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Stochastic Volatility Models: A Maximum Likelihood Approach
(Under the direction of JEM N. CORCORAN)

Volatility is inherent in all empirical financial models. It enters the model through investor behavior with implications to the price of the asset. The specific form this volatility takes is in its dependence on the past values. Financial time series display autocorrelated volatility with the variance of returns on assets changing over time. A method, due to Engle (1982), for modeling volatility is known as autoregressive conditional heteroscedasticity (ARCH). ARCH models securities returns data with a heteroscedastic error term allowing the conditional variance to be a function of the squares of previous observations on stock prices and past variances.

Stochastic Volatility (SVOL) model, an alternative to using the ARCH framework, is examined here that allows both the conditional mean and variance to be driven by separate stochastic processes. The changing variance in such models follows some latent stochastic process. This is an advantage over the ARCH class of models in that the SVOL model has the potential to parsimoniously model the volatility process itself versus an ARCH specification that models the conditional expectation of the volatility. There is evidence in the time-series literature suggesting that correlation between the errors distribution introduces the leverage effect that is important in characterizing the behavior of stock returns. The present analysis tests the hypothesis that the alternate disturbance term assumption would produce improved volatility forecasts in the securities market and ultimately superior timing of entry and exit in the stock/derivative market.

Augmenting the data set and considering correlated error terms results in improved estimates of the standard errors. However, this study, in agreement with some previous analyses, shows a lack of evidence for the leverage effect.

KEY WORDS: ARCH Models, Stochastic Volatility, Maximum Likelihood, Leverage Effect, S&P500 Index.

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CHAPTER I

INTRODUCTION

Problem Statement

The financial markets are generally highly competitive. Information flows rapidly and different volumes of instruments can change hands more or less continuously with minimal friction. The efficiency and smooth operation of these markets are not by chance. Their structure and procedure have been designed to achieve these goals. Empirical examination of the theoretical models of these markets show that they sometimes fail.

Since the theoretical models of financial markets are adopted by portfolio managers in making their investments decisions, they require empirical testing for determining their validity. However, the empirical nature of financial economics does not make it an experimental discipline. The most effective method of inference and prediction for a financial economist is employing statistical models. Financial econometric models are somewhat unique due to the presence of uncertainty in the financial sector. The random fluctuations that require the use of statistical theory to estimate and test financial models are closely related to the uncertainty on which those models are based.

Volatility is inherent in all empirical financial models. It enters the model through investor behavior with implications to the price of the asset. The specific form this volatility takes is in its dependence on the past values. Hence, we may say that most financial time series display autocorrelated volatility. Furthermore, the variance of returns on assets also tends to change over time, rendering the assumption of a homoscedastic error term in an econometric model invalid. A method, due to Engle (1982), for modeling financial data with a heteroscedastic error term is to allow the conditional variance to be a function of the squares of previous observations on stock prices and past variances. Such a modeling technique is referred to as autoregressive conditional heteroscedasticity (ARCH).

Objectives

Since the seminal work by Engle (1982), time series modeling of securities data has involved finding derivations of Engle's ARCH model to characterize the data generating process more accurately. Extensions of Engle's work have dominated this research. As early as the early 1980s, concurrently with Engle's work, research has been done on alternatives to the general ARCH framework of research. The purpose of this study is to consider one such alternative to the ARCH framework in modeling conditional heteroscedasticity in financial models. Specifically, an alternative to using the ARCH framework examined here allows both the conditional mean and variance to be driven by separate stochastic processes. The changing variance in such models follows

some latent stochastic process. This kind of assumption is often made in continuous-time theoretical models, where asset prices follow diffusions with the volatility parameters that also follow diffusions. In the literature, such models are called stochastic volatility (SVOL) models.

The specific objectives of this study are to: 1) extend the analysis in the literature by updating the data set to more current time; 2) consider an alternate disturbance term assumption for the SVOL model to account for the leverage effect. There is evidence in the time-series literature suggesting that correlation between the errors distribution introduces the leverage effect that is important in characterizing the behavior of stock returns (Nelson, 1991). It is hypothesized that the alternate disturbance term assumption would produce improved volatility forecasts in the securities market and ultimately superior timing of entry and exit in the stock/derivative market.

The overall justification for this research is that the traditional time series and regression models have unrealistic assumptions of independence and linearity about the behavior of the underlying securities and their prices. Such models are likely to produce unreliable forecasts of both return and, especially, volatility. Therefore, application of alternative models, such as the SVOL model, that address this shortcoming seems in order. This model serves as a useful tool for decision makers in the securities markets. In addition to being applied in the stock market, volatility models may also be used by hedge fund managers to compare their ability to hedge options positions that are sensitive to the term structure of volatility.

Organization of the Study

Remainder of the analysis is organized as follows. Chapter II reviews the literature tracing the historical development of stochastic volatility models. Chapter III introduces the most general form of a SVOL models as outlined by Jacqueir et al. (1994) and Fridman and Harris (1998). The model is then extended in two ways. First, the data set is extended relative to the previous studies including the financially turbulent late 1980s and 1990s. The turbulent 1930s are also examined closely. Second, the assumptions on the error terms of this model are relaxed to consider bivariate normality. Bivariate normality between the error terms of the model is expected to reflect the data generating process better for tracing the changes in stock returns. Chapter IV outlines the model estimation technique and data used in the analysis. Chapter V concludes with the findings and limitations of the present model. Also, suggestions are provided to extend the current analysis.

CHAPTER II

REVIEW OF RELATED LITERATURE

Mandelbrot (1963) and Fama (1965) have shown that time series of daily stock returns exhibit some autocorrelation for shorter lags. Profitable trading rules could not be made from these autocorrelations, which were small. Also, as empirical distributions of returns are not Gaussian distributions, the statistical significance of the usual autocorrelation estimates may even be lower. This led to the widely accepted assumption of serially uncorrelated stock returns.

Empirical evidence (see Houthakker (1961), Rocca (1969), and Mann and Heifner (1976)) had earlier suggested that daily price changes conform to highly leptokurtic distributions. Engle (1982) suggested a class of models that addressed this pricing pattern whereby large percentage changes in price tend to be followed by large percentage changes, while small percentage changes tend to be followed by relatively small changes. Domowitz and Hakkio (1985) and Milhoj (1987) applied ARCH models to foreign exchange rates.

Bollerslev (1986) extended the ARCH model by developing a technique that allows the conditional variance to be a Autoregressive and Moving

Average (ARMA) process. In terms of Engle's original work, Bollerslev included lagged values of the conditional variance of inflation rate.

Lamoureux and Lastrapes (1990) looked at heteroscedasticity in the stock returns data and compared volume and GARCH effects. They showed that the lagged squared residuals contributed little, if any, additional information about variance of the stock return process after accounting for the rate of information flow, proxied by volume.

With regards to forecasting, Akgiray (1989) forecasted volatility using both ARCH and GARCH models. He found that the ARCH and GARCH models simulated the actual pattern of stock market price volatility fairly closely. However, the GARCH specification was superior. The GARCH (1,1) process provides the best fit and forecast accuracy. Randolph and Najand (1991) tested the GARCH and Mean Reversion models in forecasting the volatility of the S&P 500 index. The implied volatility from Black's model (1976) is shown to be a better predictor of future market volatility than the historical volatility measures.

Among the earlier contributors to SVOL models was Clark (1973) who proposed an independent and identically distributed mixture model for the distribution of stock-price changes. These models have also been employed in discrete approximations to various diffusion processes of interest in the continuous-time asset-pricing literature (Hull and White, 1987; Chesney and Scott, 1989). The recent literature on volatility models has established SVOL models as an alternative to the traditional ARCH class of models. The SVOL

models arise naturally from the mixture of distributions hypothesis for the joint distribution of daily security returns and trading volume. Clark (1973), Tauchen and Pitts (1983), Harris (1987), and others have examined this quite thoroughly. Recently several papers deal with various econometric aspects of the SVOL model. Generally these papers deal with generalized method of moment (GMM) type estimators to avoid the integration problem associated with evaluating the marginal likelihood directly (Melino and Turnbull, 1990; Vetzal, 1992; Taylor, 1994). In instances where the density is known, GMM is inefficient relative to likelihood based procedures. This is particularly true for the SVOL models because the score function cannot be estimated to identify the moments to be used in a GMM strategy.

Recent Developments

Danielsson (1998) tested the leverage effects for the correlation between the returns and volatility in the bivariate SVOL models for the S&P-500 and the Tokyo stock exchange indices. He found the leverage coefficient to be non-significant in the SVOL model and moderately significant for an EGARCH specification.

Leon and Moran (1999) compared alternative time-varying volatility models for daily stock-returns using data from the Spanish equity index IBEX-35. They estimated a parametric family of models of generalized autoregressive heteroscedasticity (which nests the most popular symmetric and asymmetric GARCH Models), a semiparametric GARCH model, the generalized quadratic

ARCH model, the stochastic volatility model, the Poisson Jump Diffusion model and, a nonparametric model. Those models which use conditional standard deviation were found as a better fit than all other GARCH models. The within sample predictive power of all models was compared using a standard efficiency test. The results showed that the asymmetric behavior of responses is a statistically significant characteristic of these data. Furthermore, it was observed that specifications with a distribution that allows for fatter tails than a normal distribution do not necessarily outperform specifications with a normal distribution.

