Construction of Compactly Supported Multiwavelets

by

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(Under the direction of Ming-Jun Lai)

Abstract

This dissertation consists of three parts. In the first part, we continue the study of constructing compactly supported orthonormal B-spline multiwavelets recently presented by T.N.T. Goodman in 2003. We introduce a numerical approximation method to factorize Laurent polynomial matrices so that his computation method of orthonormal scaling functions can be simplified. We use a new inductive method of constructing corresponding multiwavelets. Explicit examples of compactly supported orthonormal wavelets using B-spline functions are included for demonstrating our constructive procedure.

In the second part of the dissertation, we construct multiscaling functions and multiwavelets in the biorthogonal setting. That is, we construct a scaling function vector by using a B-spline function and its biorthogonal dual scaling function vector with some regularity. And we provide the method for how to get the corresponding multiwavelets by unitary matrix extension. Examples of compactly supported biorthonormal B-spline wavelets are presented and their regularities are compared to the regularities of biorthogonal scaling functions constructed by Cohen, Daubechies and Feaubeau in 1992.

In the third part, we generalize the method in the univariate case to construct bivariate biorthogonal multiscaling functions and multiwavelets with specified smoothness by using the box spline functions. Finally we calculate the regularity for the comparison to the result from He and Lai in 1998.

INDEX WORDS: B-splines, Box splines, Refinable function vector, Orthonormality, Multiresolution analysis(MRA), Riesz basis, Multiscaling function, Multiwavelet, Biorthogonality, Unitary matrix extension

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Dedication

To my parents

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Chapter 1

Introduction

1.1 A Brief Overview of Wavelets

The subject of "wavelet analysis" has drawn much attention from both pure mathematicians(in harmonic analysis) and electrical engineers(in signal analysis). A wavelet is a function whose binary dilations and dyadic translations are enough to represent all the functions in $L^2(\mathbb{R})$. A particularly interesting development is the discovery of the compactly supported orthonormal wavelet basis. There exists a function $\psi \in L^2(\mathbb{R})$ such that the family of functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}, \tag{1.1}$$

constitutes an orthonormal basis for $L^2(\mathbb{R})$. The oldest example of such a basis was introduced by Haar in 1910([24]):

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The Haar function is not continuous, and its Fourier transform does not decay rapidly, corresponding to bad frequency localization. An orthonormal wavelet basis with time-frequency properties complementary to the Haar basis is given by Littlewood-Paley:

$$\psi(x) = (\pi x)^{-1}(\sin 2\pi x - \sin \pi x),$$

which has an excellent frequency localization, since its Fourier transform is compactly supported. During the 1980's, there were several constructions of orthonormal wavelet bases for

$L^2(\mathbb{R})$ that shared advantages of both the Haar basis and the Littlewood-Payley basis. The first construction was in 1982 by Stromberg([43]); his wavelets have exponential decay and C^k for arbitrary but finite k. The next example is the Meyer basis([37]) in 1985; which is C^∞. In 1987 and 1988, Battle and Lemarié([2], [34]) constructed identical families of orthonormal wavelet bases with exponentially decaying $\psi \in C^k$(k arbitrary but finite) by very different methods.

In 1986, Mallat and Meyer([36],[38]) developed the "multiresolution analysis" framework, which provided a tool for the construction of yet other bases. In 1988, Daubechies([11], [12]) discovered a whole new class of wavelets, which were not only orthogonal (like Meyer's) but compactly supported with arbitrary pre-assigned regularity. The construction starts by solving for the filter coefficients $\{h_n\}$ appearing in *the refinement equation* or *dilation equation*:

$$\phi(x) = \sum_{n=0}^{N} h_n \phi(2x - n). \tag{1.2}$$

A function $\phi(x)$ satisfying the equation (1.2) is called a *refinable function.* If the integer translates of $\phi(x)$ form an orthonormal basis of their span, then $\phi(x)$ is called *the scaling function* or *father wavelet. The wavelet function* or *mother wavelet* corresponding to $\phi(x)$ is defined by

$$\psi(x) = \sum_{n=0}^{N} (-1)^n h_{N-n} \phi(2x - n). \tag{1.3}$$

Daubechies wavelets turn the theory into a practical tool that can be easily programmed and used by any scientist with a minimum of mathematical training. Since Daubechies' seminal construction of compactly supported orthonormal wavelets, there have been many attempts to construct compactly supported orthonormal wavelets using B-spline functions due to the fact that B-splines have nice refinement properties and explicit representations. Three major research works along this direction are worth mentioning. In 1991 and 1992, Chui and Wang([8], [9]) constructed two kinds of semi-orthonormal B-spline wavelets. One of them is compactly supported although the orthonormality among the translates is lost. In 1996, Donovan, Geronimo, Hardin and Massopust([17]) initiated a fractal functional approach to construct compactly supported orthonormal wavelets from B-spline functions. Examples of C^0 and C^1 compactly supported B-spline wavelets were constructed. T.N.T. Goodman proposed another approach which will be explained later.

In many applications, symmetry of the filter coefficients is often desirable. The Haar wavelet is the only known wavelet that is compactly supported, orthogonal, and symmetric. In order to get symmetry, the orthogonality condition should be relaxed, allowing nonorthogonal wavelet bases. For this reason, the so-called "biorthogonal wavelets" were introduced by Chui and Wang([9]), and Cohen, Daubechies and Feauveau([10]). A biorthogonal wavelet consists of two dual Riesz bases $\psi_{j,k}, \widetilde{\psi}_{j,k}$, associated with two hierarchies of multiresolution analysis ladders, satisfying

$$\langle \psi_{j,k}, \widetilde{\psi}_{j',k'} \rangle = \delta_{j,j'} \delta_{k,k'}, \tag{1.4}$$

which is called *the biorthogonality condition*.

One dimensional wavelets can be extended to two or higher dimensional wavelets to process multi-dimensional signals like images. A trivial way to construct an orthonormal basis for $L^2(\mathbb{R}^2)$ is to use the tensor product generated by two one dimensional bases. The wavelets from this method are called separable wavelets. With this construction, filtering can be done on "rows" and "columns" in two dimensional array, corresponding to horizontal and vertical directions in images. Since separable wavelet filters have three favorite directions, this may cause anomalies in some applications. To get rid of this difficulty, non-separable wavelets are needed. Non-separable biorthogonal wavelet filters were constructed by He and Lai([27], [28]). Other examples of biorthogonal non-separable wavelets are also given in their papers([25], [26]).

The refinement equation (1.2) can be extended to *a vector refinement equation* of type

$$\Phi(x) = \sum_{n=0}^{N} P_n \Phi(2x - n), \tag{1.5}$$

where $\Phi(x) = (\phi_1(x), \cdots, \phi_r(x))^T$ is a function vector and P_n are $r \times r$ coefficient matrices. Since the study of refinable function vectors was initiated in early 1990's by Goodman and Lee([19], [20]), multiscaling functions and multiwavelets have been researched extensively. Multiwavelets possess some nice features that uniwavelets do not have, that is, symmetry or antisymmetry and orthogonality can be achieved simultaneously.

Many examples are available in the literature. In [14], Donovan, Geronimo and Hardin used interwining multiresolution analysis to show the existence of compactly supported orthonormal B-spline wavelets using multi-wavelet techniques. In [15], the researchers used orthogonal polynomials to construct compactly supported smooth wavelets; an example of a C^2 multiwavelets was given. Furthermore, in [16], the researchers extended the interwining multiresoluton analysis to the bivariate setting. Examples of compactly supported continuous piecewise linear spline wavelets were given. These approaches have an obvious difficulty since the number of wavelets is dependent on the size of the support of the scaling function.

Recently, another approach to multi-wavelets was given by Goodman([21]). He showed how to construct compactly supported scaling functions using B-splines of any degree and indicated how to construct associated wavelets. One of the advantages of Goodman's approach in [21] is that the number of wavelets is always 3 for B-splines of any degree. Although the construction of orthonormal scaling functions is clearly described, a constructive method of wavelets was given without any supporting examples. This is because the construction is dependent on the factorization of positive definite matrices. The technique in Hardin, Hogan and Sun([23]) was used to factorize Laurent polynomial matrices. It requires a lot of manual computation.

Following the Goodman approach, we worked through his steps and found out that the computation of orthonormal scaling functions can be simplified by introducing a numerical approximation method of factorization of Laurent polynomial matrices and a new inductive method of constructing wavelets is given so that the whole constructive procedure becomes much simpler([6]). One of the purposes of this dissertation is to describe this new and simple constructive procedure. In addition, we adopt this method to construct univariate biorthogonal multiscaling functions and multiwavelets using B-spline functions. Further we extend to the bivariate biorthogonal multiscaling functions and multiwavelets using box spline functions. One of the aims is to make these compactly supported B-spline and box spline wavelets available to wavelet analysts as well as general wavelet practitioners. That is, it is to make the support of dual functions smaller so that the application process becomes faster.

1.2 Outline and Notations

The dissertation is organized as follows. In Chapter 2, we first give basic definitions and describe a general procedure in the preliminary section. This procedure is similar to the one given in Goodman([21]). Then we explain how to factorize Laurent polynomial matrices by using a symbol approximation method similar to Lai([32]). The convergence analysis of the method in the setting of Laurent polynomial matrices is given by Geronimo and Lai([18]). This is described in Section 2.2. In Section 2.3, an inductive method for constructing compactly supported B-spline wavelets is introduced. In Section 2.4, we summarize the computational steps and present three examples of compactly supported B-spline wavelets to illustrate the computation procedure.

In Chapter 3, we first construct two refinable function vectors satisfying the biorthogonality condition of Section 3.1. Then we give a decay estimate for Fourier transforms of refinable functions to get arbitrary pre-assigned regularity in Section 3.2. In Section 3.3, we show that two refinable function vectors satisfy the Riesz basis property. In Section 3.4, we discuss computation of the associated multiwavelets by unitary marix extension. Chapter 3 ends with some examples of refinable function vectors with pre-assigned smoothness via B-spline functions.

Chapter 4 is devoted to the construction of bivariate biorthogonal multiscaling functions and multiwavelets. The method used in Chapter 3 is modified to the bivariate setting. The support widths of the multiscaling functions constructed by using the box spline functions are given for the specified smoothness.

The following is the list of notations used in this dissertation.

:= is the sign idicating "equal by definition".

$\{x \in X : P(x)\}$:= the set of elements in X satisfying the property $P(x)$.

$A \setminus B := \{a \in A : a \notin B\}$

$\mathbb{N}$:= the set of natural numbers.

$\mathbb{Z}$:= the set of integers.

$\mathbb{R}$:= the set of real numbers.

$\mathbb{C}$:= the set of complex numbers.

$\bar{z}$:= the conjugate of complex number z.

$[a, b]$ is a closed interval.

(a, b) is an open interval.

$\langle f, g\rangle$:= the inner product of f and g in Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$.

$\|u\|$:= the norm of u in Hilbert space $\mathcal{H}$.

$O_{r\times r}$:= the $r \times r$ zero matrix.

$I_{r\times r}$:= the $r \times r$ identity matrix.

$\binom{N}{k} := \frac{N!}{k!(N-k)!}$, a binomial coefficient.

$$\delta_{j,k} = \begin{cases} 1, & j = k \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbf{x} := (x_1, \cdots, x_n)$ for $n \in \mathbb{N}$.

A^T := the transpose of a matrix A.

A^* := the transpose conjugate of a matrix A.

$\det(A)$:= the determinant of a matrix A.

Chapter 2

Construction of Orthonormal B-Spline Multiwavelets

2.1 Preliminaries

2.1.1 Definitions and Concepts

Throughout this dissertation, we will use the following notations for *the inner product* and *norm* for the space $L^2(\mathbb{R}^d)$:

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}, \tag{2.1}$$

$$\|f\|^2 := \langle f, f \rangle, \tag{2.2}$$

where $f, g \in L^2(\mathbb{R}^d)$.

The Fourier transform defined below not only is a very powerful mathematical tool, but also has very significant physical interpretations in applications

Definition 2.1.1 The *Fourier transform* of a function $f \in L^2(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\omega) := \int_{\mathbb{R}^d} f(\mathbf{x})e^{-i\mathbf{x}\omega}d\mathbf{x}, \quad \omega \in \mathbb{R}^d. \tag{2.3}$$

Many useful properties are available in literature(e.g., [5],[42]). The following result is instrumental in extending the notion of Fourier transform to include $L^2(\mathbb{R}^d)$ functions.

Theorem 2.1.2 *The Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{R}^d)$ is in $L^2(\mathbb{R}^d)$, and satisfies "Parseval's Identity":*

$$\|\widehat{f}\|^2 = (2\pi)^d \|f\|^2. \tag{2.4}$$

In many interesting examples the orthonormal wavelet bases can be assocoated with a multiresolution analysis framework. The concept of multiresolution analysis was introduced by S. Mallat. We need the following definitions.

Definition 2.1.3 For a fixed integer $r \geq 1$, let $\phi_1, \cdots, \phi_r$ be compactly supported continuous functions in $L^2(\mathbb{R}^d)$ and $\Phi := (\phi_1, \cdots, \phi_r)^T$. Then Φ is called *a refinable function vector* if it satisfies *a refinement equation*:

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} A_{\mathbf{k}} \Phi(2\mathbf{x} - \mathbf{k}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{2.5}$$

where each $A_{\mathbf{k}}$ is an $r \times r$ real matrix. Φ is called *an orthonormal function vector* if it satisfies

$$\int_{\mathbb{R}^d} \phi_i(\mathbf{x}) \phi_j(\mathbf{x} - \mathbf{k}) d\mathbf{x} = \begin{cases} 1, & \text{if } i = j \text{ and } \mathbf{k} = \mathbf{0}, \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

Definition 2.1.4 A refinable function vector $\Phi(\mathbf{x})$ generates *a multiresolution analysis of* $L^2(\mathbb{R}^d)$, if it satisfies the following properties:

(1) The integer translates of the components of Φ constitute an orthonormal basis of $V_0 \subset L^2(\mathbb{R}^d)$, where

$$V_0 := \overline{\operatorname{span}_{L^2}\{\phi_l(\mathbf{x} - \mathbf{k}) : 1 \leq l \leq r, \mathbf{k} \in \mathbb{Z}^d\}}.$$

(2) If $V_j := \{f(2^j \mathbf{x}) : f \in V_0\}$ for $j \in \mathbb{Z}$, then

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots.$$

(3) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.

(4) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$.

(5) $f(\mathbf{x}) \in V_j \Leftrightarrow f(2\mathbf{x}) \in V_{j+1}, \ j \in \mathbb{Z}$.

The sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ is called *a multiresolution approximation of* $L^2(\mathbb{R}^d)$.

Definition 2.1.5 If a refinable function vector $\Phi(\mathbf{x})$ generates a multiresolution analysis of $L^2(\mathbb{R}^d)$, then $\Phi(\mathbf{x})$ is called *a multiscaling function vector.*

If we have a multiresolution approximation $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$, we define W_j to be the orthogonal complement of V_j in V_{j+1}, that is,

$$V_{j+1} = V_j \oplus W_j, \quad j \in \mathbb{Z}.$$

It follows that

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j = \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots .$$

Corresponding to a muliscaling function vector $\Phi(\mathbf{x})$, a function vector $\Psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \cdots, \psi_s(\mathbf{x}))^T$ is called *a multiwavelet function vector* if $\{\psi_l(\mathbf{x}-\mathbf{k}) : 1 \leq l \leq s, \mathbf{k} \in \mathbb{Z}^d\}$ forms an orthonormal basis for W_0, so that $\{2^{j/d}\psi_l(2^j\mathbf{x} - \mathbf{k}) : 1 \leq l \leq s, \mathbf{k} \in \mathbb{Z}^d, j \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$

The orthonormality of $\Phi(\mathbf{x})$ can be relaxed: we only need to require that $\{\phi_l(\mathbf{x}-\mathbf{k}) : 1 \leq l \leq r, \mathbf{k} \in \mathbb{Z}^d\}$ constitute a Riesz basis, which is defined below.

Definition 2.1.6 $\{u_k\}_{k\in\mathbb{Z}}$ is *a Riesz basis* in a Hilbert space $\mathcal{H}$ if it satisfies the following properties:

(i) $\{u_k\}_{k\in\mathbb{Z}}$ is a linearly independent set, i.e.,

$$\sum_{k\in\mathbb{Z}} c_k u_k = 0 \Rightarrow c_k = 0 \text{ for all } k.$$

(ii) There exist A, B with $0 < A \leq B < \infty$ such that

$$A\|u\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle u, u_k\rangle|^2 \leq B\|u\|^2, \tag{2.7}$$

for all $u \in \mathcal{H}$.

Another definition of a Riesz basis, which is useful in computational work as follows(see [10], [12]): $\{u_k\}_{k\in\mathbb{Z}}$ is a Riesz basis in a Hilbert space $\mathcal{H}$ if and only if (i') The closure of the linear span of $\{u_k\}$ is $\mathcal{H}$, i.e.,

$$\mathcal{H} = \overline{\{\sum_{k=1}^{n} c_k u_k : k \in \mathbb{Z}^+, c_1, \cdots, c_n \in \mathbb{R}\}}.$$

(ii') There exist A, B with $0 < A \leq B < \infty$ such that

$$A \sum_{k \in \mathbb{Z}} |c_k|^2 \leq \Big\| \sum_{k \in \mathbb{Z}} c_k u_k \Big\|^2 \leq B \sum_{k \in \mathbb{Z}} |c_k|^2, \tag{2.8}$$

for all $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{R})$.

Definition 2.1.7 A refinable function $\Phi(\mathbf{x})$ is said to be *stable* if its integer translates contitute a Riesz basis for $\overline{\mathrm{span}_{L^2}\{\phi_l(\mathbf{x}-\mathbf{k}) : 1 \leq l \leq r, \mathbf{k} \in \mathbb{Z}^d\}}$.

The following argument shows how to construct an orthonormal basis from a Riesz basis of V_0. $\{\phi_l(\mathbf{x}-\mathbf{k}) : 1 \leq l \leq r, \mathbf{k} \in \mathbb{Z}^d\}$ constitutes a Riesz basis if and only if they span V_0, and for all $\{c_{l,\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^d} \in \ell^2(\mathbb{R})$,

$$A \sum_{l=1}^{r} \sum_{\mathbf{k}} |c_{l,\mathbf{k}}|^2 \leq \Big\| \sum_{l=1}^{r} \sum_{\mathbf{k}} c_{l,\mathbf{k}} \phi_l(\mathbf{x}-\mathbf{k}) \Big\|^2 \leq B \sum_{l=1}^{r} \sum_{\mathbf{k}} |c_{l,\mathbf{k}}|^2, \tag{2.9}$$

where $A > 0, B < \infty$ are independent of the $c_{l,\mathbf{k}}$. From Parseval's identity in Theorem 2.1.2(or see [5], [42]) we have

$$\begin{aligned} \| \sum_{l=1}^{r} \sum_{\mathbf{k}} c_{l,\mathbf{k}} \phi_l(\mathbf{x}-\mathbf{k}) \|^2 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Big| \sum_{l=1}^{r} \sum_{\mathbf{k}} c_{l,\mathbf{k}} e^{-i\mathbf{k}\omega} \widehat{\phi}_l(\omega) \Big|^2 d\omega \\ &= \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \sum_{l=1}^{r} \sum_{m=1}^{r} \overline{h_l(\omega)}\, h_m(\omega)\, \lambda_{l,m}(\omega), \end{aligned}$$

where $h_l(\omega) = \sum_{\mathbf{k}} c_{l,\mathbf{k}} e^{-i\mathbf{k}\omega}$ and $\lambda_{l,m}(\omega) = \sum_{\mathbf{j}} \overline{\widehat{\phi}_l(\omega + 2\pi\mathbf{j})}\, \widehat{\phi}_m(\omega + 2\pi\mathbf{j})$, $1 \leq l, m \leq r$. Note that Parseval's theorem([41],[42],[44]) implies that for $l, 1 \leq l \leq r$.

$$\sum_{\mathbf{k}} |c_{l,\mathbf{k}}|^2 = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \Big| \sum_{\mathbf{k}} c_{l,\mathbf{k}} e^{-i\mathbf{k}\omega} \Big|^2 d\omega.$$

Letting

$$H := (h_1(\omega), \cdots, h_r(\omega))^T, \quad \Lambda := (\lambda_{l,m}(\omega))_{1 \leq l,m \leq r},$$

then the inequality (2.9) is equivalent to

$$A \cdot H^*H \leq H^*\Lambda H \leq B \cdot H^*H. \tag{2.10}$$

It follows that Λ is positive definite. Since Λ is also Hermitian, $\Lambda = CC^*$. Λ is $(2\pi)^d$ periodic, and so is C. Define $\Phi^{new}(\mathbf{x}) \in L^2(\mathbb{R}^d)$ by

$$\widehat{\Phi}^{new}(\omega) = C^{-1}\widehat{\Phi}(\omega). \tag{2.11}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} \Phi^{new}(\mathbf{x})\overline{\Phi^{new}(\mathbf{x}-\mathbf{k})}d\mathbf{x} &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\Phi}^{new}(\omega)\,\overline{\widehat{\Phi}^{new}(\omega)}\,e^{-i\mathbf{k}\omega}d\omega \\
&= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} C^{-1}\sum_{\mathbf{j}\in\mathbb{Z}^d}\widehat{\Phi}(\omega+2\pi\mathbf{j})\,\overline{\widehat{\Phi}(\omega+2\pi\mathbf{j})}\,(C^{-1})^*\,e^{-i\mathbf{k}\omega}d\omega \\
&= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} C^{-1}\,\Lambda\,(C^{-1})^* e^{-i\mathbf{k}\omega}d\omega \\
&= \delta_{\mathbf{0},\mathbf{k}} I_{r\times r}.
\end{aligned}$$

This implies that Φ^{new} is an orthonormal vector. On the other hand, let V_0^{new} be the space spanned by integer translates of all $\phi_l^{new}(\mathbf{x}), 1 \leq l \leq r$. We calim $V_0^{new} = V_0$. Indeed,

$$\begin{aligned}
& f = \sum_{l,\mathbf{k}} c_{l,\mathbf{k}}^{new}\phi_l^{new}(\mathbf{x}-\mathbf{k}),\ 1 \leq l \leq r,\ \{c_{l,\mathbf{k}}^{new}\}_{\mathbf{k}\in\mathbb{Z}^d} \in \ell^2(\mathbb{R}) \\
\Leftrightarrow\ & \widehat{f} = \nu\widehat{\Phi}^{new} \ \text{ with } (2\pi)^d \text{ periodic } \nu \in L^2([0,2\pi]^d) \\
\Leftrightarrow\ & \widehat{f} = \mu\widehat{\Phi} \ \text{ with } (2\pi)^d \text{ periodic } \mu = \nu\, C^{-1} \in L^2([0,2\pi]^d) \text{ (by ((2.11))} \\
\Leftrightarrow\ & f = \sum_{l,\mathbf{k}} c_{l,\mathbf{k}}\phi_l(\mathbf{x}-\mathbf{k}),\ 1 \leq l \leq r,\ \{c_{l,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \in \ell^2(\mathbb{R}).
\end{aligned}$$

As described above, a multiresolution analysis consisists of a ladder of spaces $\{V_j\}_{j\in\mathbb{Z}}$ and a special function vector $\Phi(\mathbf{x}) \in V_0$ such that (1)-(4) in Definition 2.1.4 are satisfied. The orthonormal basis condition (1) is possibly relaxed as a Riesz basis. We then try to start the construction from an appropriate choice of the refinable function vector $\Phi(\mathbf{x})$: then V_0 can be constructed from its integer translates, and from there, all the other $\{V_j\}_{j\in\mathbb{Z}}$ can be generated(see [12] for the details).

We close this subsection with the following definition, which will be used often throughout this dissertation.

Definition 2.1.8 A function $A : \mathbb{R}^d \to \mathbb{C}$ is *a trigonometric polynomial (or Laurent polynomial)* if $A(\omega) = \sum_{\mathbf{k}\in\Omega} c_{\mathbf{k}} e^{-i\omega\mathbf{k}}$, where Ω is a finite subset of $\mathbb{Z}^d$ and the coefficients $c_{\mathbf{k}}$'s are real.

Without any confusion, we use both $A(\mathbf{z})$ and $A(\omega)$ interchageably for the same trigometric polynomial where $\mathbf{z} = e^{-i\omega}$.

2.1.2 B-splines and Their Basic Properties

The name, spline function was introduced by Schönberg in 1946. A spline function is a piecewise polynomial with a certain degree of smoothness. Because of easy computer implementation and flexibility, B-spline functions are used in many applications such as interpolation, data fitting, numerical solution of ordinary and partial differential equations(finite element method), and in curve and surface fitting.

The first order (cardinal) B-spline $N_1(x)$ is the characteristic function of the unit interval $[0, 1)$, i.e.,

$$N_1(x) := \begin{cases} 1 & \text{for } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

The m^{th} order B-spline for $m \ge 2$, $N_m(x)$ is defined recursively by (integer) convolution:

$$\begin{aligned} N_m(x) &:= \int_{-\infty}^{\infty} N_{m-1}(x-t)N_1(t)dt \\ &= \int_0^1 N_{m-1}(x-t)dt \end{aligned} \tag{2.13}$$

Since N_m is the $m-$fold convolution of N_1 and $\widehat{N}_1(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)$, we see that $\widehat{N}_m(\omega) = \left(\widehat{N}_1(\omega)\right)^m$, so that

$$\widehat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m = e^{-im\frac{\omega}{2}} \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^m. \tag{2.14}$$

Many properties of N_m can be derived from the definition above. We state several important properties in the following theorem. These properties are useful to evaluate functional values, derivatives and integrals of B-splines. Most properties can be derived easily by induction, using the definition of N_m. In the theorem, $C^n(\mathbb{R})$ denotes the collection of all functions f such that $f, f', \cdots, f^{(n)}$ are continuous on $\mathbb{R}$, and we denote $C(\mathbb{R}) := C^0(\mathbb{R})$ for the convenience.

Theorem 2.1.9 *The m^{th} order (cardinal) B-spline N_m satisfies the following properties:*

1. *For every $f \in C(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} f(x)N_m(x)dx = \int_0^1 \cdots \int_0^1 f(x_1 + \cdots + x_m)dx_1 \cdots dx_m,$$

2. *For every $g \in C^m(\mathbb{R})$,*

$$\int_{-\infty}^{\infty} g^{(m)}(x)N_m(x)dx = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}g(k).$$

3. *supp $N_m = [0, m]$.*

4. *$N_m(x) \geq 0,$ for $0 < x < m$.*

5. *$\sum_{k=-\infty}^{\infty} N_m(x-k) = 1,$ for all x.*

6. *The B-splines N_m and N_{m-1} are related as follows:*

$$N_m(x) = \frac{x}{m-1}N_{m-1}(x) + \frac{m-x}{m-1}N_{m-1}(x-1).$$

7. *The B-spline N_m is symmetric with respect to the center of its support, i.e.,*

$$N_m\left(\frac{m}{2} + x\right) = N_m\left(\frac{m}{2} - x\right), x \in \mathbb{R}.$$

8. *N_m satisfies the dilation relation*

$$N_m(x) = \sum_{k=0}^{m} 2^{-m+1}\binom{m}{k}N_m(2x-k).$$

For a fixed m, let V_0^m be the space generated by the integer translates of $N_m(x)$ and let

$$V_j^m := \overline{\text{span}\{N_m(2^j x - k) : k \in \mathbb{Z}\}}.$$

Then $\{V_j^m\}_{j\in\mathbb{Z}}$ is a nested sequence of subspaces in $L^2(\mathbb{R})$(see [5], p.85), i.e., it satisfies

$$\cdots \subset V_{-1}^m \subset V_0^m \subset V_1^m \subset \cdots .$$

This nested sequence of spline subspaces also satisfies following properties(see [12], pp.141-143)

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j^m} = L^2(\mathbb{R}) \ \text{ and } \ \bigcap_{j\in\mathbb{Z}} V_j^m = \{0\}.$$

Furthermore, it satisfies that for some constants $A, B > 0$,

$$A \leq \sum_{k\in\mathbb{Z}} \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 \leq B < \infty,$$

which means that $\{N_m(x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_j^m([5], p.90). This implies that for any $j \in \mathbb{Z}$, the collection

$$\{2^{j/2} N_m(2^j x - k) : k \in \mathbb{Z}\}$$

is also a Riesz basis of V_j^m with the same Riesz bounds as those of V_0^m. Therefore, the B-spline spaces $\{V_j^m\}_{j\in\mathbb{Z}}$ constitute a multiresolution approximation.

2.2 Construction of Orthonormal Scaling Functions

For a fixed integer $r \geq 1$, let $\gamma_1, \cdots, \gamma_r$ be compactly supported continuous functions in $L^2(\mathbb{R}^d)$ and $\Gamma := (\gamma_1, \cdots, \gamma_r)^T$.

Let us denote the *Grammian matrix* associated with $\Gamma(\mathbf{x})$ by $\mathcal{G}(z) = (G_{ij}(\mathbf{z}))_{i,j=1,\cdots,r}$ where

$$G_{ij}(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathbf{z}^{\mathbf{k}} \int_{\mathbb{R}^d} \gamma_i(\mathbf{x})\gamma_j(\mathbf{x}-\mathbf{k})d\mathbf{x}$$

is a trigonometric polynomial with $\mathbf{z}^{\mathbf{k}} = e^{-i\omega\mathbf{k}}$, for all $i, j = 1, \cdots, r$. We note that $\Gamma(\mathbf{x})$ is orthonormal if and only if its Grammian matrix $\mathcal{G}$ is the identity matirx.

We suppose that $\Gamma(\mathbf{x})$ generates a space $\mathcal{S} \subset L^2(\mathbb{R}^d)$. That is, $\mathcal{S}$ comprises of all linear combinations of shifts of elements of Γ. Then for any compactly supported functions $\phi_1, \cdots, \phi_s$ in $\mathcal{S}$, there exist finitely many nonzero matrices C_k of size $s \times r$ such that

$$\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \cdots, \phi_s(\mathbf{x}))^T = \sum_{\mathbf{k}\in\mathbb{Z}^d} C_k \Gamma(\mathbf{x} - \mathbf{k}).$$

The equation above is represented in terms of the Fourier transform,

$$\widehat{\Phi}(\omega) = C(\mathbf{z})\, \widehat{\Gamma}(\omega)$$

where $C(\mathbf{z})$ denotes the $s \times r$ matrix of Laurent polynomials, i.e.,

$$C(\mathbf{z}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} C_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}.$$

A square matrix $C(\mathbf{z})$ is said to be *invertible* if $\det(C(\mathbf{z}))$ is a monomial of $\mathbf{z}$, e.g., $\alpha z_1^{m_1} \cdots z_d^{m_d}$ for a scalar $\alpha \neq 0$ and $m_1, \cdots, m_d \in \mathbb{Z}$, where $\mathbf{z} = (z_1, z_2, \cdots, z_d)^T \in \mathbb{C}^d$. It is clear that if $C(\mathbf{z})$ is invertible, Φ generates the same $\mathcal{S}$. We have the following result from [21].

Lemma 2.2.1 *Fix $d = 1$. Suppose that $\Gamma := (\gamma_1, \cdots, \gamma_r)^T$ is compactly supported and generates a space $\mathcal{S}$. Let $\mathcal{G}(z) = (G_{ij}(z))_{i,j=1,\cdots,r}$ of size $r \times r$ associated with Γ by*

$$G_{ij}(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_i(x)\, \gamma_j(x-k)\, dx$$

for all $i, j = 1, \cdots, r$ be the Grammian matrix associated with Γ. If the Grammian matrix $\mathcal{G}(z)$ is invertible, then there exists a Φ which is orthonormal and generates $\mathcal{S}$. The converse is also true.

Proof: We first show that the Grammian matrix $\mathcal{G}(z)$ associated with Γ is positive semi-definite. For any vector $a = (a_1, \cdots, a_r)^T \in \mathbb{C}^r$,

$$\begin{aligned}
\sum_{i,j=1}^{r} a_i\, G_{ij}(z)\, \bar{a}_j &= \sum_{i,j=1}^{r} a_i \cdot \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_i(x)\, \gamma_j(x-k)dx \cdot \bar{a}_j \\
&= \sum_{k\in\mathbb{Z}^d} z^k \int_{\mathbb{R}} \sum_{i=1}^{r} a_i\, \gamma_i(x) \sum_{j=1}^{r} \bar{a}_j\, \gamma_j(x-k)\, dx \\
&= \int_{\mathbb{R}} z^{-\ell}\, h(x-\ell) \sum_{k\in\mathbb{Z}^d} z^k\, \bar{h}(x-k)\, dx,
\end{aligned}$$

where $h(x) = \sum_{i=1}^{r} a_i \, \gamma_i(x)$. It then follows that

$$
\begin{aligned}
(2m+1) \sum_{i,j=1}^{r} a_i \, G_{ij}(z) \, \bar{a}_j &= \sum_{\ell=-m}^{m} \int_{\mathbb{R}} z^{-\ell} \, h(x-\ell) \sum_{k \in \mathbb{Z}^d} z^k \, \bar{h}(x-k) \, dx \\
&= \int_{\mathbb{R}} \Big| \sum_{k=-m}^{m} z^{-k} \, h(x-k) \Big|^2 \, dx \\
&\quad + \int_{\mathbb{R}} \sum_{k=-m}^{m} z^{-k} \, h(x-k) \sum_{|k|>m} z^k \, \bar{h}(x-k) \, dx \\
&= \int_{-m-n}^{m+n} \Big| \sum_{k=-m}^{m} z^{-k} \, h(x-k) \Big|^2 dx \\
&\quad + \int_{-m-n}^{m+n} \sum_{k=-m}^{m} z^{-k} \, h(x-k) \sum_{m<|k|\le m+n} z^k \, \bar{h}(x-k) \, dx
\end{aligned}
$$

where we have assumed that $h(x)$ is supported on $[-n, n]$. It follows that

$$
\begin{aligned}
\sum_{i,j=1}^{r} a_i \, G_{ij}(z) \, \bar{a}_j = {} & \frac{1}{2m+1} \int_{-m-n}^{m+n} \Big| \sum_{k=-m}^{m} z^{-k} \, h(x-k) \Big|^2 \, dx \\
& + \frac{1}{2m+1} \int_{-m-n}^{m+n} \sum_{k=-m}^{m} z^{-k} \, h(x-k) \sum_{m<|k|\le m+n} z^k \, h(x-k) \, dx
\end{aligned}
$$

The second term converges to zero as m goes to infinity while the first term is bounded below. If we take a close look at the second term, then since $h(x)$ is continuous and has the support $[-n, n]$, we have

$$
\begin{aligned}
& \frac{1}{2m+1} \Big| \int_{-m-n}^{m+n} \sum_{k=-m}^{m} z^{-k} \, h(x-k) \sum_{m<|k|\le m+n} z^{-k} \, h(x-k) \, dx \Big| \\
& \le \frac{1}{2m+1} \Big\{ \int_{-m-n}^{m+n} \sum_{k=m-n+1}^{m} |h(x-k)| \sum_{k=m+1}^{m+n} |h(x-k)| \, dx \\
& \quad + \int_{-m-n}^{m+n} \sum_{k=-m}^{n-m-1} |h(x-k)| \sum_{k=-m-n}^{-m-1} |h(x-k)| \, dx \Big\} \\
& \le \frac{2}{2m+1} \int_{m-n}^{m+n} n \|h\|_\infty \, n \|h\|_\infty \, dx \\
& = \frac{1}{2m+1} \, 4n^3 \, \|h\|_\infty^2 \to 0 \ \text{ as } m \to \infty.
\end{aligned}
$$

Since $\mathcal{G}(z)$ is symmetric and positive definite, we can find an invertible Laurent polynomial matrix $B(z)$ (cf. [23]) such that $\mathcal{G}(z) = B(z) \, B(z)^*$, where $B(z)^*$ denotes the transpose and conjugate of $B(z)$. Let

$$\widehat{\Phi}(\omega) = \big(B(z)\big)^{-1}\widehat{\Gamma}(\omega)$$

Then the computation of the Grammian matrix $\widetilde{\mathcal{G}}(z)$ associated with Φ yields

$$\widetilde{\mathcal{G}}(z) = \big(B(z)\big)^{-1}\mathcal{G}(z)\left((B(z))^{-1}\right)^{*} = I_{r\times r}$$

and so Φ is orthonormal. Morever, it follows from the definition of Φ that Φ generates the same space $\mathcal{S}$ as does Γ.

On the other hand, it is clear that the Grammian matrix of Φ has a determinant 1, when Φ is orthonormal. □

Example 2.2.2 *Let $\gamma_1(x) := (1-|x|)_+$ and $\gamma_2(x) := (x(1-x))_+(2x-1+a)$, where a is a constant to be determined and f_+ denotes the positive part of f. The determinant of the Grammian matrix $\mathcal{G}(z)$ associated with (γ_1, γ_2) is*

$$\det(\mathcal{G}(z)) = \frac{1}{25200}\left((66+210a^2)+(27-35a^2)(z+1/z)\right).$$

(Any computer algebra system may help with the computation.) We can see that $\mathcal{S}$ generated by γ_1, γ_2 has orthonormal generators ϕ_1, ϕ_2 if $a = \pm\sqrt{27/35}$.

The above Lemma 2.2.1 reveals a key for constructing orthonormal vector of scaling functions: find $\gamma_1, \cdots, \gamma_r$ which generate a space $\mathcal{S}$ such that its Grammian matrix has a constant determinant.

We now follow the steps in [21] to use B-splines for constructing an orthonormal vector of scaling functions with $r = 3$. Let N_m be the normalized B-spline of order m, in terms of the Fourier transform,

$$\widehat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m \tag{2.15}$$

Let $V_0 = \text{span}\{N_m(x-k), k\in\mathbb{Z}\}$ be the spline space. Since N_m is a refinable function, for V_1 being spanned by the integer translates of $N_m(2x-k), k\in\mathbb{Z}$, we have $V_0 \subset V_1$. Thus, letting $\gamma_1(x) = N_m(2x)$ and $\gamma_2(x) = N_m(2x-1)$, γ_1 and γ_2 generate V_1. On the other hand, by the dilation equation, there exist two finite sequences a_{2k} and a_{2k+1} such that

$$N_m(x) = \sum_{k\in\mathbb{Z}} a_{2k}\,\gamma_1(x-k) + \sum_{k\in\mathbb{Z}} a_{2k+1}\,\gamma_2(x-k). \tag{2.16}$$

The equation (2.15) yields

$$\widehat{N}_m(2\omega) = \frac{1}{2}\,A(z)\,\widehat{N}_m(\omega), \tag{2.17}$$

where $A(z) = 2\left(\frac{1+z}{2}\right)^m$. On the other hand, the Fourier transform of equation (2.16) is

$$\begin{aligned}\widehat{N}_m(\omega) =& A_0(z)\,\widehat{\gamma}_1(\omega) + A_1(z)\,\widehat{\gamma}_2(\omega)\\ =& A_0(z)\,\frac{1}{2}\,\widehat{N}_m(\frac{\omega}{2}) + A_1(z)\,\frac{1}{2}\,e^{-i\frac{\omega}{2}}\,\widehat{N}_m(\frac{\omega}{2}),\end{aligned} \tag{2.18}$$

where

$$A_0(z) = \sum_{k\in\mathbb{Z}} a_{2k}\,z^k \quad \text{and} \quad A_1(z) = \sum_{k\in\mathbb{Z}} a_{2k+1}\,z^k.$$

It follows that

$$A(z) = A_0(z^2) + z\,A_1(z^2). \tag{2.19}$$

Note that the proof of the following lemma is constructive.

