant and $f : \mathbb{T}_{h_1 H_1} \rightarrow \mathbb{R}$. Applying the Fourier transform to both sides of (4.4) gives

$$\hat{u}_t = \kappa \left(\lim_{\tau \rightarrow h} \frac{e^{i\omega\tau} - 1}{\tau} \right)^2 \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega), \quad \omega \in \hat{\mathbb{T}}_{h_1 H_1}.$$

Notice that this is now just an ordinary dynamic equation with respect to the variable t . If we let $p(\omega) = \left(\lim_{\tau \rightarrow h} \frac{e^{i\omega\tau} - 1}{\tau} \right)^2$ then according Theorem 2.37 the solution is given by

$$\hat{u}(\omega, t) = \hat{f}(\omega) e_{p(\omega)}(t, 0).$$

Thus if $\hat{f}(\omega) e_{p(\omega)}(t, 0)$ satisfies the conditions of Theorem 4.20, then

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e_{p(\omega)}(t, 0) e^{i\omega t} \Delta\omega, \quad \omega \in \hat{\mathbb{T}}_{h_1 H_1}.$$

CHAPTER 5

CONCLUSION: MOVING TOWARD A GENERAL THEORY OF TRANSFORMS

In section 3, we saw that Laplace transform for \mathbb{R} and the discrete $\tilde{\mathcal{Z}}$ -transform could both be represented as a single time scale integral. Furthermore this integral extends the notion of the Laplace transform to other closed subsets of \mathbb{R} . Similarly, in section 4, we saw that the different types of the Fourier transform could all be represented by a single time scale integral. Representing the Fourier transform like this makes it clearer that the Fourier series and Fourier integral convey the same idea but for different domain sets. Perhaps we could do something similar for other transforms. For instance, it would be interesting if one could solve the time scale analog to the Bessel equation, which might look something like

$$t^2 x^{\Delta\Delta}(t) + tx^{\Delta}(t) + (t^2 - \nu^2) x(t) = 0,$$

and use the resulting Bessel functions to extend the Hankel transform to time scales. That is if one could find the Bessel function, $J_{\nu}(x)$, of order $\nu \geq 0$, then we might define the Hankel transform of $f : \mathbb{T}_+ \rightarrow \mathbb{R}$ by

$$\mathcal{H}_{\nu}(f)(s) = \int_0^{\infty} f(x) J_{\nu}(sx) x \Delta x, \text{ for } s \geq 0.$$

In this type of manner, we could extend many know transforms to time scales and in some cases write two or more transforms as a single time scales integral.

The times scale integral also gives us a means to define what we mean by a transform. For instance, we might define integral transforms of a function f defined

on \mathbb{R} by integrating f against some function g , that is

$$\int_a^b f(t)g(t, s)dt.$$

However we exclude discrete transforms such as the Fourier series in this definition.

If we were to define a transform of a function f defined on \mathbb{T} by

$$\int_a^b f(t)g(t, s)\Delta t$$

then we would have included many more known transforms. Such a definition would allow for a general theory of transforms for functions of one variable.

Extending such a definition to higher dimensions might prove more challenging. Say, for instance, one wished to extend the definition to two dimensions. Then one must decide whether to consider functions defined on $\mathbb{T}_1 \times \mathbb{T}_2$, where \mathbb{T}_1 and \mathbb{T}_2 are two time scales, or to consider more generally functions defined on some closed subset of \mathbb{R}^2 . Choosing the former already presents a problem in that conventional analysis techniques have thus far failed in attempts to prove that the derivative of a function defined on \mathbb{T}_1 and \mathbb{T}_2 is unique. Thus one would have to develop multivariable calculus on time scales before making a definition of transforms of multivariable functions.

There are existing problems with the Laplace and Fourier transforms that should be addressed before moving on to other transforms. In particular, attention should be given to the issue of the injectivity of \mathcal{L} . Recall that in Example 3.2, we used \mathcal{L} to solve a dynamic equation for \mathbb{T}_+ and then we verified our solution directly. Although we did not know that \mathcal{L} was injective for each each time scale of this form, we were still able to obtain the correct solution by proceeding as if it were injective. Perhaps we do not need \mathcal{L} to be injective on each time scale \mathbb{T}_+ . If we could say that a dynamic equation maintains solutions of a like form over a collection of times scales \mathfrak{T} and if \mathcal{L} is injective on at least one time scale $\mathbb{T}_\alpha \in \mathfrak{T}$, then it would seem

reasonable that using the Laplace transform method would yield correct solutions for all $\mathbb{T} \in \mathfrak{T}$.

Proving a result of this nature would not only be a valuable addition to the theory of this transform but would also give one hope of extending the Fourier transform to a more general class of time scales. In order to ensure the existence of the Fourier inversion, we need our time scale to be a group. However maybe we can still employ the Fourier transform method to solving say PDEs without needing inversion. A result to this effect would motivate the extension of our current Fourier transform to perhaps all time scales. It is important that the issue of the injectivity of these transforms be confronted as early as possible because this issue is likely to present itself again in the development of additional transforms.

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