Bates (2000) in his research found that post-crash distributions inferred from S&P 500 future option prices have been strongly negatively skewed. He examined two alternate explanations: stochastic volatility and jumps. The two option pricing models are nested, and are fitted to S&P 500 futures options data over 1988-1993. The stochastic volatility model requires extreme parameters (e.g., high volatility of volatility) that are implausible given the time series properties of option prices. The stochastic volatility/jump-diffusion model fits option prices better, and generates more plausible volatility process parameters. However, its implicit distributions are inconsistent with the absence of large stock index moves over 1988-93.

CHAPTER III

THE STOCHASTIC VOLATILITY MODEL

The model proposed by Jacqueir et al. (1994) and Fridman and Harris (1998) estimates the following simple structure SVOL model,

$$\begin{aligned}y_t &= \sqrt{h_t} \varepsilon_t \\h_t &= \exp(\alpha + \delta \log h_{t-1} + \sigma_v v_t) \\ \text{where} \\ t &= 1, \dots, T\end{aligned}$$

Under the Fridman and Harris assumptions ε_t and v_t are two independent white noise processes with unit variance, and T is the number of observations. The volatility variable h_t is latent, and the only observable variable in the system is the data y_t . In modeling the log of h_t process, we ensure positive values for volatility. The parameter vector to be estimated is $\lambda = (\alpha \ \delta \ \sigma_v^2)$, the location, persistence, and scale of volatility, respectively. Assuming strict stationarity of the log-volatility process, $|\delta| < 1$, yields the following mean and variance of the process.

$$\begin{aligned}\mu_{\log h} &= \frac{\alpha}{1 - \delta} \\ \sigma_{\log h}^2 &= \frac{\sigma_v^2}{1 - \delta^2}\end{aligned}$$

Strict stationarity of h_t implies strict stationarity of the observed process y_t , with a distribution that depends on the distribution of the two disturbance terms in the system. Under the stationarity and Gaussian disturbances assumptions, h_t is lognormal, and it follows that y_t has variance, $\exp(\mu_{\log h} + \sigma_{\log h}^2/2)$ and kurtosis, $3\exp(\sigma_{\log h}^2)$. Excess kurtosis is characteristic of financial price time series, thus the excess kurtosis when $\sigma_{\log h}^2 > 0$ is a desirable property of the model (Fridman and Harris, 1998).

Advantages of SVOL Models

The first major advantage SVOL models have over the GARCH class of models is their close relationship to microeconomic theory (Tauchen and Pitts, 1983). In SVOL models volatility flow may be interpreted as an exogenous information flow to the markets, and such models allow for easy integration of volume into the model.

The second major advantage of using a SVOL model is its potential to parsimoniously model the volatility process itself versus an ARCH specification that models the conditional expectation of the volatility. This gain in precision, however, comes at the cost of analytical complexities encountered in obtaining the marginal likelihood. Discrete time SVOL models are appreciably harder to estimate than the successful ARCH family of models. Analytical difficulties of this integration problem maybe circumvented via numerical methods. Fridman and Harris (1998) have suggested one such strategy that employs the choice of an integration grid to estimate the

parameters of a SVOL model.

Fridman and Harris presented a direct method for the evaluation of the marginal likelihood function for SVOL processes. Their method enables the evaluation of the marginal likelihood through recursive numerical integration. At each time period, the intermediate integrated likelihood up-to the current time is calculated by integrating out the contemporaneous volatility disturbance term. Empirically, the T-fold integral is estimated by the product of T square matrices of dimension k, where k is the number of lattice points used to represent the volatility levels.

For theoretical work, especially with the derivatives, it is convenient to formulate the SVOL models in continuous time. However, since financial data available for estimating the parameters of such a model are recorded on a discrete interval, the empirical estimates of the parameters are obtained using a set of discrete time data. Nonetheless, the weighted-sum in the above formulation is chosen, approximating the continuous properties of the assumed latent volatility process.

The Likelihood Function

Maximum likelihood estimation of continuous dynamic latent-variable models requires conditioning the latent volatility, h_t , in the joint density of $\{y_t\}$

$$L(y; \lambda) = P_\lambda(y, h) = \int_h P_\lambda(y | h) P_\lambda(h) dh = \int_h \left[\prod_{t=1}^T P_\lambda(y_t | h_t) P_\lambda(h_t | h_{t-1}) \right] dh$$

and $\{h_t\}$ for $t = 1 \dots T$ given by the T-dimensional integral, where

$y^{\mathbf{N}} = (y_1, \dots, y_T)$, $h^{\mathbf{N}} = (h_1, \dots, h_T)$, and the exact form of $P_{\lambda}(y_t/h_t)$ and $P_{\lambda}(h_t/h_{t-1})$ is determined by the specific model. The initial volatility h_0 is distributed according to the stationary volatility distribution $P(h)$.

The multiple integral $L(y; \lambda)$ may not be decomposed into T one dimensional integrals due to the dependence in the latent variable h_t on the past values. Exact evaluation of the likelihood is possible through the Kalman filter only for the case in which both $P_{\lambda}(y_t/h_t)$ and $P_{\lambda}(h_t/h_{t-1})$ are Gaussian. However, Gaussian distributions are a poor representation of volatilities since they are defined over the complete real number line and the volatility distribution is purported to be highly skewed to the right. In the SVOL model, the non-negativity constraint on h_t makes positive distributions such as the log normal distribution a more desirable choice for $P_{\lambda}(h_t/h_{t-1})$ (Fridman and Harris, 1998).

Since the multiple integral cannot be factored into a simple product, this integration is analytically very complex. Therefore, it is evaluated directly using an iterated numeric integration procedure introduced by Kitagawa (1987). Given the algorithm procedure, we may maximize the likelihood individually over the elements of the λ vector. For ease of exposition the method must first be described as though the latent variable were discrete and then it may be generalized to a continuous case.

In the discrete model, the latent variable h takes on only one of the k values denoted by $\{l_1, \dots, l_k\}$. For each fixed h_{t-1} , $P_{\lambda}(h_t/h_{t-1})$ is the k -value discrete probability mass function for h_t . These k conditional distributions can be summarized by the k by k probability transition matrix P_t whose i - j th

element is given by $\{P_{\lambda}(h_t = l_j | h_{t-1} = l_i)\}$ for i and $j = 1, \dots, k$. In the Fridman and Harris SVOL model the matrix P_t does not depend upon time. Now define the i^{th} element of the vector L_t of conditional likelihoods given h_t , is proportional to the following,

$$f_{Y_t|H_t=l_i}(y_t) = \frac{f_{Y_t, H_t}(y_t, l_i)}{f_{H_t}(l_i)}$$

where

$$f_{Y_t, H_t}(y, h) = \int_0^{\infty} f_{Y_t, H_t|H_{t-1}}(y, h | u) f_{H_{t-1}}(u) du$$

The upper case letters stand for the random variable and the lower case letters for the realization of the random variable. The solution to the above integral involves the change of variable theorem, completing the squares, and integration by parts. The solution to the above integral matches the integral of a lognormal distribution.

Finally, define,

$$G_{t+1} = \mathbf{1}' \text{diag}(L_{t+1}) [\text{diag}(G_t) P_{t+1}]' \quad t = 0, \dots, T$$

as a 1 by k vector of forward intermediate integrated joint likelihoods. It may be shown that the i^{th} element of G_t is the joint probability $P_{\lambda}(y_1, \dots, y_t, h_t = l_i)$ for $i = 1, \dots, k$. Given this notation, the following is the forward recursive relationship for the k -level discrete volatility process, with the initial condition $G_0 = P_0 = [P_{\lambda}(h_0 = l_1), \dots, P_{\lambda}(h_0 = l_k)]$. Here $\mathbf{1}$ is a k by 1 vector of ones and $\text{diag}(X)$ is the square diagonal matrix created from the vector (X) . For the k -level discrete volatility case, the marginal likelihood is simply the sum of the

components of G_T and is given by,

$$L(\lambda) = \mathbf{1}'(G_T)$$

This discrete method may be adjusted to incorporate the continuous nature of the latent variable by evaluating the probability density at values of h_t and h_{t-1} determined by an evaluation grid for h . The resulting square matrix is then element-wise multiplied by the corresponding vector of integration weights such that the rows sum to one. The result is employed as P_{t+1} in the volatility process above. The matrix multiplication in the forward recursion compute the k numeric integrals over h_t given h_{t-1} .