Lemma 2.2.3 *There exist two Laurent polynomials $B_0(z)$ and $B_1(z)$ of degree $\leq m$ such that*

$$A_0(z)\,B_0(z) + A_1(z)\,B_1(z) = 1. \tag{2.20}$$

Proof: Recall

$$\begin{aligned}1 =& \left(\frac{1+z}{2} + \frac{1-z}{2}\right)^{2m-1}\\ =& \sum_{j=0}^{m-1} \binom{2n-1}{j} \left(\frac{1+z}{2}\right)^{2m-1-j} \left(\frac{1-z}{2}\right)^{j}\\ &+ \sum_{j=0}^{m-1} \binom{2m-1}{j} \left(\frac{1-z}{2}\right)^{2m-1-j} \left(\frac{1+z}{2}\right)^{j}\\ =:& \; A(z)\,\ell(z) + A(-z)\,\ell(-z)\end{aligned}$$

by using the binomial expansion, where $\ell(z)$ is a polynomial of degree $\leq m-1$. Thus, we have

$$\begin{aligned}1 =&(A_0(z^2) + z\,A_1(z^2))\,\ell(z) + (A_0(z^2) - zA_1(z^2))\,\ell(-z)\\ =&A_0(z^2)\,(\ell(z)+\ell(-z)) + A_1(z^2)\,z\,(\ell(z)-\ell(-z)).\end{aligned}$$

That is, $B_0(z^2) = \ell(z)+\ell(-z)$ while $B_1(z^2) = z\,(\ell(z)-\ell(-z))$. □

We now define a new spline function in terms of the Fourier transform by

$$\widehat{M}_m(\omega) = -B_1(z)\,\widehat{\gamma}_1(\omega) + B_0(z)\,\widehat{\gamma}_2(\omega). \tag{2.21}$$

Recall from (2.18) that

$$\widehat{N}_m(\omega) = A_0(z)\,\widehat{\gamma}_1(\omega) + A_1(z)\,\widehat{\gamma}_2(\omega)$$

It follows that N_m and M_m generate V_1 since

$$\begin{bmatrix}\widehat{N}_m(\omega)\\ \widehat{M}_m(\omega)\end{bmatrix} = \begin{bmatrix}A_0(z) & A_1(z)\\ -B_1(z) & B_0(z)\end{bmatrix}\begin{bmatrix}\widehat{\gamma}_1(\omega)\\ \widehat{\gamma}_2(\omega)\end{bmatrix} \tag{2.22}$$

and the determinant of the matrix $\begin{bmatrix}A_0(z) & A_1(z)\\ -B_1(z) & B_0(z)\end{bmatrix}$ is of a constant 1. Furthermore, $N_m(2x)$, $N_m(2x-1)$, $M_m(2x)$, $M_m(2x-1)$ generate V_2.

Define $\gamma_3(x) = \sum_{k\in\mathbb{Z}} \alpha_k\, M_m(2x-k)$ for some finitely many nonzero coefficients α_k. We will show how to find such α_k so that the Grammian matrix associated with $\{\gamma_1,\gamma_2,\gamma_3\}$, $\mathcal{G}(z) = \left(\sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_i(x)\,\gamma_j(x-k)\,dx\right)_{i,j=1,2,3}$ has a constant determinant. Put

$$r(z) = \sum_{k\in\mathbb{Z}} \alpha_k z^k.$$

The computation in ([21]) gives

$$4\det\mathcal{G}(z^2) = D(z)\,r(z)\,r(1/z) + D(-z)\,r(-z)\,r(-1/z), \tag{2.23}$$

where

$$D(z) = \frac{1}{2}\big(a(z^2) - zb(z^2)\big)\,\big(a(z)^2 - zb(z)^2\big), \tag{2.24}$$

with

$$a(z) = \sum_{k\in\mathbb{Z}} \int N_m(2x)\, N_m(2x-k)\, dx, \tag{2.25}$$

$$b(z) = \sum_{k\in\mathbb{Z}} \int N_m(2x)\, N_m(2x-2k-1)\, dx. \tag{2.26}$$

Let us closely look at each entry of the Grammian matrix $\mathcal{G}(z)$ as follows. First direct computation gives

$$G_{11}(z) = G_{22}(z) = a(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_1(x)\, \gamma_1(x-k)\, dx$$

Next we have

$$G_{12}(z) = b(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_1(x)\, \gamma_2(x-k)\, dx$$

Then

$$G_{21}(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_2(x)\, \gamma_1(x-k)\, dx = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_1(x)\, \gamma_1(x-k+1) dx = z\, b(z)$$

Also, we have

$$G_{31}(z) = \sum_{k\in\mathbb{Z}} z^k \int_{\mathbb{R}} \gamma_3(x)\, \gamma_1(x-k) dx = \sum_{k\in\mathbb{Z}} z^k \sum_{j\in\mathbb{Z}} \alpha_j \int_{\mathbb{R}} M_n(2x-2k-j)\, \gamma_1(x)\, dx$$

To simplify the above equation, we introduce two Laurent polynomials

$$r(z) = \sum_{j\in\mathbb{Z}} \alpha_j\, z^j \quad \text{and} \quad c(z) = \sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} \gamma_1(x)\, M_n(2x-j)\, dx.$$

Then we can see that

$$r(z)\, c(z) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \alpha_j\, z^k \int_{\mathbb{R}} M_n(2x-k+j)\, \gamma_1(x) dx$$

and hence,

$$G_{31}(z^2) = \frac{1}{2}\big(r(z)\, c(z) + r(-z)\, c(-z)\big).$$

Next we compute

$$
\begin{aligned}
G_{32}(z^2) =& \sum_{k\in\mathbb{Z}} z^{2k} \int_{\mathbb{R}} \gamma_3(x)\,\gamma_2(x-k)\,dx \\
=& \sum_{k\in\mathbb{Z}} z^{2k} \sum_{j\in\mathbb{Z}} \alpha_j \int_{\mathbb{R}} M(2x+2k+1-j)\,\gamma_1(x)\,dx \\
=& z^{-1} \sum_{k\in\mathbb{Z}} z^{2k-j+1} \sum_{j\in\mathbb{Z}} \alpha_j\, z^j \int_{\mathbb{R}} M(2x+2k-j+1)\,dx \\
=& z^{-1}\, \frac{1}{2} \big(r(z)\,c(z) - r(-z)\,c(-z) \big)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{33}(z^2) =& \sum_{k\in\mathbb{Z}} z^{2k} \int_{\mathbb{R}} \gamma_3(x)\,\gamma_3(x-k)\,dx \\
=& \sum_{k\in\mathbb{Z}} z^{2k} \sum_{i,j\in\mathbb{Z}} \alpha_i\,\alpha_j \int_{\mathbb{R}} M(2x-i)\,M(2x-2k-j)\,dx \\
=& \frac{1}{2} \sum_{k\in\mathbb{Z}} z^{2k} \sum_{i,j\in\mathbb{Z}} \alpha_i\,\alpha_j \int_{\mathbb{R}} M(x)\,M(x-2k+i-j)\,dx \\
=& \frac{1}{4} \big(d(z)\,r(z)\,r(1/z) + d(-z)\,r(-z)\,r(-1/z) \big)
\end{aligned}
$$

Note that $G_{ij}(z) = G_{ji}(1/z)$, $i,j = 1,2,3$, and $\bar z = 1/z$. It follows that

$$
4\det(G(z^2)) = \begin{vmatrix}
a(z^2) & b(z^2) & r(\bar z)c(\bar z) + r(-\bar z)c(-\bar z) \\
z^2\, b(z^2) & a(z^2) & z\,\big(r(\bar z)\,c(\bar z) - r(-\bar z)\,c(-\bar z) \big) \\
\begin{array}{c} r(z)\,c(z) \\ +\,r(-z)\,c(-z) \end{array} & \begin{array}{c} z^{-1}\big(r(z)\,c(z) \\ -\,r(-z)\,c(-z) \big) \end{array} & \begin{array}{c} d(z)\,r(z)\,r(\bar z) \\ +d(-z)\,r(-z)\,r(-\bar z) \end{array}
\end{vmatrix}.
$$

Using properties of determinant this can be simplified as below:

$$4\det(G(z^2)) = \begin{vmatrix} a(z^2) & z\,b(z^2) & r(\bar{z})\,c(\bar{z}) + r(-\bar{z})\,c(-\bar{z}) \\ z\,b(z^2) & a(z^2) & r(\bar{z})\,c(\bar{z}) - r(-\bar{z})\,c(-\bar{z}) \\ r(z)\,c(z) & r(z)\,c(z) & d(z)\,r(z)\,r(\bar{z}) \\ +r(-z)\,c(-z) & -r(-z)\,c(-z) & +d(-z)\,r(-z)\,r(-\bar{z}) \end{vmatrix}$$

$$= \begin{vmatrix} a(z^2) & z\,b(z^2) & r(\bar{z})\,c(\bar{z}) \\ z\,b(z^2) & a(z^2) & r(\bar{z})\,c(\bar{z}) \\ r(z)\,c(z) + r(-z)\,c(-z) & r(z)\,c(z) - r(-z)\,c(-z) & d(z)\,r(z)\,r(\bar{z}) \end{vmatrix}$$

$$+ \begin{vmatrix} a(z^2) & z\,b(z^2) & r(-\bar{z})\,c(-\bar{z}) \\ z\,b(z^2) & a(z^2) & -r(-\bar{z})\,c(-\bar{z})) \\ r(z)\,c(z) + r(-z)\,c(-z) & r(z)\,c(z) - r(-z)\,c(-z)) & d(-z)\,r(-z)\,r(-\bar{z}) \end{vmatrix}$$

$$= \begin{vmatrix} a(z^2) & z\,b(z^2) & r(\bar{z})\,c(\bar{z}) \\ z\,b(z^2) & a(z^2) & r(\bar{z})\,c(\bar{z}) \\ r(z)\,c(z) & r(z)\,c(z) & d(z)\,r(z)\,r(\bar{z}) \end{vmatrix}$$

$$+ \begin{vmatrix} a(z^2) & z\,b(z^2) & r(\bar{z})\,c(\bar{z}) \\ z\,b(z^2) & a(z^2) & r(\bar{z})\,c(\bar{z}) \\ r(-z)\,c(-z) & -r(-z)\,c(-z) & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a(z^2) & z\,b(z^2) & r(-\bar{z})\,c(-\bar{z}) \\ z\,b(z^2) & a(z^2) & -r(-\bar{z})\,c(-\bar{z})) \\ r(z)\,c(z) & r(z)\,c(z) & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} a(z^2) & z\,b(z^2) & r(-\bar{z})\,c(-\bar{z}) \\ z\,b(z^2) & a(z^2) & -r(-\bar{z})\,c(-\bar{z})) \\ r(-z)\,c(-z) & -r(-z)\,c(-z)) & d(-z)\,r(-z)\,r(-\bar{z}) \end{vmatrix}$$

$$= r(z)\,r(\bar{z})\,D_1(z) + r(-z)\,r(\bar{z})\,D_2(z) + r(z)\,r(-\bar{z})\,D_3(-z) + r(-z)\,r(-\bar{z})\,D_4(z),$$

where

$$D_1(z) = \begin{vmatrix} a(z^2) & z\,b(z^2) & c(\bar z) \\ z\,b(z^2) & a(z^2) & c(\bar z) \\ c(z) & c(z) & d(z) \end{vmatrix}, \qquad D_2(z) = \begin{vmatrix} a(z^2) & z\,b(z^2) & c(\bar z) \\ z\,b(z^2) & a(z^2) & c(\bar z) \\ c(-z) & -c(-z) & 0 \end{vmatrix},$$

$$D_3(z) = \begin{vmatrix} a(z^2) & z\,b(z^2) & c(-\bar z) \\ z\,b(z^2) & a(z^2) & -c(-\bar z) \\ c(z) & c(z) & 0 \end{vmatrix}, \quad D_4(z) = \begin{vmatrix} a(z^2) & z\,b(z^2) & c(\bar z) \\ z\,b(z^2) & a(z^2) & -c(-\bar z) \\ c(-z) & -c(-z) & d(-z) \end{vmatrix}.$$

It is easy to check that $D_2(z) = 0, D_3(z) = 0$, and $D_4(z) = D_1(-z)$. We thus obtain (2.23) for the determinant of $\mathcal{G}(z)$,

$$4\det(G(z^2)) = D_1(z)\, r(z)\, r(\bar z) + D_1(-z)\, r(-z)\, r(-\bar z).$$

Now it is time to simplify $D_1(z)$. A direct computation of $D_1(z)$ yields

$$\begin{aligned} D_1(z) =& d(z)\left(a(z^2)^2 - z^2\, b(z^2)^2\right) - \,2\,c(z)c(\bar z)\left(a(z^2) - z\,b(z^2)\right) \\ =& \left(a(z^2) - z\,b(z^2)\right)\cdot\Big(d(z)\left(a(z^2) + zb(z^2)\right) - \,2\,c(z)c(\bar z)\Big). \end{aligned}$$

To simplify $D_1(z)$ more, let us consider

$$\begin{aligned} a(z^2) + zb(z^2) =& \sum_{j\in\mathbb{Z}} z^{2j} \int_{\mathbb{R}} \gamma_1(x)\,\gamma_1(x-j)\,dx + \sum_{j\in\mathbb{Z}} z^{2j+1} \int_{\mathbb{R}} \gamma_1(x)\,\gamma_2(x-j)\,dx \\ =& \frac{1}{2}\sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_n(x)\,N_n(x-j)\,dx =: \frac{1}{2}E(z). \end{aligned}$$

Then $a(z^2) - zb(z^2) = \frac{1}{2}E(-z)$. The second factor of $D_1(z)$ can be written by

$$d(z)\left(a(z^2) + zb(z^2)\right) - 2c(z)c(\bar z) = \frac{1}{2}\begin{vmatrix} E(z) & 2c(z) \\ 2c(\bar z) & d(z) \end{vmatrix}.$$

Since we have

$$2c(z) = \sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_n(x)\,M_n(x+j)\,dx,$$

the above determinant is the determinant of the Grammian matrix of $\{N_n, M_n\}$ which is the same as that of the Grammian matrix of $\{\gamma_1, \gamma_2\}$. By the special relation (2.22) of $\{\widehat{N}_n, \widehat{M}_n\}$

and $\{\widehat{\psi}_1, \widehat{\psi}_2\}$, the determinant of the Grammian matrix is

$$\begin{vmatrix} a(z) & b(z) \\ zb(z) & a(z) \end{vmatrix} = a(z)^2 - zb(z)^2.$$

Thus, $D_1(z) = \frac{1}{2}\left(a(z^2) - zb(z^2)\right)\left(a(z)^2 - zb(z)^2\right)$.

Now we claim that there exists a polynomial $p(z) \geq 0$ such that

$$D(z)\,p(z) + D(-z)\,p(-z) = 1. \tag{2.27}$$

Once we have such a $p(z)$, it follows from the Riesz-Féjer lemma that there exists a polynomial $r(z)$ such that $r(z)r(1/z) = p(z)$. This $r(z)$ is the polynomial we look for such that the determinant (2.23) of Grammian matrix $\mathcal{G}(z)$ is a nonzero constant.

To prove claim (2.27), we need the following lemma from ([22]).

Lemma 2.2.4 *Let p be a polynomal of degree n with all its zeros in $[1, \infty)$ having a positive leading coefficient. Then there exists a unique polynomial q with real coefficients of degree $n-1$ such that*

$$p(x)\,q(x) + p(1-x)\,q(1-x) = 1.$$

for $x \in [0,1]$. Moreover, $(-1)^n q(x) > 0$ for $x \in (0,1)$.

Proof: Let $x_i, i = 1, \cdots, r$ be the distinct zeroes of polynomial $p(x)$ with multiplicities $m_1, \cdots, m_r$, respectively. Define a polynomial q of degree $n-1$ by requiring that

$$q^{(j)}(1-x_i) = \left(\frac{d}{dx}\right)^j \frac{1}{p(x)}\Bigg|_{x=1-x_i}$$

for $j = 0, 1, \cdots, m_i - 1$ and $i = 1, \cdots, r$. Then the polynomial

$$v(x) = p(x)\,q(x) + p(1-x)\,q(1-x)$$

is of degree $2n-1$ and satsfies

$$v^{(j)}(x_i) = v^{(j)}(1-x_i) = \begin{cases} 1, & \text{if } j = 0 \text{ and } i = 1, \cdots, r \\ 0, & \text{otherwise} \end{cases}$$

for all $j = 0, 1, \cdots, m_i - 1$ and $i = 1, \cdots, r$. These $2n$ interpolatory conditions uniquely determine a polynomal $v(x)$. Clearly, the constant 1 satisfies the interpolatory conditions and hence, $v(x) \equiv 1$.

Next we note that $(-1)^n p(x) > 0$ on $(0, 1)$. Consider $w = pq$ and observe $w^{(j)}(x_i) = 0, j = 0, 1, \cdots, m_i - 1$ and $i = 1, \cdots, r$ while $w^{(j)}(1 - x_i) = 0, j = 1, \cdots, m_i - 1$ and $i = 1, \cdots, r$ and $w(1 - x_i) = 1$ for all $i = 1, \cdots, r$. Thus w' is of degree $2n - 2$ and has $2n - 2$ zeros by Rolle's theorem outside $(0, 1)$ and hence, w' is zero free on $(0, 1)$. Since $w(x_1) = 0$ and $w(1 - x_1) = 1$, we know that $w(x) > 0$. Thus, $(-1)^n q(x) > 0$ in $(0, 1)$. This completes the proof. □

To use the lemma above, we need to examine the zeros of $D(z) = \frac{1}{2}\left(a(z^2) - zb(z^2)\right)\left(a(z)^2 - zb(z)^2\right)$. Let us simplify $a(z^2) - zb(z^2)$ a little bit more:

$$
\begin{aligned}
a(z^2) - zb(z^2) =& \sum_{j\in\mathbb{Z}} z^{-2j} \int_{\mathbb{R}} N_m(2x) N_m(2x - 2j)\, dx \\
&+ \sum_{j\in\mathbb{Z}} z^{-(2j+1)} \int_{\mathbb{R}} N_m(2x) N_m(2x - 2j - 1)\, dx \\
=& \frac{1}{2} \sum_{-j\in\mathbb{Z}} z^{j} \int_{\mathbb{R}} N_m(x) N_m(x - j)\, dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} z^{j} \int_{\mathbb{R}} N_m(x) N_m(x + j)\, dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} z^{j} \int_{\mathbb{R}} N_m(x) N_m(m - j - x)\, dx \\
=& \frac{1}{2} \sum_{j\in\mathbb{Z}} N_{2m}(m - j)\, z^{j} = \frac{1}{2} \sum_{j\in\mathbb{Z}} N_{2m}(j)\, z^{m-j}
\end{aligned}
$$

where we have used the symmetric property of B-spline functions, i.e., $N_m(x) = N_m(m - x)$. It is well-known that $E_{2m}(z) := \sum_{j\in\mathbb{Z}} N_{2m}(j) z^j$ is an Euler-Frobenius polynomial which is never zero for $z = e^{-i\omega}$ for any ω. The zeros of $E_{2m}(z)$ are in $(-\infty, 0)$ since all coefficients of $E_{2m}(z)$ are positive. By the following Lemma 2.2.5, $E_{2m}(z)$ can be written in terms of $p(x)$ with $x = \sin^2(\omega/2)$ and $p(x)$ has only zeros in $[1, +\infty)$. Next we consider $a(z)^2 - zb(z)^2$. As above,

$$\begin{aligned}
a(z) &= \sum_{j\in\mathbb{Z}} z^j \int_{\mathbb{R}} N_m(2x)N_m(2x-2j)\,dx \\
&= \sum_{j\in\mathbb{Z}} \frac{1}{2}\,N_{2m}(m+2j)\,z^j \\
&= \frac{1}{2}\Big(N_{2m}(m) + \sum_{j=1}^{[m/2]} N_{2m}(m+2j)(z^j + 1/z^j)\Big)
\end{aligned}$$

is a real polynomial in $\cos(\omega)$ which can be converted to a polynomial in terms of $x = \sin^2(\omega/2)$. So is $a(z)^2$. Similarly,

$$\begin{aligned}
b(z) =& \sum_{j\in\mathbb{Z}} z^j \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} N_m(2x)N_m(2x-2j-1)dx \\
=& \frac{1}{2}\sum_{j\in\mathbb{Z}} N_{2m}(m+2j+1)z^j \\
=& \frac{z^{-1/2}}{2} \sum_{j=-[m/2]}^{[m/2]-1} N_{2m}(m+2j+1)z^{(2j+1)/2} \\
=& \frac{1}{2z^{1/2}}\Big(N_{2m}(m+1)z^{1/2} + N_{2m}(m-1)z^{-1/2} \\
&+ N_{2m}(m+3)z^{3/2} + N_{2m}(m-3)z^{-3/2} + \cdots\Big) \\
=& \frac{1}{2z^{1/2}} \sum_{j=0}^{[m/2]-1} N_{2m}(m+2j+1)\left(z^{(2j+1)/2} + z^{-(2j+1)/2}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
zb(z)^2 &= \frac{1}{4}\left(\sum_{j=0}^{[m/2]-1} N_{2m}(m+2j+1)\left(z^{(2j+1)/2} + z^{-(2j+1)/2}\right)\right)^2 \\
&= \frac{1}{4}\sum_{i,j=0}^{[m/2]-1} N_{2m}(m+2j+1)\,N_{2m}(m+2i+1)\left(z^{i+j+1} + z^{-(i+j+1)} + z^{i-j} + z^{j-i}\right)
\end{aligned}$$

is again a real polynomial in $\cos(\omega)$ which can be converted to a polynomial in terms of $x = \sin^2(\omega/2)$ by Lemma 2.2.5 below. The zeros of $a(z)^2 - zb(z)^2$ are contained in the zeros of $a(z)$ which are located in $[1, +\infty)$. Therefore, Lemma 2.2.4 implies that a polynomial $p(z)$ exists such that

$$D(z)\,p(z) + D(-z)\,p(-z) = 1$$

and $p(z) > 0$. This completes the proof of our claim.

Lemma 2.2.5 *Let*

$$c(z) = \sum_{-m}^{m} c_j z^j$$

be a polynomial which has zeros only in $(-\infty, 0)$ *with real coefficients* c_j *and* $c_j = c_{-j}$*. Then there is a polynomial* p *of degree* m *such that*

$$p(x) = c(e^{-i\omega}), \ \textit{with } x = \sin^2(\omega/2)$$

and $p(x)$ *has only zeros in* $[1, \infty)$*.*

Proof: Clearly, $c(z)$ can be written as

$$c(z) = c_0 + \sum_{j=1}^{m} 2\, c_j \, \cos(j\omega) = \sum_{j=0}^{m} d_j \left(\frac{z + 1/z}{2} \right)^j$$

for some real coefficients $d_0, \cdots, d_m$. Then we define

$$p(x) = \sum_{j=0}^{m} d_j (1 - 2x)^j$$

Then we can see that $c(z) = p(1/2 - (z + 1/z)/4) = p(\sin 2(\omega/2))$. If $p(x) = 0$ with $x = 1/2 - (z + 1/z)/4$ for $z \in (-\infty, 0)$, then $z + 1/z \leq -2$ implies that $x \geq 1$. $\square$

A major step in the computation of orthonormal scaling function vector is to factorize the Grammian matrix $\mathcal{G}(z)$ which will be discussed in the following section. This finishes the construction steps for compactly supported orthonormal scaling functions based on B-splines.

2.3 Riesz-Féjer Factorization of Laurent Polynomial Matrices

2.3.1 Constructive Riesz-Féjer Factorization

Let $M(z)$ be a matrix of size $r \times r$ with Laurent polynomial entries $M_{ij}(z) = \sum_{k \in \mathbb{Z}} m_{ij,k} z^k$ for $i, j = 1, \cdots, r$. Here $m_{ij,k} = 0$ except for finitely many k. Let $M(z)^* = M(1/z)^T$ stand for the transpose and conjugate. Suppose that $M(z)$ is positive semi-definite in the sense that for any vector $\mathbf{x} = (x_1, \cdots, x_r)^T$ of size r,

$$\mathbf{x}^T \, M(z) \, \mathbf{x} \geq 0$$

for any $z \in \mathbb{C}$ with $|z| = 1$. The well-known Riesz-Féjer factorization of Laurent polynomial matrices is the following

Theorem 2.3.1 *Fix $r \geq 1$. Suppose that $M(z)$ is positive semi-definite. Then there exists a Laurent polynomial matrix $N(z)$ of size $r \times r$ such that*

$$M(z) = N(z)^* N(z).$$

Proof: We only consider the case $r > 1$. For $r = 1$, see a proof in [12]. First of all, we use LU decomposition and Riesz-Féjer factorization for $r = 1$ to find a matrix $A(z)$ of size $r \times r$ with rational Laurent polynomial entries such that

$$M(z) = A(z)^* A(z).$$

We now use several unitary matrix transform to cancel the poles in the entries of $A(z)$. Let a be a real number which is a pole of order n in one of entries of $A(z)$ and the pole of highest order in $A(z)$. Expand $A(z)$ in terms of $\dfrac{1-za}{z-a}$. That is,

$$A(z) = \left(\frac{1-za}{z-a}\right)^n \left(R_a + \left(\frac{z-a}{1-za}\right) Q_a(z)\right).$$

where R_a is the residual matrix of size $r \times r$. Similarly we have

$$A(z) = \left(\frac{z-a}{1-za}\right)^m \left(R_{1/a} + \left(\frac{1-za}{z-a}\right) Q_{1/a}(z)\right)$$

Since $A(z)^* A(z)$ has no nonzero pole, we conclude that $R_a^T R_{1/a} = 0$. Let P be the orthogonal projection onto the column space of R_a satisfying $PR_a = R_a, PR_{1/a} = 0$. It is easy to check that

$$E(z) = I - P + \left(\frac{z-a}{1-za}\right) P$$

is a unitary matrix. Then

$$E(z)A(z) = \left(\frac{1-za}{z-a}\right)^{n-1} \left(R_a + (I-P)Q_a(z) + \left(\frac{z-a}{1-za}\right) PQ_a(z)\right).$$

That is, the pole at $z = a$ is reduced by one. Next we claim that $E(z)A(z)$ does not increase the pole at $z = 1/a$. Indeed,

$$\begin{aligned}E(z)A(z) =& E(z)\left(\frac{z-a}{1-za}\right)^m \left(R_{1/a} + \left(\frac{1-za}{z-a}\right) Q_{1/a}(z)\right)\\ =& \left(\frac{z-a}{1-za}\right)^m \left(R_{1/a} + PQ_{1/a}(z) + \left(\frac{1-za}{z-a}\right)(I-P)Q_{1/a}(z)\right)\end{aligned}$$

which shows that the pole at $z = 1/a$ remains the same. Next we consider a pole which is a complex number. Let a be a nonzero complex number which is a pole of order n in one entry of $A(z)$, where n is the highest order of pole at $z = a$. It follows that $\bar{a}$ is also a pole of order n. As above, we have

$$\begin{aligned}A(z) &= \left(\frac{1-za}{z-a}\right)^n \left(R_a + \left(\frac{z-a}{1-za}\right) Q_a(z)\right)\\ A(z) &= \left(\frac{z-a}{1-za}\right)^n \left(R_{1/a} + \left(\frac{1-za}{z-a}\right) Q_{1/a}(z)\right).\end{aligned}$$

Since $A(1/z)^T A(z)$ is a matrix with polynomial entries, we know that $R_a^T R_{1/a} = 0$. Since $\text{Im}(a) \neq 0$, we use the following unitary matrix

$$E(z) = Q_1 + \left(\frac{z-a}{1-za} + \frac{z-\bar{a}}{1-z\bar{a}}\right) Q_2 + \left(\frac{z-a}{1-za}\frac{z-\bar{a}}{1-z\bar{a}}\right) Q_3$$

with $Q_1 + 2Q_2 + Q_3 = I$ to cancel the pole at $z = a$ and $z = 1/a$. The proof of $E(z)$ is a unitary matrix is given in Lemma 2.3.2. Here, Q_1, Q_2, Q_3 are scalar matrices which satisfy the following property:

$$(Q_1 + i\lambda Q_2) R_a = 0 \text{ and } (Q_3 + i\lambda Q_2) R_{1/a} = 0$$

with $\lambda = 2\text{Im}(a)/(1-|a|^2) \neq 0$. A constructive method to find Q_1, Q_2 and Q_3 is given in the proof of Lemma 2.3.3. It follows that in the neighborhood of $z = a$,

$$\frac{z-\bar{a}}{1-z\bar{a}} = \frac{z-a+a-\bar{a}}{1-(z-a)\bar{a}-|a|^2} = i\lambda + O(|z-a|)$$

and hence,

$$\begin{aligned}
E(z)A(z) = &\left(\frac{1-za}{z-a}\right)^n \left\{ Q_1 + \frac{z-\bar{a}}{1-z\bar{a}} Q_2 R_a \right. \\
&+ \left(\frac{z-a}{1-za}\right)\left(Q_1 Q_a(z) + Q_2 R_a + \frac{z-\bar{a}}{1-z\bar{a}} Q_3 R_a\right) \\
&\left. + \left(\frac{z-a}{1-za}\right) 2 \left(\frac{z-\bar{a}}{1-z\bar{a}}\right) Q_3 Q_a(z) \right\} \\
= &\left(\frac{1-za}{z-a}\right)^n \left\{ (Q_1 + i\lambda Q_2) R_a + \left(\frac{z-\bar{a}}{1-z\bar{a}}\right) R(z) \right\} \\
= &\left(\frac{1-za}{z-a}\right)^{n-1} R(z)
\end{aligned}$$

for a matrix $R(z)$ with rational polynomial entries which is analytic at $z = a$. Similar for $z = \bar{a}$. However, in a neighborhood of $z = 1/a$, we have

$$\frac{z-\bar{a}}{1-z\bar{a}} = \frac{z - \frac{1}{a} + \frac{1}{a} - \bar{a}}{1 - az - z(\bar{a} - a)} = \frac{1}{i\lambda} + O(|1-az|)$$

and

$$\begin{aligned}
E(z)A(z) = &\left\{ Q_1 + \frac{1}{i\lambda} Q_2 + \left(\frac{z-a}{1-za}\right)(Q_2 + \frac{1}{i\lambda} Q_3) + O(|\frac{1-az}{z-a}|) \right\} \\
&\times \left(\frac{z-a}{1-za}\right)^n \left(R_{1/a} + \left(\frac{1-za}{z-a}\right) Q_{1/a}(z)\right) \\
= &\left(\frac{z-a}{1-za}\right)^n \left\{ (Q_1 + \frac{1}{i\lambda} Q_2) R_{1/a} \right. \\
&+ \left(\frac{z-a}{1-za}\right)(Q_2 + \frac{1}{i\lambda} Q_3) R_{1/a} + \left(\frac{1-za}{z-a}\right)(Q_1 + \frac{1}{i\lambda} Q_2) Q_{1/a}(z) \\
&\left. + (Q_2 + \frac{1}{i\lambda} Q_3) Q_{1/a}(z) + O\left(\left|\frac{1-az}{z-a}\right|\right) \right\} \\
= &\left(\frac{z-a}{1-za}\right)^n \left(R(z) + \left(\frac{1-za}{z-a}\right) Q(z)\right)
\end{aligned}$$

for some matrices $R(z)$ and $Q(z)$ with rational polynomial entries. Here we have used the fact that

$$(Q_2 + \frac{1}{i\lambda} Q_3)\, R_{1/a} = 0.$$

This shows that the pole at $z = 1/a$ is not increased after the multiplication of $E(z)$ to $A(z)$. Similar for $z = 1/\bar{a}$.

We therefore use these transforms to eliminate all the poles in $A(z)$. Let $C(z)$ be the product of all unitary transforms. Then

$$M(z) = A(z)^* A(z) = (C(z)A(z))^* \, C(z)A(z)$$

with $N(z) := C(z)A(z)$ which has no pole and hence $N(z)$ is the desirable Laurent polynomial matrix. □

Lemma 2.3.2 *Let a be a complex number with nonzero imaginary part $Im(a)$. And let Q_1, Q_2, Q_3 be three scalar matrices such that $Q_1 + 2Q_2 + Q_3 = I$ and $(Q_1 + i\lambda Q_2)(Q_3 + i\lambda Q_2)^T = I$, where $\lambda = 2Im(a)/(1 - |a|^2) \neq 0$. Then*

$$E(z) = Q_1 + \left(\frac{z-a}{1-za} + \frac{z-\bar{a}}{1-z\bar{a}}\right) Q_2 + \left(\frac{z-a}{1-za}\frac{z-\bar{a}}{1-z\bar{a}}\right) Q_3$$

is a unitary matrix.

Proof: Let

$$T(z) = (z-a)(z-1/a)(z-\bar{a})(z-1/\bar{a})\big(E(z)^* E(z) - I\big)$$

be a polynomial matrix of degree at most 4. It is clear that $T(z)$ is equal to zero at $z = a, 1/a, \bar{a}, 1/\bar{a}$. That is, $T(z)$ has four distinct zeros. Note that

$$T(1) = (1-a)(1-1/a)(1-\bar{a})(1-1/\bar{a})\big((Q_1 + 2Q_2 + Q_3)^* (Q_1 + 2Q_2 + Q_3) - I\big) = 0.$$

It follows that polynomial $T(z)$ of degree 4 has 5 distinct zeros. That is, $T(z) \equiv 0$ or $E(z)$ is a unitary matrix. □

Lemma 2.3.3 *Suppose that λ is a nonzero real number and W is a linear subspace of $\mathbb{C}^N$. Then there exist $N \times N$ matrices Q_1, Q_2, Q_3 with real entries such that*

(1) $Q_1 + 2Q_2 + Q_3 = I$*;*

(2) $(Q_1 + i\lambda Q_2)w = 0$ *for all* $w \in W$*;*

(3) $(Q_3 - i\lambda Q_2)v = 0$ *for all* $v \in W^\perp$*.*

Proof: Without loss of generality, we may assume that $W \neq \{0\}$. Let B be a matrix whose columns form an orthonormal basis for $W^\perp$. It follows that the standard ℓ^2 norm $\|B\|_2$ of B satisfies $\|B\|_2 \leq 1$, where $\|B\|_2 = \max_{\|x\|_2=1} \|Bx\|_2$. Thus, the following matrix

$$\begin{bmatrix} I & \frac{-1}{1-\lambda i}\bar{B}^T\bar{B} \\ \frac{-1}{1+\lambda i}B^T B & I \end{bmatrix}$$

is invertible. Hence, the following matrix equation

$$[X, Y] \begin{bmatrix} I & \frac{-1}{1-\lambda i}\bar{B}^T\bar{B} \\ \frac{-1}{1+\lambda i}B^T B & I \end{bmatrix} = \left[\frac{\lambda i}{1+\lambda i}B,\ \frac{-\lambda i}{1-\lambda i}\bar{B}\right]$$

has a solution (X, Y). In fact, $Y = \bar{X}$ and

$$\frac{\lambda i}{1+\lambda i}B - X + \frac{1}{1+\lambda i}\bar{X}B^T B = 0. \tag{2.28}$$

Define

$$Q_1 = \frac{1}{2}\,(X\bar{B}^T + \bar{X}B^T), \quad Q_2 = \frac{1}{2\lambda i}\,(X\bar{B}^T - \bar{X}B^T), \quad Q_3 = I - Q_1 - 2Q_2.$$

We claim that these matrices Q_1, Q_2, Q_3 satisfy the properties mentioned in Lemma 2.3.3 We only need to prove (2) and (3). It is clear that

$$(Q_1 + i\lambda Q_2)\, w = X\bar{B}^T\, w = 0, \quad \forall w \in W$$

and

$$\begin{aligned}(Q_3 - i\lambda Q_2)B =& B - \left(1 + \frac{1}{i\lambda}\right) X\bar{B}^T B + \frac{1}{\lambda i}\bar{X}B^T B \\ =& B - \left(1 + \frac{1}{i\lambda}\right) X + \frac{1}{\lambda i}\bar{X}B^T B = 0\end{aligned}$$

by (2.28). It follows that $(Q_3 - i\lambda Q_2)\, v = 0$ for all $v \in W^\perp$ since B contains the orthonormal basis for $W^\perp$. $\square$

2.3.2 A Computational Method for the Matrix Riesz-Féjer Factorization

Let $\gamma_1, \gamma_2, \gamma_3$ be the three compactly supported functions defined in the previous section. Since the determinant of the Grammian matrix $\mathcal{G}(z)$ associated with $\Gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ is a nonzero monomial, it can be factored into $\mathcal{G}(z) = B(z)B(z)^*$ with invertible polynomial matrix $B(z)$, where $B(z)^*$ stands for the transpose and conjugate of $B(z)$. In this section we discuss a computational method for the matrix factorization. Although the method ([23]) in subsection 2.3.1 is constructive, it requires a technique to factor a positive definite Hermitian matrix into matrices with rational Laurent polynomials, a method to identify the location of poles, an expansion of the rational entries into a special format, and construction of unitary matrices to cancel these poles. It is really not an easy task. To simplify the factorization, we describe a straightforward computational method to do such factorizations.

The basic ideas are as follows. Let $\mathcal{A}$ be a bi-infinite matrix with entries $A_{ij} = c_{i-j}$, where $\{c_j\}$ is a finite sequence. Let x be a bi-infinite sequence. Then $y = \mathcal{A}x$ is another bi-infinite sequence. Formally, the discrete Fourier transform of y can be given by

$$Y(\omega) = \sum_j y_j e^{-ij\omega} = A(\omega)X(\omega)$$

with $X(\omega) = \sum\limits_k x_k e^{-ik\omega}$ and $A(\omega) = \sum\limits_k c_k e^{-ik\omega}$. This is an identification of the bi-infinite matrix $\mathcal{A}$ and Laurent polynomial $A(\omega)$. If $A(\omega)$ is symmetric, i.e., $A(-\omega) = A(\omega)$ and positive, we know that it can be factored into a polynomial B in $e^{-i\omega}$ such that $A(\omega) = B(\omega)B(-\omega)$ by Riesz-Féjer factorization. Then the matrix $\mathcal{A}$ can be factored into a product of two matrices $\mathcal{B}\mathcal{B}^T$, where $\mathcal{B}$ is a lower-trangular bi-infinite matrix. This is indeed the case as discussed in [32].

For a positive definite Laurent polynomial matrix $M(\omega)$ of size $r \times r$, we write it as

$$M(\omega) = \sum_{k=-p}^{p} m_k e^{-ik\omega}$$

with $r \times r$ matrix coefficients m_k. For simplicity, we assume that m_k are matrices with real entries. Then we can identify $M(\omega)$ with a bi-infinite block matrix $\mathcal{M} = [\mathcal{M}_{ij}]_{-\infty<i,j<\infty}$ with

$\mathcal{M}_{ij} = m_{i-j}$. When $M(\omega) = N(\omega)N(-\omega)$ with $N(\omega) = \sum_{k=0}^{p} n_k e^{-ik\omega}$, $\mathcal{M} = \mathcal{N}\mathcal{N}^T$ with a bi-infinite matrix $\mathcal{N}$. The converse is also true.

Our computational method is to compute the Cholesky decomposition of a central section $\mathcal{M}_\ell := [\mathcal{M}_{ij}]_{1\le i,j\le \ell}$ of the bi-infinite matrix $\mathcal{M}$ with $\ell > 1$ being an integer. That is, let

$$[\mathcal{M}_{ij}]_{1\le i,j\le \ell} = \mathcal{N}_\ell \mathcal{N}_\ell^T$$

with lower triangular matrix $\mathcal{N}_\ell = [a_{ij}^\ell]_{1\le i,j\le r\ell}$. Let

$$\begin{aligned} n_0^\ell &= [a_{ij}^\ell]_{r(\ell-1)+1\le i,j\le r\ell}, \\ n_1^\ell &= [a_{ij}^\ell]_{\substack{r(\ell-1)+1\le i\le r\ell \\ r(\ell-2)+1\le j\le r(\ell-1)}}, \\ &\vdots \end{aligned}$$

and define $N_\ell(\omega) = \sum_{k=0}^{p} n_k^\ell e^{-ik\omega}$, then $N_\ell(\omega)$ converges to $N(\omega)$ as $\ell \to +\infty$. (See [18] for a proof.)

Example 2.3.4

$$M(\omega) = \begin{bmatrix} 8 + z + 1/z & 1 + z \\ 1 + 1/z & 1 \end{bmatrix}$$

is a Hermitian and positive definite matrix. Then $M(\omega) = m_0 + m_1 z + m_{-1} 1/z$ *with three matrices* m_{-1}, m_0, m_1 *of size* 2×2*. Let* $\mathcal{M}$ *be of size of* 20×20 *with* m_0 *on the diagonal blocks* m_1 *on the upper diagonal blocks, and* m_{-1} *on the sub-diagonal blocks. The remaining entries are zero. Using computer software MATLAB, we find*

$$N_\ell(\omega) = \begin{bmatrix} 2.64575131106459 & 0 \\ 0.377964473009 & 0.9258200997725 \end{bmatrix} + \begin{bmatrix} 0.3779644730092 & 0.9258200997725 \\ 0 & 0 \end{bmatrix} z.$$

It can be verified by using MAPLE to see that $M(\omega) = N_\ell(\omega)N_\ell(-\omega)^T + o(1)$ *with* $o(1) = 10^{-9}$.

Our main result in this section is the following :

Theorem 2.3.5 *Let $\mathcal{M}$ be a bi-infinite matrix associated with a positive definite Hermitian matrix $M(\omega)$ and let $\mathcal{N}_\ell$ be the Cholesky factorization of the central section $\mathcal{M}_\ell$. Then the block entry $[a_{ij}]_{r(\ell-1)+1\le i,j\le r\ell}$ converges to m_0 exponentially fast. Similar for the other block entries.*

We refer the reader to [18] for the detailed proof of convergence and more numerical examples.

2.4 Construction of the associated Wavelets

For the Grammian matrix $\mathcal{G}(z)$ associated with $\gamma_1, \gamma_2, \gamma_3$, let

$$\mathcal{G}(z) = B(z)^* \, B(z) \tag{2.29}$$

be the Riesz-Fejér factorization as discussed in the previous section. Letting

$$\widehat{\Phi}(z) = B(z)^{-1} \, \widehat{\Gamma}(z), \tag{2.30}$$

we know that the Grammian matrix of Φ is $B(z)^{-1}\, \mathcal{G}(z)\, (B(z)^{-1})^*$ which is the identity matrix and hence $\Phi = (\phi_1, \phi_2, \phi_3)$ is an orthonormal refinable function vector. In this section we discuss how to compute the associated wavelets. We begin with

Lemma 2.4.1 Φ *is refinable. That is, letting*

$$\widetilde{\Phi}(x) = \sqrt{2}(\phi_1(2x), \phi_2(2x), \phi_3(2x), \phi_1(2x-1), \phi_2(2x-1), \phi_3(2x-1))^T,$$

there exist matrix coefficients p_k of size 3×6 such that

$$\Phi(x) = \sum_{k\in\mathbb{Z}} p_k \, \widetilde{\Phi}(x-k) \quad \text{or} \quad \widehat{\Phi}(\omega) = P(z)\, \widehat{\widetilde{\Phi}}(\omega), \tag{2.31}$$

where $P(z)$ is a matrix mask of size 3×6.

Proof: Indeed, since Γ is refinable,

$$\widehat{\Gamma}(2\omega) = C(z)\, \widehat{\Gamma}(\omega) \tag{2.32}$$

for a matrix mask $C(z)$ of size 3×3. If we denote by

$$\widetilde{\Gamma}(x) = (\gamma_1(2x), \gamma_2(2x), \gamma_3(2x), \gamma_1(2x-1), \gamma_2(2x-1), \gamma_3(2x-1))^T, \tag{2.33}$$

the dilation relation of Γ can be rewritten in terms of $\widetilde{\Gamma}$. That is, by (2.32),

$$\begin{aligned}\Gamma(x) &= \sum_{k\in\mathbb{Z}} c_k\,\Gamma(2x-k)\\ &= \sum_{k\in 2\mathbb{Z}} c_k\,\Gamma(2x-k) + \sum_{k\in 2\mathbb{Z}} c_{k+1}\,\Gamma(2x-k-1)\\ &= \sum_{k\in\mathbb{Z}} \widetilde{c}_k\,\widetilde{\Gamma}(x-k).\end{aligned}$$

In terms of the Fourier transform, $\widehat{\Gamma}(z) = \widetilde{C}(z)\widehat{\widetilde{\Gamma}}(z)$ with matrix $\widetilde{C}(z)$ of size 3×6.

By (2.30), $\Phi(x) = \sum_{k\in\mathbb{Z}} b_k\Gamma(x-k)$ for matrix coefficients b_k of size 3×3. It follows that

$$\Phi(2x) = \sum_{k\in\mathbb{Z}} b_k\,\Gamma(2x-k), \quad \Phi(2x-1) = \sum_{k\in\mathbb{Z}} b_k\,\Gamma(2x-k-1)$$

and

$$\widetilde{\Phi}(x) = \sum_{k\in\mathbb{Z}} \widetilde{b}_k\,\widetilde{\Gamma}(x-k).$$

In terms of the Fourier transform, $\widehat{\widetilde{\Phi}}(z) = \widetilde{B}(z)\,\widehat{\widetilde{\Gamma}}(z)$. Note that $\widetilde{B}(z)$ is invertible because that both $\widetilde{\Phi}$ and $\widetilde{\Gamma}$ generates the same space $\mathcal{S}_1$. We have

$$\widehat{\widetilde{\Gamma}}(\omega) = \widetilde{B}(z)^{-1}\,\widehat{\widetilde{\Phi}}(\omega).$$

Therefore, we have

$$\begin{aligned}\widehat{\Phi}(\omega) =& B(z)^{-1}\,\widehat{\Gamma}(\omega) = B(z)^{-1}\,\widetilde{C}(z)\,\widehat{\widetilde{\Gamma}}(\omega)\\ =& B(z)^{-1}\,\widetilde{C}(z)\,\widetilde{B}(z)^{-1}\,\widehat{\widetilde{\Phi}}(\omega)\end{aligned}$$

which completes the proof. □

Next we have the relation among matrix coefficients p_k in (2.31):

$$\sum_{i\in\mathbb{Z}} p_i p_i^T = I_{3\times 3}, \tag{2.34}$$

since the dilation equation (2.31) and the orthonormality of $\phi_i, i = 1, 2, 3$ implies

$$\begin{aligned}
I_{3\times 3} &= \int_{\mathbb{R}} \Phi(x)\Phi(x)^T\,dx \\
&= \int_{\mathbb{R}} \sum_{i\in\mathbb{Z}} p_i \widetilde{\Phi}(x-i) \cdot \sum_{j\in\mathbb{Z}} (\widetilde{\Phi}(x-j))^T p_j^T\,dx \\
&= \sum_{i,j\in\mathbb{Z}} p_i \cdot \int_{\mathbb{R}} \widetilde{\Phi}(x-i)(\widetilde{\Phi}(x-j))^T dx \cdot p_j^T \\
&= \sum_{i,j\in\mathbb{Z}} p_i \cdot \delta_{2i,2j} I_{6\times 6} \cdot p_j^T \\
&= \sum_{i\in\mathbb{Z}} p_i p_i^T.
\end{aligned}$$

Since Φ is of compact support, we may assume that only $m+1$ terms $p_0, p_1, \cdots, p_m$ are nonzero matrix coefficients. Then, we further have

$$\sum_{i=k}^{m} p_i\, p_{i-k}^T = 0, \text{ for } k = 1, \cdots, m. \tag{2.35}$$

Indeed, for $k = 1, \cdots, m$ we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \Phi(x)\Phi(x-k)^T dx \\
&= \int_{\mathbb{R}} \sum_{i\in\mathbb{Z}} p_i \widetilde{\Phi}(x-i) \cdot \sum_{j\in\mathbb{Z}} (\widetilde{\Phi}(x-j))^T p_j^T\,dx \\
&= \sum_{i,j=1}^{m} p_i \cdot \int_{\mathbb{R}} \widetilde{\Phi}(x-i)(\widetilde{\Phi}(x-j-k))^T dx \cdot p_j^T \\
&= \sum_{i,j=1}^{m} p_i \cdot \delta_{2i,2j+2k} I_{6\times 6} \cdot p_j^T \\
&= \sum_{i=k}^{m} p_i p_{i-k}^T.
\end{aligned}$$

In particular, we have

$$p_m\, p_0^T = 0 \tag{2.36}$$

We now use induction on m to show how to construct three compactly supported orthonormal wavelets $\psi_1, \psi_2, \psi_3 \in \mathcal{S}_1$ such that letting

$$\mathcal{W} := \operatorname{span}\{\psi_1(\cdot - i),\, \psi_2(\cdot - j),\, \psi_3(\cdot - k),\, i, j, k \in \mathbb{Z}\},$$

$\mathcal{W}$ is the orthogonal completement of $\mathcal{S}$ in $\mathcal{S}_1$. That is, $\mathcal{S}_1 = \mathcal{S} \oplus \mathcal{W}$. More precisely, let $\Psi = (\psi_1, \psi_2, \psi_3)^T$. (2.31) and the Fourier transform of Ψ give

$$\widehat{\Phi}(\omega) = P(z)\,\widehat{\widehat{\Phi}}(\omega), \quad \widehat{\Psi}(\omega) = \; Q(z)\,\widehat{\widehat{\Phi}}(\omega)$$

where $Q(z) = \sum_{i\in\mathbb{Z}} q_i z^i$ is a Laurent polynomial matrix of size 3×6. The orthogonal completementness and the orthonormality of ψ_1, ψ_2, ψ_3 imply that the matrix

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$$

is a unitary matrix. That is, $Q(z)$ is an unitary extension of $P(z)$.

It is trivial when $m = 0$. Indeed, in this case, $P(z) = p_0$ is a scalar matrix. We simply choose $Q(z)$ to be a scalar matrix which is an orthonormal extension of p_0. Assume that for $m \geq 1$, when $P_m(z) = \sum_{k=0}^{m} p_k z^k$ is an orthonormal matrix of 3×6, we can find $Q_m(z)$ such that

$$\begin{bmatrix} P_m(z) \\ Q_m(z) \end{bmatrix}$$

is unitary. We now consider the case of $m+1$: $P_{m+1}(z) = \sum_{k=0}^{m+1} p_k z^k$ satisfying orthonormal properties in (3.4) and (3.5). In particular, (3.6) implies that there exists a unitary matrix U_0 of size 6×6 such that $p_0\, U_0 = \left[0_{3\times3},\, \widetilde{p}_0^b\right]$ and $p_{m+1}\, U_0 = \left[\widetilde{p}_{m+1}^a,\, 0_{3\times3}\right]$, where $\widetilde{p}_0^b$ is of size 3×3 and the same for $\widetilde{p}_{m+1}^a$. Writing $p_k U_0 = \left[\widetilde{p}_k^a,\, \widetilde{p}_k^b\right]$ with $\widetilde{p}_k^a$ and $\widetilde{p}_k^b$ being of size 3×3. Then

$$P_{m+1}(z) U_0 = \left[\sum_{k=1}^{m+1} \widetilde{p}_k^a z^k,\, \sum_{k=0}^{m} \widetilde{p}_k^b z^k \right]$$

Let

$$U_1 := \begin{bmatrix} \frac{1}{z} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix}.$$

Then it follows that

$$\begin{aligned} P_{m+1}(z) U_0 U_1 &= \left[\sum_{k=0}^{m} \widetilde{p}_{k+1}^a z^k,\, \sum_{k=0}^{m} \widetilde{p}_k^b z^k \right] \\ &= \sum_{k=0}^{m} \left[\widetilde{p}_{k+1}^a,\, \widetilde{p}_k^b\right] z^k. \end{aligned}$$

That is, $\widetilde{P}_m(z) := P_{m+1}(z)\,U_0\,U_1$ has only $m+1$ terms and is unitary. By induction, we can find an unitary extension $\widetilde{Q}_m(z)$ such that

$$\begin{bmatrix} P_{m+1}(z)U_0U_1 \\ \widetilde{Q}_m(z) \end{bmatrix}$$

is unitary. Clearly,

$$\begin{bmatrix} P_{m+1}(z)\,U_0\,U_1 \\ \widetilde{Q}_m(z) \end{bmatrix} U_1^* U_0^* = \begin{bmatrix} P_{m+1}(z) \\ \widetilde{Q}_m(z)\,U_1^*\,U_0^* \end{bmatrix}$$

is also unitary. It follows that $Q_{m+1}(z) := \widetilde{Q}_m(z)\,U_2^*\,U_1^*$ is an unitary extension of $P_{m+1}(z)$. This completes the induction procedure. Therefore, we conclude the following :

Theorem 2.4.2 *Given refinable orthonormal functions ϕ_1, ϕ_2, ϕ_3, we can construct three associated wavelets ψ_1, ψ_2, ψ_3 such that $\psi_i(x-k)$ is orthogonal to $\phi_j(x-m)$ for all $i, j = 1, 2, 3$ and $m \in \mathbb{Z}$, $\psi_i(x-k)$'s are orthonormal among each other for all $i = 1, 2, 3$ and $k \in \mathbb{Z}$, and the linear span of $\psi_i(x-k), i = 1, 2, 3$ and $k \in \mathbb{Z}$ forms a subspace of $\mathcal{S}_1$ which is an orthogonal complement of $\mathcal{S}$ in $\mathcal{S}_1$.*

2.5 Examples

In this section we want to provide a few examples based on the construction method in the previous sections. Recall that the B-spline function of order m, $N_m(x)$ has the mask

$$A(z) = A_0(z^2) + zA_1(z^2) = 2\left(\frac{1+z}{2}\right)^m.$$

Thus $\gamma_1(x) = N_m(2x), \gamma_2(x) = N_m(2x-1)$. Then we have

$$\widehat{N}_m(\omega) = A_0(z)\,\widehat{\gamma}_1(\omega) + A_1(z)\,\widehat{\gamma}_2(\omega).$$

Our computation to find a scaling function vector $\Phi(x)$ and coreesponding wavelet functon vector $\Psi(x)$ can be organized in the following four major steps:

Step 1. Computation of $M_m(x)$.

First we find $B_0(z), B_1(z)$ satisfying the equation

$$A_0(z)\, B_0(z) + A_1(z)\, B_1(z) = 1$$

as in Lemma 2.2.3. And then we define a new spline function $M_m(x)$ in terms of the Fourier transform according to (2.21), i.e.,

$$\widehat{M}_m(\omega) = -B_1(z)\, \widehat{\gamma}_1(\omega) + B_0(z)\, \widehat{\gamma}_2(\omega).$$

Step 2. Computation of $\gamma_3(x)$.

We have to begin with the computation of the determinant of $\mathcal{G}(z)$, which is the Grammian matrix of $\{\gamma_1(x), \gamma_2(x), \gamma_3(x)\}$. The formula of $\det(\mathcal{G}(z))$ is obtained as

$$4 \det \mathcal{G}(z^2) = D(z)r(z)r(1/z) + D(-z)r(-z)r(-1/z),$$

where

$$D(z) = \frac{1}{2}\big(a(z^2) - zb(z^2)\big)\,\big(a(z)^2 - zb(z)^2\big).$$

The formula for $a(z)$ and $b(z)$ are given as follows,

$$a(z) = \sum_{k\in\mathbb{Z}} \int N_m(2x)N_m(2x-k)\, dx,$$

$$b(z) = \sum_{k\in\mathbb{Z}} \int N_m(2x)N_m(2x-2k-1)\, dx.$$

Then by Lemma 2.2.4 we find a polynomial $p(z) \geq 0$ such that

$$D(z)p(z) + D(-z)p(-z) = 1.$$

A straightforward computation (cf. [32]) gives $r(z) = \sum \alpha_k z^k$ such that $p(z) = r(z)r(1/z)$, and so we get $\gamma_3(x) = \sum \alpha_k M_m(2x-k)$.

Step 3. Computation of ϕ_1, ϕ_2, ϕ_3.

We need to find the entries for the Grammian matrix $\mathcal{G}(z)$ using $\gamma_1, \gamma_2, \gamma_3$. Then by using the method in section 3.2 we factorize into $\mathcal{G}(z) = B(z)B(z)^*$ with the help from the computer software MAPLE. Therefore we can define the orthonormal scaling vector $\Phi = (\phi_1, \phi_2, \phi_3)^T$ in terms of the Fourier transform :

$$\widehat{\Phi}(\omega) = B(z)^{-1}\,\widehat{\Gamma}(\omega).$$

Step 4. Computation of the associated wavelets ψ_1, ψ_2, ψ_3.

In this step, we follow Lemma 2.4.1 so that we have the dilation relation for ϕ_1, ϕ_2, ϕ_3 :

$$\widehat{\Phi}(\omega) = P(z)\,\widehat{\widetilde{\Phi}}(\omega)$$

where $P(z) = B(z)^{-1}\,\widetilde{C}(z)\,\widetilde{B}(z)^{-1}$. Then by induction on m, i.e., steps in the proof of Theorem 2.4.2 we find the unitary extension $Q(z)$ of $P(z)$. Hence we define ψ_1, ψ_2, ψ_3 in the Fourier transform:

$$\widehat{\Psi}(\omega) = Q(z)\,\widehat{\widetilde{\Phi}}(\omega),$$

where $\Psi = (\psi_1, \psi_2, \psi_3)^T$.

Following steps as the above, we obtain three orthonormal scaling functions ϕ_1, ϕ_2, ϕ_3 and the corresponding wavelet functions ψ_1, ψ_2, ψ_3 for $m = 2, 3, 4$ described in next subsections.

2.5.1 $m = 2$: Linear B-spline case

Using the linear B-spline function, Step 1 - Step 3 above gives the following three scaling functions

$$\begin{aligned}
\phi_1(x) =& \sqrt{3}\,N_2(2x),\\
\phi_2(x) =& \frac{\sqrt{165}}{11}\,N_2(2x) - \frac{4\sqrt{33}\,(2+\sqrt{5})}{33}\,N_2(4x) + \frac{4\sqrt{33}\,(2-\sqrt{5})}{33}\,N_2(4x-2),\\
\phi_3(x) =& \sum_{j=0}^{2} \alpha_j\,N_2(2x-j) + \sum_{k=0}^{3} \beta_k\,N_2(4x-2k)
\end{aligned}$$

where the coefficients $\alpha_j' s$ and $\beta_k' s$ in $\phi_3(x)$ are defined as follows:

$$\alpha_0 = -\frac{\sqrt{231}\,(3+2\sqrt{5})}{154}, \qquad \beta_0 = \frac{\sqrt{231}\,(3+2\sqrt{5})}{231},$$

$$\alpha_1 = \frac{\sqrt{231}}{7}, \qquad \beta_1 = -\frac{\sqrt{231}\,(4-\sqrt{5})\,(2-\sqrt{5})}{231},$$

$$\alpha_2 = -\frac{\sqrt{231}\left(3-2\sqrt{5}\right)}{154}, \qquad \beta_2 = -\frac{\sqrt{231}\left(13+6\sqrt{5}\right)}{231},$$

$$\beta_3 = \frac{\sqrt{231}\left(4+\sqrt{5}\right)\left(2-\sqrt{5}\right)}{231}.$$

The first wavelet function associated with the above scaling functions are obtained as

$$\psi_1(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\,\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\,\phi_2\left(2x-k\right) + \sum_{l=0}^{3} \gamma_l \sqrt{2}\,\phi_3\left(2x-l\right)$$

where the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ are defined as follows:

$$\alpha_0 = -0.000458857008, \quad \beta_0 = -0.000732123098, \quad \gamma_0 = -0.00439249500,$$

$$\alpha_1 = 0.008233854626, \quad \beta_1 = 0.004743860499, \quad \gamma_1 = 0.01293269968,$$

$$\alpha_2 = 0.03396060301, \quad \beta_2 = 0.09061226061, \quad \gamma_2 = 0.5436434207,$$

$$\alpha_3 = -0.6093982729, \quad \beta_3 = 0.03009628816, \quad \gamma_3 = 0.5679245459.$$

Similarly, the second wavelet function has the form of

$$\psi_2(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\,\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\,\phi_2\left(2x-k\right) + \sum_{l=0}^{2} \gamma_l \sqrt{2}\,\phi_3\left(2x-l\right)$$

with the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\alpha_0 = -0.01131064902, \quad \beta_0 = -0.01804655319, \quad \gamma_0 = -0.1082733148,$$

$$\alpha_1 = 0.2029613542, \quad \beta_1 = 0.1169343393, \quad \gamma_1 = 0.3187860801,$$

$$\alpha_2 = -0.0001241837662, \quad \beta_2 = 0.8977104423, \quad \gamma_2 = -0.1590117827,$$

$$\alpha_3 = 0.002228387187, \quad \beta_3 = -0.01256974349.$$

The third wavelet function is given by

$$\psi_3(x) = \sum_{j=0}^{3} \alpha_j \sqrt{2}\,\phi_1\left(2x-j\right) + \sum_{k=0}^{3} \beta_k \sqrt{2}\,\phi_2\left(2x-k\right) + \sum_{l=0}^{2} \gamma_l \sqrt{2}\,\phi_3\left(2x-l\right)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{aligned}
&\alpha_0 = 0.0009518791574, && \beta_0 = 0.001518757925, && \gamma_0 = 0.009112042237,\\
&\alpha_1 = -0.01708077780, && \beta_1 = -0.009840934872, && \gamma_1 = -0.02682833007,\\
&\alpha_2 = 0.007635145592, && \beta_2 = -0.06338384300, && \gamma_2 = -0.3802819728,\\
&\alpha_3 = -0.1370071236, && \beta_3 = -0.8697092234, && \gamma_3 = 0.2737702148.
\end{aligned}$$

The graphs for three linear B-spline scaling functions and wavelet functions can be seen from Fig. 2.1. For the application, the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions ψ_1, ψ_2, ψ_3 are also provided as

$$P(z) = \sum_{j=0}^{1} p_j\, z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{1} q_j\, z^j$$

where the matrix coefficients $p_j's$ and $q_j's$ are as follows :

$$p_0 = \begin{bmatrix} 0.5303300858 & 0.09401789265 & 0.5640760745 & 0.5303300858 & -0.332383539 & 0.0 \\ -0.846628498 & 0.06338685030 & 0.3803000119 & 0.2904419884 & -0.224092936 & 0.0 \\ -0.025088411 & -0.0400294773 & -0.240163545 & 0.4501932763 & 0.2593747642 & 0.7071067810 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.000275454 & -0.408778340 & 0.0473162416 & 0.0049428373 & -0.027881239 & 0.0 \end{bmatrix}$$

$$q_0 = \begin{bmatrix} -0.000458857 & -0.000732123 & -0.004392495 & 0.0082338546 & 0.0047438604 & 0.0129326996 \\ -0.011310649 & -0.018046553 & -0.108273314 & 0.2029613542 & 0.1169343393 & 0.3187860801 \\ 0.0009518791 & 0.0015187579 & 0.0091120422 & -0.017080777 & -0.009840934 & -0.026828330 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.0339606030 & 0.0906122606 & 0.5436434207 & -0.609398272 & 0.0300962881 & 0.5679245459 \\ -0.000124183 & 0.8977104423 & -0.159011782 & 0.0022283871 & -0.012569743 & 0.0 \\ 0.0076351455 & -0.063383843 & -0.380281972 & -0.137007123 & -0.869709223 & 0.2737702148 \end{bmatrix}.$$

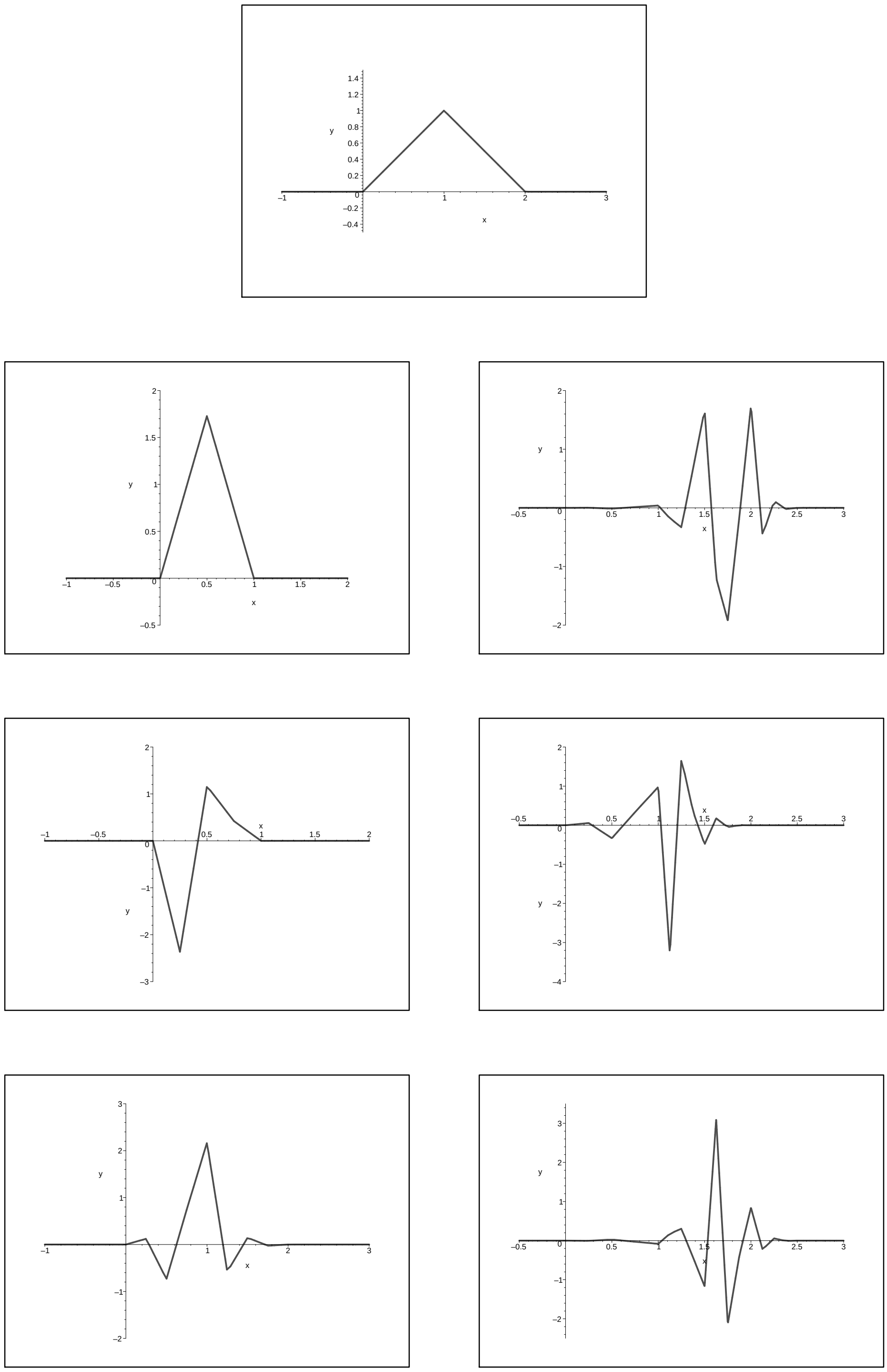

Figure 2.1. The scaling functions ϕ_1, ϕ_2, ϕ_3 in the left column and the asociated wavelet functions ψ_1, ψ_2, ψ_3 in the right column with the linear B-spline function $N_2(x)$ on the top.

2.5.2 $m = 3$: Quadratic B-spline case

Now we use the quadratic B-spline function N_3 to obtain the following scaling functions. The same computation algorithm gives the first scaling function ϕ_1 which has the form of

$$\phi_1(x) = \sum_{j=0}^{5} \alpha_j N_3(2x - j) + \sum_{k=4}^{13} \beta_k N_3(4x - k)$$

where $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$$\begin{array}{lll} \alpha_0 = 1.912780893, & \beta_4 = -0.2493006029, & \beta_{10} = 0.01320417254, \\ \alpha_1 = 0, & \beta_5 = -0.08310020090, & \beta_{11} = 0.004401390844, \\ \alpha_2 = 0.1423331085, & \beta_6 = -0.04069437830, & \beta_{12} = 0.0004398176573, \\ \alpha_3 = -0.07936182709, & \beta_7 = -0.01356479276, & \beta_{13} = 0.0001466058857, \\ \alpha_4 = -0.01604382797, & \beta_8 = 0.07955030613, & \\ \alpha_5 = -0.0005827378251, & \beta_9 = 0.02651676869. & \end{array}$$

Similarily ϕ_2 is obtained as

$$\phi_2(x) = \sum_{j=0}^{5} \alpha_j N_3(2x - j) + \sum_{k=4}^{13} \beta_k N_3(4x - k)$$

with $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$$\begin{array}{lll} \alpha_0 = -1.357081867, & \beta_4 = -4.443679097, & \beta_{10} = 0.02245652519, \\ \alpha_1 = 4.031609464, & \beta_5 = 1.481226365, & \beta_{11} = 0.007485508396, \\ \alpha_2 = 1.039045691, & \beta_6 = -0.7253602925, & \beta_{12} = 0.0007492812684, \\ \alpha_3 = -0.1062083099, & \beta_7 = -0.2417867641, & \beta_{13} = 0.0002497604227, \\ \alpha_4 = -0.02726113500, & \beta_8 = 0.1136754828, & \\ \alpha_5 = -0.00100088449, & \beta_9 = 0.03789182756. & \end{array}$$

Finally the third scaling function has obtained as

$$\phi_3(x) = \sum_{j=0}^{5} \alpha_j N_3(2x - j) + \sum_{k=0}^{12} \beta_k N_3(4x - k)$$

with $\alpha_j's$ and $\beta_k's$ defined as follows:

$$
\begin{array}{lll}
\alpha_0 = 2.444184403, & \beta_0 = -3.273033858, & \beta_6 = 0.2682118763, \\
\alpha_1 = -1.614417716, & \beta_1 = -1.091011285, & \beta_7 = 0.08940395872, \\
\alpha_2 = -0.3295492918, & \beta_2 = -0.5342709824, & \beta_8 = 0.005554086345, \\
\alpha_3 = -0.008453078355, & \beta_3 = -0.1780903274, & \beta_9 = 0.001851362115, \\
\alpha_4 = 0.0006435488021 & \beta_4 = 1.625509509, & \beta_{10} = -0.0005359258598, \\
\alpha_5 = 0.00002549272240 & \beta_5 = 0.5418365024, & \beta_{11} = -0.0001786419532, \\
 & & \beta_{12} = -0.00001837002370.
\end{array}
$$

Using the unitary extension algorithm, we have the wavelet functions associated with the above scaling functions. The first wavelet function is obtained as

$$\psi_1(x) = \sum_{j=0}^{5} \alpha_j \sqrt{2}\phi_1 (2x - j) + \sum_{k=0}^{5} \beta_k \sqrt{2}\phi_2 (2x - k) + \sum_{l=0}^{5} \gamma_l \sqrt{2}\phi_3 (2x - l)$$

where the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ are given by:

$$
\begin{array}{lll}
\alpha_0 = -0.0003062253733, & \beta_0 = -0.0009007471920, & \gamma_0 = 0, \\
\alpha_1 = 0.01246118727, & \beta_1 = 0.01113491465, & \gamma_1 = 0.001246236311, \\
\alpha_2 = -0.05420976520, & \beta_2 = 0.1380902192, & \gamma_2 = -0.01548641273, \\
\alpha_3 = -0.2681617806, & \beta_3 = -0.5672682747, & \gamma_3 = -0.1903506842, \\
\alpha_4 = 0, & \beta_4 = -0.5276284620, & \gamma_4 = -0.4716883893, \\
\alpha_5 = 0.08569854953, & \beta_5 = -0.1210110593, & \gamma_5 = -0.1570470319.
\end{array}
$$

The second wavelet function is

$$\psi_2(x) = \sum_{j=0}^{5} \alpha_j \sqrt{2}\phi_1 (2x - j) + \sum_{k=0}^{5} \beta_k \sqrt{2}\phi_2 (2x - k) + \sum_{l=0}^{5} \gamma_l \sqrt{2}\phi_3 (2x - l)$$

with $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = 0, & \beta_0 = -0.0004330240360, & \gamma_0 = 0, \\
\alpha_1 = 0.005162474047, & \beta_1 = 0.004781085640, & \gamma_1 = 0.0005879009950, \\
\alpha_2 = -0.02259767455, & \beta_2 = 0.05676958760, & \gamma_2 = -0.006599104846, \\
\alpha_3 = -0.1085045180, & \beta_3 = -0.2322047695, & \gamma_3 = -0.07833780565, \\
\alpha_4 = -0.03976697815, & \beta_4 = 0.7119356252, & \gamma_4 = -0.4966254880, \\
\alpha_5 = -0.2307637715, & \beta_5 = 0.3258511211, & \gamma_5 = -0.09920870542.
\end{array}$$

Finally the third wavelet function has the form

$$\psi_3(x) = \sum_{j=0}^{5} \alpha_j \sqrt{2}\phi_1 (2x - j) + \sum_{k=0}^{5} \beta_k \sqrt{2}\phi_2 (2x - k) + \sum_{l=0}^{5} \gamma_l \sqrt{2}\phi_3 (2x - l)$$

where $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$$\begin{array}{lll}
\alpha_0 = 0, & \beta_0 = -0.0001599180187, & \gamma_0 = 0, \\
\alpha_1 = 0.001792952322, & \beta_1 = 0.001685246499, & \gamma_1 = 0.0002171148813, \\
\alpha_2 = -0.007203042950, & \beta_2 = 0.02075884756, & \gamma_2 = -0.002320509445, \\
\alpha_3 = -0.05758550293, & \beta_3 = -0.09736113015, & \gamma_3 = -0.02872173604, \\
\alpha_4 = 0.09983056480, & \beta_4 = 0.1948900260, & \gamma_4 = -0.1888528064, \\
\alpha_5 = 0.2573657082, & \beta_5 = -0.3634145170, & \gamma_5 = 0.8390259756.
\end{array}$$

The graphs for three quadratic B-spline scaling and wavelet functions can be seen from Fig. 4.2. And the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions ψ_1, ψ_2, ψ_3 are

$$P(z) = \sum_{j=0}^{2} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{2} q_j z^j$$

where the matrix coefficients $p_j's$ and $q_j's$ are as follows :

$$p_0 = \begin{bmatrix} 0.3552913428 & 0.2516129760 & 0.0 & 0.8027328340 & -0.050135660 & -0.368667639 \\ -0.252072488 & -0.178514647 & 0.0 & 0.1793314476 & 0.5659003882 & 0.2615627177 \\ -0.891722034 & 0.1301621724 & 0.0 & 0.3279007596 & -0.194119464 & -0.108795722 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.0020893579 & -0.045619592 & -0.158795175 & 0.0146786383 & -0.007900767 & 0.0001620848 \\ 0.002453825 & -0.193477010 & -0.664386546 & 0.0407356939 & -0.034351855 & -0.043087136 \\ 0.0003609633 & 0.0489391883 & 0.1673889663 & -0.007003601 & 0.0088201391 & 0.0176140477 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0.0000202218 & 0.0023289399 & 0.0079844069 & -0.000321256 & 0.0004210025 & 0.0008752574 \\ 0.0000398405 & 0.0039729864 & 0.0145016142 & -0.000540177 & 0.0007155982 & 0.0014886743 \\ 0.0 & 0.0000993182 & -0.000700788 & 0.0000116601 & -0.000017589 & -0.000037064 \end{bmatrix}$$

$$q_0 = \begin{bmatrix} -0.000306225 & -0.000900747 & 0.0 & 0.0124611872 & 0.0111349146 & 0.0012462363 \\ 0.0 & -0.000433024 & 0.0 & 0.0051624740 & 0.0047810856 & 0.0005879009 \\ 0.0 & -0.000159918 & 0.0 & 0.0017929523 & 0.0016852464 & 0.0002171148 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} -0.054209765 & 0.138090219 & -0.0154864127 & -0.268161780 & -0.567268274 & -0.190350684 \\ -0.022597674 & 0.056769587 & -0.0065991048 & -0.108504518 & -0.232204769 & -0.078337805 \\ -0.007203042 & 0.020758847 & -0.0023205094 & -0.057585502 & -0.097361130 & -0.028721736 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.0 & -0.527628462 & -0.471688389 & 0.0856985495 & -0.121011059 & -0.157047031 \\ -0.039766978 & 0.7119356252 & -0.496625488 & -0.230763771 & 0.3258511211 & -0.099208705 \\ 0.0998305648 & 0.1948900260 & -0.188852806 & 0.2573657082 & -0.363414517 & 0.8390259756 \end{bmatrix}.$$

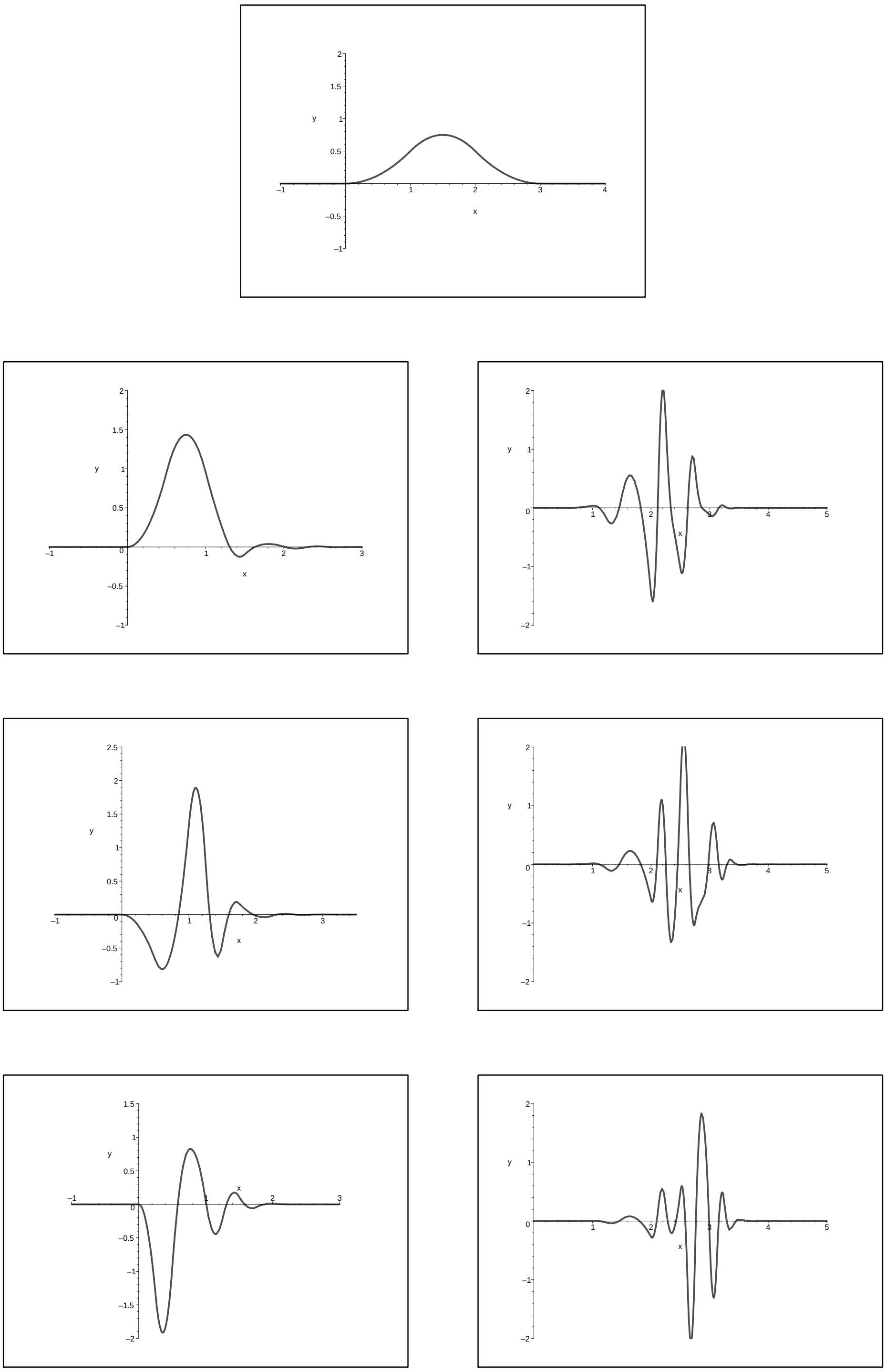

Figure 2.2. The scaling functions ϕ_1, ϕ_2, ϕ_3 in the left column and the asociated wavelet functions ψ_1, ψ_2, ψ_3 in the right column with the quadratic B-spline function $N_3(x)$ on the top.