Maximum likelihood may now be invoked via nonlinear optimization techniques. The choice of the iteration grid involves assuming the stationarity of the volatility process. As suggested by Fridman and Harris, to adequately cover the range of integration, the grid is centered at $\mu_{\log h}$ and it is stretched out $3\sigma_{\log h}$ in either direction. The integral is evaluated every 0.3 times within the integration range which ensures at least five significantly nonzero grid evaluation points. Furthermore, to ensure the numerical stability of the procedure both the recursive likelihood evaluation and the iterative optimization algorithm have to be adjusted. The adjustment is made through numerically scaling the vector of the intermediate integrated joint likelihoods at each iteration by dividing every element of the G_T matrix by its maximal element. Also, the probability transition matrix weighting scheme mentioned earlier has the stabilizing effect on the optimization search over the parameter space and in particular of keeping the scale parameter, σ_v^2 , under control.

From the estimated vector of parameters, the latent volatility sequence, h_t , may be filtered, smoothed, and predicted using the conditional expectation $E_{\lambda}(h_t/y)$ taking the appropriate conditioning sequence y for each case. However, as pointed out by Fridman and Harris, in practice the volatility estimates are derived by evaluating the conditional expectation $E_{\lambda}(h_t/y)$ at λ_{MLE} .

In the literature, a comparison of simulation experiments based on direct maximum likelihood principles to that of the Bayesian Monte Carlo Markov Chain (MCMC), Method of Moments (MM), and Quasi-Maximum Likelihood (QML) shows that the results of the former approach are comparable with those of Bayesian MCMC with a diffused prior. However, in comparison to MM and QML approaches, the ML has better estimates on account of both bias and variability. Concerning reasonable parameter values, past empirical evidence from stock data suggests that the values of δ are between 0.8 and 0.995 (Jacquier, et al. 1994).

CHAPTER IV

MODEL ESTIMATION

Since the stationarity assumption, $|\delta| < 1$, for the stock return data set is fundamental to the SVOL model estimation, it is tested using the augmented Dickey Fuller test. Augmented Dickey-Fuller is the appropriate test statistic since it allows the errors to be non i.i.d. Results of the test validate the stationarity assumption for this data set.

Following the ARCH literature (see Bollerslev, Chou, and Kroner, 1992) it may be natural to investigate whether the conditional distribution is normal. Results in Gallant, Hsieh, and Tauchen (1994) show that the conditional distribution is non-normal for stock returns. There is evidence in the time-series literature suggesting that correlation between the errors distribution introduces the leverage effect that is important in characterizing the behavior of stock returns (Nelson, 1991). Stock returns are negatively correlated with changes in returns volatility – volatility tends to rise in response to “bad news” and fall in response to “good news” (Black, 1976). Introducing a negative correlation between the errors of the variance and the stock returns produces this leverage effect.

Although the economic reasons for such phenomenon are not known, Black (1976) and Christie (1982) note that financial and operating leverage play a role. The major contribution of the current analysis is relaxing the assumption of independence of the error terms. Instead the present model is a bivariate normal version of a SVOL model where the error terms (ε_t, v_{t+1}) are distributed bivariate normal with correlation coefficient, ρ .

In the discrete model, the latent variable h takes on only one of the k values denoted by $\{l_1, \dots, l_k\}$. For each fixed h_{t-1} , $P_{\lambda}(h_t|h_{t-1})$ is the k -value discrete probability mass function for h_t . These k conditional distributions can be summarized by the k by k probability transition matrix P_t where $P_t = \{P_{\lambda}(h_t = l_i|h_{t-1}=l_j)\}$ for $i, j = 1, \dots, k$. Unlike the Fridman and Harris SVOL model the matrix P_t now does depend upon time.

Furthermore, updating the data set extends Fridman and Harris analysis. While Fridman and Harris fitted the SVOL model using 1980 - 1987 S&P 500-index daily geometric return data, the present analysis extends the data set up to December 1997 (4552 observations). The systematic calendar effects in the mean and variance of the index were removed using the same program as used by Fridman and Harris. This program and the 1980 - 1987 data were available from Tauchen's public access web site at the Duke University Business School. The remainder of the data was collected from the Center for Research on Securities Prices (CRSP).

Since volatility is the driving force of the SVOL model, it is of interest to also evaluate the behavior of the parameter around the highly volatile 1930s. The Fridman and Harris analysis is extended retrospectively by estimating parameters for the years 1928 – 1939, inclusive (3024 observations). These data were also available from the Tauchen's public access site and were rid of the calendar effects in the mean and variance of the index in the same manner as above. The data are summarized graphically in figures 1a and 1b.

The numerical algorithm was written in Fortran-90[®] (Appendix B) and it solves for the optimal parameters via a non-linear grid search method. The standard errors were solved analytically by the method outlined below.

The asymptotic standard errors for a sufficiently large sample size may be well approximated by the following distribution:

$$\hat{\lambda} \approx N(\lambda_0, T^{-1} I^{-1})$$

where λ_0 is the true parameter vector and the matrix I^{-1} is the inverse of the Fisher information matrix. The information matrix may be estimated via the Hessian matrix of the log likelihood function (Hamilton, 1994).

In the absence of regularity restrictions, the asymptotic variance covariance of maximum likelihood estimators of the distribution parameters can be obtained as outlined above. Unfortunately, the lognormal distribution is subject to regularity problems which may mean that the asymptotic variance covariance matrix thus found may apply to local maximum likelihood estimates

(Crow and Shimizu, 1988). However, they have tended to perform well in simulation studies (Harter and Moore, 1966; Cohen, et al., 1985). These asymptotic standard errors are stated parenthetically in the above table. The present analysis, for the most part, presents smaller standard errors that were found via numerical evaluation of the Hessian.

There are certain drawbacks of computing derivatives numerically which include enhanced computation required to evaluate the function and its gradient and a possible inability to ensure the negative definiteness of a Hessian thus computed (Greene, 1997). Despite these drawbacks, evaluating standard errors numerically for analytically complex likelihoods is an inevitable and a well-accepted practice in this research area .

The resulting parameters and the asymptotic standard errors for the two data series 1928 – 1939 and 1987 – 1997 are reported in the tables in the following chapter for both the independence as well as the bivariate normal model.

CHAPTER V

FINDINGS, LIMITATIONS, AND FURTHER RESEARCH

This study is based on the assumption that financial time series data are not independent, and hence traditional regression models will provide inefficient results. The objectives of this study are to: 1) extend the analysis in the literature by updating the data set to more current time; 2) consider an alternate disturbance term assumption for the SVOL model to account for the leverage effect. It is hypothesized that the alternate disturbance term assumption would produce improved volatility forecasts in the securities market and ultimately superior timing of entry and exit in the stock/derivative market.

Findings

Although the parameter values in the table below do not deviate significantly from the Fridman and Harris findings for the 1980 – 1987 data, they tend to be larger in magnitude. This is particularly true of the volatility parameter, σ_v , for the 1928 – 1939 data series. These findings are further summarized graphically in figures 2 through 7 (Appendix A). Standard errors

of the parameter estimates also tend to be smaller in the models with an augmented data set and relaxed assumptions regarding the error term.

Table 1: Estimation Results for the Daily S&P 500 Geometric-Return Series 1980 -1997 (n=4552)

Parameter	Analysis Assuming Independent Errors		Analysis Assuming Dependent Errors	
	Parameter Estimate	Standard Error	Parameter Estimate	Standard Error
α	-0.051	0.0052	-0.063	0.0021
δ	0.968	0.0027	0.967	0.0074
σ_v	0.12	0.0211	0.09	0.0042
ρ	0	0	-0.08	0.105

In congruence with the Fridman and Harris finding, the present analysis also favors a negative drift model for the independent as well as dependent model. This is seen by a statistically significant non-zero value associated with the location parameter in both models. This finding is in conformity with some of the earlier studies and identifies declining risk in the S&P500 Index through the 1990s. The persistence parameter turns out to be fairly close to that estimated by Fridman and Harris albeit with appreciably lower standard error. This highlights the crucial role of persistence in driving the dynamics of this model. The lower standard deviation estimate for the present data series relative to the 1980 – 1987 data may be explained by the longer horizon data series that dampens the effects of the 1980s shocks. The standard deviation of the log-volatility is significantly different from zero, further supporting the

stochastic nature of the volatility process inherent in the SVOL model. The leverage parameter, ρ , is not significantly different from zero given the relatively large standard error of the estimate. In the following, table 2, contains the same parameters estimated for the 1928 – 1939 data series.

Table 2: Estimation Results for the Daily S&P 500 Geometric-Return Series 1928 -1939 (n=3024)

Parameter	Analysis Assuming Independent Errors		Analysis Assuming Dependent Errors	
	Parameter Estimate	Standard Error	Parameter Estimate	Standard Error
α	-0.038	0.0062	-0.042	0.0061
δ	0.984	0.0031	0.981	0.0023
σ_v	0.192	0.0302	0.181	0.0241
ρ	0	0	-0.06	0.172

A similar pattern to the 1980 – 1997 data emerges for this data set as well. Once again the ρ parameter fails to play itself out in better explaining the data generating process. The dramatic difference is in the magnitude of the standard deviation of the log-volatility. For the 1928 – 1939 data set this parameter is almost twice the size of its 1980 – 1997 data counterpart. Again, this finding confirms the larger role volatility played through the decade of 1930s. A graphical comparison of these two data sets in figure 1(a) and 1(b) is also very illustrative of these differences.

Conclusions

This study has reinforced previous works that have found the maximum likelihood approach better on account of both bias and variability. Furthermore, the Danielsson (1998) findings are validated with a statistically non-significant leverage parameter coefficient, ρ . The ρ parameter appears negatively, however, with large standard errors relative to the parameter values. This notion of an insignificant effect of the leverage parameter is also highlighted graphically in the appendix. The ρ parameter is plotted against the likelihood in figure 8 where it is evident that the likelihood is uninfluenced by the values of the ρ parameter.

Although the findings rendering the leverage effect insignificant are in agreement with past research, they violate a theoretical principal in finance. The leverage effect is important in appropriately characterizing the stock returns behavior. Findings in the current analysis are limited in only considering Gaussian disturbances. Alternate error term distributions may yield different results.

In summary, the results for the present analysis do not deviate dramatically from those of Fridman and Harris. The present analysis favors a negative drift term for both the models. This finding is in congruence with some of the earlier studies. Furthermore, persistence parameter turns out to be very close to that estimated by Fridman and Harris. This highlights the crucial role of persistence in driving the dynamics of this model. There is an appreciably lower standard error estimates for the present data series relative to

the 1980 – 1987 data. This may be explained by the longer horizon data series that dampens the effects of the 1980s shocks. The standard deviation of the log-volatility is significantly different from zero, further supporting the stochastic nature of the volatility process inherent in the SVOL model. And finally the leverage parameter, ρ , is not significantly different from zero given the relatively large standard error of the estimator.

Limitations and Further Research

The present analysis is limited in considering only Gaussian error terms of the SVOL model. Several interesting possibilities exist where one may consider introducing modifications to the SVOL model. The error term in the model could be considering kurtotic v for the model. Such an assumption better characterizes the sharp jumps in the volatility process such as in the 1980s. Likewise, a student's-t distribution assumption on ϵ explains large jumps in the price through large random fluctuations in the price volatility process. Furthermore, the alternate distributional assumptions may better assist in capturing the leverage effect.

The augmented Dickey-Fuller result validated the assumption of stationarity in this data set. However, given that empirical estimates of most stock returns data yield a close to one value for the persistence parameter, δ , it would be prudent to examine the data for the existence of unit root in a structural change framework. A test for the existence of unit root may be conducted against a one-time shift in the unconditional mean for the data set

following the shock to the market in October 1987. Since this is a single shock at a known time, the appropriate testing procedure would be as outlined in Perron (1990). However, there is need for caution in case of structural breaks. When there are structural shifts several Dickey-Fuller test statistics are biased in favor of non-rejection of a unit root (Enders, 1995).

Considering a multivariate form of the stochastic volatility model (MSVOL) will also enrich the analysis. The MSVOL models have an advantage over their multivariate GARCH models in their parsimonious specification (Danielsson, 1998). However, necessary adjustments are needed to the univariate SVOL estimation technique since volatility in such models is a dynamic latent variable.

On a lighter note, one must be cautioned that predicting the future using the past has been compared with driving a car blindfolded while following directions given by a person looking out of the back window. However, since this is the best that may be done it is necessary to do so with an appreciation of the potential pitfalls of this strategy. Specifically, it is rather unscrupulous to associate supernatural powers to the person in the back seat as is often done by many forecasting procedures and personnel.