2.5.3 $m = 4$: Cubic B-spline case

Finally, using the cubic B-spline function, N_4 gives the following scaling functions :

$$\phi_1(x) = \sum_{j=0}^{7} \alpha_j N_3(2x - j) + \sum_{k=4}^{16} \beta_k N_3(4x - k)$$

with the coeffcients $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$\alpha_0 = 2.047361858$, $\beta_4 = -0.005761704746$, $\beta_{12} = 0.008672283344$,

$\alpha_1 = 0$, $\beta_5 = -0.004609363797$, $\beta_{13} = 0.007721741696$,

$\alpha_2 = -0.1624533715$, $\beta_6 = -0.08308449674$, $\beta_{14} = 0.002164408424$,

$\alpha_3 = 0.4546458474$, $\beta_7 = -0.06554572464$, $\beta_{15} = 0.0001871784003$,

$\alpha_4 = 0.02093253113$, $\beta_8 = -0.3256052423$, $\beta_{16} = 0.00004813776044$,

$\alpha_5 = -0.01491339515$, $\beta_9 = -0.2473750490$,

$\alpha_6 = -0.0004199338466$, $\beta_{10} = -0.06674323113$,

$\alpha_7 = 0.00006724572218$, $\beta_{11} = -0.003919575091$.

$$\phi_2(x) = \sum_{j=0}^{7} \alpha_j N_3(2x - j) + \sum_{k=4}^{16} \beta_k N_3(4x - k)$$

with the coeffcients $\alpha_j' s$ and $\beta_k' s$ defined as follows:

$\alpha_0 = -1.422340146$, $\beta_4 = -0.06019396928$, $\beta_{12} = 0.04411863502$,

$\alpha_1 = 3.133241228$, $\beta_5 = -0.04815517542$, $\beta_{13} = 0.04535506727$,

$\alpha_2 = -2.155096355$, $\beta_6 = -0.8680044995$, $\beta_{14} = 0.01271778136$,

$\alpha_3 = 4.847978891$, $\beta_7 = -0.6847725646$, $\beta_{15} = 0.001103211630$,

$\alpha_4 = 0.1818024290$, $\beta_8 = -3.402501408$, $\beta_{16} = 0.0002837193665$,

$\alpha_5 = -0.08830568379$, $\beta_9 = -2.585046614$,

$\alpha_6 = -0.002265414769$, $\beta_{10} = -0.7091376490$,

$\alpha_7 = 0.00009356915410$, $\beta_{11} = -0.05030079621$.

$$\phi_3(x) = \sum_{j=0}^{6} \alpha_j N_3(2x - j) + \sum_{k=0}^{16} \beta_k N_3(4x - k)$$

with the coeffcients $\alpha_j's$ and $\beta_k's$ defined as follows:

$\alpha_0 = 0.3708734612$, $\beta_0 = 0.1798049852$, $\beta_9 = -2.601033845$,

$\alpha_1 = -10.86060055$, $\beta_1 = 0.1438439881$, $\beta_{10} = -0.7163380139$,

$\alpha_2 = -2.688916817$, $\beta_2 = 2.592810177$, $\beta_{11} = -0.05286364210$,

$\alpha_3 = 4.905934976$, $\beta_3 = 2.045479345$, $\beta_{12} = 0.03330331585$,

$\alpha_4 = 0.1735299986$, $\beta_4 = 10.10617927$, $\beta_{13} = 0.03721538111$,

$\alpha_5 = -0.07265482128$, $\beta_5 = 7.675847548$, $\beta_{14} = 0.01043738398$,

$\alpha_6 = -0.001805007946$, $\beta_6 = 1.290416379$, $\beta_{15} = 0.0009068309569$,

$\beta_7 = -0.5028364074$, $\beta_{16} = 0.0002332150039$.

$\beta_8 = -3.377001408$,

The wavelet functions associated with the above scaling functions are :

$$\psi_1(x) = \sum_{j=1}^{7} \alpha_j \sqrt{2}\phi_1(2x - j) + \sum_{k=1}^{7} \beta_k \sqrt{2}\phi_2(2x - k) + \sum_{l=1}^{7} \gamma_l \sqrt{2}\phi_3(2x - l)$$

with the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$\alpha_1 = -0.00004781192773$, $\beta_1 = -0.00004079145383$, $\gamma_1 = 0$,

$\alpha_2 = -0.002882971910$, $\beta_2 = -0.006875470238$, $\gamma_2 = -0.00001486447257$,

$\alpha_3 = -0.004353343447$, $\beta_3 = -0.04195913083$, $\gamma_3 = -0.002398832949$,

$\alpha_4 = 0.04073890858$, $\beta_4 = -0.01984221762$, $\gamma_4 = -0.01495097012$,

$\alpha_5 = 0.05265946433$, $\beta_5 = 0.2872416818$, $\gamma_5 = -0.01011081711$,

$\alpha_6 = 0.01280299015$, $\beta_6 = 0.2278488512$, $\gamma_6 = -0.6505816899$,

$\alpha_7 = 0.0006264187940$, $\beta_7 = -0.0006748493335$, $\gamma_7 = -0.6596687075$,

$$\psi_2(x) = \sum_{j=0}^{7} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=0}^{7} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=0}^{7} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$\alpha_0 = -0.00001443062762$, $\beta_0 = -0.00002854942961$, $\gamma_0 = 0$,

$\alpha_1 = 0.0005203769086$, $\beta_1 = 0.0003647536138$, $\gamma_1 = -0.00001002001414$,

$\alpha_2 = 0.03214154669$, $\beta_2 = 0.07679563964$, $\gamma_2 = 0.0001355920316$,

$\alpha_3 = 0.05228295363$, $\beta_3 = 0.4723191704$, $\gamma_3 = 0.02678216224$,

$\alpha_4 = -0.2395777302$, $\beta_4 = 0.7402831469$, $\gamma_4 = 0.1683325379$,

$\alpha_5 = -0.1788855688$, $\beta_5 = 0.03649664622$, $\gamma_5 = 0.2938945458$,

$\alpha_6 = 0.06001241338$, $\beta_6 = -0.04775755830$, $\gamma_6 = -0.08421641470$,

$\alpha_7 = -0.004491933329$, $\beta_7 = 0.004839220159$, $\gamma_7 = -0.007479139119$,

$$\psi_3(x) = \sum_{j=1}^{7} \alpha_j \sqrt{2}\phi_1\left(2x-j\right) + \sum_{k=1}^{7} \beta_k \sqrt{2}\phi_2\left(2x-k\right) + \sum_{l=1}^{7} \gamma_l \sqrt{2}\phi_3\left(2x-l\right)$$

with the coefficients $\alpha_j's$, $\beta_k's$ and $\gamma_l's$ defined as follows:

$\alpha_1 = -0.00004205017244$, $\beta_1 = -0.00003037744814$, $\gamma_1 = 0$,

$\alpha_2 = -0.002674936735$, $\beta_2 = -0.006408569926$, $\gamma_2 = -0.00001130783746$,

$\alpha_3 = -0.004184674332$, $\beta_3 = -0.03921648320$, $\gamma_3 = -0.002234727942$,

$\alpha_4 = 0.03047580223$, $\beta_4 = -0.03648961583$, $\gamma_4 = -0.01397509120$,

$\alpha_5 = 0.03538338810$, $\beta_5 = 0.1558149332$, $\gamma_5 = -0.01570116072$,

$\alpha_6 = -0.02820551623$, $\beta_6 = -0.2200752201$, $\gamma_6 = -0.6831368125$,

$\alpha_7 = 0.0004966118760$, $\beta_7 = -0.0005350067050$, $\gamma_7 = 0.6739835824$.

The graphs of the C^2 cubic spline scaling functions and associated wavelet functions can be seen from Figure 4.3. And for the applicants, the masks associated with scaling functions ϕ_1, ϕ_2, ϕ_3 and wavelet functions ψ_1, ψ_2, ψ_3 are providded in the following:

$$P(z) = \sum_{j=0}^{3} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{3} q_j z^j$$

where the matrix coefficients $p_j's$ and $q_j's$ are as follows :

$$p_0 = \begin{bmatrix} 0.2676276275 & -0.1323568653 & 0.0996048260 & -0.049091354 & 0.0233274267 & -0.004245095 \\ 0.8400785659 & 0.02826010795 & 0.0613682814 & -0.302428179 & 0.0867358876 & 0.0560255374 \\ 0.197299666 & 0.4819353231 & -0.241001568 & -0.230491161 & 0.0338867101 & 0.0907006419 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0.2676276275 & -0.1323568653 & 0.0996048260 & -0.049091354 & 0.0233274267 & -0.004245095 \\ 0.8400785659 & 0.02826010795 & 0.0613682814 & -0.302428179 & 0.0867358876 & 0.0560255374 \\ 0.197299666 & 0.4819353231 & -0.241001568 & -0.230491161 & 0.0338867101 & 0.0907006419 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0.0089114979 & -0.0078765941 & 0.0048407190 & -0.000455648 & 0.0007018501 & -0.00078793952 \\ 0.072061128 & -0.0597016449 & 0.0369156652 & -0.005030108 & 0.0057868995 & -0.00571638390 \\ 0.064631440 & -0.0527503132 & 0.0326388593 & -0.004778213 & 0.0052121789 & -0.00499654634 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} -0.000063062 & 0.00005158167 & -0.000031102 & 0.0 & 0.0 & 0.0 \\ -0.000110162 & 0.00009114914 & -0.000049567 & 0.0000171108 & 0.0 & 0.0 \\ -0.000020402 & 0.00001636063 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$q_0 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & -0.000047811 & -0.000040791 & 0.0 \\ -0.000014430 & -0.0000285494 & 0.0 & 0.0005203769 & 0.0003647536 & -0.00001002001 \\ 0.0 & 0.0 & 0.0 & -0.000042050 & -0.000030377 & 0.0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} -0.002882971 & -0.0068754702 & -0.000014864 & -0.004353343 & -0.041959130 & -0.0023988329 \\ 0.0321415466 & 0.07679563964 & 0.0001355920 & 0.0522829536 & 0.4723191704 & 0.02678216224 \\ -0.002674936 & -0.0064085699 & -0.000011307 & -0.004184674 & -0.039216483 & -0.0022347279 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.0407389085 & -0.019842217 & -0.014950970 & 0.0526594643 & 0.2872416818 & 0.0101108171 \\ -0.239577730 & 0.7402831469 & 0.1683325379 & -0.178885568 & 0.0364966462 & 0.2938945458 \\ 0.0304758022 & -0.036489615 & -0.013975091 & 0.0353833881 & 0.155814933 & -0.015701160 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0.0128029901 & 0.2278488512 & -0.650581689 & 0.0006264187 & -0.000674849 & -0.6596687075 \\ 0.0600124133 & -0.047757558 & -0.084216414 & -0.004491933 & 0.0048392201 & -0.0074791391 \\ -0.028205516 & -0.220075220 & -0.683136812 & 0.0004966118 & -0.000535006 & 0.6739835824 \end{bmatrix}.$$

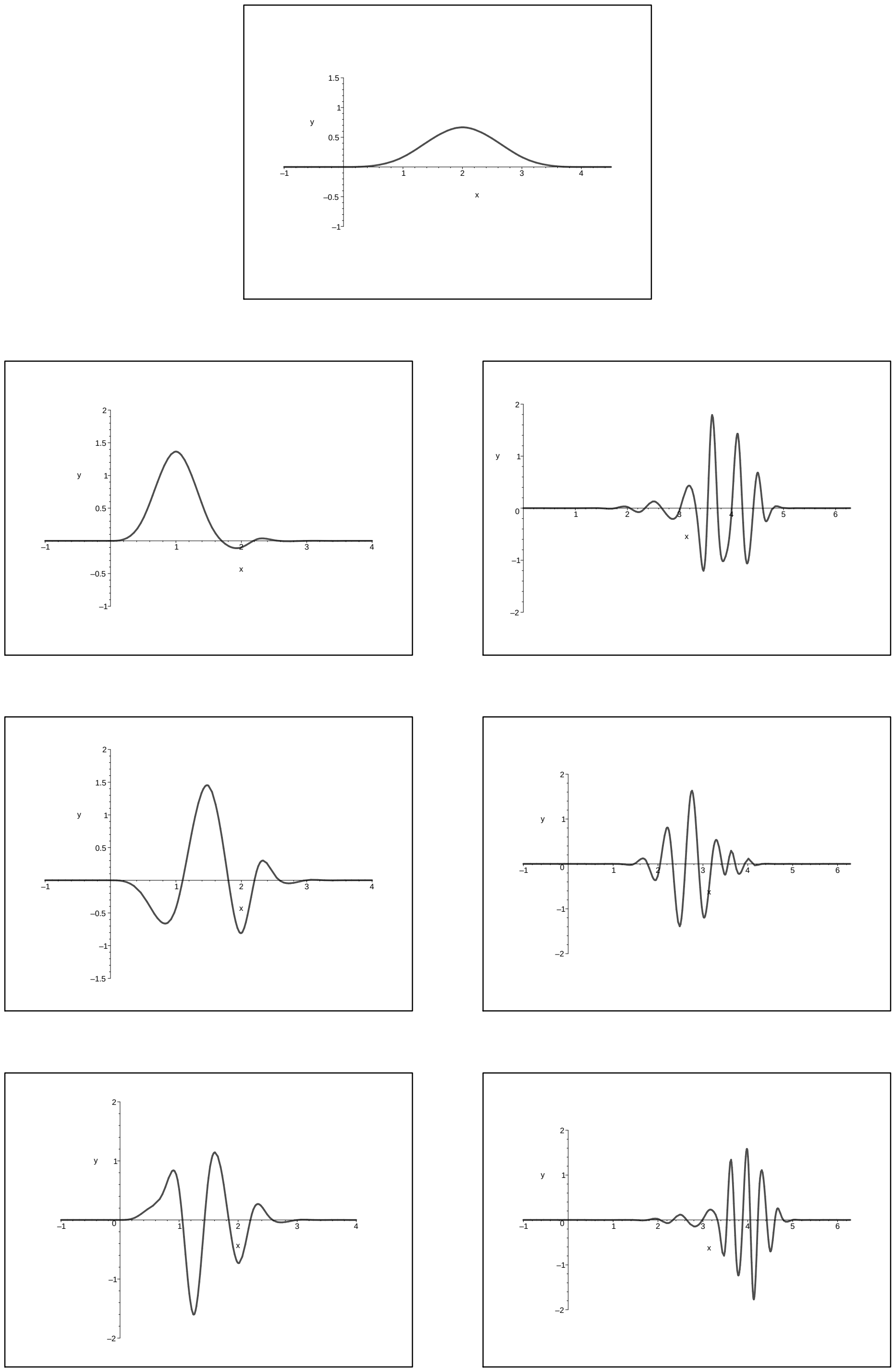

Figure 2.3. The scaling functions ϕ_1, ϕ_2, ϕ_3 in the left column and the asociated wavelet functions ψ_1, ψ_2, ψ_3 in the right column with the cubic B-spline function $N_4(x)$ on the top.

Chapter 3

Construction of Biorthogonal B-spline Multiwavelets

Fix integers $r \geq 1$ and $d \geq 1$. Let $\phi_1, \cdots, \phi_r$ and $\widetilde{\phi}_1, \cdots, \widetilde{\phi}_r$ be compactly supported continuous real-valued functions in $\mathbb{R}^d$ and let

$$\Phi = (\phi_1, \cdots, \phi_r)^T \text{ and } \widetilde{\Phi} = (\widetilde{\phi}_1, \cdots, \widetilde{\phi}_r)^T.$$

A pair of two scaling function vectors Φ and $\widetilde{\Phi}$ is said to be *biorthogonal* if

$$\langle \Phi(\mathbf{x}), \widetilde{\Phi}(\mathbf{x} - \mathbf{k}) \rangle = \delta_{\mathbf{0},\mathbf{k}} I_{r\times r},$$

where δ stands for the Kronecker delta, and $I_{r\times r}$ is the identity matrix. Similarly, a pair of two wavelet function vectors Ψ and $\widetilde{\Psi}$, associated with Φ and $\widetilde{\Phi}$ respectively, is said to be *biorthogonal* if

$$\langle \Phi(\mathbf{x}), \widetilde{\Psi}(\mathbf{x} - \mathbf{k}) \rangle = \langle \Psi(\mathbf{x}), \widetilde{\Phi}(\mathbf{x} - \mathbf{k}) \rangle = O_{r\times r},$$

$$\langle \Psi(\mathbf{x}), \widetilde{\Psi}(\mathbf{x} - \mathbf{k}) \rangle = \delta_{\mathbf{0},\mathbf{k}} I_{r\times r}, \quad \mathbf{k} \in \mathbb{Z}^d,$$

where $O_{r\times r}$ and $I_{r\times r}$ denote zero and idendtity matrices respectively.

The wavelet function vector Ψ corresponding to a multiscaling function vector Φ satisfies the refinement equation:

$$\Psi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} Q_{\mathbf{k}} \Phi(2\mathbf{x} - \mathbf{k}),$$

where $Q_{\mathbf{k}}$ is a $r \times r$ coefficient matrix.

Fix $d = 1$. And we take the Fourier transform of Φ and Ψ, we have

$$\widehat{\Phi}(2\xi) = P(\xi)\,\widehat{\Phi}(\xi),\ P(z) = \frac{1}{2}\sum_j P_k z^j,$$

$$\widehat{\Psi}(2\xi) = Q(\xi)\,\widehat{\Psi}(\xi),\ Q(z) = \frac{1}{2}\sum_j Q_k z^j,$$

where $P(z)$ and $Q(z)$ are Laurent polynomial matrices called *the matrix symbols* of $\Phi(x)$ and $\Psi(x)$, respectively. Here we note that $P(z)$ and $P(\xi)$ are interchangeable without any confusion. Similarly, let $\widetilde{P}(z)$ and $\widetilde{Q}(z)$ be the matrix symbols of $\widetilde{\Phi}(x)$ and $\widetilde{\Psi}(x)$, respectively. Then in terms of the matrix symbols $P(z), Q(z), \widetilde{P}(z)$, and $\widetilde{Q}(z)$, the biorthogonality conditions are represented as:

$$P(z)\widetilde{P}^*(z) + P(-z)\widetilde{P}^*(-z) = I_{r\times r}, \tag{3.1}$$

$$P(z)\widetilde{Q}^*(z) + P(-z)\widetilde{Q}^*(-z) = O_{r\times r}, \tag{3.2}$$

$$\widetilde{P}(z)Q^*(z) + \widetilde{P}(-z)Q^*(-z) = O_{r\times r}, \tag{3.3}$$

$$Q(z)\widetilde{Q}^*(z) + Q(-z)\widetilde{Q}^*(-z) = I_{r\times r}. \tag{3.4}$$

Let N_m be the B-spline function whose Fourier transform is

$$\widehat{N}_m(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^m.$$

For simplicity, we denote N_m by N. Then we have the mask of N,

$$\left(\frac{1+z}{2}\right)^m, \quad z = e^{-i\xi}$$

so that it satisfies the following refinement equation:

$$N(x) = \sum_j a_j N(2x-j), \tag{3.5}$$

where

$$A(z) := \sum_j a_j z^j = 2\left(\frac{1+z}{2}\right)^m. \tag{3.6}$$

Let V_0 be the space generated by $N(x)$ in the sense that V_0 comprises all finite linear combinations of integer translates of $N(x)$, and define

$$V_j = \left\{ f(2^j x) : f \in V_0 \right\}, \text{ for } j = \pm 1, \pm 2, \cdots.$$

It is well known that $\{V_j : j \in \mathbb{Z}\}$ constitutes a multiresolution approximation. Suppose that $\widetilde{V}_j$ is a dual space of V_j, i.e., $\{\widetilde{V}_j : j \in \mathbb{Z}\}$ constitutes another multiresolution approximation. In this chapter we want to find two scaling function vectors $\Phi = (\phi_1, \phi_2, \phi_3)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ such that $\Phi, \widetilde{\Phi}$ generate $\mathcal{S}, \widetilde{\mathcal{S}}$ respectivly, where

$$V_1 \subset \mathcal{S} \subset V_2 \text{ and } \widetilde{V}_1 \subset \widetilde{\mathcal{S}} \subset \widetilde{V}_2.$$

And also we construct corresponding wevelet function vectors $\Psi = (\psi_1, \psi_2, \psi_3)^T$, $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)^T$. That is, letting $\mathcal{T}, \widetilde{\mathcal{T}}$ be the spaces generated by $\Psi(x), \widetilde{\Psi}(x)$, respectively, it turns out that $\mathcal{S} \perp \widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{S}} \perp \mathcal{T}$.

3.1 Construction of Refinable Function Vectors Φ and $\widetilde{\Phi}$

First we begin with putting

$$\phi_1(x) = N(2x), \quad \phi_2(x) = N(2x-1).$$

From the refinement equation (3.5),

$$\begin{aligned} N(x) &= \sum_j a_j N(2x-j) \\ &= \sum_j a_{2j} N(2x-2j) + a_{2j+1} N(2x-2j-1) \\ &= \sum_j a_{2j} \phi_1(x-j) + \sum_j a_{2j+1} \phi_2(x-j). \end{aligned}$$

We write $A(z)$ in its polyphase form :

$$A(z) = A_0(z^2) + zA_1(z^2),$$

where

$$A_0(z^2) = \frac{A(z) + A(-z)}{2}, \quad A_1(z^2) = \frac{A(z) - A(-z)}{2z}.$$

Then

$$\widehat{N}(\xi) = A_0(z)\widehat{\phi}_1(\xi) + A_1(z)\widehat{\phi}_2(\xi), \tag{3.7}$$

and so

$$\widehat{\phi}_1(2\xi) = \frac{1}{2}\widehat{N}(\xi) = \frac{1}{2}A_0(z)\widehat{\phi}_1(\xi) + \frac{1}{2}A_1(z)\widehat{\phi}_2(\xi), \tag{3.8}$$

$$\widehat{\phi}_2(2\xi) = \frac{z}{2}\widehat{N}(\xi) = \frac{z}{2}A_0(z)\widehat{\phi}_1(\xi) + \frac{z}{2}A_1(z)\widehat{\phi}_2(\xi). \tag{3.9}$$

We consider

$$B(z) := \frac{1}{2}\left(B_0(z^2) + zB_1(z^2)\right) = \frac{1}{2}\left(\frac{1+z}{2}\right)^{\widetilde{m}-m} H_{\widetilde{m}}(z), \quad \widetilde{m} \geq 1 \tag{3.10}$$

where

$$H_{\widetilde{m}}(z) = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(\frac{1+z}{2}\right)^{\widetilde{m}-1-k} \left(\frac{1-z}{2}\right)^{k}. \tag{3.11}$$

Then we have $A_0(z)B_0(z) + zA_1(z)B_1(z) = 1$, since

$$\begin{aligned}
1 &= \left(\frac{1+z}{2}\right)^{\widetilde{m}} H_{\widetilde{m}}(z) + \left(\frac{1-z}{2}\right)^{\widetilde{m}} H_{\widetilde{m}}(-z) \\
&= A(z)B(z) + A(-z)B(-z) \\
&= \frac{1}{2}A_0(z^2)B_0(z^2) + \frac{z}{2}A_0(z^2)B_1(z^2) + \frac{z}{2}A_1(z^2)B_0(z^2) + \frac{z^2}{2}A_1(z^2)B_1(z^2) \\
&+ \frac{1}{2}A_0(z^2)B_0(z^2) - \frac{z}{2}A_0(z^2)B_1(z^2) - \frac{z}{2}A_1(z^2)B_0(z^2) + \frac{z^2}{2}A_1(z^2)B_1(z^2) \\
&= A_0(z^2)B_0(z^2) + z^2A_1(z^2)B_1(z^2).
\end{aligned}$$

So the matrix

$$P_1(z) := \begin{bmatrix} A_0(z) & A_1(z) \\ -zB_1(z) & B_0(z) \end{bmatrix} \tag{3.12}$$

is of determinant 1.

Now we define $M \in V_1$ in its Fourier transform

$$\widehat{M}(\xi) = -z\, B_1(z)\, \widehat{\phi}_1(\xi) + B_0(z)\, \widehat{\phi}_2(\xi). \tag{3.13}$$

Since $(\phi_1, \phi_2)^T$ generates V_1, hence $(N, M)^T$ generates V_1. We put

$$f_1(x) = M(2x), \quad f_2(x) = M(2x-1),$$

then we can see that $(\phi_1, \phi_2, f_1, f_2)^T$ generates V_2.

We now define

$$\phi_3(x) = \sum_{j=-\infty}^{\infty} \alpha_j M(2x - j), \tag{3.14}$$

with a Laurent polynomial

$$R(z) = \frac{1}{2} \sum_{j=-\infty}^{\infty} \alpha_j z^j, \tag{3.15}$$

then ϕ_3 is a function in V_2 whose Fourier transform is

$$\widehat{\phi}_3(2\xi) = R(z)\widehat{M}(\xi) = -zR(z)B_1(z)\widehat{\phi}_1(\xi) + R(z)B_0(z)\widehat{\phi}_2(\xi). \tag{3.16}$$

So we have

$$\widehat{\Phi}(2\xi) = P(z)\widehat{\Phi}(\xi),$$

where

$$P(z) = \begin{bmatrix} \frac{1}{2}A_0(z) & \frac{1}{2}A_1(z) & 0 \\ z\frac{1}{2}A_0(z) & z\frac{1}{2}A_1(z) & 0 \\ -zR(z)B_1(z) & R(z)B_0(z) & 0 \end{bmatrix} \tag{3.17}$$

To construct the dual scaling function vector $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ we only need to find the matrix symbol $\widetilde{P}(z)$ satisfying the first biorthogonality condition (3.1):

$$P(z)\widetilde{P}^*(z) + P(-z)\widetilde{P}^*(-z) = I_{3\times 3}.$$

The well known Quillen-Suslin's Theorem guarantees the existence of such dual matrix symbol $\widetilde{P}(z)$. For convenience we state it below.

Theorem 3.1.1 *Suppose that a matrix A of size $r \times l$ with $r < l$ with entries over $\mathcal{R} := \mathbb{C}[z_1, \cdots, z_s]$ is unimodular. Then there exists a unimodular matrix U of size $l \times l$ over $\mathcal{R}$ such that*

$$AU = [I_{r\times r}, 0, \cdots, 0]_{r\times l}.$$

In Theorem 3.1.1, a matrix A is *unimodular* if the maximal minors of A generate the unit ideal i.e., letting $\det(A_\alpha)$ denote a minor of A of size $r \times r$, there exist polynomials $c_\alpha(\mathbf{z}) \in \mathcal{R}$ such that

$$\sum_\alpha c_\alpha(\mathbf{z}) \det(A_\alpha) = 1.$$

Here, we note that we work on the case: $r = 3, s = 1, l = 6$ in the theorem above. Then we can easily check that $\operatorname{rank}([P(z), P(-z)]) = 3$, or $[P(z), P(-z)]$ is of full rank for all $z \in \mathbb{C} \setminus \{0\}$. We express $P(z), \widetilde{P}(z)$ in polyphase form as follows:

$$P(z) = p_0(z^2) + zp_1(z^2), \quad \widetilde{P}(z) = \widetilde{p}_0(z^2) + z\widetilde{p}_1(z^2).$$

Then the equation (3.1) is equivalent to

$$p_0(z^2)\,\widetilde{p}_0(z^2) + p_1(z^2)\,\widetilde{p}_1(z^2) = \frac{1}{2}\, I_{3\times3},$$

i.e., the search for $\widetilde{P}(z)$ is replaced by the search for $\widetilde{p}_0(z^2), \widetilde{p}_1(z^2)$. Since

$$\begin{bmatrix} P(z) & P(-z) \end{bmatrix} = \begin{bmatrix} p_0(z^2) & p_1(z^2) \end{bmatrix} \cdot \Lambda(z),$$

where $\Lambda(z) = \begin{bmatrix} I_{3\times3} & I_{3\times3} \\ zI_{3\times3} & -zI_{3\times3} \end{bmatrix}$ is invertible, the full rankness of $[P(z),\ P(-z)]$ is equivalent to the full rankness of $[p_0(z^2),\ p_1(z^2)]$. We claim the following lemma.

Lemma 3.1.2 *Let A be the matrix $[p_0(z^2),\ p_1(z^2)]$ of size 3×6. Then, A is of full rank, i.e., rank$(A) = 3$ if and only if A is unimodular in $\mathbb{C}(z)$.*

Proof: The full rankness implies that there exists one minor of size 3×3 whose determinant is not equal to zero. Thus all minors of A have no common zeros. By Hilbert-Nullstellensatz theorem, the maximal minors generate the unit ideal of $\mathbb{C}(z)$.

Conversely, A is unimodular implies that 1 can be generated by all minors of A. Hence at least one of minors is not zero for every z. This implies that A is of full rank. $\square$

Therefore, applying Theorem 3.1.1 to the polyphase matrix A in Lemma 3.1.2 ensures the existence of a dual matrix symbol $\widetilde{P}(z)$.

We now write

$$P(z)=\begin{bmatrix} X(z) & O \\ Y(z) & 0 \end{bmatrix}, \quad \widetilde{P}(z)=\begin{bmatrix} \widetilde{X}(z) & \widetilde{V}(z) \\ \widetilde{Y}(z) & \widetilde{W}(z) \end{bmatrix}.$$

where $X(z), \widetilde{X}(z)$ are 2×2 polynomial matrices, $Y(z), \widetilde{Y}(z)$ are 1×2 polynomial matrices $\widetilde{V}(z)$ is a 2×1 polynomial matrix, and $\widetilde{W}(z)$ is a polynomial. Then the first biorthogonality condition (1.1) which is restated in the above becomes

$$X(z)\widetilde{X}^*(z)+X(-z)\widetilde{X}^*(-z)=I_{2\times 2}, \tag{3.18}$$

$$X(z)\widetilde{Y}^*(z)+X(-z)\widetilde{Y}^*(-z)=O_{2\times 1}, \tag{3.19}$$

$$\widetilde{X}(z)Y^*(z)+\widetilde{X}(-z)Y^*(-z)=O_{1\times 2}, \tag{3.20}$$

$$Y(z)\widetilde{Y}^*(z)+Y(-z)\widetilde{Y}^*(-z)=1. \tag{3.21}$$

Theorem 3.1.3 *Define*

$$\widetilde{X}(z)=\begin{bmatrix} B_0(\bar{z}) & \bar{z}B_1(\bar{z}) \\ zB_0(\bar{z}) & B_1(\bar{z}) \end{bmatrix}, \quad \widetilde{Y}(z) \;=\; \begin{bmatrix} -A_1(\bar{z}) & A_0(\bar{z}) \end{bmatrix}.$$

For any 2×1 Laurent polynomial matrix $\widetilde{V}$ and any Laurent polynomial $\widetilde{W}$, if we take $R(z)$ such that

$$R(z)+R(-z)=1,$$

then $X(z), \widetilde{X}(z), Y(z), \widetilde{Y}(z)$ satisfy the above equations in (3.18) – (3.21).

Proof: The straightforward matrix computation gives

$$X\widetilde{X}^*(z)=\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\bar{z} \\ \frac{1}{2}z & \frac{1}{2} \end{bmatrix},$$

$$X\widetilde{Y}^*(z)=\begin{bmatrix} 0 & 0 \end{bmatrix}^T, \quad Y\widetilde{X}^*(z)=\begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$Y\widetilde{Y}^*(z)=zR(z)B_1(z)A_1(z)+R(z)B_0(z)A_0(z)=R(z).$$

Therefore, the equations (3.18) – (3.21) hold. □

We define two refinable function vectors Φ and $\widetilde{\Phi}$ by

$$\widehat{\Phi}(\xi) = \prod_{j=1}^{\infty} P(\frac{\xi}{2^j})\,\widehat{\Phi}(0), \quad \widehat{\widetilde{\Phi}}(\xi) = \prod_{j=1}^{\infty} \widetilde{P}(\frac{\xi}{2^j})\,\widehat{\widetilde{\Phi}}(0),$$

where $\widehat{\Phi}(0) = (1,1,0)^T, \widehat{\widetilde{\Phi}}(0) = (1,1,0)^T$, which are the right eigenvectors of $P(1), \widetilde{P}(1)$, respectively. We study the regularity of Φ and $\widetilde{\Phi}$ in next section.

3.2 Regularity of Scaling Function Vectors

First we recall the following lemma to ensure the uniform convergence of the above infinite matrix products(See [3] for a proof).

Lemma 3.2.1 *The infinite matrix product $\prod_{j=1}^{\infty} P(\frac{\xi}{2^j})$ converges uniformly on any compact set to a continuous matrix-valued function if and only if $P(0)$ is similar to $\begin{bmatrix} I_s & 0 \\ 0 & J \end{bmatrix}$, where the eigenvalues of J are $\lambda_{s+1}, \cdots, \lambda_r$ with $|\lambda_{s+1}|, \cdots, |\lambda_r| < 1$ for $1 \le s \le r$.*

Theorem 3.2.2 *Let $R(z) = \frac{1}{2}$. And we take $\widetilde{V} = [0,0]^T$, $\widetilde{W} = 0$. Then $P(1)$, $\widetilde{P}(1)$ have a simple eigenvalue 1, with all other eigenvalues less than 1.*

Proof:

From $\frac{1}{2}B_0(z^2) = \dfrac{B(z)+B(-z)}{2}, \frac{1}{2}B_1(z^2) = \dfrac{B(z)-B(-z)}{2z}$, we have

$$B_0(1) = B(1) + B(-1) = \frac{1}{2}, \quad B_1(1) = B(1) - B(-1) = \frac{1}{2}.$$

Similarly, $A_0(z^2) = \dfrac{A(z)+A(-z)}{2}, A_1(z^2) = \dfrac{A(z)-A(-z)}{2z}$ gives

$$A_0(1) = \frac{A(1)+A(-1)}{2} = 1, \quad A_1(1) = \frac{A(1)-A(-1)}{2} = 1.$$

Therefore the matrices

$$P(1) = \begin{bmatrix} \frac{1}{2}\cdot A_0(1) & \frac{1}{2}\cdot A_1(1) & 0 \\ \frac{1}{2}\cdot A_0(1) & \frac{1}{2}\cdot A_1(1) & 0 \\ -\frac{1}{2}\cdot B_1(1) & \frac{1}{2}\cdot B_0(1) & 0 \end{bmatrix}, \quad \widetilde{P}(1) = \begin{bmatrix} B_0(1) & 1\cdot B_1(1) & 0 \\ 1\cdot B_1(1) & B_1(1) & 0 \\ -A_1(1) & A_0(1) & 0 \end{bmatrix}$$

have eigenvalues ≤ 1. □

By Lemma 3.2.1 and Theorem 3.2.2, we see that the two infinite matrix products $\prod_{j=1}^{\infty} P(\frac{\xi}{2^j})$ and $\prod_{j=1}^{\infty} \widetilde{P}(\frac{\xi}{2^j})$ converge uniformly.

Let $\widetilde{A}^*(z) = B(z)$, i.e.,

$$\widetilde{A}^*(z) = \frac{B_0(z^2)}{2} + z\frac{B_1(z^2)}{2} = \frac{1}{2}\left(\frac{1+z}{2}\right)^{\widetilde{m}-m} H_{\widetilde{m}}(z),$$

then it satisfies

$$A(z)\widetilde{A}^*(z) + A(-z)\widetilde{A}^*(-z) = 1.$$

And we construct $\widetilde{N}$ by

$$\widehat{\widetilde{N}}(\xi) = \prod_{j=1}^{\infty} \widetilde{A}(\frac{\xi}{2^j}), \tag{3.22}$$

then we have the following lemma.

Lemma 3.2.3 *Let $\widetilde{m}$ be large enough. Then $\widetilde{N}(x)$ is a well defined compactly supported L^2 function. Furthermore, for any $\alpha \geq 0$, $\widetilde{N} \in C^\alpha(\mathbb{R})$ if $\widetilde{m}$ is sufficiently large, that is,*

$$\widetilde{m} > \frac{\alpha + m + 1 + \frac{1}{2}\log_2 3}{1 + \frac{1}{2}\log_2 3}.$$

Proof: Since $\left|\frac{1+z}{2}\right| = \left|\cos\frac{\xi}{2}\right|$ and $\left|\frac{1-z}{2}\right| = \left|\sin\frac{\xi}{2}\right|$, we have

$$\begin{aligned} |H_{\widetilde{m}}(z)| &= \left|\sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(\frac{1+z}{2}\right)^{\widetilde{m}-1-k} \left(\frac{1-z}{2}\right)^k\right| \\ &\leq \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left|\cos\frac{\xi}{2}\right|^{\widetilde{m}-1-k} \left|\sin\frac{\xi}{2}\right|^k \end{aligned}$$

$$\leq C \left\{ \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(\cos^2 \frac{\xi}{2} \right)^{\widetilde{m}-1-k} \left(\sin^2 \frac{\xi}{2} \right)^{k} \right\}^{\frac{1}{2}}$$

$$= C \left\{ \mathbf{P}_{\widetilde{m}} \left(\sin^2 \frac{\xi}{2} \right) \right\}^{\frac{1}{2}}, \tag{3.23}$$

where $\mathbf{P}_{\widetilde{m}}$ is the polynomial

$$\mathbf{P}_{\widetilde{m}}(y) = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} (1-y)^{\widetilde{m}-1-k} y^k. \tag{3.24}$$

Similarly we have

$$\left| H_{\widetilde{m}}(e^{-i2\xi}) \right| \leq C \left\{ \mathbf{P}_{\widetilde{m}} \left(\sin^2 \xi \right) \right\}^{\frac{1}{2}}$$

$$= C \left\{ \mathbf{P}_{\widetilde{m}} \left(4 \sin^2 \frac{\xi}{2} \left(1 - \sin^2 \frac{\xi}{2} \right) \right) \right\}^{\frac{1}{2}}$$

By applying Lemmas 7.1.1 $\sim$ 7.1.8 in [12], we have

$$\left| \prod_{j=1}^{\infty} \widetilde{A}(\frac{\xi}{2^j}) \right| \leq C(1+|\xi|)^{-\widetilde{m}+m+\frac{1}{2}\log_2 \mathbf{P}_{\widetilde{m}}(\frac{3}{4})}$$

$$\leq C(1+|\xi|)^{-\widetilde{m}+m+\frac{1}{2}(\widetilde{m}-1)\log_2 3}.$$

For the last step in the above we have used that $\mathbf{P}_{\widetilde{m}}(\frac{3}{4}) \leq 3^{\widetilde{m}-1}$. Indeed,

$$\mathbf{P}_{\widetilde{m}} \left(\frac{3}{4} \right) = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(1 - \frac{3}{4} \right)^{\widetilde{m}-1-k} \left(\frac{3}{4} \right)^{k}$$

$$= \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \frac{3^k}{4^{\widetilde{m}-1}}$$

$$= \frac{1}{4^{\widetilde{m}-1}} \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} 3^k$$

$$\leq \left(\frac{3}{4} \right)^{\widetilde{m}-1} \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k}$$

$$= \left(\frac{3}{4} \right)^{\widetilde{m}-1} \frac{1}{2} (1+1)^{2\widetilde{m}-1} = 3^{\widetilde{m}-1}$$

from the easy computations :

$$\sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{2\widetilde{m}-1-k} = \sum_{k=\widetilde{m}}^{2\widetilde{m}-1} \binom{2\widetilde{m}-1}{k},$$

$$\sum_{k=0}^{2\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} = (1+1)^{2\widetilde{m}-1}.$$

Therefore, if we choose $\widetilde{m}$ so that $-\widetilde{m} + m + \frac{1}{2}(\widetilde{m}-1)\log_2 3 < -\frac{1}{2}$, then $\widetilde{N} \in L^2(\mathbb{R})$. By choosing $\widetilde{m}$ even larger, i.e., for $\alpha \geq 0$, $-\widetilde{m} + m + \frac{1}{2}(\widetilde{m}-1)\log_2 3 < -\alpha - 1$, we can make $\widetilde{N} \in C^\alpha(\mathbb{R})$. Finally, the lemma below borrowed from Deslauriers and Dubuc([12], p.176) proves that $\widetilde{N}$ is compactly supported. □

Lemma 3.2.4 *If* $\Gamma(\xi) = \sum_{n=N_1}^{N_2} \gamma_n e^{-in\xi}$, *with* $\sum_{n=N_1}^{N_2} \gamma_n = 1$, *then* $\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)$ *is an entire function of exponential type. In particular, it is the Fourier transform of a distribution with support in* $[N_1, N_2]$.