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APPENDIX A – FIGURES

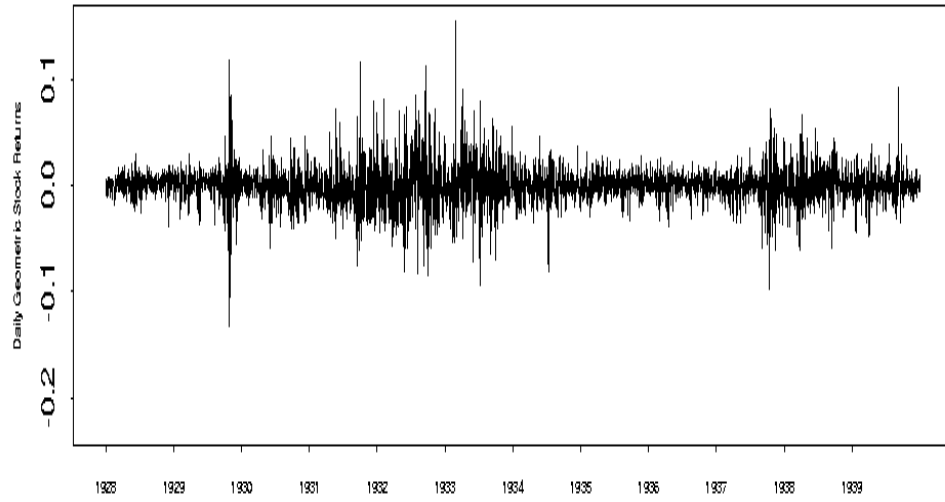


Figure 1a: S&P Index - Daily Log Change 1928-1939

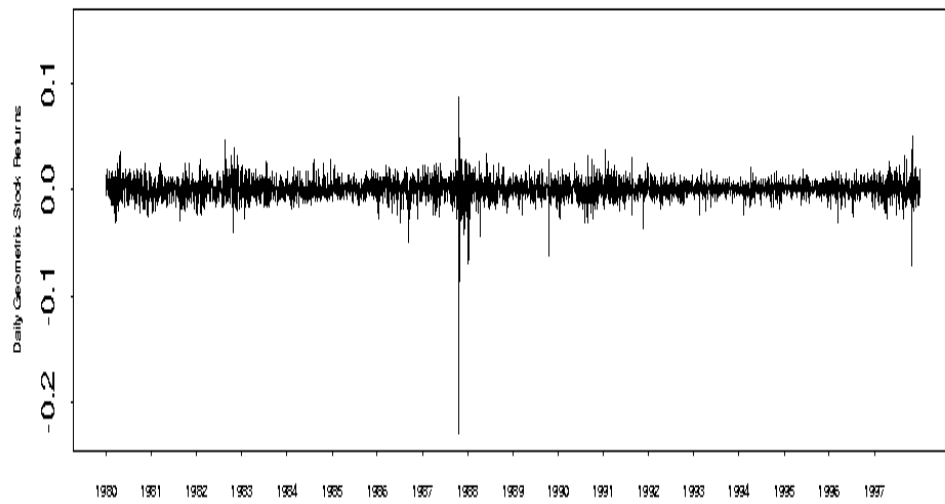


Figure 1b: S&P Index - Daily Log Change 1980-1997

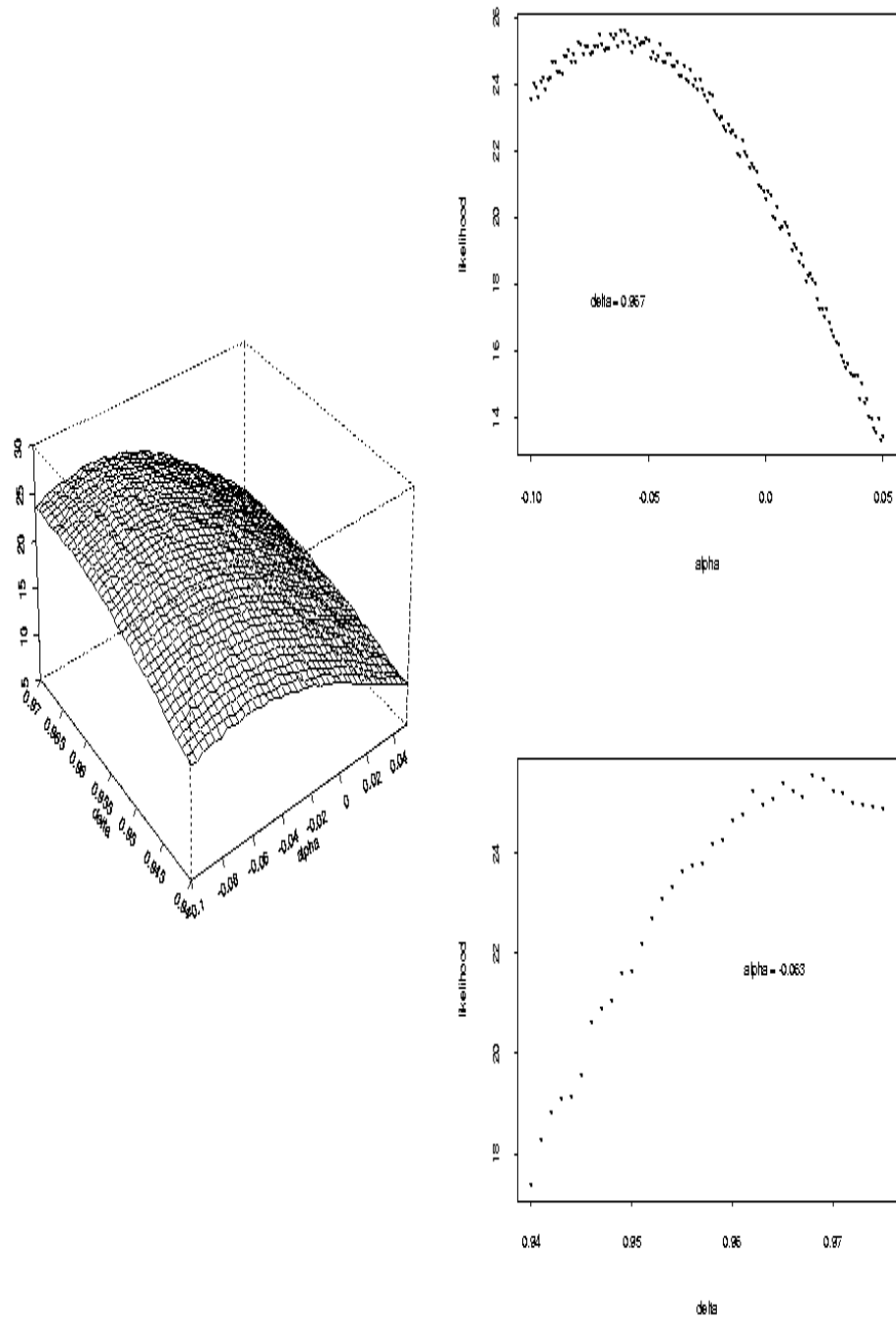


Figure 2: Likelihood as a function of α and δ , 1980-1997

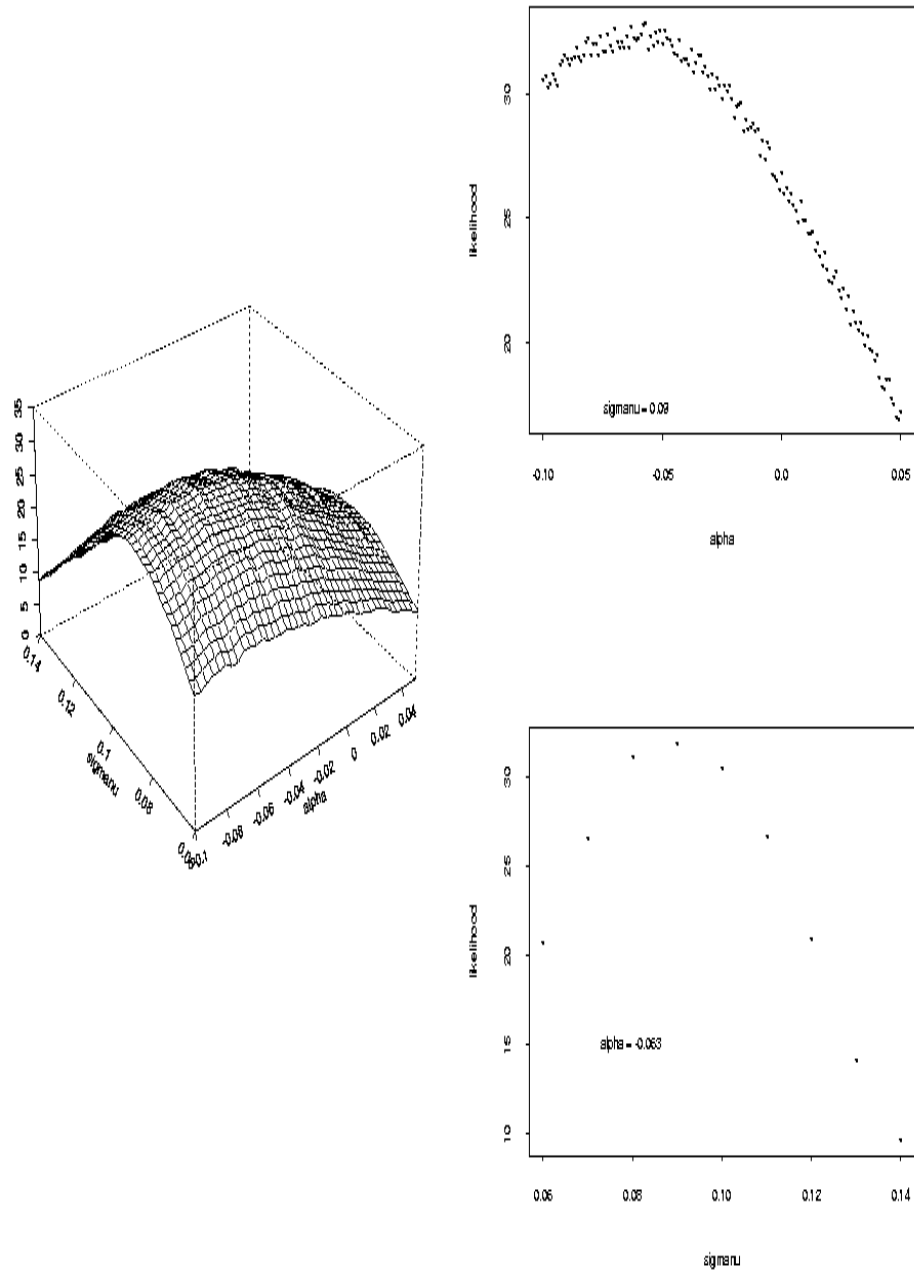


Figure 3: Likelihood as a function of α and σ_u , 1980-1997

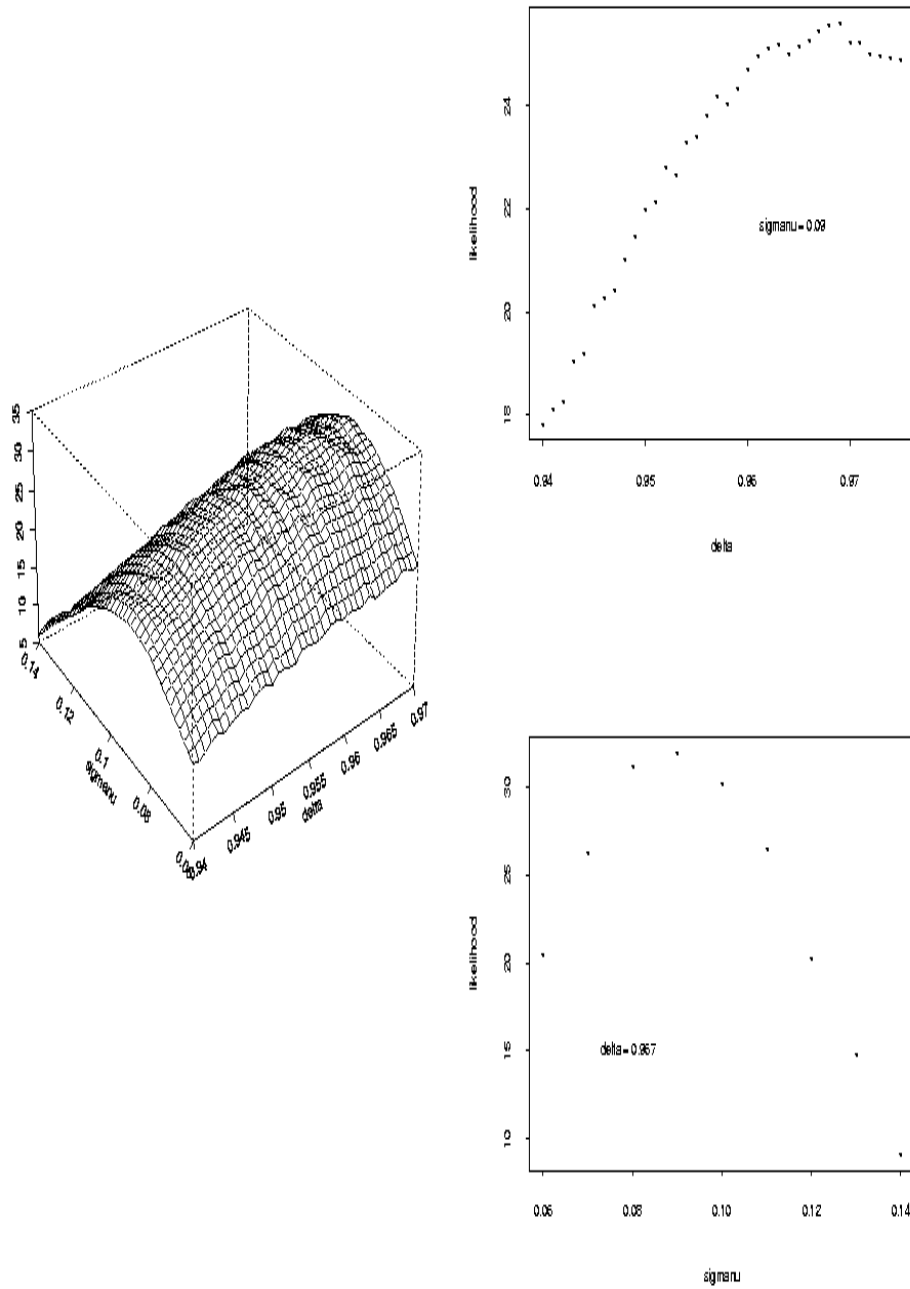


Figure 4: Likelihood as a function of delta and sigma_u, 1980-1997

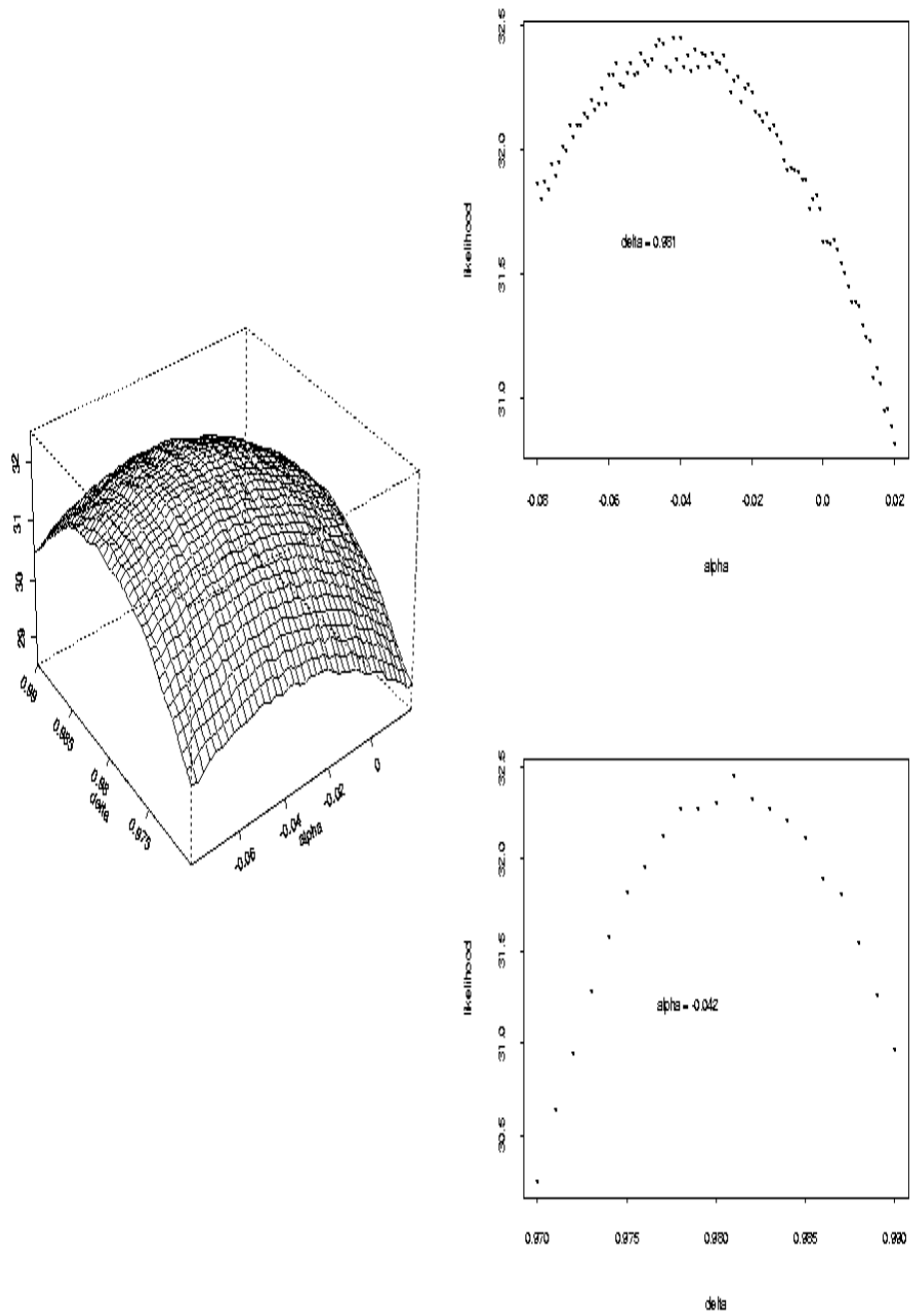


Figure 5: Likelihood as a function of alpha and delta, 1928-1939

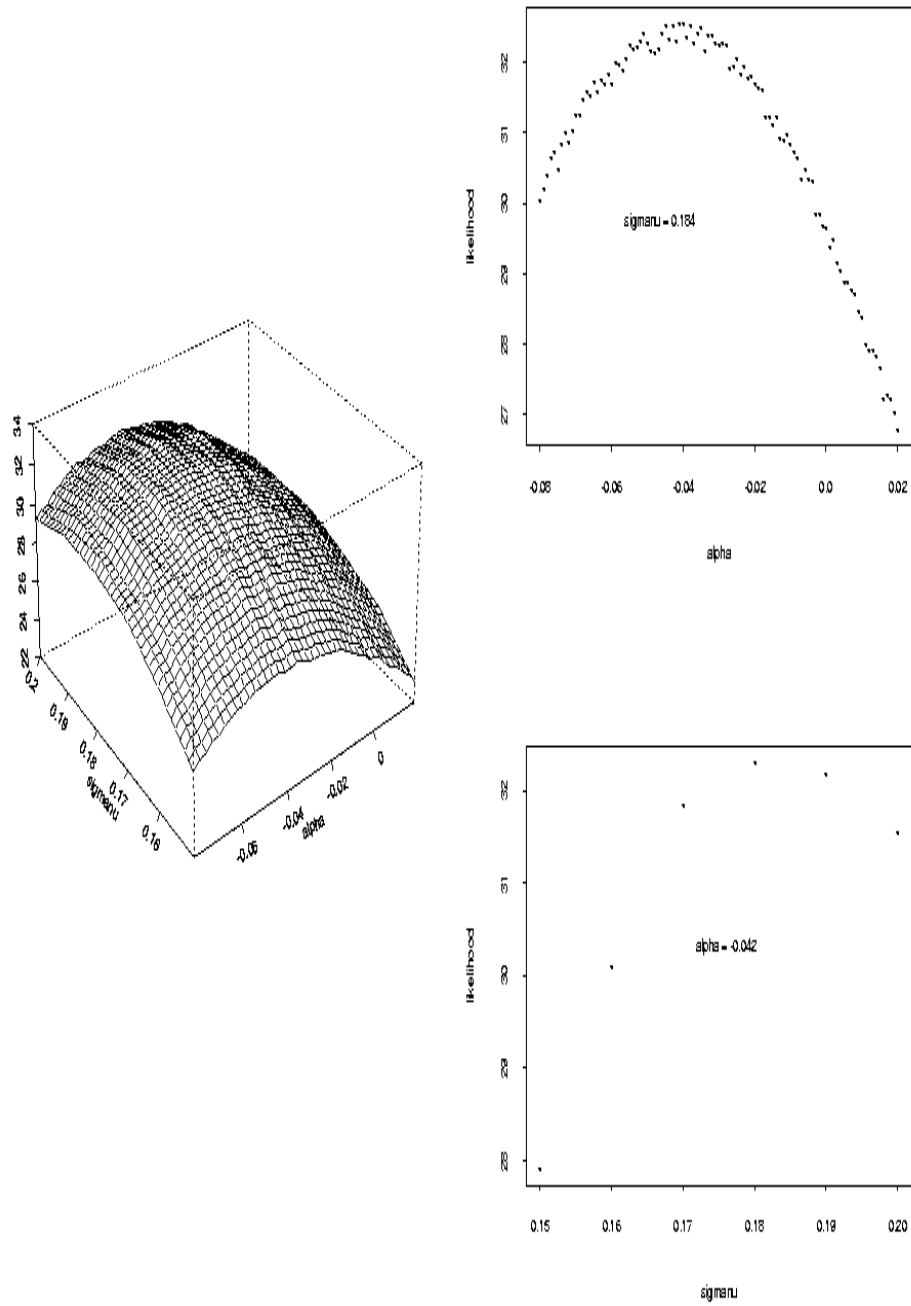


Figure 6: Likelihood as a function of α and σ_u , 1928-1939

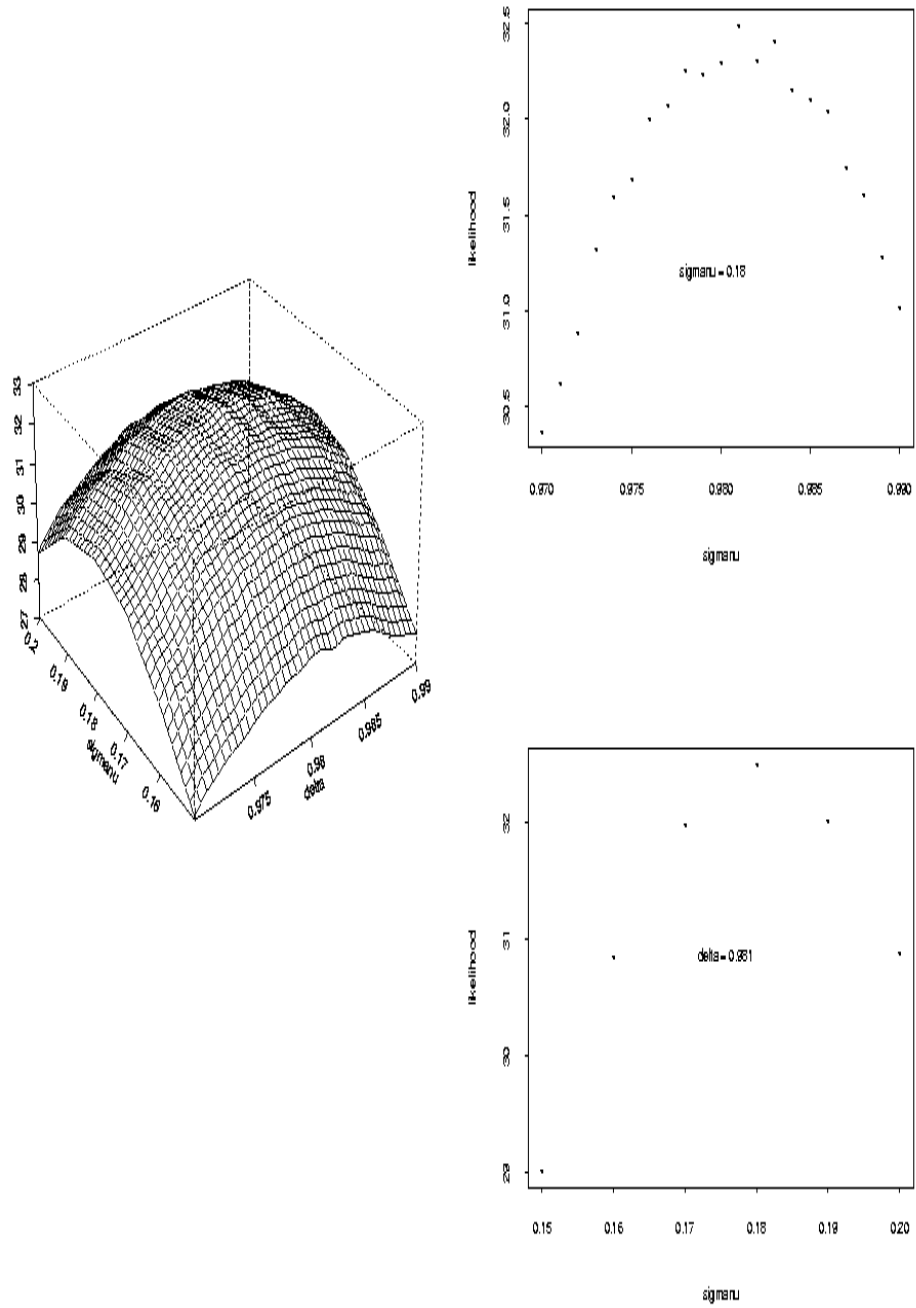


Figure 7: Likelihood as a function of delta and sigma_μ, 1928-1939

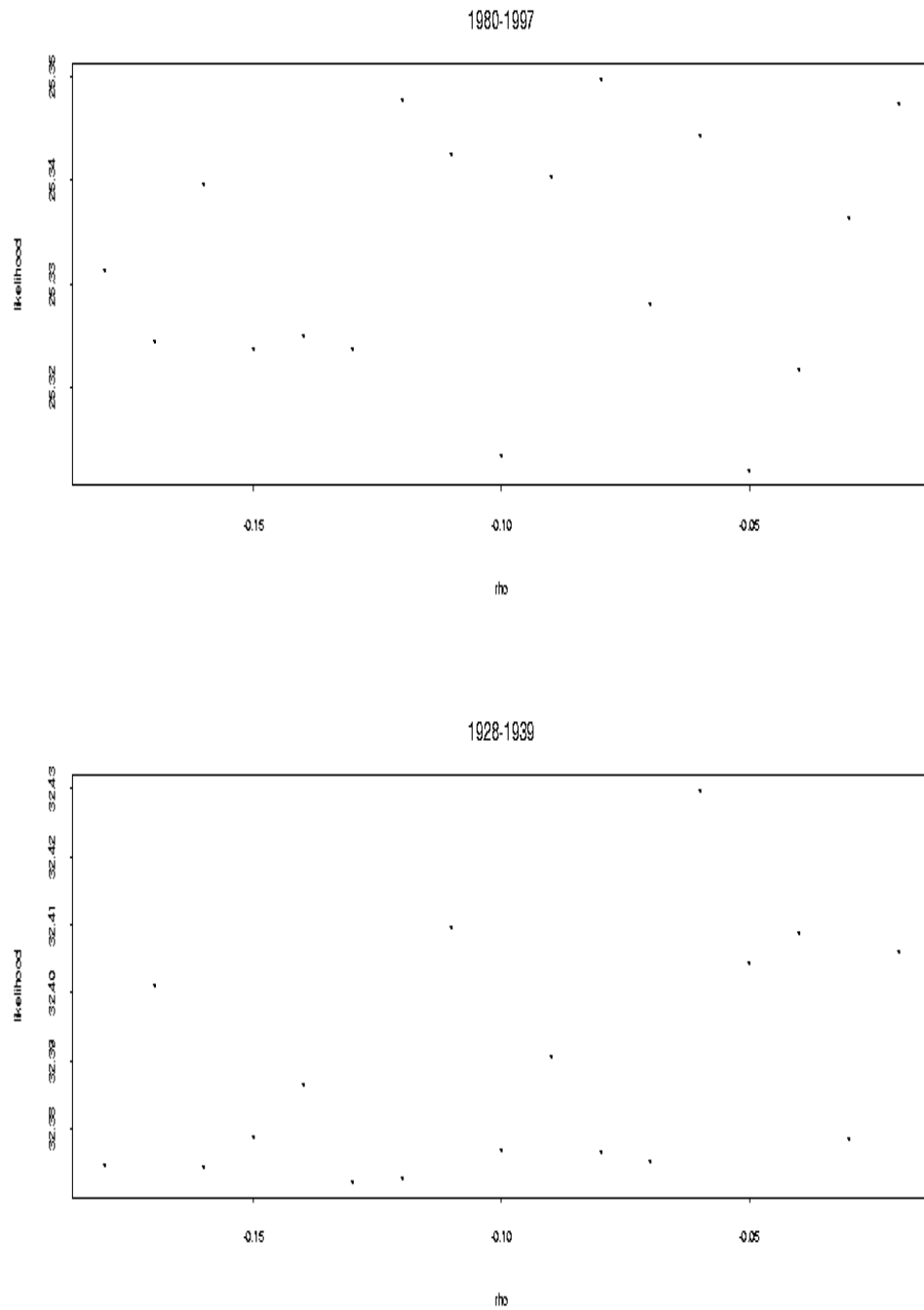


Figure 8: Likelihood as a function of rho

**APPENDIX B – FORTRAN 90 PROGRAM FOR PARAMETER
ESTIMATION**

```
PROGRAM MLE
```

```
IMPLICIT NONE
```

```
INTEGER, PARAMETER :: DP=SELECTED_REAL_KIND(14)
```

```
INTEGER :: I,J,K,maxtime, numleftpts,t,BIGLOOP
```

```
INTEGER :: InputStatus, OpenStatus
```

```
INTEGER :: alphaloop, deltaloop,rholoop
```

```
INTEGER, PARAMETER :: length = 4552
```

```
REAL(KIND=DP),PARAMETER :: pi = 3.141592654
```

```
REAL(KIND=DP) :: alpha, delta, sigmanu, rho, mulogh, sigma2logh, &  
increment, gridcenter, gridstretch, leftendpt, hold, likelihood, &  
right, left, cdf, cdf,accumlog,top,bottom,c1,c2,c3,d,hold1,hold2,&  
hold3,astart,aend,ainc,dstart,dend,dinc,sstart,send,sint,rstart,rend, rinc
```

```
REAL(KIND=DP), DIMENSION(:),ALLOCATABLE :: grid,l,ones,F,W,holdvec
```

```
REAL(KIND=DP), DIMENSION(:,,:), ALLOCATABLE :: P,holdmat,dF,dW
```

```
REAL(KIND = DP), DIMENSION(length) :: y
```

```
PRINT *, "Enter starting value for alpha: " READ *, astart  
PRINT *, "Enter ending value for alpha: " READ *, aend  
PRINT *, "Enter alpha grid search increment: " READ *, aint  