Define

$$\widetilde{\phi}_1(x) = 4\,\widetilde{N}(2x), \quad \widetilde{\phi}_2(x) = 4\,\widetilde{N}(2x-1),$$

or in the Fourier transform,

$$\widehat{\widetilde{\phi}}_1(\xi) = 2\,\widehat{\widetilde{N}}\left(\frac{\xi}{2}\right), \quad \widehat{\widetilde{\phi}}_2(\xi) = 2\,e^{-i\frac{\xi}{2}}\,\widehat{\widetilde{N}}\left(\frac{\xi}{2}\right).$$

Since

$$\widehat{\widetilde{N}}(2\xi) = 2\,\widetilde{A}(z)\,\widehat{\widetilde{N}}(\xi) = \left(B_0(\bar{z}^2) + \bar{z}\,B_1(\bar{z}^2)\right)\,\widehat{\widetilde{N}}(\xi),$$

we have

$$\widehat{\widetilde{N}}(\xi) = \frac{1}{2}\,B_0(\bar{z})\,\widehat{\widetilde{\phi}}_1(\xi) + \frac{1}{2}\,\bar{z}\,B_1(\bar{z})\,\widehat{\widetilde{\phi}}_2(\xi), \tag{3.25}$$

and so

$$\widehat{\widetilde{\phi}}_1(2\xi) = B_0(\bar{z})\,\widehat{\widetilde{\phi}}_1(\xi) + \bar{z}\,B_1(\bar{z})\,\widehat{\widetilde{\phi}}_2(\xi) \tag{3.26}$$

$$\widehat{\widetilde{\phi}}_2(2\xi) = zB_0(\bar{z})\,\widehat{\widetilde{\phi}}_1(\xi) + B_1(\bar{z})\,\widehat{\widetilde{\phi}}_2(\xi) \tag{3.27}$$

And we define $\widetilde{M}(x)$ and $\widetilde{\phi}_3(x)$ in the Fourier transform,

$$\widehat{\widetilde{M}}(\xi) = -\frac{1}{2}\, A_1(\bar{z})\, \widehat{\widetilde{\phi}}_1(\xi) + \frac{1}{2}\, A_0(\bar{z})\, \widehat{\widetilde{\phi}}_2(\xi), \tag{3.28}$$

$$\widehat{\widetilde{\phi}}_3(2\xi) = 2\, \widehat{\widetilde{M}}(\xi) = -A_1(\bar{z})\widehat{\widetilde{\phi}}_1(\xi) + A_0(\bar{z})\widehat{\widetilde{\phi}}_2(\xi). \tag{3.29}$$

Then we have the following

Theorem 3.2.5 *If $\widetilde{m}$ is sufficiently large, then two scaling function vectors $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$ and $\{\widetilde{\phi}_1(x), \widetilde{\phi}_2(x), \widetilde{\phi}_3(x)\}$ are compactly supported L^2 functions with high regularities.*

Proof: From the definitions, we see that ϕ_1, ϕ_2 are just dilations of standard B-spline function $N(x)$. And $\widetilde{\phi}_1, \widetilde{\phi}_2$ are dilations of $\widetilde{N}(x)$ which is an L^2 function by Lemma 3.2.3. We consider ϕ_3 and $\widetilde{\phi}_3$. Let us recall

$$\widehat{\phi}_3(\xi) = -\frac{1}{2}e^{-i\frac{\xi}{2}}B_1(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2}) + \frac{1}{2}B_0(\frac{\xi}{2})\widehat{\phi}_2(\frac{\xi}{2}),$$

$$\widehat{\widetilde{\phi}}_3(\xi) = -A_1(-\frac{\xi}{2})\widehat{\widetilde{\phi}}_1(\frac{\xi}{2}) + A_0(-\frac{\xi}{2})\widehat{\widetilde{\phi}}_2(\frac{\xi}{2}).$$

Then we get

$$\left|\widehat{\phi}_3(\xi)\right| \le \frac{1}{2}\left|B_1(\frac{\xi}{2})\widehat{\phi}_1(\frac{\xi}{2})\right| + \frac{1}{2}\left|B_0(\frac{\xi}{2})\widehat{\phi}_2(\frac{\xi}{2})\right|.$$

$$\left|\widehat{\widetilde{\phi}}_3(\xi)\right| \le \left|A_1(-\frac{\xi}{2})\widehat{\widetilde{\phi}}_1(\frac{\xi}{2})\right| + \left|A_0(-\frac{\xi}{2})\widehat{\widetilde{\phi}}_2(\frac{\xi}{2})\right|.$$

By applying (3.8), (3.9) and (3.26), (3.27),

$$\widehat{\phi}_2(\xi) = e^{-i\frac{\xi}{2}}\widehat{\phi}_1(\xi), \quad \widehat{\widetilde{\phi}}_2(\xi) = e^{-i\frac{\xi}{2}}\widehat{\widetilde{\phi}}_1(\xi).$$

Since $B(z)$ is bounded, $B_1(\xi)$ and $B_0(\xi)$ are bounded, that is

$$|B_0(\xi)|, |B_1(\xi)| \le K_B, \quad \forall\, \xi \in [0, 2\pi],$$

where $K_B \ge 0$ is a constant. It follows that

$$\left|\widehat{\phi}_3(\xi)\right| \le \frac{1}{2}K_B\left|\widehat{\phi}_1(\frac{\xi}{2})\right| + \frac{1}{2}K_B\left|\widehat{\phi}_1(\frac{\xi}{2})\right| \le K_B\left|\widehat{\phi}_1(\frac{\xi}{2})\right|.$$

Similarly we can find a constant $K_A \ge 0$ such that

$$|A_0(\xi)|, |A_1(\xi)| \le K_A, \quad \forall\, \xi \in [0, 2\pi],$$

since $A(z)$ is bounded. So we have

$$\left|\widehat{\widetilde{\phi}}_3(\xi)\right| \leq K_A \left|\widehat{\widetilde{\phi}}_1(\frac{\xi}{2})\right| + K_A \left|\widehat{\widetilde{\phi}}_2(\frac{\xi}{2})\right| \leq 2K_A \left|\widehat{\widetilde{\phi}}_1(\frac{\xi}{2})\right|.$$

Therefore, the proof is done. □

3.3 Riesz Basis Property

In this section we show that two refinable function vectors $\Phi = (\phi_1, \phi_2, \phi_3)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ are biorthogonal dual to each other. Since N is a standard B-spline function, it is already known that $\{V_k : k \in \mathbb{Z}\}$ constitutes a multiresolution analysis. Let $\widetilde{V}_0$ be the space generated by the integer translates of $\widetilde{N}$, and for $k \in \mathbb{Z}$, let $\widetilde{V}_k = \{f(2^j x) : f \in \widetilde{V}_0\}$. Then it remains to show that $\{\widetilde{V}_k : k \in \mathbb{Z}\}$ constitutes another multiresolution approximation.

First we consider the Riesz Basis property of $\widetilde{N}$. According to Daubechies([12]), the Riesz basis conditon is equivalent to

$$0 < C_1 < \sum_{l\in\mathbb{Z}} \left|\widehat{\widetilde{N}}(\xi + 2\pi l)\right|^2 \leq C_2 < \infty. \tag{3.30}$$

To prove (3.30), we need the following lemma.

Lemma 3.3.1 *For any sufficiently large* $\widetilde{m}$,

$$\sum_{l\in\mathbb{Z}} \left|\widehat{N}(\xi + 2\pi l)\widehat{\widetilde{N}}(\xi + 2\pi l)\right| \geq C > 0. \tag{3.31}$$

Proof: We first recall that

$$\begin{aligned}
|A(\xi)\widetilde{A}(\xi)| &= \left|2\left(\frac{1+e^{-i\xi}}{2}\right)^m \cdot \frac{1}{2}\left(\frac{1+e^{-i\xi}}{2}\right)^{\widetilde{m}-m} H_{\widetilde{m}}(e^{-i\xi})\right| \\
&= \left|\frac{1+e^{-i\xi}}{2}\right|^{\widetilde{m}} \left|H_{\widetilde{m}}(e^{-i\xi})\right| \\
&= \left|\cos\frac{\xi}{2}\right|^{\widetilde{m}} \left|H_{\widetilde{m}}(e^{-i\xi})\right|.
\end{aligned}$$

For the last equality above we have used a simple computation,

$$\left|\frac{1+e^{-i\xi}}{2}\right| = \left|\frac{e^{-i\frac{\xi}{2}}(e^{i\frac{\xi}{2}} + e^{-i\frac{\xi}{2}})}{2}\right| = \left|\frac{e^{-i\frac{\xi}{2}}\, 2\cos\frac{\xi}{2}}{2}\right| = \left|\cos\frac{\xi}{2}\right|.$$

Since the sum of the inequality (3.31) is $2\pi-$periodic, we show that (3.31) holds for $\xi \in [-\pi, \pi]$. Note that from (3.11), we have

$$H_{\widetilde{m}}(e^{-i\xi}) = e^{i\frac{\xi}{2}\widetilde{m}} \sum_{k=0}^{\frac{\widetilde{m}}{2}-1} \binom{\widetilde{m}-1}{k} \left(\cos^2\frac{\xi}{2}\right)^{\frac{\widetilde{m}}{2}-1-k} \left(\sin^2\frac{\xi}{2}\right)^{k}. \tag{3.32}$$

Indeed,

$$\begin{aligned}
1 &= \left(\frac{1+z}{2}\right)^{\widetilde{m}} H_{\widetilde{m}}(z) + \left(\frac{1-z}{2}\right)^{\widetilde{m}} H_{\widetilde{m}}(-z) \\
&= e^{-i\frac{\xi}{2}\widetilde{m}} \left(\cos\frac{\xi}{2}\right)^{\widetilde{m}} \cdot e^{i\frac{\xi}{2}\widetilde{m}} \sum_{k=0}^{\frac{\widetilde{m}}{2}-1} \binom{\widetilde{m}-1}{k} \left(\cos^2\frac{\xi}{2}\right)^{\frac{\widetilde{m}}{2}-1-k} \left(\sin^2\frac{\xi}{2}\right)^{k} \\
&+ e^{-i\frac{\xi}{2}\widetilde{m}} \left(i\sin\frac{\xi}{2}\right)^{\widetilde{m}} \cdot e^{i\frac{\xi}{2}\widetilde{m}}(i)^{\widetilde{m}} \sum_{k=0}^{\frac{\widetilde{m}}{2}-1} \binom{\widetilde{m}-1}{k} \left(\sin^2\frac{\xi}{2}\right)^{\frac{\widetilde{m}}{2}-1-k} \left(\cos^2\frac{\xi}{2}\right)^{k}
\end{aligned}$$

In the above we have assumed that $\widetilde{m}$ is an even positive integer. Then, the well known Bezout's Theorem([12]) implies the uniqueness of such polynomial, i.e., $H_{\widetilde{m}}(e^{-i\xi})$ is equivalent to (3.32). So we have

$$\left|H_{\widetilde{m}}(e^{-i\xi})\right| = \left|\sum_{k=0}^{\frac{\widetilde{m}}{2}-1} \binom{\widetilde{m}-1}{k} \left(\cos^2\frac{\xi}{2}\right)^{\frac{\widetilde{m}}{2}-1-k} \left(\sin^2\frac{\xi}{2}\right)^{k}\right|.$$

Since $\cos^2\frac{\xi}{2} \geq 0$, $\sin^2\frac{\xi}{2} \geq 0$, we have

$$\left|H_{\widetilde{m}}(e^{-i\xi})\right| \geq \left|\cos\frac{\xi}{2}\right|^{\widetilde{m}-2},$$

and hence

$$\prod_{j=1}^{\infty} \left|H_{\widetilde{m}}(e^{-i\frac{\xi}{2^j}})\right| \geq \prod_{j=1}^{\infty} \left|\cos\frac{\xi}{2^{j+1}}\right|^{\widetilde{m}-2} = \left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^{\widetilde{m}-2}.$$

Here we note that $\frac{\sin x}{x} \geq \frac{2}{\pi}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore we have that for $\xi \in [-\pi, \pi]$,

$$\begin{aligned}
|\widehat{N}(\xi)\widehat{\widetilde{N}}(\xi)| &= \prod_{j=1}^{\infty} \left|\cos\frac{\xi}{2^{j+1}}\right|^{\widetilde{m}} \cdot \prod_{j=1}^{\infty} \left|H_{\widetilde{m}}(e^{-i\frac{\xi}{2^j}})\right| \\
&\geq \left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^{\widetilde{m}} \cdot \left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^{\widetilde{m}-2} \\
&\geq \left(\frac{2}{\pi}\right)^{2\widetilde{m}-2} > 0.
\end{aligned}$$

This completes the proof of Lemma 3.3.1. □

By the same argument in the proof of Lemma 3.3.1, we also have that

$$\sum_{l\in\mathbb{Z}} \left|\hat{\widetilde{N}}(\xi+2\pi l)\right|^2 \geq C_1 > 0. \tag{3.33}$$

And from Lemma 3.2.3, we have

$$\sum_{l\in\mathbb{Z}} \left|\hat{\widetilde{N}}(\xi+2\pi l)\right|^2 \leq C_2. \tag{3.34}$$

Let $\widetilde{V}_0$ be the space generated by $\widetilde{N}(x)$, i.e., $\widetilde{V}_0 = \text{span}\{\widetilde{N}(x-j) : j \in \mathbb{Z}\}$, then by (3.33), (3.34), the integer translates of $\widetilde{N}$ constitute a Riesz basis of $\widetilde{V}_0$. And for $k \in \mathbb{Z}$, letting

$$\widetilde{V}_k = \{f(2^j x) : f \in \widetilde{V}_0\},$$

we can show that $\bigcup_{k\in\mathbb{Z}} \widetilde{V}_k$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{k\in\mathbb{Z}} \widetilde{V}_k = \{0\}$. Thus, $\widetilde{N}$ generates a multiresolution analysis of $L^2(\mathbb{R})$.

On the other hand, the matrix

$$\widetilde{P}_1(z) := \begin{bmatrix} \frac{1}{2}B_0(\bar{z}) & \frac{1}{2}\bar{z}B_1(\bar{z}) \\ -\frac{1}{2}A_1(\bar{z}) & \frac{1}{2}A_0(\bar{z}) \end{bmatrix} \tag{3.35}$$

has a constant determinant. And since $(\widetilde{\phi}_1, \widetilde{\phi}_2)^T$ generates $\widetilde{V}_1$, $(\widetilde{N}, \widetilde{M})^T$ generates $\widetilde{V}_1$. And it is easy to check that matrices $P_1(z)$ and $\widetilde{P}_1(z)$ satisfy

$$P_1(z)\widetilde{P}_1^*(z) + P_1(-z)\widetilde{P}_1^*(-z) = 1.$$

If we put

$$\widetilde{f}_1(x) = 4\widetilde{M}(2x), \quad \widetilde{f}_2(x) = 4\widetilde{M}(2x-1),$$

then we can see that $(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{f}_1, \widetilde{f}_2)^T$ generates $\widetilde{V}_2$ which is dual to V_2. Putting everything together, we have the following theorm.

Theorem 3.3.2 *Let $N(x)$ be a B-spline function of order m with its mask*

$$A(z) = A_0(z^2) + zA_1(z^2) = 2\left(\frac{1+z}{2}\right)^m.$$

and let

$$\widetilde{A}^*(z) = \frac{1}{2}\left(B_0(z^2) + zB_1(z^2)\right) = \frac{1}{2}\left(\frac{1+z}{2}\right)^{\widetilde{m}-m} H_{\widetilde{m}}(z), \quad \widetilde{m} > m,$$

be a Laurent polynomial, where

$$H_{\widetilde{m}}(z) = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(\frac{1+z}{2}\right)^{\widetilde{m}-1-k} \left(\frac{1-z}{2}\right)^{k},$$

Define a function $\widetilde{N}(x)$ *in its Fourier transform*

$$\widehat{\widetilde{N}}(\xi) = \prod_{j=1}^{\infty} \widetilde{A}(\frac{\xi}{2^j}),$$

and define two function vectors $\Phi = (\phi_1, \phi_2, \phi_3)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ *by*

$$\widehat{\phi}_1(2\xi) = \frac{1}{2}\widehat{N}(\xi), \quad \widehat{\phi}_2(2\xi) = \tfrac{1}{2}\, e^{-i\xi}\widehat{N}(\xi), \quad \widehat{\phi}_3(2\xi) = \frac{1}{2}\widehat{M}(\xi),$$

$$\widehat{\widetilde{\phi}}_1(2\xi) = 2\widehat{\widetilde{N}}(\xi), \quad \widehat{\widetilde{\phi}}_2(2\xi) = 2e^{-i\xi}\widehat{\widetilde{N}}(\xi), \quad \widehat{\widetilde{\phi}}_3(2\xi) = 2\widehat{\widetilde{M}}(\xi),$$

where

$$\widehat{M}(\xi) = -zB_1(z)\widehat{\phi}_1(\xi) + B_0(z)\widehat{\phi}_2(\xi),$$

$$\widehat{\widetilde{M}}(\xi) = -\frac{1}{2}A_1(\bar{z})\widehat{\widetilde{\phi}}_1(\xi) + \frac{1}{2}A_0(\bar{z})\widehat{\widetilde{\phi}}_2(\xi).$$

Then for sufficiently large $\widetilde{m}$, $\widetilde{N}(x)$ *generates a multiresolution analysis, i.e.,* $\{\widetilde{V}_j : j \in \mathbb{Z}\}$ *constitutes a multiresolution approximation. Moreover, two scaling function vectors* $\Phi, \widetilde{\Phi}$ *generate* $\mathcal{S}, \widetilde{\mathcal{S}}$ *respectivly, such that*

$$V_1 \subset \mathcal{S} \subset V_2 \text{ and } \widetilde{V}_1 \subset \widetilde{\mathcal{S}} \subset \widetilde{V}_2,$$

where $V_j, \widetilde{V}_j$ *are defined as*

$$V_0 = span\{N(x-k) : k \in \mathbb{Z}\}, \quad V_j = \{f(2^j x) : f \in V_0\} \text{ for } j \in \mathbb{Z}$$

$$\widetilde{V}_0 = span\{\widetilde{N}(x-k) : k \in \mathbb{Z}\}, \quad \widetilde{V}_j = \{f(2^j x) : f \in \widetilde{V}_0\} \text{ for } j \in \mathbb{Z}.$$

3.4 Matrix Extension for Wavelet Function Vectors

We are now ready to construct two wavelet function vectors $\Psi = (\psi_1, \psi_2, \psi_3)^T$, $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)^T$ associated with the dual scaling function vectors $\Phi = (\phi_1, \phi_2, \phi_3)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ that we have constructed in previous sections. To define wavelet functions $\Psi, \widetilde{\Psi}$ we mainly need to find the matrix symbols $Q(z), \widetilde{Q}(z)$, associated with the matrix symbols $P(z), \widetilde{P}(z)$. That is, we find matrix blocks $Q(z), \widetilde{Q}(z)$ such that

$$\begin{bmatrix} P(z) & P(-z) \\ Q(z) & Q(-z) \end{bmatrix} \begin{bmatrix} \widetilde{P}^*(z) & \widetilde{Q}^*(z) \\ \widetilde{P}^*(-z) & \widetilde{Q}^*(-z) \end{bmatrix} = I_{6\times 6}, \tag{3.36}$$

which is equivalent to the biorthogonality conditions: (3.1)∼(3.4).

We rewrite $[P(z),\ P(-z)]$ and $[\widetilde{P}(z),\ \widetilde{P}(-z)]$ in terms of polyphase form. That is, if we write

$$P(z) = p_0(z^2) + zp_1(z^2), \quad \widetilde{P}(z) = \widetilde{p}_0(z^2) + z\widetilde{p}_1(z^2),$$

$$Q(z) = q_0(z^2) + zq_1(z^2), \quad \widetilde{Q}(z) = \widetilde{q}_0(z^2) + z\widetilde{q}_1(z^2),$$

then (3.36) is equvalent to

$$\begin{bmatrix} \mathcal{P}(z) \\ \mathcal{Q}(z) \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{P}}^*(z) & \widetilde{\mathcal{Q}}^*(z) \end{bmatrix} = \frac{1}{2} I_{6\times 6}, \tag{3.37}$$

where

$$\mathcal{P}(z) = [p_0(z^2),\ p_1(z^2)], \quad \widetilde{\mathcal{P}}(z) = [\widetilde{p}_0(z^2),\ \widetilde{p}_1(z^2)],$$

$$\mathcal{Q}(z) = [q_0(z^2),\ q_1(z^2)], \quad \widetilde{\mathcal{Q}}(z) = [\widetilde{q}_0(z^2),\ \widetilde{q}_1(z^2)].$$

For the case $r \geq 1$, the univariate matrix extension is treated in [33], where r is the number of rows of the matrices $\mathcal{P}(z), \widetilde{\mathcal{P}}(z)$. For $r = 1$, several constructive methods for bivariate matrix extensions associated with box spline biorthogonal wavelets are available in literature. For $r \geq 1$, the matrix extenasion procedure for the bivariate case is given in an unpublished manuscript of Chui, He and Lai([7]).

The restriction of the procedure in [7] to univariate case is quite constructive, so we want to introduce the general method briefly.

Assume that we need to extend $\mathcal{P}(z), \widetilde{\mathcal{P}}(z)$ of size $r \times m$. The construction proceedes by induction on r. Without loss of generality, we rewrite $\mathcal{P}(z) = [p^1(z), \cdots, p^m(z)]$, where $p^j(z)$ is the j^{th} column of $\mathcal{P}(z)$ for $1 \le j \le m$. Similarly, we write $\widetilde{\mathcal{P}}(z) = [\widetilde{p}^1(z), \cdots, \widetilde{p}^m(z)]$.

For $r = 1$, we have

$$p^1(z)\widetilde{p}^1(z)^* + \cdots + p^m(z)\widetilde{p}^m(z)^* = 1, \tag{3.38}$$

which is clear from the biorthogonality condition (3.1). Since $p^1(z), \cdots, p^m(z)$ have no common zeros in $\mathbb{C} \setminus \{0\}$, by using the algorithm in Theorem 2.1 on [35] or Theorem 3.1 on [7], we can find an $m \times m$ invertible matrix $U(z)$ such that

$$\begin{bmatrix} p^1(z) & \cdots & p^m(z) \end{bmatrix} U(z) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times m}. \tag{3.39}$$

Then the above becomes

$$\begin{bmatrix} p^1(z) & \cdots & p^m(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1\times m} (U(z))^{-1}, \tag{3.40}$$

i.e, the first row of $(U(z))^{-1}$ is $[p^1(z), \cdots, p^m(z)]$. Now we look at

$$(U(z))^{-1} \begin{bmatrix} \widetilde{p}^1(z)^* \\ \widetilde{p}^2(z)^* \\ \vdots \\ \widetilde{p}^m(z^*) \end{bmatrix} = \begin{bmatrix} 1 \\ h_1(z) \\ \vdots \\ h_{m-1}(z) \end{bmatrix}, \tag{3.41}$$

where $h_1(z), \cdots, h_{m-1}(z)$ are polynomials. Multiplying both sides of above (3.41) by

$$L(z) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -h_1(z) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -h_{m-1}(z) & 0 & \cdots & 1 \end{bmatrix}, \tag{3.42}$$

we have

$$L(z)\,(U(z))^{-1} \begin{bmatrix} \widetilde{p}^1(z)^* \\ \widetilde{p}^2(z)^* \\ \vdots \\ \widetilde{p}^m(z^*) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.43}$$

Put $\mathcal{M}(z) := L(z)\,(U(z))^{-1}$, then $\mathcal{M}(z)$ is invertible and its first row is still $[p^1(z), \cdots, p^m(z)]$. So we have

$$\begin{bmatrix} \widetilde{p}^1(z)^* \\ \widetilde{p}^2(z)^* \\ \vdots \\ \widetilde{p}^m(z^*) \end{bmatrix} = (\mathcal{M}(z))^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.44}$$

i.e., $[\widetilde{p}^1(z)^*, \cdots, \widetilde{p}^m(z)^*]^T$ is the first column of $(\mathcal{M}(z))^{-1}$. Let $(\widetilde{\mathcal{M}}(z))^* = (\mathcal{M}(z))^{-1}$, then we have the matrix extension :

$$\mathcal{M}(z)\,\widetilde{\mathcal{M}}(z)^* = I_{m\times m}. \tag{3.45}$$

Now we assume that we have obtained a matrix extension for $r = l < m$. Consider the case of $r = l+1$. Suppose that $\mathcal{P}_{l+1}(z)$ and $\widetilde{\mathcal{P}}_{l+1}(z)$ are polynomial matrices of size $(l+1)\times m$ satisfying

$$\mathcal{P}_{l+1}(z)\,\widetilde{\mathcal{P}}^*_{l+1}(z) = I_{(l+1)\times(l+1)}. \tag{3.46}$$

Denote the first l rows of $\mathcal{P}_{l+1}(z)$ by $\mathcal{P}_l(z)$, and similarly the first l rows of $\widetilde{\mathcal{P}}_{l+1}(z)$ by $\widetilde{\mathcal{P}}_l(z)$. Then we have

$$\mathcal{P}_l(z)\,\widetilde{\mathcal{P}}^*_l(z) = I_{l\times l}. \tag{3.47}$$

This is obvious from the equation (3.1). By the induction assumption, we can find two polynomial matrices $\mathcal{Q}_l(z)$ and $\widetilde{\mathcal{Q}}_l(z)$ of size $(m-l)\times m$ such that

$$\begin{bmatrix} \mathcal{P}_l(z) \\ \mathcal{Q}_l(z) \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{P}}^*_l(z) & \widetilde{\mathcal{Q}}^*_l(z) \end{bmatrix} = I_{m\times m}. \tag{3.48}$$

Now we claim that there exist polynomial matrices $\mathcal{Q}_{l+1}(z)$ and $\widetilde{\mathcal{Q}}_{l+1}(z)$ of size $(m-l-1)\times m$ satisfying

$$\begin{bmatrix} \mathcal{P}_{l+1}(z) \\ \mathcal{Q}_{l+1}(z) \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{P}}^*_{l+1}(z) & \widetilde{\mathcal{Q}}^*_{l+1}(z) \end{bmatrix} = I_{m\times m}. \tag{3.49}$$

First we take a look at

$$\mathcal{P}_{l+1}(z)\left[\begin{array}{cc}\widetilde{\mathcal{P}}_l^*(z) & \widetilde{\mathcal{Q}}_l^*(z)\end{array}\right]=\left[\begin{array}{cc}I_{l\times l} & O\\ O & H(z)\end{array}\right]_{(l+1)\times m}, \tag{3.50}$$

where $H(z)=[h_1(z),\cdots,h_{m-l}(z)]$ with polynomial entries. Note that the matrix $\mathcal{P}(z)$ is of full rank and $\det(\left[\widetilde{\mathcal{P}}_l^*(z),\widetilde{\mathcal{Q}}_l^*(z)\right])\neq 0$ for all $z\in\mathbb{C}\setminus\{0\}$. This implies that $(m-l)$ polynomials $h_1(z),\cdots,h_{m-l}(z)$ do not have any common zeros in $\mathbb{C}\setminus\{0\}$. By using the algorithm in Theorem 2.1 on [35], we find an invertible polynomial matrix $V(z)$ of size $(m-l)\times(m-l)$ such that

$$H(z)\,V(z)=\left[\begin{array}{cccc}1 & 0 & \cdots & 0\end{array}\right]_{1\times(m-l)}. \tag{3.51}$$

Multiplying both sides of (3.49) by $\left[\begin{array}{cc}I_{l\times l} & O\\ O & V(z)\end{array}\right]_{m\times m}$ on the right , we have

$$\mathcal{P}_{l+1}(z)\left[\begin{array}{cc}\widetilde{\mathcal{P}}_l^*(z) & \widetilde{\mathcal{Q}}_l^*(z)V(z)\end{array}\right]=\left[\begin{array}{cc}I_{(l+1)\times(l+1)} & O\end{array}\right]_{(l+1)\times m}. \tag{3.52}$$

Let $W(z)=\left[\widetilde{\mathcal{P}}_l^*(z),\ \widetilde{\mathcal{Q}}_l^*(z)V(z)\right]$. We claim that $W(z)$ is invertible, so that $(W(z))^{-1}$ is also a polynomial matrix. Indeed, by the definition of $W(z)$ we have

$$W(z)=\left[\begin{array}{cc}\widetilde{\mathcal{P}}_l^*(z) & \widetilde{\mathcal{Q}}_l^*(z)V(z)\end{array}\right]=\left[\begin{array}{cc}\widetilde{\mathcal{P}}_l^*(z) & \widetilde{\mathcal{Q}}_l^*(z)\end{array}\right]\left[\begin{array}{cc}I_{l\times l} & O\\ O & V(z)\end{array}\right]_{m\times m}.$$

Since $\left[\widetilde{\mathcal{P}}_l^*(z),\ \widetilde{\mathcal{Q}}_l^*(z)\right]$ is an invertible polynomial matrix by the induction assumption(see (3.48)), and also $V(z)$ is an invertible polynomial matrix in our construction(see (3.51)), hence $W(z)$ is invertible in a polynomial ring. Therefore, (3.52) becomes

$$\mathcal{P}_{l+1}(z)=\left[\begin{array}{cc}I_{(l+1)\times(l+1)} & O\end{array}\right]_{(l+1)\times m}\cdot(W(z))^{-1}, \tag{3.53}$$

i.e., $\mathcal{P}_{l+1}(z)$ is the first $l+1$ rows of matrix $(W(z))^{-1}$. So we may put $(W(z))^{-1}=\left[\mathcal{P}_{l+1}(z),\ \mathcal{Q}'_{l+1}(z)\right]^T$. And we note that

$$(W(z))^{-1}\,\widetilde{\mathcal{P}}_l^*(z)=\left[\begin{array}{c}\mathcal{P}_{l+1}(z)\\ \mathcal{Q}'_{l+1}(z)\end{array}\right]\widetilde{\mathcal{P}}_l^*(z)=\left[\begin{array}{c}I_{l\times l}\\ 0\end{array}\right]_{(l+1)\times l}. \tag{3.54}$$

Then, by the assumption (3.46),

$$(W(z))^{-1}\,\widetilde{\mathcal{P}}^*_{l+1}(z) = \begin{bmatrix} \mathcal{P}_{l+1}(z) \\ \mathcal{Q}'_{l+1}(z) \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{P}}^*_l(z) & \widetilde{p}^{l+1}(z)^* \end{bmatrix} = \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 1 \\ 0 & G \end{bmatrix}_{m\times(l+1)} \tag{3.55}$$

where $G(z) = [g_1(z), \cdots, g_{m-l-1}(z)]^T$ with polynomial entries. Let

$$M(z) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -g_1(z) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -g_{m-l-1}(z) & 0 & \cdots & 1 \end{bmatrix}_{(m-l)\times(m-l)} . \tag{3.56}$$

Multiplying both sides of (3.55) by $\begin{bmatrix} I_{l\times l} & 0 \\ 0 & M(z) \end{bmatrix}$, we get

$$\begin{bmatrix} \mathcal{P}_{l+1}(z) \\ M'\mathcal{Q}'_{l+1}(z) \end{bmatrix} \widetilde{\mathcal{P}}^*_{l+1}(z) = \begin{bmatrix} I_{(l+1)\times(l+1)} \\ O \end{bmatrix}_{m\times(l+1)} , \tag{3.57}$$

where $M'(z) := \left[O_{(m-l-1)\times l},\, G(z),\, I_{(m-l-1)\times(m-l-1)}\right]_{(m-l-1)\times m}$. Let $\mathcal{N}(z) := [\mathcal{P}_{l+1}(z),\, \mathcal{Q}_{l+1}(z)]^T$, where $\mathcal{Q}_{l+1}(z) = M'(z)\mathcal{Q}'_{l+1}(z)$. By the same argument as $W(z)$, $\mathcal{N}(z)$ is invertible in a polynomial ring, and so $(\mathcal{N}(z))^{-1}$ is a polynomial matrix. Thus we have

$$\widetilde{\mathcal{P}}^*_{l+1}(z) = (\mathcal{N}(z))^{-1} \cdot \begin{bmatrix} I_{(l+1)\times(l+1)} \\ O \end{bmatrix}_{m\times(l+1)} , \tag{3.58}$$

i.e., $\widetilde{\mathcal{P}}_{l+1}(z)$ is the first $l+1$ rows of $((\mathcal{N}(z))^{-1})^*$. Let $\widetilde{\mathcal{N}}(z)^* = (\mathcal{N}(z))^{-1}$, then we have the matrix extension

$$\mathcal{N}(z)\,\widetilde{\mathcal{N}}(z)^* = I_{m\times m}, \tag{3.59}$$

which is equivalent to (3.49).

Summarizing the discussion above, we finally have the following theorem.

Theorem 3.4.1 *Let* $\Phi = (\phi_1, \phi_2, \phi_3)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ *be two scaling function vectors defined in Theorem 3.3.2. Let*

$$\begin{bmatrix} P(z) & P(-z) \\ Q(z) & Q(-z) \end{bmatrix}, \quad \begin{bmatrix} \widetilde{P}(z) & \widetilde{P}(-z) \\ \widetilde{Q}(z) & \widetilde{Q}(-z) \end{bmatrix}$$

be matrix extensions of $[P(z),\, P(-z)], [\widetilde{P}(z), \widetilde{P}(-z)]$ *by the above construction method. Define two function vectors* $\Psi, \widetilde{\Psi}$ *in terms of the Fourier Transform by*

$$\widehat{\Psi}(2\xi) = Q(z)\,\widehat{\Phi}(\xi), \quad \widehat{\widetilde{\Psi}}(2\xi) = \widetilde{Q}(z)\,\widehat{\widetilde{\Phi}}(\xi).$$

Then $\Psi = (\psi_1, \psi_2, \psi_3)^T$ *and* $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)^T$ *are wavelet function vectors associated with* Φ *and* $\widetilde{\Phi}$ *respectively, satisfying (3.1)-(3.4).*

3.5 Examples

We construct two scaling function vectors Φ and $\widetilde{\Phi}$ by using B-spline functions. Moreover we consider their regularities of dual scaling functions.

Example 3.5.1 *For the Laurent polynomial mask* $A(z) = 2\left(\frac{1+z}{2}\right)$ *associated with the constant B-spline* $N(x) = N_1(x)$ *we have*

$$A_0(z) = 1, \quad A_1(z) = 1,$$

which satisfy $A(z) = A_0(z^2) + zA_1(z^2)$*. Recall that from (3.10) and (3.11),*

$$B(z) := \frac{1}{2}\left(B_0(z^2) + zB_1(z^2)\right) = \frac{1}{2}\left(\frac{1+z}{2}\right)^{\widetilde{m}-m} H_{\widetilde{m}}(z), \quad \widetilde{m} \geq 1$$

where

$$H_{\widetilde{m}}(z) = \sum_{k=0}^{\widetilde{m}-1} \binom{2\widetilde{m}-1}{k} \left(\frac{1+z}{2}\right)^{\widetilde{m}-1-k} \left(\frac{1-z}{2}\right)^k$$

Consider $\widetilde{m} = 3$*, then we two Laurent polynomials*

$$B_0(z) = 1 - \frac{7}{8}\,z + \frac{3}{8}\,z^2, \quad B_1(z) = \frac{7}{8} - \frac{3}{8}\,z$$

such that $A_0(z)B_0(z) + zA_1(z)B_1(z) = 1$. Then we have the 3×3 matrix symbols $P(z)$ and $\widetilde{P}(z)$ of Φ, $\widetilde{\Phi}$, respectively as follows:

$$P(z) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2}\,z & \frac{1}{2}\,z & 0 \\ -\frac{7}{16}\,z + \frac{3}{16}\,z^2 & \frac{1}{2} - \frac{7}{16}\,z + \frac{3}{16}\,z^2 & 0 \end{bmatrix},$$

$$\widetilde{P}(z) = \begin{bmatrix} \frac{3}{8}\,z^{-2} - \frac{7}{8}\,z^{-1} + 1 & -\frac{3}{8}\,z^{-2} + \frac{7}{8}\,z^{-1} & 0 \\ \frac{3}{8}\,z^{-1} - \frac{7}{8} + \,z & -\frac{3}{8}\,z^{-1} + \frac{7}{8} & 0 \\ -1 & -1 & 0 \end{bmatrix},$$

satisfying the first biorthogonality condition (3.1) $P(z)\widetilde{P}^*(z) + P(-z)\widetilde{P}^*(-z) = I_{3\times 3}$. So we obtain refinable functions ϕ_1, ϕ_2, ϕ_3 from the matrix symbol $P(z)$:

$$\begin{aligned}
\widehat{\phi}_1(2\omega) &= \frac{1}{2}\,\widehat{\phi}_1(\omega) + \frac{1}{2}\,\widehat{\phi}_2(\omega), \\
\widehat{\phi}_2(2\omega) &= \frac{1}{2}\,z\,\widehat{\phi}_1(\omega) + \frac{1}{2}\,z\,\widehat{\phi}_2(\omega), \\
\widehat{\phi}_3(2\omega) &= \Big(-\frac{7}{16}\,z + \frac{3}{16}\,z^2\Big)\,\widehat{\phi}_1(\omega) + \Big(\frac{1}{2} - \frac{7}{16}\,z + \frac{3}{16}\,z^2\Big)\,\widehat{\phi}_2(\omega),
\end{aligned}$$

and similarly, the matrix symbol $\widetilde{P}(z)$ yields the dual refinable functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3$ in the Fourier transform :

$$\begin{aligned}
\widehat{\widetilde{\phi}}_1(2\omega) &= \Big(\frac{3}{8}\,z^{-2} - \frac{7}{8}\,z^{-1} + 1\Big)\,\widehat{\widetilde{\phi}}_1(\omega) + \Big(-\frac{3}{8}\,z^{-2} + \frac{7}{8}\,z^{-1}\Big)\,\widehat{\widetilde{\phi}}_2(\omega), \\
\widehat{\widetilde{\phi}}_2(2\omega) &= \Big(\frac{3}{8}\,z^{-1} - \frac{7}{8} + \,z\Big)\,\widehat{\widetilde{\phi}}_1(\omega) + \Big(-\frac{3}{8}\,z^{-1} + \frac{7}{8}\Big)\,\widehat{\widetilde{\phi}}_2(\omega), \\
\widehat{\widetilde{\phi}}_3(2\omega) &= -\,\widehat{\widetilde{\phi}}_1(\omega) - \widehat{\widetilde{\phi}}_2(\omega)
\end{aligned}$$

Since we have defined $\phi_1(x) = N_1(2x)$, $\phi_2(x) = N_1(2x-1)$, the scaling funtion vector $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is obtained as

$$\phi_1(x) = \phi_1(2x) + \phi_2(2x) = N_1(4x) + N_1(4x-1),$$

$$\phi_2(x) = \phi_1(2x-1) + \phi_2(2x-1) = N_1(4x-2) + N_1(4x-3),$$

$$\begin{aligned}\phi_3(x) &= -\frac{7}{8}\,\phi_1(2x-1) + \frac{3}{8}\,\phi_1(2x-2) + \phi_2(2x) - \frac{7}{8}\,\phi_2(2x-1) + \frac{3}{8}\,\phi_2(2x-2) \\ &= N_1(4x-1) - \frac{7}{8}\,N_1(4x-2) - \frac{7}{8}\,N_1(4x-3) + \frac{3}{8}\,N_1(4x-4) + \frac{3}{8}\,N_1(4x-5)\end{aligned}$$

Similarily, the definition of $\widetilde{\phi}_1(x) = 4\widetilde{N}_1(2x)$, $\widetilde{\phi}_2(x) = 4\widetilde{N}_1(2x-1)$ *with a function* $\widetilde{N}_1$ *defined by the equation (3.22) gives the dual scaling funtion vector* $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$, *where*

$$\begin{aligned}\widetilde{\phi}_1(x) &= \frac{3}{4}\,\widetilde{\phi}_1(2x+2) - \frac{7}{4}\,\widetilde{\phi}_1(2x+1) + 2\,\widetilde{\phi}_1(2x) - \frac{3}{4}\,\widetilde{\phi}_2(2x+2) + \frac{7}{4}\,\widetilde{\phi}_2(2x+1) \\ &= 3\,\widetilde{N}_1(4x+4) - 3\,\widetilde{N}_1(4x+3) - 7\,\widetilde{N}_1(4x+2) + 7\,\widetilde{N}_1(4x+1) + 8\,\widetilde{N}_1(4x) \\ \widetilde{\phi}_2(x) &= \frac{3}{4}\,\widetilde{\phi}_1(2x+1) - \frac{7}{4}\,\widetilde{\phi}_1(2x) + 2\,\widetilde{\phi}_1(2x-1) - \frac{3}{4}\,\widetilde{\phi}_2(2x+1) + \frac{7}{4}\widetilde{\phi}_2(2x) \\ &= 3\,\widetilde{N}_1(4x+2) - 3\,\widetilde{N}_1(4x+1) - 7\,\widetilde{N}_1(4x) + 7\,\widetilde{N}_1(4x-1) + 8\,\widetilde{N}_1(4x-2) \\ \widetilde{\phi}_3(x) &= -2\widetilde{\phi}_1(2x) - 2\widetilde{\phi}_2(2x) = -8\widetilde{N}_1(4x) - 8\widetilde{N}_2(4x-1)\end{aligned}$$