PRINT *, "Enter starting value for delta: " READ *, dstart  
PRINT *, "Enter ending value for delta: " READ *, dend  
PRINT *, "Enter delta grid search increment: " READ *, dint  
PRINT *, "Enter starting value for sigmanu: " READ *, sstart  
PRINT *, "Enter ending value for sigmanu: " READ *, send  
PRINT *, "Enter sigmanu grid search increment: " READ *, sint  
PRINT *, "Enter starting value for rho: " READ *, rstart  
PRINT *, "Enter ending value for rho: " READ *, rend  
PRINT *, "Enter rho grid search increment: " READ *, rint
```

```
! Read in data
```

```
OPEN(8, file="smalldata", status="old")
```

```
READ(8, *, IOSTAT=InputStatus) y
```

```
DO alphaloop = astart,aend,aint
```

```
DO deltaloop = dstart,dend,dint
```

```
DO sigmanuloop = sstart,send,sint
```

```
DO rholoop = rstart,rend,rint
```

```
mulogh = alpha/(1-delta)
```

```
sigma2logh = (sigmanu**2)/(1-delta**2)
```

```
increment = sigmanu/2
```

```

gridcenter = mulogh
gridstretch = 3*sqrt(sigma2logh)
numleftpts = (gridstretch/increment)
numleftpts = numleftpts + 1
leftendpt = gridcenter-(increment*numleftpts)

IF(.NOT. ALLOCATED(grid)) ALLOCATE(grid(2*numleftpts+1))
IF(.NOT. ALLOCATED(l)) ALLOCATE(l(2*numleftpts+1))

IF(deltaloop == 1) THEN
  grid(1) = leftendpt

  DO I = 1,2*numleftpts+1
    grid(I) = grid(I-1) + increment
    l(I) = exp(grid(I))
  END DO

  K = 2*numleftpts + 1
END IF

! Define the *initial* P matrix
IF(.NOT. ALLOCATED(P))ALLOCATE(P(K,K))

DO I = 1,K
  DO J = 1,K

    left = (1/sigmanu)*(log(l(j)-(increment/2.0))-alpha-&
      (delta*log(l(i))))
    right = (1/sigmanu)*(log(l(j)+(increment/2.0))-alpha-&
      (delta*log(l(i))))

    call nprob(left,cdf)
    call nprob(right,cdf)

    P(i,j) = cdf - cdf

  END DO
END DO

! Normalize the rows of P
DO I = 1,K
  P(I,:) = P(I:)/sum(P(I,:))
END DO

! define a vector of ones

```

```

IF(.NOT. ALLOCATED(ones)) ALLOCATE(ones(K))
ones = 1
! def. terminal time, T, as maxtime
maxtime = length

IF(.NOT. ALLOCATED(W))ALLOCATE(W(K))
IF(.NOT. ALLOCATED(F))ALLOCATE(F(K))