By using the matrix extension algorithm we have discussed in section 4, the 3×3 *matrix symbols* $Q(z)$ *and* $\widetilde{Q}(z)$ *of* Ψ, $\widetilde{\Psi}$ *are obtained:*

$$Q(z) = \begin{bmatrix} \frac{7}{8}\,z^2 - \frac{3}{8}\,z^3 & -z + \frac{7}{8}\,z^2 - \frac{9}{16}\,z^3 & \\ \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5 & \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5 & -\frac{3}{2}\,z - \frac{1}{16}\,z^3 + \frac{3}{128}\,z^5 \\ 0 & 0 & \frac{1}{2} \\ \frac{7}{16}\,z^2 - \frac{3}{16}\,z^3 & -\frac{1}{2}\,z + \frac{7}{16}\,z^2 - \frac{3}{8}\,z^3 & \\ \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5 & \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5 & -z - \frac{1}{8}\,z^3 + \frac{3}{128}\,z^5 \end{bmatrix},$$

$$\widetilde{Q}(z) = \begin{bmatrix} -\frac{3}{64}\,z^{-3} + \frac{1}{4}\,z^{-1} + 2\,z & \frac{3}{64}\,z^{-3} - \frac{1}{4}\,z^{-1} - 2\,z & \frac{3}{8}\,z^{-1} + z \\ 0 & 0 & 1 \\ \frac{3}{64}\,z^{-3} - \frac{1}{8}\,z^{-1} - 3\,z & -\frac{3}{64}\,z^{-3} + \frac{1}{8}\,z^{-1} + 3\,z & -\frac{3}{8}\,z^{-1} - 2\,z \end{bmatrix},$$

which satisfy the biorthogonality conditions (3.2)-(3.4). This gives three wavelet functions ψ_1, ψ_2, ψ_3 *in the Fourier transform as follows*

$$\begin{aligned}
\widehat{\psi}_1(2\omega) &= \Big(\frac{7}{8}\,z^2 - \frac{3}{8}\,z^3 + \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5\Big)\,\widehat{\phi}_1(\omega) \\
&\quad + \Big(-z + \frac{7}{8}\,z^2 - \frac{9}{16}\,z^3 + \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5\Big)\,\widehat{\phi}_2(\omega) \\
&\quad + \Big(-\frac{3}{2}\,z - \frac{1}{16}\,z^3 + \frac{3}{128}\,z^5\Big)\,\widehat{\phi}_3(\omega) \\
\widehat{\psi}_2(2\omega) &= \frac{1}{2}\,\widehat{\phi}_3(\omega), \\
\widehat{\psi}_3(2\omega) &= \Big(\frac{7}{16}\,z^2 - \frac{3}{16}\,z^3 + \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5\Big)\,\widehat{\phi}_1(\omega) \\
&\quad + \Big(-\frac{1}{2}\,z + \frac{7}{16}\,z^2 - \frac{3}{8}\,z^3 + \frac{21}{128}\,z^4 - \frac{9}{128}\,z^5\Big)\,\widehat{\phi}_2(\omega) \\
&\quad + \Big(-z - \frac{1}{8}\,z^3 + \frac{3}{128}\,z^5\Big)\,\widehat{\phi}_3(\omega)
\end{aligned}$$

and also dual wavelet functions $\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3$ *are defined in the Fourier transform*

$$\begin{aligned}
\widehat{\widetilde{\psi}}_1(2\omega) &= \Big(-\frac{3}{64}\,z^{-3} + \frac{1}{4}\,z^{-1} + 2\,z\Big)\,\widehat{\widetilde{\phi}}_1(\omega) + \Big(\frac{3}{64}\,z^{-3} - \frac{1}{4}\,z^{-1} - 2\,z\Big)\,\widehat{\widetilde{\phi}}_2(\omega) \\
&\quad + \Big(\frac{3}{8}\,z^{-1} + z\Big)\,\widehat{\widetilde{\phi}}_3(\omega), \\
\widehat{\widetilde{\psi}}_2(2\omega) &= \widehat{\widetilde{\phi}}_3(\omega), \\
\widehat{\widetilde{\psi}}_3(2\omega) &= \Big(\frac{3}{64}\,z^{-3} - \frac{1}{8}\,z^{-1} - 3\,z\Big)\,\widehat{\widetilde{\phi}}_1(\omega) + \Big(-\frac{3}{64}\,z^{-3} + \frac{1}{8}\,z^{-1} + 3\,z\Big)\,\widehat{\widetilde{\phi}}_2(\omega) \\
&\quad + \Big(-\frac{3}{8}\,z^{-1} - 2\,z\Big)\,\widehat{\widetilde{\phi}}_3(\omega).
\end{aligned}$$

Therefore we have a wavelet function vector $\Psi = (\psi_1, \psi_2, \psi_3)^T$ *associated with the scaling function vector* $\Phi = (\phi_1, \phi_2, \phi_3)^T$ *where*

$$\begin{aligned}
\psi_1(x) &= \frac{7}{4}\,\phi_1(2x-2) - \frac{3}{4}\,\phi_1(2x-3) + \frac{21}{64}\,\phi_1(2x-4) - \frac{9}{64}\,\phi_1(2x-5) \\
&\quad - 2\,\phi_2(2x-1) + \frac{7}{4}\,\phi_2(2x-2) - \frac{9}{8}\,\phi_2(2x-3) + \frac{21}{64}\,\phi_2(2x-4) - \frac{9}{64}\,\phi_2(2x-5) \\
&\quad - 3\,\phi_2(2x-1) - \frac{1}{8}\,\phi_3(2x-3) + \frac{3}{64}\,\phi_3(2x-5), \\
\psi_2(x) &= \phi_3(2x),
\end{aligned}$$

$$\begin{aligned}\psi_3(x) = & \frac{7}{8}\,\phi_1(2x-2) - \frac{3}{8}\,\phi_1(2x-3) + \frac{21}{64}\,\phi_1(2x-4) - \frac{9}{64}\,\phi_1(2x-5) \\ & - \phi_2(2x-1) + \frac{7}{8}\,\phi_2(2x-2) - \frac{3}{4}\,\phi_2(2x-3) + \frac{21}{64}\,\phi_2(2x-4) - \frac{9}{64}\,\phi_2(2x-5) \\ & - 2\,\phi_3(2x-1) - \frac{1}{4}\,\phi_3(2x-3) + \frac{3}{64}\,\phi_3(2x-5).\end{aligned}$$

Similarily, we can find the wavelet function vector $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)^T$ *associated with the scaling function vector* $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$*, which is defined by*

$$\begin{aligned}\widetilde{\psi}_1(x) = & -\frac{3}{8}\,\phi_1(2x+3) + \frac{1}{2}\,\phi_1(2x+1) + 4\;\phi_1(2x-1) \\ & + \frac{3}{8}\;\phi_2(2x+3) - \frac{1}{2}\,\phi_2(2x+1) - 4\,\phi_2(2x-1) + \frac{3}{4}\,\phi_3(2x+1) + 2\;\phi_3(2x-1), \\ \widetilde{\psi}_2(x) = & \; 2\,\widetilde{\phi}_3(2x), \\ \widetilde{\psi}_3(x) = & \; \frac{3}{8}\,\phi_1(2x+3) - \frac{1}{4}\,\phi_1(2x+1) - 6\,\phi_1(2x-1) \\ & + -\frac{3}{32}\,\phi_2(2x+3) + \frac{1}{4}\,\phi_2(2x+1) + 6\,\phi_2(2x-1) - \frac{3}{4}\,\widetilde{\phi}_3(2x+1) - 4\,\widetilde{\phi}_3(2x-1).\end{aligned}$$

□

Example 3.5.2 *For the Laurent polynomial mask* $A(z) = 2\left(\frac{1+z}{2}\right)^2$ *associated with the linear B-spline* $N(x) = N_2(x)$ *we have*

$$A_0(z) = \frac{1}{2} + \frac{1}{2}z, \quad A_1(z) = 1,$$

which satisfy $A(z) = A_0(z^2) + zA_1(z^2)$*. Choose* $\widetilde{m} = 6$ *on the equations (3.10) and (3.11), then we two Laurent polynomials*

$$\begin{aligned}B_0(z) = & \, 2 - \frac{309}{64}\,z + \frac{409}{64}\,z^2 - \frac{259}{64}\,z^3 + \frac{63}{64}\,z^4, \\ B_1(z) = & \, \frac{181}{128} - \frac{25}{32}\,z - \frac{75}{64}\,z^2 + \frac{49}{32}\,z^3 - \frac{63}{128}\,z^4,\end{aligned}$$

such that $A_0(z)B_0(z) + zA_1(z)B_1(z) = 1$*. Then we have the* 3×3 *matrix symbols* $P(z)$ *and* $\widetilde{P}(z)$ *of* Φ *and* $\widetilde{\Phi}$*, respectively as follows:*

$$P(z) = \begin{bmatrix} \frac{1}{4} + \frac{1}{4}\,z & \frac{1}{2} & 0 \\ \frac{1}{4}\,z + \frac{1}{4}\,z^2 & \frac{1}{2}\,z & 0 \\ -\frac{181}{256}\,z + \frac{25}{64}\,z^2 + \frac{75}{128}\,z^3 & 1 - \frac{309}{128}\,z + \frac{409}{128}\,z^2 & \\ -\frac{49}{64}\,z^4 + \frac{63}{256}\,z^5 & -\frac{259}{128}\,z^3 + \frac{63}{128}\,z^4 & 0 \end{bmatrix},$$

$$\widetilde{P}(z) = \begin{bmatrix} \frac{63}{64}\,z^{-4} - \frac{259}{64}\,z^{-3} + \frac{409}{64}\,z^{-2} & -\frac{63}{128}\,z^{-5} + \frac{49}{32}\,z^{-4} - \frac{75}{64}\,z^{-3} & \\ -\frac{309}{64}\,z^{-1} + 2 & -\frac{25}{32}\,z^{-2} + \frac{181}{128}\,z^{-1} & 0 \\ \frac{63}{64}\,z^{-3} - \frac{259}{64}\,z^{-2} + \frac{409}{64}\,z^{-1} & -\frac{63}{128}\,z^{-4} + \frac{49}{32}\,z^{-3} - \frac{75}{64}\,z^{-2} & \\ -\frac{309}{64} + 2\,z & -\frac{25}{32}\,z^{-1} + \frac{181}{128} & 0 \\ -1 & \frac{1}{2}\,z^{-1} + \frac{1}{2} & 0 \end{bmatrix},$$

satisfying the first biorthogonality condition (3.1) $P(z)\widetilde{P}^*(z) + P(-z)\widetilde{P}^*(-z) = I_{3\times 3}$. *So we obtain refinable functions* ϕ_1, ϕ_2, ϕ_3 *from the matrix symbol* $P(z)$*:*

$$\begin{aligned}
\widehat{\phi}_1(2\omega) &= \Big(\frac{1}{4} + \frac{1}{4}\,z\Big)\,\widehat{\phi}_1(\omega) + \frac{1}{2}\,\widehat{\phi}_2(\omega), \\
\widehat{\phi}_2(2\omega) &= \Big(\frac{1}{4}\,z + \frac{1}{4}\,z^2\Big)\,\widehat{\phi}_1(\omega) + \frac{1}{2}\,z\,\widehat{\phi}_2(\omega), \\
\widehat{\phi}_3(2\omega) &= \Big(-\frac{181}{256}\,z + \frac{25}{64}\,z^2 + \frac{75}{128}\,z^3 - \frac{49}{64}\,z^4 + \frac{63}{256}\,z^5\Big)\,\widehat{\phi}_1(\omega) \\
&\quad + \Big(1 - \frac{309}{128}\,z + \frac{409}{128}\,z^2 - \frac{259}{128}\,z^3 + \frac{63}{128}\,z^4\Big)\,\widehat{\phi}_2(\omega),
\end{aligned}$$

and the matrix symbol $\widetilde{P}(z)$ *gives dual refinable functions* $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3$ *in the Fourier transform :*

$$\begin{aligned}
\widehat{\widetilde{\phi}}_1(2\omega) &= \Big(\frac{63}{64}\,z^{-4} - \frac{259}{64}\,z^{-3} + \frac{409}{64}\,z^{-2} - \frac{309}{64}\,z^{-1} + 2\Big)\,\widehat{\widetilde{\phi}}_1(\omega) \\
&\quad + \Big(-\frac{63}{128}\,z^{-5} + \frac{49}{32}\,z^{-4} - \frac{75}{64}\,z^{-3} - \frac{25}{32}\,z^{-2} + \frac{181}{128}\,z^{-1}\Big)\,\widehat{\widetilde{\phi}}_2(\omega), \\
\widehat{\widetilde{\phi}}_2(2\omega) &= \Big(\frac{63}{64}\,z^{-3} - \frac{259}{64}\,z^{-2} + \frac{409}{64}\,z^{-1} - \frac{309}{64} + 2\,z\Big)\,\widehat{\widetilde{\phi}}_1(\omega) \\
&\quad + \Big(-\frac{63}{128}\,z^{-4} + \frac{49}{32}\,z^{-3} - \frac{75}{64}\,z^{-2} - \frac{25}{32}\,z^{-1} + \frac{181}{128}\Big)\,\widehat{\widetilde{\phi}}_2(\omega),
\end{aligned}$$

$$\widehat{\widetilde{\phi}}_3(2\omega) = -\,\widehat{\widetilde{\phi}}_1(\omega) + \Big(\frac{1}{2}\,z^{-1} + \frac{1}{2}\Big)\,\widehat{\widetilde{\phi}}_2(\omega)$$

Since we have defined $\phi_1(x) = N_2(2x)$, $\phi_2(x) = N_2(2x-1)$, *we have the scaling funtion vector* $\Phi = (\phi_1, \phi_2, \phi_3)^T$, *where*

$$\begin{aligned}
\phi_1(x) &= \frac{1}{2}\,\phi_1(2x) + \frac{1}{2}\,\phi_1(2x-1) + \phi_2(2x) \\
&= \frac{1}{2}\,N_2(4x) + N_2(4x-1) + \frac{1}{2}\,N_2(4x-2), \\
\phi_2(x) &= \frac{1}{2}\,\phi_1(2x-1) + \frac{1}{2}\,\phi_1(2x-2) + \phi_2(2x-1) \\
&= \frac{1}{2}\,N_2(4x-2) + N_2(4x-3) + \frac{1}{2}\,N_2(4x-4), \\
\phi_3(x) &= -\frac{181}{128}\,\phi_1(2x-1) + \frac{25}{32}\,\phi_1(2x-2) + \frac{75}{64}\,\phi_1(2x-3) - \frac{49}{32}\,\phi_1(2x-4) \\
&\quad + \frac{63}{128}\,\phi_1(2x-5) + \phi_2(2x) - \frac{309}{64}\,\phi_2(2x-1) + \frac{409}{64}\,\phi_2(2x-2) \\
&\quad - \frac{259}{64}\,\phi_2(2x-3) + \frac{63}{64}\,\phi_2(2x-4) \\
&= N_2(4x-1) - \frac{181}{128}\,N_2(4x-2) - \frac{309}{64}\,N_2(4x-3) + \frac{25}{32}\,N_2(4x-4) \\
&\quad + \frac{409}{64}\,N_2(4x-5) + \frac{75}{64}\,N_2(4x-6) - \frac{259}{64}\,N_2(4x-7) - \frac{49}{32}\,N_2(4x-8) \\
&\quad + \frac{63}{64}\,N_2(4x-9) + \frac{63}{128}\,N_2(4x-10).
\end{aligned}$$

Similarily, the definition of $\widetilde{\phi}_1(x) = 4\widetilde{N}_2(2x)$, $\widetilde{\phi}_2(x) = 4\widetilde{N}_2(2x-1)$ *with a function* $\widetilde{N}_2$ *defined by the equation (3.22) gives the dual scaling funtion vector* $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T$ *with the following explicit forms:*

$$\begin{aligned}
\widetilde{\phi}_1(x) &= \frac{63}{32}\,\widetilde{\phi}_1(2x+4) - \frac{259}{32}\,\widetilde{\phi}_1(2x+3) + \frac{409}{32}\,\widetilde{\phi}_1(2x+2) - \frac{309}{32}\,\widetilde{\phi}_1(2x+1) \\
&\quad + 4\,\widetilde{\phi}_1(2x) - \frac{63}{64}\,\widetilde{\phi}_2(2x+5) + \frac{49}{16}\,\widetilde{\phi}_2(2x+4) - \frac{75}{32}\,\widetilde{\phi}_2(2x+3) \\
&\quad - \frac{25}{16}\,\widetilde{\phi}_2(2x+2) + \frac{181}{64}\,\widetilde{\phi}_2(2x+1) \\
&\quad + \frac{181}{16}\,\widetilde{N}_2(4x+1) + 16\,\widetilde{N}_2(4x)
\end{aligned}$$

$$= -\frac{63}{16}\,\widetilde{N}_2(4x+9) + \frac{63}{8}\,\widetilde{N}_2(4x+8) + \frac{49}{4}\,\widetilde{N}_2(4x+7) - \frac{259}{8}\,\widetilde{N}_2(4x+6)$$
$$- \frac{75}{8}\,\widetilde{N}_2(4x+5) + \frac{409}{8}\,\widetilde{N}_2(4x+4) - \frac{25}{4}\,\widetilde{N}_2(4x+3) - \frac{309}{8}\,\widetilde{N}_2(4x+2)$$

$$\widetilde{\phi}_2(x) = \frac{63}{32}\,\widetilde{\phi}_1(2x+3) - \frac{259}{32}\,\widetilde{\phi}_1(2x+2) + \frac{409}{32}\,\widetilde{\phi}_1(2x+1) - \frac{309}{32}\,\widetilde{\phi}_1(2x)$$
$$+ 4\,\widetilde{\phi}_1(2x-1) - \frac{63}{64}\,\widetilde{\phi}_2(2x+4) + \frac{49}{16}\,\widetilde{\phi}_2(2x+3) - \frac{75}{32}\,\widetilde{\phi}_2(2x+2)$$
$$- \frac{25}{16}\,\widetilde{\phi}_2(2x+1) + \frac{181}{64}\,\widetilde{\phi}_2(2x)$$
$$= -\frac{63}{16}\,\widetilde{N}_2(4x+7) + \frac{63}{8}\,\widetilde{N}_2(4x+6) + \frac{49}{4}\,\widetilde{N}_2(4x+5) - \frac{259}{8}\,\widetilde{N}_2(4x+4)$$
$$- \frac{75}{8}\,\widetilde{N}_2(4x+3) + \frac{409}{8}\,\widetilde{N}_2(4x+2) - \frac{25}{4}\,\widetilde{N}_2(4x+1) - \frac{309}{8}\,\widetilde{N}_2(4x)$$
$$+ \frac{181}{16}\,\widetilde{N}_2(4x-1) + 16\,\widetilde{N}_2(4x-2)$$

$$\widetilde{\phi}_3(x) = -2\,\widetilde{\phi}_1(2x) + \widetilde{\phi}_2(2x+1) + \widetilde{\phi}_2(2x) = 4\,\widetilde{N}_2(4x+1) - 8\,\widetilde{N}_2(4x) + 4\,\widetilde{N}_2(4x-1).$$

Matrix symbols $Q(z), \widetilde{Q}(z)$ associated with Ψ, $\widetilde{\Psi}$ respectively are as follows

$$Q(z) = \sum_{k=0}^{15} q_k\, z^k, \quad \widetilde{Q}(z) = \sum_{k=-14}^{1} \widetilde{q}_k\, z^k,$$

where $q_k, \widetilde{q}_k$ are 3×3 real coefficient matrices given by

$$q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \widetilde{q}_{-14} = \begin{bmatrix} 0 & \frac{3969}{65536} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$q_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & -1/4 & -3 \end{bmatrix}, \qquad \widetilde{q}_{-13} = \begin{bmatrix} -\frac{3969}{32768} & \frac{3969}{65536} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{181}{1024} & \frac{309}{512} & 0 \end{bmatrix}, \qquad \widetilde{q}_{-12} = \begin{bmatrix} 0 & \frac{36099}{65536} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{3969}{16384} & 0 \end{bmatrix},$$

$$q_3 = \begin{bmatrix} 0 & \frac{281}{128} & \frac{971}{64} \\ 0 & 0 & 0 \\ -\frac{25}{256} & -\frac{281}{128} & -\frac{6257}{512} \end{bmatrix}, \qquad \widetilde{q}_{-11} = \begin{bmatrix} -\frac{36099}{32768} & \frac{36099}{65536} & 0 \\ 0 & 0 & 0 \\ \frac{3969}{8192} & -\frac{3969}{16384} & 0 \end{bmatrix},$$

$$q_4 = \begin{bmatrix} -\frac{50861}{32768} & -\frac{86829}{16384} & 0 \\ 0 & 0 & 0 \\ \frac{110215}{131072} & \frac{254087}{65536} & 0 \end{bmatrix}, \qquad \widetilde{q}_{-10} = \begin{bmatrix} 0 & -\frac{46661}{32768} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{24003}{8192} & 0 \end{bmatrix},$$

$$q_5 = \begin{bmatrix} \frac{7025}{8192} & \frac{70641}{16384} & \frac{175895}{16384} \\ 0 & 0 & 0 \\ -\frac{11603}{32768} & -\frac{131667}{65536} & -\frac{72389}{65536} \end{bmatrix}, \qquad \widetilde{q}_{-9} = \begin{bmatrix} \frac{46661}{16384} & -\frac{46661}{32768} & 0 \\ 0 & 0 & 0 \\ \frac{24003}{4096} & -\frac{24003}{8192} & 0 \end{bmatrix},$$

$$q_6 = \begin{bmatrix} \frac{13097}{4096} & \frac{34135}{16384} & 0 \\ 0 & 0 & 0 \\ -\frac{354053}{131072} & -\frac{111193}{32768} & 0 \end{bmatrix}, \qquad \widetilde{q}_{-8} = \begin{bmatrix} 0 & -\frac{362079}{32768} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{6337}{2048} & 0 \end{bmatrix},$$

$$q_7 = \begin{bmatrix} -\frac{22419}{8192} & -\frac{131875}{16384} & -\frac{282003}{8192} \\ 0 & 0 & 0 \\ \frac{34005}{16384} & \frac{237165}{32768} & \frac{1822729}{65536} \end{bmatrix}, \qquad \widetilde{q}_{-7} = \begin{bmatrix} \frac{362079}{16384} & -\frac{362079}{32768} & \frac{63}{512} \\ 0 & 0 & 0 \\ \frac{6337}{1024} & -\frac{6337}{2048} & 0 \end{bmatrix},$$

$$q_8 = \begin{bmatrix} -\frac{11397}{16384} & \frac{109081}{16384} & 0 \\ 0 & 0 & 0 \\ \frac{90611}{65536} & -\frac{73277}{16384} & 0 \end{bmatrix}, \qquad \widetilde{q}_{-6} = \begin{bmatrix} 0 & \frac{1822729}{65536} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{282003}{8192} & 0 \end{bmatrix},$$

$$q_9 = \begin{bmatrix} \frac{15379}{8192} & -\frac{47565}{16384} & \frac{6337}{2048} \\ 0 & 0 & 0 \\ -\frac{17139}{8192} & \frac{2705}{16384} & -\frac{362079}{32768} \end{bmatrix}, \quad \widetilde{q}_{-5} = \begin{bmatrix} -\frac{1822729}{32768} & \frac{1822729}{65536} & \frac{157}{512} \\ 0 & 0 & 0 \\ -\frac{282003}{4096} & \frac{282003}{8192} & -\frac{63}{128} \end{bmatrix},$$

$$q_{10} = \begin{bmatrix} -\frac{1953}{2048} & \frac{16317}{16384} & 0 \\ 0 & 0 & 0 \\ \frac{35475}{65536} & \frac{30065}{32768} & 0 \end{bmatrix}, \quad \widetilde{q}_{-4} = \begin{bmatrix} 0 & -\frac{72389}{65536} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{175895}{16384} & 0 \end{bmatrix},$$

$$q_{11} = \begin{bmatrix} \frac{3087}{8192} & -\frac{3969}{16384} & \frac{24003}{8192} \\ 0 & 0 & 0 \\ \frac{3059}{16384} & -\frac{17829}{32768} & -\frac{46661}{32768} \end{bmatrix}, \quad \widetilde{q}_{-3} = \begin{bmatrix} \frac{72389}{32768} & -\frac{72389}{65536} & -\frac{1319}{512} \\ 0 & 0 & 0 \\ \frac{175895}{8192} & -\frac{175895}{16384} & -\frac{173}{64} \end{bmatrix},$$

$$q_{12} = \begin{bmatrix} -\frac{3969}{32768} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{19341}{131072} & \frac{16317}{65536} & 0 \end{bmatrix}, \quad \widetilde{q}_{-2} = \begin{bmatrix} 0 & -\frac{6257}{512} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{971}{64} & 0 \end{bmatrix},$$

$$q_{13} = \begin{bmatrix} 0 & 0 & \frac{3969}{16384} \\ 0 & 0 & 0 \\ \frac{3087}{32768} & -\frac{3969}{65536} & \frac{36099}{65536} \end{bmatrix}, \quad \widetilde{q}_{-1} = \begin{bmatrix} \frac{6257}{256} & -\frac{6257}{512} & \frac{715}{512} \\ 0 & 0 & 0 \\ \frac{971}{32} & -\frac{971}{64} & \frac{281}{128} \end{bmatrix},$$

$$q_{14} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3969}{131072} & 0 & 0 \end{bmatrix}, \quad \widetilde{q}_{0} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix},$$

$$q_{15} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3969}{131072} & 0 & 0 \end{bmatrix}, \quad \widetilde{q}_{1} = \begin{bmatrix} 6 & -3 & 1/4 \\ 0 & 0 & 0 \\ 4 & -2 & 0 \end{bmatrix}.$$

And also it is easy to check that $Q(z), \widetilde{Q}(z)$ satisfy the equations (3.2)-(3.4) by using any computation software. □

We now recall the result about the regularity of Daubechies's scaling funtions ${}_N\phi$ from ([40]). That is, for Daubechies's scaling fucntion ${}_N\phi$, $3 \leq N \leq 9$, values for the largest exponent $\alpha(N)$ such that

$$\int_{-\infty}^{\infty} (1+|\omega|)^{\alpha(N)} \left|{}_N\widehat{\phi}(\omega)\right| d\omega < \infty,$$

are listed in the following:

N	3	4	5	6	7	8	9
$\alpha(N)$	1.0831	1.6066	1.9424	2.1637	2.4348	2.7358	3.0432

Table 3.1 Regularity of ${}_N\phi$

Consider $N(x) = N_1(x)$, the constant B-spline function. If we choose $\widetilde{m} = 3$ as in Example 3.5.1, from the definition of $\widetilde{A}^*(z) = B(z)$ we have

$$\widetilde{A}^*(z) = \frac{1}{2}\left(\frac{1+z}{2}\right)^2 H_3(z).$$

Then the definition of $\widetilde{N}$ and proof of Lemma 3.2.3 gives

$$\begin{aligned}
\int_{\mathbb{R}} \left|\widehat{\widetilde{N}}(\xi)\right| d\xi &= \int_{\mathbb{R}} \prod_{j=1}^{\infty} \left|\widetilde{A}(e^{-i\frac{\xi}{2^j}})\right| d\xi \\
&= \int_{\mathbb{R}} \frac{1}{2} \prod_{j=1}^{\infty} \left|\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\right|^2 \left|H_3(e^{-i\frac{\xi}{2^j}})\right| d\xi \\
&\leq C \int_{\mathbb{R}} \prod_{j=1}^{\infty} \left|\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\right|^2 \left[\mathbf{P}_3\left(\sin^2 \frac{\xi}{2^{j+1}}\right)\right]^{\frac{1}{2}} d\xi \\
&\leq C \int_{\mathbb{R}} (1+|\xi|) \left|{}_3\widehat{\phi}(\xi)\right| d\xi \\
&\leq C \int_{\mathbb{R}} (1+|\xi|)^{\alpha(3)} \left|{}_3\widehat{\phi}(\xi)\right| d\xi \; < \infty.
\end{aligned}$$

So we have $\widetilde{N} \in C^0$, which implies $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)^T \in C^0$. By the same argument, we can make $\widetilde{\Phi} \in C^1, C^2$ if we chose $\widetilde{m} = 6, 9$, respectively. Similarly, for linear spline function, $N_2(x)$, we can find $\widetilde{m} = 6, 9$ to make $\widetilde{\Phi} \in C^0, C^1$ respectively. Finally for $N_3(x)$, we can easily check that $\widetilde{m} = 9$ makes $\widetilde{\Phi} \in C^0$.

By the computation above we can also estimate the size of support of dual functions which is related to the number of coefficients by Lemma 3.2.4. In particular, for the linear B-spline function, the numbers of coeffients of dual functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3$ are $10, 10$ and 3, while the dual scaling function ${}_{2,6}\widetilde{\phi}$ from Cohen, Daubechies and Feaubeau ([10]) needs 13 coefficients for the same regularity C^0. And for C^1, the numbers of coefficients of $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3$ are $16, 16$ and 3 coefficients, respectivley, while the number of coefficients of ${}_{2,8}\widetilde{\phi}$ is 19.

Chapter 4

Construction of Biorthogonal Box-Spline Multiwavelets

In this chapter, we extend the results of Chapter 3 to the bivariate setting. That is, we are going to construct two multiscaling function vectors $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \cdots, \phi_r(\mathbf{x}))^T$, $\widetilde{\Phi}(\mathbf{x}) = (\widetilde{\phi}_1(\mathbf{x}), \cdots, \widetilde{\phi}_r(\mathbf{x}))^T$, by using box spline functions. And also we are interested in the construction of associated multiwavelet function vectors $\Psi(\mathbf{x}) = (\psi_1(\mathbf{x}), \cdots, \psi_s(\mathbf{x}))^T$, $\widetilde{\Psi}(\mathbf{x}) = (\widetilde{\psi}_1(\mathbf{x}), \cdots, \widetilde{\psi}_s(\mathbf{x}))^T$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Note that the number of wavelet functions can not be consistent with the number of the scaling fuctions in bivariate setting, i.e, $r \neq s$.

4.1 Box splines and basic properties

Box splines can be interpreted as a multivariate extension of univariate B-splines. Because of their useful geometric interpolation, multivariate box splines have been used for surface design. Thus multivariate box splines are considerd as important class of refinable functions.

A very comprehensive treatment of box splines and their general theory is given in the book by de Boor, Höllig and Riemenschneider ([13]) who also give valuable information on many references. Among a couple of equivalent definitions, we introduce the inductive definition.

Let us assume that $s \geq d$ and $\mathbf{v}_1, \cdots, \mathbf{v}_d$ are linearly independent vectors in $\mathbb{R}^d$. *A d-variate box spline* $B_\kappa(\mathbf{x}) := B(\mathbf{x}|\mathbf{v}_1 \cdots \mathbf{v}_\kappa), \kappa = d+1, \cdots, s$ are defined by successive convolutions([4],[13])

$$B_d(\mathbf{x}) := \begin{cases} 1/\det[\mathbf{v}_1 \cdots \mathbf{v}_d] & \text{if } \mathbf{x} \in [\mathbf{v}_1 \cdots \mathbf{v}_d][0,1)^d \\ 0 & \text{otherwise,} \end{cases}$$
$$B_\kappa(\mathbf{x}) := \int_0^1 B_{\kappa-1}(\mathbf{x} - t\mathbf{v}_\kappa)\, dt, \quad \kappa > d.$$

From the definition it follows that the restricted box spline $B(y) := B_s(\mathbf{x}+y\mathbf{v}_r)$ is piecewise constant in y if $\mathbf{v}_r \notin \text{span}\{\mathbf{v}_1, \cdots, \check{\mathbf{v}}_r, \cdots, \mathbf{v}_s\}$. If $\mathbf{v}_r \in \text{span}\{\mathbf{v}_1, \cdots, \check{\mathbf{v}}_r, \cdots, \mathbf{v}_s\}$, $B(y)$ is continuous since it can be obtained by convolution from $B^*(y) = B_s(\mathbf{x} + y\mathbf{v}_r|\mathbf{v}_1, \cdots, \check{\mathbf{v}}_r, \cdots, \mathbf{v}_s)$,

$$B(y) = \int_0^1 B^*(y-t)\, dt = \int_{y-1}^{y} B^*(t)\, dt$$
$$= \int_{-\infty}^{y} B^*(t) - B^*(t-1)\, dt$$

Further, the *directional derivative with respect to* $\mathbf{v}_r$ is defined by

$$D_{\mathbf{v}_r} B_s(\mathbf{x}) = B'(y)|_{y=0} = B_s^*(\mathbf{x}) - B_s^*(\mathbf{x} - \mathbf{v}_r).$$

Similar to B-splines, we summarize some properties of multivariate box splines $B_k(\mathbf{x})$. For more properties and detailed proof, see references(eg.,[4],[13]).

Theorem 4.1.1 *Multivariate box splines $B_s(\mathbf{x}), s \geq d$ satisfy the following properties:*

1. $\int_{\mathbb{R}^d} B_s(\mathbf{x})\, d\mathbf{x} = 1.$

2. *For all* $f \in C(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} B_s(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_{[0,1)^s} f([\mathbf{v}_1, \cdots, \mathbf{v}_s]\mathbf{t})\, d\mathbf{t}.$$

3. $B_s(\mathbf{x}) > 0$, *for* $\mathbf{x} \in [\mathbf{v}_1, \cdots, \mathbf{v}_s][0,1)^s$.

4. *supp* $B_s(\mathbf{x}) = [\mathbf{v}_1, \cdots, \mathbf{v}_s][0,1]^s$.

5. *The Fourier transform of* $B_s(\mathbf{x})$ *is*

$$\widehat{B}_s(\omega) = \prod_{j=1}^{s} \frac{1 - e^{-i\omega\cdot\mathbf{v}_j}}{i\omega \cdot \mathbf{v}_j}.$$

6. *For* $s > d$, *and* $f \in C^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} B_s(\mathbf{x})\, D_{\mathbf{v}_j} f(\mathbf{x})\, d\mathbf{x} = -\int_{[0,1)^s} D_{\mathbf{v}_j} B_s(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x}, \quad 1 \le j \le s.$$

7. *There exists a finite sequence* $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ *such that*

$$B_s(\mathbf{x}) = \sum_{\mathbf{k}]in\mathbb{Z}^d} B_s(2\mathbf{x} - \mathbf{k}).$$

For a fixed s, let V_0^m be the space generated by the integer translates of $B_s(\mathbf{x})$ and let V_j^s be the space generated by dyadic dilations and intger translations of $B_s(\mathbf{x})$, i.e., for $j \in \mathbb{Z}$,

$$V_j^s := \overline{\text{span}\{B_s(2^j\mathbf{x} - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^d\}}.$$

Then $\{V_j^s\}_{j\in\mathbb{Z}}$ satisfies following properties(see [13], p.125)

$$\cdots \subset V_{-1}^s \subset V_0^s \subset V_1^s \subset \cdots,$$
$$\overline{\bigcup_{j\in\mathbb{Z}} V_j^s} = L^2(\mathbb{R}^d), \quad \bigcap_{j\in\mathbb{Z}} V_j^s = \{0\}.$$

It is known that the box spline spaces $\{V_j^s\}_{j\in\mathbb{Z}}$ constitute a multiresolution approximation of $L^2(\mathbb{R}^d)$ in dimensions $d = 2, 3$(see [39], pp. 133-149).

4.2 Construction of Bivariate Scaling Function Vectors

We fix $r = 5$. Similarly to univariate case, we take the Fourier transform of Φ and Ψ, then we have

$$\widehat{\Phi}(2\xi, 2\zeta) = P(z, w)\, \widehat{\Phi}(\xi, \zeta),\ P(z, w) = \frac{1}{4}\sum_{j,k} P_{j,k} z^j w^k,$$
$$\widehat{\Psi}(2\xi, 2\zeta) = Q(z, w)\, \widehat{\Phi}(\xi, \zeta),\ Q(z, w) = \frac{1}{4}\sum_{j,k} Q_{j,k} z^j w^k,$$

where the polynomial matrices $P(z,w)$ and $Q(z,w)$ are matrix symbols of $\Phi(x,y)$ and $\Psi(x,y)$, respectively. Similarly, let $\widetilde{P}(z,w)$ and $\widetilde{Q}(z,w)$ be the matrix symbols of $\widetilde{\Phi}(x,y)$ and $\widetilde{\Psi}(x,y)$, respectively. Then in terms of the matrix symbols $P(z,w), Q(z,w), \widetilde{P}(z,w)$, and $\widetilde{Q}(z,w)$, the bivariate biorthogonality conditions are represented as:

$$P\widetilde{P}^*(z,w) + P\widetilde{P}^*(-z,w) + P\widetilde{P}^*(z,-w) + P\widetilde{P}^*(-z,-w) = I_{r\times r}, \tag{4.1}$$

$$P\widetilde{Q}^*(z,w) + P\widetilde{Q}^*(-z,w) + P\widetilde{Q}^*(z,-w) + P\widetilde{Q}^*(-z,-w) = O_{r\times r}, \tag{4.2}$$

$$\widetilde{P}Q^*(z,w) + \widetilde{P}Q^*(-z,w) + \widetilde{P}Q^*(z,-w) + \widetilde{P}Q^*(-z,-w) = O_{r\times r}, \tag{4.3}$$

$$Q\widetilde{Q}^*(z,w) + Q\widetilde{Q}^*(-z,w) + Q\widetilde{Q}^*(z,-w) + Q\widetilde{Q}^*(-z,-w) = I_{r\times r}. \tag{4.4}$$

Let $B_{l,m,n}$ be the bivariate box spline function whose Fourier transform is

$$\widehat{B}_{l,m,n}(\xi,\zeta) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^l \left(\frac{1-e^{-i\zeta}}{i\zeta}\right)^m \left(\frac{1-e^{-i(\xi+\zeta)}}{i(\xi+\zeta)}\right)^n.$$

For simplicity, we denote $B_{l,m,n}$ by N. Then from the mask of N, we have

$$A(z,w) := \sum_{j=0}^{l+m} \sum_{k=0}^{m+n} a_{j,k} z^j w^k = 4\left(\frac{1+z}{2}\right)^l \left(\frac{1+w}{2}\right)^m \left(\frac{1+zw}{2}\right)^n$$

where $z = e^{-i\xi}$ and $w = e^{-i\zeta}$, so that it satisfies the following refinement equation:

$$N(x,y) = \sum_{j,k} a_{j,k} N(2x-j, 2y-k). \tag{4.5}$$

Let V_0 be the space generated by $N(x,y)$, and define

$$V_j = \left\{ f(2^j x, 2^j y) : f \in V_0 \right\}, \text{ for } j \in \mathbb{Z}.$$

It is known that the box spline function N generates a multiresolution analysis of $L^2(\mathbb{R}^2)$ (See Section 4.1).