IF(.NOT. ALLOCATED(dF))ALLOCATE(dF(K,K))
IF(.NOT. ALLOCATED(dW))ALLOCATE(dW(K,K))
IF(.NOT. ALLOCATED(holdmat))ALLOCATE(holdmat(K,K))
IF(.NOT. ALLOCATED(holdvec))ALLOCATE(holdvec(K))

dF = 0
dW = 0
accumlog = 1

DO I = 1,K
  F(I)= (1/sqrt(2*pi*sigma2logh))*(1/I)*&
    exp((-1/(2*sigma2logh))*((log(1/I))-mulogh)**2))
END DO

DO t = 1,maxtime

  ! Define the P matrix
  IF(t > 1) THEN
    DO I = 1,K
      DO J = 1,K

        left = ((1/sigmanu)*(log(l(j))-(increment/2.0))-alpha-&
          (delta*log(l(i))))-(rho*y(t-1)/sqrt(l(i)))/&
          sqrt(1-rho**2)
        right = ((1/sigmanu)*(log(l(j))+(increment/2.0))-alpha-&
          (delta*log(l(i))))-(rho*y(t-1)/sqrt(l(i)))/&
          sqrt(1-rho**2)

        ! Evaluate cdf of standard normal
        call nprob(left,cdf)
        call nprob(right,cdf)

        P(i,j) = cdf - cdf

      END DO
    END DO

    ! Normalize the rows of P
    DO I = 1,K

```

```

        P(I,:) = P(I,+)/sum(P(I,:))
    END DO
END IF
! Define the W vector (L in Fridman,Harris)

DO I = 1,K

    c1 = (-rho*y(t)*delta)/(sigmanu*sqrt(l(I))*(1-rho**2))
    c2 = -1.0/(2.0*(sigmanu**2)*(1-rho**2))
    c3 = -1.0/(2.0*sigma2logh)
    d = log(l(I))-alpha

    hold = (1/sqrt(1-rho**2))*(1/sigmanu)*(1/sqrt(sigma2logh))

    hold1 = exp((-1.0/(2.0*(1-rho**2)))*(((y(t)**2)/l(I))-&
        ((2.0*rho*y(t)/(sigmanu*sqrt(l(I))))*(log(l(I))-&
        alpha))))

    hold2 = exp(c2*d+c3*(mulogh**2)-(((c1-2.0*c2*d*delta-&
        2.0*c3*mulogh)**2)/(4.0*(c2*(delta**2)+c3))))

    hold3 = (sigma2logh*(sigmanu**2)*(1-rho**2))/&
        ((delta**2)*sigma2logh+(sigmanu**2)*(1-rho**2))

    top = hold*hold1*hold2*hold3

    bottom = (1/sqrt(sigma2logh))*exp((-1.0/(2*(sigma2logh)))*&
        (log(l(I))-mulogh)**2)

    W(I) = top/bottom

    ! while in this loop, diagonalize F
    dF(I,I) = F(I)

END DO

! Equation (4) of Fridman-Harris (without the transpose on the P matrix-- we defined P to
! be the transpose of their matrix)

holdmat = matmul(dF,P)
holdmat = TRANSPOSE(holdmat)
F = matmul(W,holdmat)
IF(t < maxtime) THEN
    accumlog = accumlog*maxval(F)