Let us begin with putting

$$\phi_1(x,y) = N(2x,2y), \qquad \phi_2(x,y) = N(2x-1,2y),$$

$$\phi_3(x,y) = N(2x,2y-1), \qquad \phi_4(x,y) = N(2x-1,2y-1),$$

then from the refinement equation (4.5),

$$
\begin{aligned}
N(x,y) =& \sum_{j,k} a_{j,k} N(2x-j, 2y-k) \\
=& \sum_{j,k} a_{2j,2k} N(2x-2j, 2y-2k) + \sum_{j,k} a_{2j+1,2k} N(2x-2j-1, 2y-2k) \\
&+ \sum_{j,k} a_{2j,2k+1} N(2x-2j, 2y-2k-1) \\
&+ \sum_{j,k} a_{2j+1,2k+1} N(2x-2j-1, 2y-2k-1) \\
=& \sum_{j,k} a_{2j,2k} \phi_1(x-j, y-k) + \sum_{j,k} a_{2j+1,2k} \phi_2(x-j, y-k) \\
&+ \sum_{j,k} a_{2j,2k+1} \phi_3(x-j, y-k) + \sum_{j,k} a_{2j+1,2k+1} \phi_4(x-j, y-k).
\end{aligned}
$$

We write $A(z,w)$ in its polyphase form :

$$
A(z,w) = A_0(z^2,w^2) + zA_1(z^2,w^2) + wA_2(z^2,w^2) + zwA_3(z^2,w^2),
$$

where the polyphase terms are expressed as follows:

$$
\begin{aligned}
A_0(z^2,w^2) &= \frac{1}{4}(A(z,w) + A(-z,w) + A(z,-w) + A(-z,-w)), \\
A_1(z^2,w^2) &= \frac{1}{4z}(A(z,w) - A(-z,w) + A(z,-w) - A(-z,-w)), \\
A_2(z^2,w^2) &= \frac{1}{4w}(A(z,w) + A(-z,w) - A(z,-w) - A(-z,-w)), \\
A_3(z^2,w^2) &= \frac{1}{4zw}(A(z,w) - A(-z,w) - A(z,-w) + A(-z,-w)).
\end{aligned}
$$

Then the Fourier transform of box spline function becomes

$$
\begin{aligned}
\widehat{N}(\xi,\zeta) =& A_0(z,w)\widehat{\phi}_1(\xi,\zeta) + A_1(z,w)\widehat{\phi}_2(\xi,\zeta) \\
&+ A_2(z,w)\widehat{\phi}_3(\xi,\zeta) + A_3(z,w)\widehat{\phi}_4(\xi,\zeta),
\end{aligned} \tag{4.6}
$$

and also from the definition of ϕ_1, ϕ_2, ϕ_3, and ϕ_4, we have

$$
\begin{aligned}
\widehat{\phi}_1(2\xi,2\zeta) &= \frac{1}{4}\widehat{N}(\xi,\zeta), & \widehat{\phi}_2(2\xi,2\zeta) &= \frac{1}{4}z\widehat{N}(\xi,\zeta), \\
\widehat{\phi}_3(2\xi,2\zeta) &= \frac{1}{4}w\widehat{N}(\xi,\zeta), & \widehat{\phi}_4(2\xi,2\zeta) &= \frac{1}{4}zw\widehat{N}(\xi,\zeta).
\end{aligned}
$$

Consider a bivariate trigonometric polynomial

$$\begin{aligned} B(z,w) &= \frac{1}{4}\left(B_0(z^2,w^2) + zB_1(z^2,w^2) + wB_2(z^2,w^2) + zwB_3(z^2,w^2)\right) \\ &= \frac{1}{4}\left(\frac{1+z}{2}\right)^{\widetilde{n}-l}\left(\frac{1+w}{2}\right)^{\widetilde{n}-m}\left(\frac{1+zw}{2}\right)^{\widetilde{m}-n} H(z,w)D(zw), \end{aligned} \tag{4.7}$$

where

$$H(z,w) = \sum_{k=0}^{\widetilde{n}-1}\binom{2\widetilde{n}-1}{k}\left(\frac{1+z}{2}\frac{1+w}{2}\right)^{\widetilde{n}-1-k}\left(\frac{1-z}{2}\frac{1-w}{2}\right)^{k}, \tag{4.8}$$

$$D(zw) = \sum_{k=0}^{L-1}\binom{2L-1}{k}\left(\frac{1+zw}{2}\right)^{L-1-k}\left(\frac{1-zw}{2}\right)^{k},\ L = 2\widetilde{n}+\widetilde{m}-1. \tag{4.9}$$

Then we have

$$A_0B_0(z,w) + zA_1B_1(z,w) + wA_2B_2(z,w) + zwA_3B_3(z,w) = 1. \tag{4.10}$$

Indeed,

$$\begin{aligned} 1 &= \left(\frac{1+zw}{2} + \frac{1-zw}{2}\right)^{2L-1} \\ &= \left(\frac{1+zw}{2}\right)^{L} D(zw) + \left(\frac{1-zw}{2}\right)^{L} D(-zw) \\ &= \left(\frac{1+zw}{2}\right)^{2\widetilde{n}-1}\left(\frac{1+zw}{2}\right)^{\widetilde{m}} D(zw) + \left(\frac{1-zw}{2}\right)^{2\widetilde{n}-1}\left(\frac{1-zw}{2}\right)^{\widetilde{m}} D(-zw) \end{aligned}$$

Here we note that

$$\frac{1+zw}{2} = \frac{1+z}{2}\cdot\frac{1+w}{2} + \frac{1-z}{2}\cdot\frac{1-w}{2}.$$

Then it follows that

$$\begin{aligned} \left(\frac{1+zw}{2}\right)^{2\widetilde{n}-1} &= \left(\frac{1+z}{2}\cdot\frac{1+w}{2} + \frac{1-z}{2}\cdot\frac{1-w}{2}\right)^{2\widetilde{n}-1} \\ &= \left(\frac{1+z}{2}\cdot\frac{1+w}{2}\right)^{\widetilde{n}} H(z,w) + \left(\frac{1-z}{2}\cdot\frac{1-w}{2}\right)^{\widetilde{n}} H(-z,-w), \end{aligned}$$

and similarly

$$\begin{aligned} \left(\frac{1-zw}{2}\right)^{2\widetilde{n}-1} &= \left(\frac{1-z}{2}\cdot\frac{1+w}{2} + \frac{1+z}{2}\cdot\frac{1-w}{2}\right)^{2\widetilde{n}-1} \\ &= \left(\frac{1-z}{2}\cdot\frac{1+w}{2}\right)^{\widetilde{n}} H(-z,w) + \left(\frac{1+z}{2}\cdot\frac{1-w}{2}\right)^{\widetilde{n}} H(z,-w). \end{aligned}$$

From the above two equations, we have

$$\begin{aligned}
1 =& \left(\frac{1+z}{2}\right)^{\widetilde{n}} \left(\frac{1+w}{2}\right)^{\widetilde{n}} \left(\frac{1+zw}{2}\right)^{\widetilde{n}} H(z,w)D(zw) \\
&+ \left(\frac{1-z}{2}\right)^{\widetilde{n}} \left(\frac{1+w}{2}\right)^{\widetilde{n}} \left(\frac{1-zw}{2}\right)^{\widetilde{n}} H(-z,w)D(-zw) \\
&+ \left(\frac{1+z}{2}\right)^{\widetilde{n}} \left(\frac{1-w}{2}\right)^{\widetilde{n}} \left(\frac{1-zw}{2}\right)^{\widetilde{n}} H(z,-w)D(-zw) \\
&+ \left(\frac{1-z}{2}\right)^{\widetilde{n}} \left(\frac{1-w}{2}\right)^{\widetilde{n}} \left(\frac{1+zw}{2}\right)^{\widetilde{n}} H(-z,-w)D(zw) \\
=& AB(z,w) + AB(-z,w) + AB(z,-w) + AB(-z,-w) \\
=& A_0B_0(z^2,w^2) + z^2A_1B_1(z^2,w^2) + w^2A_2B_2(z^2,w^2) + z^2w^2A_3B_3(z^2,w^2)
\end{aligned}$$

From the application of Theorem 3.1 in [7], we have the following theorm.

Theorem 4.2.1 *There exists a set of bivariate trogonometric polynomials $G_{j,k}(z,w)$ for $1 \le j \le 3, 0 \le k \le 3$ such that the matrix*

$$P_1(z,w) := \begin{bmatrix} A_0(z,w) & A_1(z,w) & A_2(z,w) & A_3(z,w) \\ G_{1,0}(z,w) & G_{1,1}(z,w) & G_{1,2}(z,w) & G_{1,3}(z,w) \\ G_{2,0}(z,w) & G_{2,1}(z,w) & G_{2,2}(z,w) & G_{2,3}(z,w) \\ G_{3,0}(z,w) & G_{3,1}(z,w) & G_{3,2}(z,w) & G_{3,3}(z,w) \end{bmatrix}$$

has a monomial determinant, i.e., $\det P_1(z,w) = \alpha z^\mu w^\nu$ *for some* μ, ν, *and a constant* $\alpha \neq 0$, *and furthermore it satisfies*

$$P_1(z,w) \cdot [\ B_0 \quad zB_1 \quad wB_2 \quad zwB_3\ ]^T = [\ 1 \quad 0 \quad 0 \quad 0\ ]^T.$$

Proof: From the expression of $A(z,w)$, it is clear that $A(z,w), A(-z,w)$, $A(z,-w)$, $A(-z,-w)$ have no commom zeros. So it follows that A_0, A_1, A_2, A_3 have no common zeros. Moreover we have that the first three polyphase terms A_0, A_1, A_2 have no common zeros([25]). By the well-known Hilbert-Nullstellensatz Theorem in ([29], p.292), there exist polynomials p_0, p_1, p_2 such that

$$p_0 A_0 + p_1 A_1 + p_2 A_2 = 1. \tag{4.11}$$

We define the polynomial matrix U_1 as

$$U_1 = \begin{bmatrix} p_0(1 - A_3) & 0 & 0 & 1 \\ p_1(1 - A_3) & 0 & 1 & 0 \\ p_2(1 - A_3) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

then the determinant of U_1 is obviously 1, and we get

$$[\ A_0 \quad A_1 \quad A_2 \quad A_3\ ] \cdot U_1 = [\ 1 \quad A_2 \quad A_1 \quad A_0\ ].$$

Next we define the polynomial matrix U_2 having the determinant 1 by

$$U_2 = \begin{bmatrix} 1 & -A_2 & -A_1 & -A_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and put $U := U_1 U_2$, then we can easily check that

$$[\ A_0 \quad A_1 \quad A_2 \quad A_3\ ] \cdot U = [\ 1 \quad 0 \quad 0 \quad 0\ ].$$

Since the determinant of U is 1, the definition of inverse function gives

$$U^{-1} = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ 0 & 0 & 1 & -p_2(1 - A_3) \\ 0 & 1 & 0 & -p_1(1 - A_3) \\ 1 & 0 & 0 & -p_0(1 - A_3) \end{bmatrix},$$

and also it is straightforward to check that

$$U^{-1} \cdot \begin{bmatrix} B_0 \\ zB_1 \\ wB_2 \\ zwB_3 \end{bmatrix} = \begin{bmatrix} 1 \\ wB_2 - zwB_3 p_2(1 - A_3) \\ zB_1 - zwB_3 p_1(1 - A_3) \\ B_0 - zwB_3 p_0(1 - A_3) \end{bmatrix}.$$

For computational efficiency, we denote

$$h_1 := wB_2 - zwB_3p_2(1 - A_3), \tag{4.12}$$

$$h_2 := zB_1 - zwB_3p_1(1 - A_3), \tag{4.13}$$

$$h_3 := B_0 - zwB_3p_0(1 - A_3). \tag{4.14}$$

Now we take the polynomial matrix

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -h_1 & 1 & 0 & 0 \\ -h_2 & 0 & 1 & 0 \\ -h_3 & 0 & 0 & 1 \end{bmatrix},$$

then we have

$$LU^{-1} = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ -h_1A_0 & -h_1A_1 & -h_1A_2 + 1 & -h_1A_3 - p_2(1 - A_3) \\ -h_2A_0 & -h_2A_1 + 1 & -h_2A_2 & -h_2A_3 - p_1(1 - A_3) \\ -h_3A_0 + 1 & -h_3A_1 & -h_3A_2 & -h_3A_3 - p_0(1 - A_3) \end{bmatrix},$$

Putting $P_1(z, w) = LU^{-1}$, $P_1(z, w)$ becomes the matrix extension for $[A_0, A_1, A_2, A_3]$. We clearly see that $\det(P_1(z, w)) = 1$, and furthermore

$$P_1(z, w) \cdot [\ B_0 \quad zB_1 \quad wB_2 \quad zwB_3\ ]^T = [\ 1 \quad 0 \quad 0 \quad 0\ ]^T.$$

□

Now we define $M_1, M_2, M_3 \in V_1$ by

$$\begin{aligned} \widehat{M_1}(\xi, \zeta) =& - h_1A_0(z, w) \cdot \widehat{\phi_1}(\xi, \zeta) - h_1A_1(z, w) \cdot \widehat{\phi_2}(\xi, \zeta) \\ & - (h_1A_2 - 1)(z, w) \cdot \widehat{\phi_3}(\xi, \zeta) - \big(h_1A_3 + p_2(1 - A_3)\big)(z, w) \cdot \widehat{\phi_4}(\xi, \zeta), \end{aligned}$$

$$\begin{aligned} \widehat{M_2}(\xi, \zeta) =& - h_2A_0(z, w) \cdot \widehat{\phi_1}(\xi, \zeta) - (h_2A_1 - 1)(z, w) \cdot \widehat{\phi_2}(\xi, \zeta) \\ & - h_2A_2(z, w) \cdot \widehat{\phi_3}(\xi, \zeta) - \big(h_2A_3 + p_1(1 - A_3)\big)(z, w) \cdot \widehat{\phi_4}(\xi, \zeta), \end{aligned}$$

$$\begin{aligned} \widehat{M_3}(\xi, \zeta) =& - (h_3A_0 - 1)(z, w) \cdot \widehat{\phi_1}(\xi, \zeta) - h_3A_1(z, w) \cdot \widehat{\phi_2}(\xi, \zeta) \\ & - h_3A_2(z, w) \cdot \widehat{\phi_3}(\xi, \zeta) - \big(h_3A_3 + p_0(1 - A_3)\big)(z, w) \cdot \widehat{\phi_4}(\xi, \zeta). \end{aligned}$$

Since $(\phi_1, \phi_2, \phi_3, \phi_4)^T$ generates V_1 and $\det(P_1(z,w)) = 1$ by by Theorm 4.2.1, hence $(N, M_1, M_2, M_3)^T$ also generates V_1. We put

$$M_{11}(x,y) = M_1(2x, 2y), \qquad M_{12}(x,y) = M_1(2x-1, 2y),$$

$$M_{13}(x,y) = M_1(2x, 2y-1), \qquad M_{14}(x,y) = M_1(2x-1, 2y-1),$$

and similarily, put $M_{21}, \cdots, M_{24}$ and $M_{31}, \cdots, M_{34}$, for M_2 and M_3, respectively. Then we can see that $(\phi_1, \cdots, \phi_4, M_{11}, M_{12}, \cdots, M_{34})^T$ generates V_2.

We take $M(x,y) = M_1(x,y)$, and define

$$\phi_5(x,y) = \sum_{j,k} \alpha_{j,k} M(2x-j, 2y-k), \tag{4.15}$$

with a bivariate Laurent polynomial,

$$R(z,w) = \frac{1}{4} \sum_{j,k} \alpha_{j,k} z^j w^k, \tag{4.16}$$

then we have ϕ_5 in terms of the Fourier transform,

$$\begin{aligned}
\widehat{\phi}_5(2\xi, 2\zeta) &= R(z,w)\, \widehat{M}(\xi,\zeta) \\
&= -(Rh_1A_0)(z,w) \cdot \widehat{\phi}_1(\xi,\zeta) - (Rh_1A_1)(z,w) \cdot \widehat{\phi}_2(\xi,\zeta) \\
&\quad - \left(R(h_1A_2 - 1)\right)(z,w) \cdot \widehat{\phi}_3(\xi,\zeta) \\
&\quad - \Big(R\left(h_1A_3 + p_2(1-A_3)\right)\Big)(z,w) \cdot \widehat{\phi}_4(\xi,\zeta),
\end{aligned} \tag{4.17}$$

By the recursive use of equation (4.6), we have

$$\widehat{\Phi}(2\xi, 2\zeta) = P(z,w)\, \widehat{\Phi}(\xi,\zeta),$$

where the matrix symbol $P(z,w)$ is of the form:

$$P(z,w) = \begin{bmatrix}
\frac{1}{4}A_0 & \frac{1}{4}A_1 & \frac{1}{4}A_2 & \frac{1}{4}A_3 & 0 \\
\frac{1}{4}zA_0 & \frac{1}{4}zA_1 & \frac{1}{4}zA_2 & \frac{1}{4}zA_3 & 0 \\
\frac{1}{4}wA_0 & \frac{1}{4}wA_1 & \frac{1}{4}wA_2 & \frac{1}{4}wA_3 & 0 \\
\frac{1}{4}zwA_0 & \frac{1}{4}zwA_1 & \frac{1}{4}zwA_2 & \frac{1}{4}zwA_3 & 0 \\
-Rh_1A_0 & -Rh_1A_1 & -R(h_1A_2 - 1) & -R\big(h_1A_3 + p_2(1-A_3)\big) & 0
\end{bmatrix}$$

To construct the dual scaling function vector $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4, \widetilde{\phi}_5)^T$ we only need to find the matrix symbol $\widetilde{P}(z,w)$ satisfying the first biorthogonality condition (4.1):

$$P\widetilde{P}^*(z,w) + P\widetilde{P}^*(-z,w) + P\widetilde{P}^*(z,-w) + P\widetilde{P}^*(-z,-w) = I_{5\times 5}.$$

Theorem 3.1.1 in bivariate setting guarantees the existence of such a $\widetilde{P}(z,w)$. We now find it explicitly. To make our computation easy, we write

$$P(z,w) = \begin{bmatrix} X(z,w) & O \\ Y(z,w) & 0 \end{bmatrix}, \quad \widetilde{P}(z,w) = \begin{bmatrix} \widetilde{X}(z,w) & \widetilde{V}(z,w) \\ \widetilde{Y}(z,w) & \widetilde{W}(z,w) \end{bmatrix}.$$

where $X(z,w), \widetilde{X}(z,w)$ are 4×4 polynomial matrices, $Y(z,w), \widetilde{Y}(z,w)$ are 1×4 polynomial matrices, $\widetilde{V}(z,w)$ is 4×1 a polynomial matrix, and $\widetilde{W}(z,w)$ is a polynomial. Then the first biorthogonality condition (4.1) can be expressed in four matrix equations as follows:

$$X\widetilde{X}^*(z,w) + X\widetilde{X}^*(-z,w) + X\widetilde{X}^*(z,-w) + X\widetilde{X}^*(-z,-w) = I_{4\times 4}, \tag{4.18}$$

$$X\widetilde{Y}^*(z,w) + X\widetilde{Y}^*(-z,w) + X\widetilde{Y}^*(z,-w) + X\widetilde{Y}^*(-z,-w) = O_{4\times 1}, \tag{4.19}$$

$$\widetilde{X}Y^*(z,w) + \widetilde{X}Y^*(-z,w) + \widetilde{X}Y^*(z,-w) + \widetilde{X}Y^*(-z,-w) = O_{1\times 4}, \tag{4.20}$$

$$Y\widetilde{Y}^*(z,w) + Y\widetilde{Y}^*(-z,w) + Y\widetilde{Y}^*(z,-w) + Y\widetilde{Y}^*(-z,-w) = 1. \tag{4.21}$$

According to the special structure of $P(z,w)$, the process to find its dual$\widetilde{P}(z,w)$ depends on the choice of $\widetilde{X}(z,w)$ and $\widetilde{Y}(z,w)$ as in the theorem below.

Theorem 4.2.2 *Define*

$$\widetilde{X}^*(z,w) = \begin{bmatrix} B_0(z,w) & \bar{z}B_0(z,w) & \bar{w}B_0(z,w) & \bar{z}\bar{w}B_0(z,w) \\ zB_1(z,w) & B_1(z,w) & z\bar{w}B_1(z,w) & \bar{w}B_1(z,w) \\ wB_2(z,w) & \bar{z}wB_2(z,w) & B_2(z,w) & \bar{z}B_2(z,w) \\ zwB_3(z,w) & wB_3(z,w) & zB_3(z,w) & B_3(z,w) \end{bmatrix},$$

and

$$\widetilde{Y}^*(z,w) = \begin{bmatrix} -p_0A_2(1-A_3)(z,w) \\ -p_1A_2(1-A_3)(z,w) \\ \big((p_0A_0 + p_1A_1)(1-A_3) + A_3\big)(z,w) \\ -A_2(z,w) \end{bmatrix}.$$

For any 4×1 *bivariate Laurent polynomial matrix* $\widetilde{V}$ *and any Laurent polynomial* $\widetilde{W}$*, if we take* $R(z,w)$ *such that*

$$R(z,w) + R(-z,w) + R(z,-w) + R(-z,-w) = 1\,,$$

then $X(z,w), \widetilde{X}(z,w), Y(z,w), \widetilde{Y}(z,w)$ *satisfy the biorthogonality conditions (4.18)* $\sim$ *(4.21) which are defined as the above.*

Proof: By using (4.10), (4.11), and (4.12) $\sim$ (4.14), we get

$$X\widetilde{X}^*(z,w) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4}\bar{z} & \frac{1}{4}\bar{w} & \frac{1}{4}\bar{z}\bar{w} \\ \frac{1}{4}z & \frac{1}{4} & \frac{1}{4}z\bar{w} & \frac{1}{4}\bar{w} \\ \frac{1}{4}w & \frac{1}{4}\bar{z}w & \frac{1}{4} & \frac{1}{4}\bar{z} \\ \frac{1}{4}zw & \frac{1}{4}w & \frac{1}{4}z & \frac{1}{4} \end{bmatrix},$$

$$X\widetilde{Y}^*(z,w) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad Y\widetilde{X}^*(z,w) \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

And we have

$$\begin{aligned} Y\widetilde{Y}^*(z,w) &= R(z,w)\left(p_0A_0(z,w) + p_1A_1(z,w) + p_2A_2(z,w)\right) \\ &\quad - R(z,w)A_3(z,w)\left(p_0A_0(z,w) + p_1A_1(z,w) + p_2A_2(z,w)\right) \\ &\quad + R(z,w)A_3(z,w) \\ &= R(z,w) - R(z,w)A_3(z,w) + R(z,w)A_3(z,w) = R(z,w). \end{aligned}$$

This completes the proof. □

For the simplicity of computation we take $\widetilde{V} = O_{4\times 1}, \widetilde{W} = 0$. And we define two refinable function vectors by

$$\widehat{\Phi}(\xi,\zeta) = P(\frac{\xi}{2},\frac{\zeta}{2})\,\widehat{\Phi}(\frac{\xi}{2},\frac{\zeta}{2}) = \prod_{j=1}^{\infty} P(\frac{\xi}{2^j},\frac{\zeta}{2^j})\,\widehat{\Phi}(0,0),$$

$$\widehat{\widetilde{\Phi}}(\xi,\zeta) = \widetilde{P}(\frac{\xi}{2},\frac{\zeta}{2})\,\widehat{\widetilde{\Phi}}(\frac{\xi}{2},\frac{\zeta}{2}) = \prod_{j=1}^{\infty} \widetilde{P}(\frac{\xi}{2^j},\frac{\zeta}{2^j})\,\widehat{\widetilde{\Phi}}(0,0),$$

where $\widehat{\Phi}(0,0)$, $\widehat{\widetilde{\Phi}}(0,0)$ are the right eigenvectors of $P(1,1)$, $\widetilde{P}(1,1)$, respectively. Then we need to show that two bivariate infinite matrix products, $\prod_{j=1}^{\infty} P(\frac{\xi}{2^j},\frac{\zeta}{2^j})$ and $\prod_{j=1}^{\infty} \widetilde{P}(\frac{\xi}{2^j},\frac{\zeta}{2^j})$ are well defined in $L^2(\mathbb{R}^2)$. Moreover we will study their smoothness in next section.

4.3 Smoothness of the Scaling Function Vectors

Similarily to one variable case we have the following:

Theorem 4.3.1 *Let* $R(z,w) = \frac{1}{4}$*. And we take* $\widetilde{V} = [0,0,0,0]^T$ *,* $\widetilde{W} = 0$*. Then* $P(1,1)$*,* $\widetilde{P}(1,1)$ *have a simple eigenvalue* 1*, with all other eigenvalues less than* 1*.*

Proof:

From the following polyphase terms of $B(z,w)$,

$$\begin{aligned}
\frac{1}{4}B_0(z^2,w^2) &= \frac{1}{4}(B(z,w)+B(-z,w)+B(z,-w)+B(-z,-w)),\\
\frac{1}{4}B_1(z^2,w^2) &= \frac{1}{4z}(B(z,w)+B(-z,w)+B(z,-w)+B(-z,-w)),\\
\frac{1}{4}B_2(z^2,w^2) &= \frac{1}{4w}(B(z,w)+B(-z,w)+B(z,-w)+B(-z,-w)),\\
\frac{1}{4}B_3(z^2,w^2) &= \frac{1}{4zw}(B(z,w)+B(-z,w)+B(z,-w)+B(-z,-w)),
\end{aligned}$$

we have

$$\begin{aligned}
B_0(1,1) &= B(1,1)+B(-1,1)+B(1,-1)+B(-1,-1) = \frac{1}{4},\\
B_1(1,1) &= B(1,1)+B(-1,1)+B(1,-1)+B(-1,-1) = \frac{1}{4},\\
B_2(1,1) &= B(1,1)+B(-1,1)+B(1,-1)+B(-1,-1) = \frac{1}{4},\\
B_3(1,1) &= B(1,1)+B(-1,1)+B(1,-1)+B(-1,-1) = \frac{1}{4}.
\end{aligned}$$

Similarly, the polyphase terms of $A(z,w)$

$$\begin{aligned}
A_0(z^2,w^2) &= \frac{1}{4}(A(z,w)+A(-z,w)+A(z,-w)+A(-z,-w)),\\
A_1(z^2,w^2) &= \frac{1}{4z}(A(z,w)-A(-z,w)+A(z,-w)-A(-z,-w)),\\
A_2(z^2,w^2) &= \frac{1}{4w}(A(z,w)+A(-z,w)-A(z,-w)-A(-z,-w)),\\
A_3(z^2,w^2) &= \frac{1}{4zw}(A(z,w)-A(-z,w)-A(z,-w)+A(-z,-w)).
\end{aligned}$$

gives

$$A_0(1,1) = \frac{1}{4}(A(1,1) + A(-1,1) + A(1,-1) + A(-1,-1)) = 1,$$
$$A_1(1,1) = \frac{1}{4}(A(1,1) - A(-1,1) + A(1,-1) - A(-1,-1)) = 1,$$
$$A_2(1,1) = \frac{1}{4}(A(1,1) + A(-1,1) - A(1,-1) - A(-1,-1)) = 1,$$
$$A_3(1,1) = \frac{1}{4}(A(1,1) - A(-1,1) - A(1,-1) + A(-1,-1)) = 1.$$

Then by the straightforward computation we see that $P(1,1)$ has eigenvalues

$$\frac{1}{4}A_0(1,1) + \frac{1}{4}A_1(1,1) + \frac{1}{4}A_2(1,1) + \frac{1}{4}A_3(1,1) = 1,\ 0$$

and also $\widetilde{P}(1,1)$ has eigenvalues

$$B_0(1,1) + B_1(1,1) + B_2(1,1) + B_3(1,1) = 1,\ 0.$$

Therefore, our proof of theorem is done. $\square$

By Lemma 3.2.1 in multivariate setting and Theorem 4.3.1, two infinite matrix products $\prod_{j=1}^{\infty} P(\frac{\xi}{2^j}, \frac{\zeta}{2^j})$ and $\prod_{j=1}^{\infty} \widetilde{P}(\frac{\xi}{2^j}, \frac{\zeta}{2^j})$ are well defined.

Let $\widetilde{A}^*(z,w) = B(z,w)$, i.e.,

$$\begin{aligned}\widetilde{A}^*(z,w) &= \frac{1}{4}\left(B_0(z^2,w^2) + zB_1(z^2,w^2) + wB_2(z^2,w^2) + zwB_3(z^2,w^2)\right)\\ &= \frac{1}{4}\left(\frac{1+z}{2}\right)^{\tilde{n}-l}\left(\frac{1+w}{2}\right)^{\tilde{n}-m}\left(\frac{1+zw}{2}\right)^{\tilde{m}-n} H(z,w)D(zw), \qquad (4.22)\end{aligned}$$

where $H(z,w), D(zw)$ are the same polynomials as in (4.8),(4.9) respectively, then it clearly satisfies the equation below,

$$A\widetilde{A}^*(z,w) + A\widetilde{A}^*(-z,w) + A\widetilde{A}^*(z,-w) + A\widetilde{A}^*(-z,-w) = 1. \qquad (4.23)$$

And we define $\widetilde{N}$ by

$$\widehat{\widetilde{N}}(\xi,\zeta) = \prod_{j=1}^{\infty} \widetilde{A}(\frac{\xi}{2^j}, \frac{\zeta}{2^j}), \qquad (4.24)$$

then we have the following lemma.

Lemma 4.3.2 *Let $\widetilde{m}, \widetilde{n}$ be large enough. Then $\widetilde{N}(x,y)$ is a well defined compactly supported L^2 function. Moreover, for any $\alpha \geq 0$, $\widetilde{N} \in C^\alpha(\mathbb{R})$ if $\widetilde{m}, \widetilde{n}$ are sufficiently large, that is,*

$$\widetilde{n} > \frac{\alpha + 1 - \frac{1}{2}\log_2 3 + \max(l,m)}{1 - \frac{1}{2}\log_2 3}, \ \widetilde{m} > \frac{n + (\widetilde{n}-1)\log_2 3}{1 - \frac{1}{2}\log_2 3}.$$

Proof: Recall two simple equations $\left|\frac{1+z}{2}\right| = \left|\cos\frac{\xi}{2}\right|$ and $\left|\frac{1-z}{2}\right| = \left|\sin\frac{\xi}{2}\right|$. Then we have

$$\begin{aligned}
|H(z,w)| &= \left|\sum_{k=0}^{\widetilde{n}-1} \binom{2\widetilde{n}-1}{k} \left(\frac{1+z}{2}\frac{1+w}{2}\right)^{\widetilde{n}-l-k} \left(\frac{1-z}{2}\frac{1-w}{2}\right)^{k}\right| \\
&\leq \sum_{k=0}^{\widetilde{n}-1} \binom{2\widetilde{n}-1}{k} \left|\cos\frac{\xi}{2}\cos\frac{\zeta}{2}\right|^{\widetilde{n}-l-k} \left|\sin\frac{\xi}{2}\sin\frac{\zeta}{2}\right|^{k} \\
&\leq \left\{\sum_{k=0}^{\widetilde{n}-1} \binom{2\widetilde{n}-1}{k} \left(\cos^2\frac{\xi}{2}\right)^{\widetilde{n}-l-k} \left(\sin^2\frac{\xi}{2}\right)^{k}\right\}^{\frac{1}{2}} \\
&\quad \cdot \left\{\sum_{k=0}^{\widetilde{n}-1} \binom{2\widetilde{n}-1}{k} \left(\cos^2\frac{\zeta}{2}\right)^{\widetilde{n}-l-k} \left(\sin^2\frac{\zeta}{2}\right)^{k}\right\}^{\frac{1}{2}} \\
&= \left\{\mathbf{P}_{\widetilde{n}}\left(\sin^2\frac{\xi}{2}\right)\mathbf{P}_{\widetilde{n}}\left(\sin^2\frac{\zeta}{2}\right)\right\}^{\frac{1}{2}},
\end{aligned}$$

where $\mathbf{P}_{\widetilde{m}}$ is the same polynomial (3.23) as in the proof of Lemma 3.2.3. From the inequality above we have

$$\begin{aligned}
|H(e^{-i2\xi}, e^{-i2\zeta})| &\leq \left\{\mathbf{P}_{\widetilde{n}}\left(\sin^2\xi\right)\mathbf{P}_{\widetilde{n}}\left(\sin^2\zeta\right)\right\}^{\frac{1}{2}} \\
&= \left\{\mathbf{P}_{\widetilde{n}}\left(4\sin^2\frac{\xi}{2}\left(1-\sin^2\frac{\xi}{2}\right)\right)\cdot\mathbf{P}_{\widetilde{n}}\left(4\sin^2\frac{\zeta}{2}\left(1-\sin^2\frac{\zeta}{2}\right)\right)\right\}^{\frac{1}{2}}.
\end{aligned}$$

By applying Lemma 7.1.1 $\sim$ Lemma 7.1.8 in [12], we have

$$\begin{aligned}
&\left|\prod_{j=1}^{\infty}\left(\frac{1+z}{2}\right)^{\widetilde{n}-l}\left(\frac{1+w}{2}\right)^{\widetilde{n}-m} H(z,w)\right| \\
&\leq C\left(1+|\xi|\right)^{-\widetilde{n}+l+\frac{1}{2}\log_2 \mathbf{P}_{\widetilde{n}}(\frac{3}{4})}\left(1+|\zeta|\right)^{-\widetilde{n}+m+\frac{1}{2}\log_2 \mathbf{P}_{\widetilde{n}}(\frac{3}{4})} \\
&\leq C\left\{\left(1+|\xi|\right)\left(1+|\zeta|\right)\right\}^{-\widetilde{n}+\max(l,m)+\frac{1}{2}\log_2 \mathbf{P}_{\widetilde{n}}(\frac{3}{4})} \\
&\leq C\left\{\left(1+|\xi|\right)\left(1+|\zeta|\right)\right\}^{-\widetilde{n}+\max(l,m)+\frac{1}{2}(\widetilde{n}-1)\log_2 3}.
\end{aligned}$$

where we have used $\mathbf{P}_{\widetilde{n}}(\frac{3}{4}) \le 3^{\widetilde{n}-1}$ for the last step in the above.

Now we consider the rest terms. Since

$$\begin{aligned}
|D(zw)| &= \left| \sum_{k=0}^{L-1} \binom{2L-1}{k} \left(\frac{1+zw}{2}\right)^{L-l-k} \left(\frac{1-zw}{2}\right)^{k} \right| \\
&\le \sum_{k=0}^{L-1} \binom{2L-1}{k} \left|\cos\frac{\xi+\zeta}{2}\right|^{L-l-k} \left|\sin\frac{\xi+\zeta}{2}\right|^{k} \\
&\le C \left\{ \sum_{k=0}^{L-1} \binom{2L-1}{k} \left(\cos^2\frac{\xi+\zeta}{2}\right)^{L-l-k} \left(\sin^2\frac{\xi+\zeta}{2}\right)^{k} \right\}^{\frac{1}{2}} \\
&= C \left\{ \mathbf{P}_L \left(\sin^2\frac{\xi+\zeta}{2}\right) \right\}^{\frac{1}{2}},
\end{aligned}$$

we have

$$\begin{aligned}
\left| \prod_{j=1}^{\infty} \left(\frac{1+zw}{2}\right)^{\widetilde{m}-n} D(zw) \right| &\le C\left(1+|\xi+\zeta|\right)^{-\widetilde{m}+n+\frac{1}{2}\log_2 \mathbf{P}_L(\frac{3}{4})} \\
&\le C\left(1+|\xi+\zeta|\right)^{-\widetilde{m}+n+\frac{1}{2}(L-1)\log_2 3}.
\end{aligned}$$

For any fixed $\widetilde{n}$, if we choose $\widetilde{m}$ large enough i.e.,

$$-\widetilde{m}+n+\frac{1}{2}(L-1)\log_2 3 \le 0,$$

then we have

$$\left| \prod_{j=1}^{\infty} \left(\frac{1+zw}{2}\right)^{\widetilde{m}-n} D(zw) \right| \le C,$$

where C is a positive constant. And we choose $\widetilde{n}$ large enough so that

$$-\widetilde{n}+\max(l,m)+\frac{1}{2}\left(\widetilde{n}-1\right)\log_2 3 < -\frac{1}{2},$$

then $\widetilde{N}$ becomes an L^2 function. If we choose $\widetilde{n}$ even larger, i.e., for any α, $-\widetilde{n}+\max(l,m)+\frac{1}{2}\left(\widetilde{n}-1\right)\log_2 3 < -\alpha-1$, then we have $\widetilde{N} \in C^{\alpha}$. Finally, we can show that $\widetilde{N}$ is compactly supported from generalization of Lemma 6.2.2 in [12] to bivariate setting. $\square$

Define new functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4$ by dilations and translations of $\widetilde{N}$, i.e.,

$$\widetilde{\phi}_1(x,y) = 4^2\widetilde{N}(2x,2y), \qquad \widetilde{\phi}_2(x,y) = 4^2\widetilde{N}(2x-1,2y),$$
$$\widetilde{\phi}_3(x,y) = 4^2\widetilde{N}(2x,2y-1), \qquad \widetilde{\phi}_4(x,y) = 4^2\widetilde{N}(2x-1,2y-1),$$

or in the Fourier transform,

$$\widehat{\widetilde{\phi}}_1(\xi,\zeta) = 4\,\widehat{\widetilde{N}}\left(\frac{\xi}{2},\frac{\zeta}{2}\right), \qquad \widehat{\widetilde{\phi}}_2(\xi,\zeta) = 4\,\widehat{\widetilde{N}}\left(\frac{\xi}{2},\frac{\zeta}{2}\right)e^{-i\frac{\xi}{2}},$$
$$\widehat{\widetilde{\phi}}_3(\xi,\zeta) = 4\,\widehat{\widetilde{N}}\left(\frac{\xi}{2},\frac{\zeta}{2}\right)e^{-i\frac{\zeta}{2}}, \qquad \widehat{\widetilde{\phi}}_4(\xi,\zeta) = 4\,\widehat{\widetilde{N}}\left(\frac{\xi}{2},\frac{\zeta}{2}\right)e^{-i\frac{\xi}{2}}e^{-i\frac{\zeta}{2}}.$$

Since

$$\begin{aligned}\widehat{\widetilde{N}}(2\xi,2\zeta) &= 4\,\widetilde{A}(z,w)\,\widehat{\widetilde{N}}(\xi,\zeta)\\ &= \left(B_0(\bar{z}^2,\bar{w}^2) + \bar{z}B_1(\bar{z}^2,\bar{w}^2) + \bar{w}B_2(\bar{z}^2,\bar{w}^2) + \bar{z}\bar{w}B_3(\bar{z}^2,\bar{w}^2)\right)\widehat{\widetilde{N}}(\xi,\zeta),\end{aligned}$$

we have

$$\begin{aligned}\widehat{\widetilde{N}}(\xi,\zeta) = &\frac{1}{4}\,B_0(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \frac{1}{4}\,\bar{z}\,B_1(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+ \frac{1}{4}\,\bar{w}\,B_2(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \frac{1}{4}\,\bar{z}\bar{w}\,B_3(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_4(\xi,\zeta),\end{aligned} \tag{4.25}$$

and also we easily get

$$\begin{aligned}\widehat{\widetilde{\phi}}_1(2\xi,2\zeta) = &B_0(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \bar{z}\,B_1(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+ \bar{w}\,B_2(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \bar{z}\bar{w}\,B_3(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_4(\xi,\zeta),\end{aligned} \tag{4.26}$$

$$\begin{aligned}\widehat{\widetilde{\phi}}_2(2\xi,2\zeta) = &z\,B_0(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + B_1(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+ z\bar{w}\,B_2(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \bar{w}\,B_3(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_4(\xi,\zeta),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\widehat{\widetilde{\phi}}_3(2\xi,2\zeta) = &w\,B_0(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \bar{z}w\,B_1(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+ B_2(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \bar{z}\,B_3(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_4(\xi,\zeta),\end{aligned} \tag{4.28}$$

$$\begin{aligned}\widehat{\widetilde{\phi}}_4(2\xi,2\zeta) = &zw\,B_0(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + w\,B_1(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+ z\,B_2(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + B_3(\bar{z},\bar{w})\,\widehat{\widetilde{\phi}}_4(\xi,\zeta).\end{aligned} \tag{4.29}$$

And we define $\widetilde{M}(x,y)$ and $\widetilde{\phi}_5(x,y)$ in the Fourier transform,

$$\begin{aligned}\widehat{\widetilde{M}}(\xi,\zeta) = &-\frac{1}{4}\,(p_0A_2(1-A_3))\,(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_1(\xi,\zeta) - \frac{1}{4}\,(p_1A_2(1-A_3))\,(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &+\frac{1}{4}\big((p_0A_0+p_1A_1)(1-A_3)+A_3\big)(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_3(\xi,\zeta)\\ &-\frac{1}{4}\,A_2(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_4(\xi,\zeta),\end{aligned}\tag{4.30}$$

$$\begin{aligned}\widehat{\widetilde{\phi}}_5(2\xi,2\zeta) &= 4\widehat{\widetilde{M}}(\xi,\zeta)\\ &= -\,(p_0A_2(1-A_3))\,(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_1(\xi,\zeta) - (p_1A_2(1-A_3))\,(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_2(\xi,\zeta)\\ &\quad+\big((p_0A_0+p_1A_1)(1-A_3)+A_3\big)(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_3(\xi,\zeta)\\ &\quad-A_2(\bar{z},\bar{w})\cdot\widehat{\widetilde{\phi}}_4(\xi,\zeta).\end{aligned}\tag{4.31}$$

Then we have the following theorem:

Theorem 4.3.3 *$\{\phi_1,\phi_2,\phi_3,\phi_4,\phi_5\}$ and $\{\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_3,\widetilde{\phi}_4,\widetilde{\phi}_5\}$ are compactly supported L^2 functions with arbitrary smoothness.*

Proof: From the definitions, we know that $\phi_1,\phi_2,\phi_3,\phi_4$ are dilations of a box spline function $N(x,y)$. And also $\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_3,\widetilde{\phi}_4$ are dilations of $\widetilde{N}(x,y)$ which is an L^2 function by Lemma 4.3.2. So we only consider ϕ_5 and $\widetilde{\phi}_5$. First we look at ϕ_5 of the Fourier form in (4.17):

$$\begin{aligned}\widehat{\phi}_5(\xi,\zeta) = &-\frac{1}{4}\,(h_1A_0)(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_1(\frac{\xi}{2},\frac{\zeta}{2}) - \frac{1}{4}\,(h_1A_1)(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_2(\frac{\xi}{2},\frac{\zeta}{2})\\ &-\frac{1}{4}\,(h_1A_2-1)\,(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_3(\frac{\xi}{2},\frac{\zeta}{2})\\ &-\frac{1}{4}\,(h_1A_3+p_2(1-A_3))\,(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_4(\frac{\xi}{2},\frac{\zeta}{2}).\end{aligned}$$

Then we get

$$\begin{aligned}\widehat{\phi}_5(\xi,\zeta) \le &\frac{1}{4}\left|(h_1A_0)(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_1(\frac{\xi}{2},\frac{\zeta}{2})\right| + \frac{1}{4}\left|(h_1A_1)(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_2(\frac{\xi}{2},\frac{\zeta}{2})\right|\\ &+\frac{1}{4}\left|(h_1A_2-1)\,(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_3(\frac{\xi}{2},\frac{\zeta}{2})\right|\\ &+\frac{1}{4}\left|(h_1A_3+p_2(1-A_3))\,(e^{-i\frac{\xi}{2}},e^{-i\frac{\zeta}{2}})\cdot\widehat{\phi}_4(\frac{\xi}{2},\frac{\zeta}{2})\right|\end{aligned}$$

By the definitions of ϕ_1, ϕ_2, ϕ_3, and ϕ_4, we get

$$\widehat{\phi}_2(\xi,\zeta) = e^{-i\frac{\xi}{2}}\widehat{\phi}_1(\xi,\zeta), \quad \widehat{\phi}_3(\xi,\zeta) = e^{-i\frac{\zeta}{2}}\widehat{\phi}_1(\xi,\zeta), \quad \widehat{\phi}_4(\xi,\zeta) = e^{-i\frac{\xi}{2}}e^{-i\frac{\zeta}{2}}\widehat{\phi}_1(\xi,\zeta).$$

Since $A(z,w)$ is bounded, all polyphase terms $A_0(\xi,\zeta)$, $A_1(\xi,\zeta)$, $A_2(\xi,\zeta)$ and $A_3(\xi,\zeta)$ are bounded, that is

$$|A_0(\xi,\zeta)|\,,\ |A_1(\xi,\zeta)|\,,\ |A_2(\xi,\zeta)|\,,\ |A_3(\xi,\zeta)| \le K_A, \quad \forall\, \xi \in [0,2\pi],$$

where $K_A \ge 0$ is a constant. And also the polynomials $h_1(\xi,\zeta), p_2(\xi,\zeta)$ are bounded. It follows that

$$\left|\widehat{\phi}_5(\xi,\zeta)\right| \le K \left|\widehat{\phi}_1(\frac{\xi}{2},\frac{\zeta}{2})\right|,$$

where $K \ge 0$ is a constant. The same argument can be applied to $\widetilde{\phi}_5$. □

4.4 Biorthogonality and the Riesz Basis Property

Let $\widetilde{V}_0$ be the space generated by $\widetilde{N}$, i.e.,

$$\widetilde{V}_0 = \operatorname{span}\{\widetilde{N}(x-j, y-k) : j,k \in \mathbb{Z}\}.$$

And also, for $l \in \mathbb{Z}$ we let

$$\widetilde{V}_l = \{f(2^l x, 2^l y) : f \in \widetilde{V}_0\}.$$

First we prove that the integer translates of $\widetilde{N}$ constitute a Riesz basis of $\widetilde{V}_0$, and furthermore, $\widetilde{N}$ generates a multiresolution analysis of $L^2(\mathbb{R}^2)$. To this end, we first recall the Riesz basis condition in bivariate setting:

$$0 < C_1 < \sum_{l\in\mathbb{Z}^2} \left|\widehat{\widetilde{N}}((\xi,\zeta)+2\pi l)\right|^2 \le C_2 < \infty. \tag{4.32}$$

To prove (4.32) we need following lemma. The proof of the following lemma is the extension of Lemma 3.3.1 to bivariate setting or it is the result of He and Lai([25]).