    F = F/maxval(F)
END IF

```

```

END DO!t

likelihood = sum(F)*accumlog
WRITE '(1X,f8.5,a,f8.5,a,f8.5,a,f8.5,a,es25.10e3)',&
alpha loop,' ',delta loop,' ',sigmanuloop,' ',rho loop,&
' ',likelihood

END DO ! rho loop
END DO ! sigmanuloop
END DO ! delta loop
END DO ! alpha loop

END PROGRAM MLE

! The subroutine follows

SUBROUTINE nprob(z, p)
! For a given number (z) of standard deviations from the mean, the
! probabilities to the left (p) are calculated,
! Programmer - Alan J. Miller
! Latest revision - 16 August 1996
! Revised 15 March 2000 – Jem N. Corcoran & Irfan Y. Tareen
! This version is compatible with the ELF90 subset of Fortran 90.
! REFERENCE: ADAMS,A.G. AREAS UNDER THE NORMAL CURVE,
! ALGORITHM 39, COMPUTER J., VOL. 12, 197-8, 1969.

IMPLICIT NONE
INTEGER, PARAMETER :: doubleprec = SELECTED_REAL_KIND(15, 60)
REAL(doubleprec), INTENT(IN) :: z
REAL(doubleprec), INTENT(OUT) :: p
REAL(doubleprec) :: q,pdf
! Local variables
REAL(doubleprec) :: a0 = 0.5D0, a1 = 0.398942280444D0, a2 = &
0.399903438504D0, &
a3 = 5.75885480458D0,a4 = 29.8213557808D0,a5 = 2.62433121679D0,&
a6 = 48.6959930692D0,a7 = 5.92885724438D0,b0 = 0.398942280385D0, &
b1 = 3.8052D-8,b2 = 1.00000615302D0,b3 = 3.98064794D-4,b4 = 1.98615381364D0,
b5 = 0.151679116635D0, b6 = 5.29330324926D0, &
b7 = 4.8385912808D0, b8 = 15.1508972451D0, b9 = 0.742380924027D0, &
b10 = 30.789933034D0, b11 = 3.99019417011D0, zero = 0.
D0, one = 1.D0, zabs, y
zabs = ABS(z)
IF (zabs < 12.7D0) THEN

```

```

y = a0*z*z
pdf = EXP(-y)*b0
! Z BETWEEN -1.28 AND +1.28
IF (zabs < 1.28) THEN
  q = a0 - zabs*(a1-a2*y/(y+a3-a4/(y+a5+a6/(y+a7))))
  IF(z < zero) THEN
    p = q
    q = one - p
  ELSE
    p = one - q
  END IF
  RETURN
END IF
! ZABS BETWEEN 1.28 AND 12.7
q = pdf/(zabs-b1+b2/(zabs+b3+b4/(zabs-b5+b6/(zabs+b7-b8/ &
(zabs+b9+b10/(zabs+b11))))))
IF(z < zero) THEN
  p = q
  q = one - p
ELSE
  p = one - q
END IF
RETURN
! Z FAR OUT IN TAIL
ELSE
pdf = zero
IF(z < zero) THEN
  p = zero
  q = one
ELSE
  p = one
  q = zero
END IF
RETURN
END IF
END SUBROUTINE nprob

```