Lemma 4.4.1 *For any sufficiently large $\widetilde{m}$ and $\widetilde{n}$*

$$\sum_{l\in\mathbb{Z}^2}\left|\widehat{N}((\xi,\zeta)+2\pi l)\,\widehat{\widetilde{N}}((\xi,\zeta)+2\pi l)\right|\geq C>0. \tag{4.33}$$

Proof: From the definitins of $A(z,w)$ and $\widetilde{A}(z,w)$, we get

$$\begin{aligned}
&|A(\xi,\zeta)\,\widetilde{A}(\xi,\zeta)|\\
&=\left|\left(\frac{1+e^{-i\xi}}{2}\right)^{\widetilde{n}}\left(\frac{1+e^{-i\zeta}}{2}\right)^{\widetilde{n}}\left(\frac{1+e^{-i\xi}e^{-i\zeta}}{2}\right)^{\widetilde{m}}H(e^{-i\xi},e^{-i\zeta})D(e^{-i\xi}e^{-i\zeta})\right|\\
&=\left|\cos\frac{\xi}{2}\right|^{\widetilde{n}}\left|\cos\frac{\zeta}{2}\right|^{\widetilde{n}}\left|\cos\frac{\xi+\zeta}{2}\right|^{\widetilde{m}}\left|H(e^{-i\xi},e^{-i\zeta})\right|\left|D(e^{-i(\xi+\zeta)})\right|\\
&\geq\left|\cos\frac{\xi}{2}\right|^{\widetilde{n}}\left|\cos\frac{\zeta}{2}\right|^{\widetilde{n}}\left|\cos\frac{\xi+\zeta}{2}\right|^{\widetilde{m}}\left|H(e^{-i\xi},e^{-i\zeta})\right|
\end{aligned}$$

The last inequality above comes from a simple fact: $\left|D(e^{-i(\xi+\zeta)})\right|\geq 1$ for $(\xi,\zeta)\in\mathbb{R}^2$. Indeed, we have $D(e^{-i(\xi+\zeta)})=\mathbf{P}_L(y)$ by putting $y=\frac{1-e^{i(\xi+\zeta)}}{2}$, where $\mathbf{P}_L(y)$ is a polynomial in (3.23). Then it is known that $\mathbf{P}_L(y)$ is an increasing function on $[0,1]$ in [12] and $\mathbf{P}_L(0)=1$.

Since the sum of the inequality (4.33) is 2π-periodic, we know that (4.33) holds for $(\xi,\zeta)\in[-\pi,\pi]\times[-\pi,\pi]$. Since $\left|\widehat{\widetilde{N}}(\xi,\zeta)\right|=\left|\widehat{\widetilde{N}}(-\xi,-\zeta)\right|$, it is enough to consider (4.33) for $(\xi,\zeta)\in[-\pi,\pi]\times[0,\pi]$. Note that

$$\left|H(e^{-i\xi},e^{-i\zeta})\right|=\left|\sum_{k=0}^{\widetilde{n}-1}\binom{2\widetilde{n}-1}{k}\left(\cos\frac{\xi}{2}\,\cos\frac{\zeta}{2}\right)^{\widetilde{n}-1-k}\left(-\sin\frac{\xi}{2}\,\sin\frac{\zeta}{2}\right)^{k}\right|.$$

First, consider (4.33) on $(\xi,\zeta)\in[-\pi,0]\times[0,\pi]$. Since $\cos\frac{\xi}{2}$, $\cos\frac{\zeta}{2}$, $-\sin\frac{\xi}{2}$, $\sin\frac{\zeta}{2}\geq 0$ on this interval, we have

$$\left|H(e^{-i\xi},e^{-i\zeta})\right|\geq\left|\cos\frac{\xi}{2}\,\cos\frac{\zeta}{2}\right|^{\widetilde{n}-1}.$$

The simple fact that $\frac{\sin x}{x}\geq\frac{2}{\pi}$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$ implies

$$\begin{aligned}
\prod_{j=1}^{\infty}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta}{2^j}})\right|&\geq\prod_{j=1}^{\infty}\left|\cos\frac{\xi}{2^{j+1}}\,\cos\frac{\zeta}{2^{j+1}}\right|^{\widetilde{n}-1}\\
&=\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\,\frac{\sin\frac{\zeta}{2}}{\frac{\zeta}{2}}\right|^{\widetilde{n}-1}\\
&\geq\left(\frac{2}{\pi}\right)^{2\widetilde{n}-2}.
\end{aligned}$$

Thus we have, for $(\xi,\zeta)\in[-\pi,0]\times[0,\pi]$,

$$\begin{aligned}
&\left|\widehat{N}(\xi,\zeta)\widehat{\widetilde{N}}(\xi,\zeta)\right| \\
&\geq \prod_{j=1}^{\infty}\left|\cos\frac{\xi}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\zeta}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\xi+\zeta}{2^{j+1}}\right|^{\widetilde{m}}\cdot\prod_{j=1}^{\infty}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta}{2^j}})\right| \\
&\geq \left(\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right)^{\widetilde{n}}\left(\frac{\sin\frac{\zeta}{2}}{\frac{\zeta}{2}}\right)^{\widetilde{n}}\left(\frac{\sin\frac{\xi+\zeta}{2}}{\frac{\xi+\zeta}{2}}\right)^{\widetilde{m}}\left(\frac{2}{\pi}\right)^{2\widetilde{n}-2} \\
&\geq \left(\frac{2}{\pi}\right)^{4\widetilde{n}+\widetilde{m}-2} > 0.
\end{aligned}$$

Now consider (4.33) on $(\xi,\zeta)\in[0,\pi]\times[0,\pi]$. Note that $|H(1,1)|=1$ and $H(e^{-i\xi},e^{-i\zeta})$ is entire. Then, there exists $\delta>0$ such that for $(\xi,\zeta)\in[0,\delta]\times[0,\delta]$, $|H(e^{-i\xi},e^{-i\zeta})|\geq\frac{1}{2}$. And by Mean Value Theorem in bivariate setting, we get

$$|H(e^{-i\xi},e^{-i\zeta})-1|\leq C(|\xi|+|\zeta|)$$

or equivalently

$$|H(e^{-i\xi},e^{-i\zeta})|\leq 1-C(|\xi|+|\zeta|).$$

On the other hand, there exists $k_0\in\mathbb{Z}$ such that $C\left(|\frac{\xi}{2^j}|+|\frac{\zeta}{2^j}|\right)\leq\frac{1}{2}$ for $j\geq k_0$. Since $1-x\geq e^{-2x}$ for $0\leq x\leq\frac{1}{2}$, we have

$$\begin{aligned}
\prod_{j=1}^{\infty}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta}{2^j}})\right| &= \prod_{j=1}^{k_0}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta}{2^j}})\right|\prod_{j=k_0}^{\infty}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta}{2^j}})\right| \\
&\geq \left(\frac{1}{2}\right)^{k_0}\cdot\prod_{j=1}^{\infty}\left(1-C\,\frac{|\xi|+|\zeta|}{2^{k_0+j}}\right) \\
&\geq \left(\frac{1}{2}\right)^{k_0}\cdot\prod_{j=1}^{\infty}e^{-2\left(C\frac{|\xi|+|\zeta|}{2^{k_0+j}}\right)} \\
&= \left(\frac{1}{2}\right)^{k_0}\cdot e^{-2C\,\frac{|\xi|+|\zeta|}{2^{k_0}}\,\sum_{j=1}^{\infty}\frac{1}{2^j}} \\
&\geq \left(\frac{1}{2}\right)^{k_0}\cdot e^{-1}.
\end{aligned}$$

Therefore for $(\xi,\zeta)\in[0,\delta]\times[0,\delta]$,

$$\left|\widehat{N}(\xi,\zeta)\widehat{\widetilde{N}}(\xi,\zeta)\right| \geq \left(\frac{2}{\pi}\right)^{2\widetilde{n}+\widetilde{m}} \left(\frac{1}{2}\right)^{k_0} \cdot e^{-1} > 0.$$

Finally, for the proof of (4.33) on $(\xi,\zeta)\in[0,\pi]\times[\delta,\pi]$, we consider the following term in the summation of (4.33):

$$\left|\widehat{N}(\xi,\zeta-2\pi)\,\widehat{\widetilde{N}}(\xi,\zeta-2\pi)\right|.$$

From $0\leq\xi\leq\pi$ and $\delta\leq\zeta\leq\pi$, we easily get

$$0\leq\frac{\xi}{2}\leq\frac{\pi}{2},\quad -\pi\leq\frac{\delta-2\pi}{2}\leq\frac{\zeta-2\pi}{2}\leq-\frac{\pi}{2},\quad \frac{\delta}{2}-\pi\leq\frac{\xi+\zeta-2\pi}{2}\leq 0,$$

and then, from the property of the function $\frac{\sin x}{x}$

$$\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^{\widetilde{n}} \geq \left(\frac{2}{\pi}\right)^{\widetilde{n}},\ \left|\frac{\sin\frac{\zeta-2\pi}{2}}{\frac{\zeta-2\pi}{2}}\right|^{\widetilde{n}} \geq \left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{\widetilde{n}},\ \left|\frac{\sin\frac{\xi+\zeta-2\pi}{2}}{\frac{\xi+\zeta-2\pi}{2}}\right|^{\widetilde{m}} \geq \left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{\widetilde{m}}.$$

Since $\cos\frac{\xi}{2^{j+1}},\cos\frac{\zeta-2\pi}{2^{j+1}},\sin\frac{\xi}{2^{j+1}},-\sin\frac{\zeta-2\pi}{2^{j+1}}\geq 0$ for $j\geq 1$, hence

$$\begin{aligned}
&\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta-2\pi}{2^j}})\right| \\
&=\left|\sum_{k=0}^{\widetilde{n}-1}\binom{2\widetilde{n}-1}{k}\left(\cos\frac{\xi}{2^{j+1}}\,\cos\frac{\zeta-2\pi}{2^{j+1}}\right)^{\widetilde{n}-1-k}\left(-\sin\frac{\xi}{2^{j+1}}\,\sin\frac{\zeta-2\pi}{2^{j+1}}\right)^{k}\right| \\
&\geq\left|\cos\frac{\xi}{2^{j+1}}\,\cos\frac{\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{n}-1}.
\end{aligned}$$

Therefore it follows that for $(\xi,\zeta)\in[0,\pi]\times[\delta,\pi]$,

$$\begin{aligned}
&\left|\widehat{N}(\xi,\zeta-2\pi)\widehat{\widetilde{N}}(\xi,\zeta-2\pi)\right| \\
&\geq\prod_{j=1}^{\infty}\left|\cos\frac{\xi}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\xi+\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{m}}\prod_{j=1}^{\infty}\left|H(e^{-i\frac{\xi}{2^j}},e^{-i\frac{\zeta-2\pi}{2^j}})\right| \\
&\geq\prod_{j=1}^{\infty}\left|\cos\frac{\xi}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{n}}\left|\cos\frac{\xi+\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{m}}\left|\cos\frac{\xi}{2^{j+1}}\,\cos\frac{\zeta-2\pi}{2^{j+1}}\right|^{\widetilde{n}-1} \\
&=\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^{\widetilde{n}}\left|\frac{\sin\frac{\zeta-2\pi}{2}}{\frac{\zeta-2\pi}{2}}\right|^{\widetilde{n}}\left|\frac{\sin\frac{\xi+\zeta-2\pi}{2}}{\frac{\xi+\zeta-2\pi}{2}}\right|^{\widetilde{m}}\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\,\frac{\sin\frac{\zeta-2\pi}{2}}{\frac{\zeta-2\pi}{2}}\right|^{\widetilde{n}-1} \\
&\geq\left(\frac{2}{\pi}\right)^{2\widetilde{n}-1}\left(\frac{\sin\frac{\delta}{2}}{\pi-\frac{\delta}{2}}\right)^{2\widetilde{n}+\widetilde{m}-1} > 0.
\end{aligned}$$

Similarly (4.33) holds for $(\xi,\zeta) \in [\delta,\pi]\times[0,\pi]$. This completes the proof of Lemma 4.4.1.

□

By the same argument in the proof of Lemma 4.4.1, we can prove that

$$\sum_{l\in\mathbb{Z}^2}\left|\widehat{\widetilde{N}}((\xi,\zeta)+2\pi l)\right|^2 \geq C_1 > 0. \tag{4.34}$$

And Lemma 4.3.2 implies

$$\sum_{l\in\mathbb{Z}^2}\left|\widehat{\widetilde{N}}((\xi,\zeta)+2\pi l)\right|^2 \leq C_2. \tag{4.35}$$

By (4.34) and (4.35), $\{\widetilde{N}(x-j,y-k) : j,k\in\mathbb{Z}\}$ constitutes a Riesz basis for $\widetilde{V}_0$. Thus, we conclude that $\widetilde{N}(x,y)$ generates a multiresolution analysis of $L^2(\mathbb{R}^2)$ by the following lemma by Jia and Shen([31]).

Lemma 4.4.2 *Let Φ be a compactly supported refinable function vector. If $\Phi \in (L^2(\mathbb{R}^2))^r$, then*

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^2) \ \textit{and} \ \bigcap_{j\in\mathbb{Z}} V_j = \{0\},$$

where for each $j\in\mathbb{Z}$, V_j is the space generated by the integer translates of $\Phi(2^jx,2^jy)$.

Consider a bivariate Laurent polynomial matrix which is of the form:

$$\widetilde{P}_1^*(z,w) := \begin{bmatrix} \frac{1}{4}\,B_0(z,w) & \widetilde{G}_{1,0}^* & \widetilde{G}_{2,0}^* & \widetilde{G}_{3,0}^* \\ \frac{1}{4}\,zB_1(z,w) & \widetilde{G}_{1,1}^* & \widetilde{G}_{2,1}^* & \widetilde{G}_{3,1}^* \\ \frac{1}{4}\,wB_2(z,w) & \widetilde{G}_{1,2}^* & \widetilde{G}_{2,2}^* & \widetilde{G}_{3,2}^* \\ \frac{1}{4}\,zwB_2(z,w) & \widetilde{G}_{1,3}^* & \widetilde{G}_{2,3}^* & \widetilde{G}_{3,3}^* \end{bmatrix},$$

where $\widetilde{G}_{j,k}, 1\leq j\leq 3, 0\leq k\leq 3$ are defined as follows:

$$\widetilde{G}_{1,0}^* = -\frac{1}{4}p_0A_2(1-A_3), \qquad \widetilde{G}_{1,1}^* = -\frac{1}{4}p_1A_2(1-A_3),$$

$$\widetilde{G}_{1,2}^* = \frac{1}{4}\left((p_0A_0+p_1A_1)(1-A_3)+A_3\right), \qquad \widetilde{G}_{1,3}^* = -\frac{1}{4}A_2,$$

$$\widetilde{G}^*_{2,0} = -\frac{1}{4}p_0A_1(1-A_3), \qquad \widetilde{G}^*_{2,1} = \frac{1}{4}\left((p_0A_0+p_2A_2)(1-A_3)+A_3\right),$$
$$\widetilde{G}^*_{2,2} = -\frac{1}{4}p_2A_1(1-A_3), \qquad \widetilde{G}^*_{2,3} = -\frac{1}{4}A_1,$$
$$\widetilde{G}^*_{3,0} = \frac{1}{4}\left((p_1A_1+p_2A_2)(1-A_3)+A_3\right), \qquad \widetilde{G}^*_{3,1} = -\frac{1}{4}p_1A_0(1-A_3),$$
$$\widetilde{G}^*_{3,2} = -\frac{1}{4}p_2A_0(1-A_3), \qquad \widetilde{G}^*_{3,3} = -\frac{1}{4}A_0.$$

Now we define $\widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3 \in \widetilde{V}_1$ in the Fourier transform,

$$\begin{aligned}
\widehat{\widetilde{M}}_1(\xi,\zeta) &= \widetilde{G}_{1,0}(z,w)\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \widetilde{G}_{1,1}(z,w)\,\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad + \widetilde{G}_{1,2}(z,w)\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \widetilde{G}_{1,3}(z,w)\,\widehat{\widetilde{\phi}}_4(\xi,\zeta) \\
&= -\frac{1}{4}\left(p_0A_2(1-A_3)\right)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_1(\xi,\zeta) - \frac{1}{4}\left(p_1A_2(1-A_3)\right)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad + \frac{1}{4}\big((p_0A_0+p_1A_1)(1-A_3)+A_3\big)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_3(\xi,\zeta) \\
&\quad - \frac{1}{4}A_2(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_4(\xi,\zeta), \\
\widehat{\widetilde{M}}_2(\xi,\zeta) &= \widetilde{G}_{2,0}(z,w)\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \widetilde{G}_{2,1}(z,w)\,\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad + \widetilde{G}_{2,2}(z,w)\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \widetilde{G}_{2,3}(z,w)\,\widehat{\widetilde{\phi}}_4(\xi,\zeta) \\
&= -\frac{1}{4}p_0A_1(1-A_3)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_1(\xi,\zeta) \\
&\quad + \frac{1}{4}\left((p_0A_0+p_2A_2)(1-A_3)+A_3\right)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad - \frac{1}{4}p_2A_1(1-A_3)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_3(\xi,\zeta) - \frac{1}{4}A_1(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_4(\xi,\zeta), \\
\widehat{\widetilde{M}}_3(\xi,\zeta) &= \widetilde{G}_{3,0}(z,w)\,\widehat{\widetilde{\phi}}_1(\xi,\zeta) + \widetilde{G}_{3,1}(z,w)\,\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad + \widetilde{G}_{3,2}(z,w)\,\widehat{\widetilde{\phi}}_3(\xi,\zeta) + \widetilde{G}_{3,3}(z,w)\,\widehat{\widetilde{\phi}}_4(\xi,\zeta) \\
&= \frac{1}{4}\left((p_1A_1+p_2A_2)(1-A_3)+A_3\right)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_1(\xi,\zeta) \\
&\quad - \frac{1}{4}p_1A_0(1-A_3)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_2(\xi,\zeta) \\
&\quad - \frac{1}{4}p_2A_0(1-A_3)(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_3(\xi,\zeta) - \frac{1}{4}A_0(\bar z,\bar w)\cdot\widehat{\widetilde{\phi}}_4(\xi,\zeta),
\end{aligned}$$

From the above definitions, clearly we know $\widetilde{M}_1 = \widetilde{M}$. And it is easy to check that $\widetilde{P}^*_1(z,w) = \frac{1}{4}\left[P_1(z,w)\right]^{-1}$. Since $P_1(z,w)$ has a constant determinant by Theorm 4.2.1, the matrix $\widetilde{P}_1(z,w)$ also has a constant determinant. Since $\left(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4\right)^T$ generates $\widetilde{V}_1$, hence $\left(\widetilde{N}, \widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3\right)^T$ generates $\widetilde{V}_1$. And it is easy to check that matrices $P_1(z,w)$ and $\widetilde{P}_1(z,w)$ satisfy

$$P_1\widetilde{P}_1^*(z,w) + P_1\widetilde{P}_1^*(-z,w) + P_1\widetilde{P}_1^*(z,-w) + P_1\widetilde{P}_1^*(-z,-w) = 1.$$

We put

$$\widetilde{M}_{11}(x,y) = \widetilde{M}_1(2x,2y), \qquad \widetilde{M}_{12}(x,y) = \widetilde{M}_1(2x-1,2y),$$

$$\widetilde{M}_{13}(x,y) = \widetilde{M}_1(2x,2y-1), \qquad \widetilde{M}_{14}(x,y) = \widetilde{M}_1(2x-1,2y-1),$$

and by the same way, define $\widetilde{M}_{21}, \cdots, \widetilde{M}_{24}$ and $\widetilde{M}_{31}, \cdots, \widetilde{M}_{34}$, for $\widetilde{M}_2$ and $\widetilde{M}_3$, respectively. Then we can see that $\left(\widetilde{\phi}_1, \cdots, \widetilde{\phi}_4, \widetilde{M}_{11}, \widetilde{M}_{12}, \cdots, \widetilde{M}_{34}\right)^T$ generates $\widetilde{V}_2$. Summarizing the discussion above gives the following theorem.

Theorem 4.4.3 *Let $N(x,y)$ be the bivariate box spline function of order (l,m,n), and let $\widetilde{A}^*(z,w)$ be a Laurent polynomial of the form*

$$\widetilde{A}^*(z,w) = \frac{1}{4}\left(\frac{1+z}{2}\right)^{\widetilde{n}-l}\left(\frac{1+w}{2}\right)^{\widetilde{n}-m}\left(\frac{1+zw}{2}\right)^{\widetilde{m}-n} H(z,w)D(zw),$$

where $H(z,w), D(zw)$ are defined as follows

$$H(z,w) = \sum_{k=0}^{\widetilde{n}-1}\binom{2\widetilde{n}-1}{k}\left(\frac{1+z}{2}\frac{1+w}{2}\right)^{\widetilde{n}-1-k}\left(\frac{1-z}{2}\frac{1-w}{2}\right)^{k},$$

$$D(zw) = \sum_{k=0}^{L-1}\binom{2L-1}{k}\left(\frac{1+zw}{2}\right)^{L-1-k}\left(\frac{1-zw}{2}\right)^{k}, \ L = 2\widetilde{n}+\widetilde{m}-1.$$

Define a function $\widetilde{N}(x,y)$ by

$$\widehat{\widetilde{N}}(\xi,\zeta) = \prod_{j=1}^{\infty}\widetilde{A}(\frac{\xi}{2^j},\frac{\zeta}{2^j}),$$

and define two function vectors $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1,\widetilde{\phi}_2,\widetilde{\phi}_3,\widetilde{\phi}_4,\widetilde{\phi}_5)^T$ in terms of the Fourier transform

$$\widehat{\phi}_1(2\xi,2\zeta) = \frac{1}{4}\widehat{N}(\xi,\zeta), \qquad \widehat{\widetilde{\phi}}_1(2\xi,2\zeta) = 4\widehat{\widetilde{N}}(\xi,\zeta),$$

$$\widehat{\phi}_2(2\xi, 2\zeta) = \frac{1}{4} e^{-i\xi} \widehat{N}(\xi, \zeta), \qquad \widehat{\widetilde{\phi}}_2(2\xi, 2\zeta) = 4 e^{-i\xi} \widehat{\widetilde{N}}(\xi, \zeta),$$

$$\widehat{\phi}_3(2\xi, 2\zeta) = \frac{1}{4} e^{-i\zeta} \widehat{N}(\xi, \zeta), \qquad \widehat{\widetilde{\phi}}_3(2\xi, 2\zeta) = 4 e^{-i\zeta} \widehat{\widetilde{N}}(\xi, \zeta),$$

$$\widehat{\phi}_4(2\xi, 2\zeta) = \frac{1}{4} e^{-i\xi} e^{-i\zeta} \widehat{N}(\xi, \zeta), \qquad \widehat{\widetilde{\phi}}_4(2\xi, 2\zeta) = 4 e^{-i\xi} e^{-i\zeta} \widehat{\widetilde{N}}(\xi, \zeta),$$

$$\widehat{\phi}_5(2\xi, 2\zeta) = \frac{1}{4} \widehat{M}(2\xi, 2\zeta), \qquad \widehat{\widetilde{\phi}}_5(2\xi, 2\zeta) = 4 \widehat{\widetilde{M}}(\xi, \zeta),$$

with

$$\begin{aligned}
\widehat{M}(\xi, \zeta) = & - h_1 A_0(z, w) \cdot \widehat{\phi}_1(\xi, \zeta) - h_1 A_1(z, w) \cdot \widehat{\phi}_2(\xi, \zeta) \\
& - (h_1 A_2 - 1)(z, w) \cdot \widehat{\phi}_3(\xi, \zeta) - \big(h_1 A_3 + p_2(1 - A_3)\big)(z, w) \cdot \widehat{\phi}_4(\xi, \zeta), \\
\widehat{\widetilde{M}}(\xi, \zeta) = & - \frac{1}{4} \left(p_0 A_2 (1 - A_3)\right)(\bar{z}, \bar{w}) \cdot \widehat{\widetilde{\phi}}_1(\xi, \zeta) - \frac{1}{4} \left(p_1 A_2 (1 - A_3)\right)(\bar{z}, \bar{w}) \cdot \widehat{\widetilde{\phi}}_2(\xi, \zeta) \\
& + \frac{1}{4} \big((p_0 A_0 + p_1 A_1)(1 - A_3) + A_3\big)(\bar{z}, \bar{w}) \cdot \widehat{\widetilde{\phi}}_3(\xi, \zeta) - \frac{1}{4} A_2(\bar{z}, \bar{w}) \cdot \widehat{\widetilde{\phi}}_4(\xi, \zeta).
\end{aligned}$$

where p_j's and h_j's, $j = 1, 2, 3$ satisfy (4.11)-(4.14). Then, for sufficiently large $\widetilde{m}$ and $\widetilde{n}$, $\widetilde{N}(x, y)$ generates a multiresolution analysis of $L^2(\mathbb{R}^2)$. And further, two function vectors $\Phi, \widetilde{\Phi}$ generate $\mathcal{S}, \widetilde{\mathcal{S}}$ respectivly, with

$$V_1 \subset \mathcal{S} \subset V_2 \ \textit{and} \ \widetilde{V}_1 \subset \widetilde{\mathcal{S}} \subset \widetilde{V}_2,$$

where for $j \in \mathbb{Z}$, $V_j, \widetilde{V}_j$ are defined as

$$V_0 = span\{N(x - l, y - k) : l, k \in \mathbb{Z}\}, \quad V_j = \{f(2^j x, 2^j y) : f \in V_0\},$$

$$\widetilde{V}_0 = span\{\widetilde{N}(x - l, y - k) : l, k \in \mathbb{Z}\}, \quad \widetilde{V}_j = \{f(2^j x, 2^j y) : f \in \widetilde{V}_0\}.$$

4.5 Matrix Extension for Wavelet Function Vectors

We have constructed a class of compactly supported multiscaling functions. Now we need to construct multiwavelet function vectors $\Psi = (\psi_1, \cdots, \psi_{15})^T$, $\widetilde{\Psi} = (\widetilde{\psi}_1, \cdots, \widetilde{\psi}_{15})^T$ associated with the dual scaling function vectors $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4, \widetilde{\phi}_5)^T$, respectively. To construct wavelet functions $\Psi, \widetilde{\Psi}$, we need to find matrix symbols $Q(z,w)$, $\widetilde{Q}(z,w)$, associated with matrix symbols $P(z,w)$ and $\widetilde{P}(z,w)$, where these four matrix symbols satisfy (4.1)-(4.4). Similarly to the univariate case in Chapter 3, it turns out to be a matrix extension problem.

In general, we write $\mathcal{P}(\mathbf{z}) = [P(\omega + \pi\mathbf{k}), \mathbf{k} \in \{0,1\}^d]$ to be a block matrix of size $r \times r2^d$ with trogonometric polynomials as its entries. In the same way, $\widetilde{\mathcal{P}}(\mathbf{z}) = [\widetilde{P}(\omega + \pi\mathbf{k}), \mathbf{k} \in \{0,1\}^d]$. Then we need to find the block matrices $\mathcal{Q}(\mathbf{z})$ and $\widetilde{\mathcal{Q}}(\mathbf{z})$ of size $r(2^d - 1) \times r2^d$ such that

$$\begin{bmatrix} \mathcal{P}(\mathbf{z}) \\ \mathcal{Q}(\mathbf{z}) \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{P}}^*(\mathbf{z}) & \widetilde{\mathcal{Q}}^*(\mathbf{z}) \end{bmatrix} = I_{r2^d \times r2^d}. \tag{4.36}$$

For $r \geq 1$ and $d = 1$, the matrix extension problem was treated in [33]. For $r = 1$ and $d > 1$, several extension methods related to box spline biorthogonal wavelets are available in the literature, e.g., in [30] and [25]. The case of $r > 1$ and $d > 1$, can be found in an unpublished manuscript [7].

As usual, once we find $\mathcal{Q}(\mathbf{z})$ and $\widetilde{\mathcal{Q}}(\mathbf{z})$, we define Ψ and $\widetilde{\Psi}$ in terms of the Fourier transform by

$$\widehat{\Psi}(\omega) = \mathcal{Q}(\frac{\omega}{2})\,\widehat{\Phi}(\frac{\omega}{2}), \quad \widehat{\widetilde{\Psi}}(\omega) = \widetilde{\mathcal{Q}}(\frac{\omega}{2})\,\widehat{\widetilde{\Phi}}(\frac{\omega}{2}).$$

Therefore we have the following theorem:

Theorem 4.5.1 *Let* $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T$, $\widetilde{\Phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4, \widetilde{\phi}_5)^T$ *be two scaling function vectors defined in Theorem 4.4.3. Let*

$$\begin{bmatrix} \mathcal{P}(\mathbf{z}) \\ \mathcal{Q}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} P(z,w) & P(-z,w) & P(z,-w) & P(-z,-w) \\ Q(z,w) & Q(-z,w) & Q(z,-w) & Q(-z,-w) \end{bmatrix}_{20\times 20},$$

$$\begin{bmatrix} \widetilde{\mathcal{P}}(\mathbf{z}) \\ \widetilde{\mathcal{Q}}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} \widetilde{P}(z,w) & \widetilde{P}(-z,w) & \widetilde{P}(z,-w) & \widetilde{P}(-z,-w) \\ \widetilde{Q}(z,w) & \widetilde{Q}(-z,w) & \widetilde{Q}(z,-w) & \widetilde{Q}(-z,-w) \end{bmatrix}_{20\times 20},$$

be matrix extensions of $\mathcal{P}(\mathbf{z}), \widetilde{\mathcal{P}}(\mathbf{z})$*, which are obtained by the method above, where* $\mathbf{z} = (z, w) = (e^{-i\xi}, e^{-i\zeta})$*. Define two function vectors* $\Psi, \widetilde{\Psi}$ *in terms of the Fourier Transform by*

$$\widehat{\Psi}(2\xi, 2\zeta) = Q(\omega, \zeta)\,\widehat{\Phi}(\omega, \zeta), \quad \widehat{\widetilde{\Psi}}(2\omega, 2\zeta) = \widetilde{Q}(\omega, \zeta)\,\widehat{\widetilde{\Phi}}(\omega, \zeta).$$

Then $\Psi = (\psi_1, \cdots, \psi_{15})^T$ *and* $\widetilde{\Psi} = (\widetilde{\phi}_1, \cdots, \widetilde{\phi}_{15})^T$ *are wavelet function vectors associated with* Φ *and* $\widetilde{\Phi}$ *respectively, satisfying (4.1)-(4.4).*

Remark 4.5.2 In the above construction, we are on $r = 5$ and $d = 2$. The algorithm from [7] is based on induction on $r = 5$, which is the number of rows in the matrix $\mathcal{P}(\mathbf{z})$ of size 5×20. Even this algorithm is quite constructive, it is not the easy work in bivariate setting.

4.6 Regularity

Similar to the univariate case, we compute the required numbers for $\widetilde{n}, \widetilde{m}$ to give $\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3, \widetilde{\phi}_4, \widetilde{\phi}_5$ some smoothness .

Consider $N(x, y) = B_{1,1,1}(x, y)$, the box spline function of order $(1, 1, 1)$. We choose $\widetilde{n} = 3$ in the definition of $\widetilde{A}^*(z) = B(z)$, then we have

$$\widetilde{A}^*(z, w) = \frac{1}{4}\left(\frac{1+z}{2}\right)^2 \left(\frac{1+w}{2}\right)^2 \left(\frac{1+zw}{2}\right)^{\widetilde{m}-1} H(z, w) D(zw),$$

where

$$H(z, w) = \sum_{k=0}^{2} \binom{5}{k} \left(\frac{1+z}{2}\frac{1+w}{2}\right)^{2-k} \left(\frac{1-z}{2}\frac{1-w}{2}\right)^{k},$$

$$D(zw) = \sum_{k=0}^{4+\widetilde{m}} \binom{9+2\widetilde{m}}{k} \left(\frac{1+zw}{2}\right)^{4+\widetilde{m}-k} \left(\frac{1-zw}{2}\right)^{k}.$$

The definition of $\widetilde{N}$ implies

$$\left|\widehat{\widetilde{N}}(\xi,\zeta)\right| = \prod_{j=1}^{\infty}\left|\widetilde{A}(e^{-i\frac{\xi}{2^j}}, e^{-i\frac{\zeta}{2^j}})\right|$$

$$= \prod_{j=1}^{\infty}\left|\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\cdot\frac{1+e^{-i\frac{\zeta}{2^j}}}{2}\right|^2 \left|H(e^{-i\frac{\xi}{2^j}},\, e^{-i\frac{\zeta}{2^j}})\right|$$

$$\cdot\prod_{j=1}^{\infty}\left|\frac{1+e^{-i\frac{\xi}{2^j}}e^{-i\frac{\zeta}{2^j}}}{2}\right|^{\widetilde{m}-1} D(e^{-i\frac{\xi}{2^j}}e^{-i\frac{\zeta}{2^j}}).$$

It follows from Table 3.1 that

$$\int_{\mathbb{R}^2}\prod_{j=1}^{\infty}\left|\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\cdot\frac{1+e^{-i\frac{\zeta}{2^j}}}{2}\right|^2 \left|H(e^{-i\frac{\xi}{2^j}}, e^{-i\frac{\zeta}{2^j}})\right| d\xi d\zeta$$

$$\leq C\int_{\mathbb{R}^2}\prod_{j=1}^{\infty}\left|\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\cdot\frac{1+e^{-i\frac{\zeta}{2^j}}}{2}\right|^2 \left[\mathbf{P}_3\left(\sin^2\frac{\xi}{2^{j+1}}\right)\cdot\mathbf{P}_3\left(\sin^2\frac{\zeta}{2^{j+1}}\right)\right]^{\frac{1}{2}} d\xi d\zeta$$

$$\leq C\int_{\mathbb{R}^2}(1+|\xi|)\;|{}_3\widehat{\phi}(\xi)|\cdot(1+|\zeta|)\;|{}_3\widehat{\phi}(\zeta)|\,d\xi d\zeta$$

$$= C\int_{\mathbb{R}}(1+|\xi|)\;|{}_3\widehat{\phi}(\xi)|\,d\xi\cdot\int_{\mathbb{R}}(1+|\zeta|)\;|{}_3\widehat{\phi}(\zeta)|\,d\zeta$$

$$\leq C\int_{\mathbb{R}}(1+|\xi|)^{\alpha(3)}\;|{}_3\widehat{\phi}(\xi)|\,d\xi\cdot\int_{\mathbb{R}}(1+|\zeta|)^{\alpha(3)}\;|{}_3\widehat{\phi}(\zeta)|\,d\zeta\;<\infty.$$

By Lemma 4.3.2, we choose $\widetilde{m}$ such that $\widetilde{m} > (1+2\log_2 3)/(1-\frac{1}{2}\log_2 3) = 20.0942$, this implies

$$\prod_{j=1}^{\infty}\left(\frac{1+e^{-i\frac{\xi+\zeta}{2^j}}}{2}\right)^{\widetilde{m}-1} D(e^{-i\frac{\xi+\zeta}{2^j}}) \leq C.$$

Thus, the choice of $\widetilde{n}=3, \widetilde{m}=21$ is needed for $\widetilde{N}\in C^0$, it follows that $\widetilde{\Phi}=(\widetilde{\phi}_1,\cdots,\widetilde{\phi}_5)^T\in C^0$. By the same argument, we can make $\widetilde{\Phi}\in C^1$ or C^2 with the appropriate choice of $\widetilde{n}$ and $\widetilde{m}$. For a given box spline function, required numbers of $\widetilde{n}$ and $\widetilde{m}$ for some regularity are listed in the table below:

$N = B_{l,m,n}$	$\widetilde{\Phi}$	$\widetilde{\Phi}$	$\widetilde{\Phi}$
	C^0	C^1	C^2
$(1,1,1)$	$\widetilde{n}=3, \widetilde{m}=21$	$\widetilde{n}=6, \widetilde{m}=44$	$\widetilde{n}=9, \widetilde{m}=66$
$(2,2,1)$	$\widetilde{n}=6, \widetilde{m}=44$	$\widetilde{n}=9, \widetilde{m}=66$	

Table 4.1 Regularity of $\widetilde{\Phi}$

This can be compared to the result by He and Lai([25]):

$B_{l,m,n}$	$\widetilde{B}_{l,m,n}$	$\widetilde{B}_{l,m,n}$	$\widetilde{B}_{l,m,n}$
	C^0	C^1	C^2
$(1,1,1)$	$\widetilde{n}=3, \widetilde{m}=25$	$\widetilde{n}=6, \widetilde{m}=47$	$\widetilde{n}=9, \widetilde{m}=71$
$(2,2,1)$	$\widetilde{n}=6, \widetilde{m}=47$	$\widetilde{n}=9, \widetilde{m}=71$	

Table 4.2 Regularity of $\widetilde{B}_{l,m,n}$

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