ASYMPTOTIC BEHAVIOR OF ARITHMETIC EQUIVARIANTS IN NON-ARCHIMEDEAN DYNAMICS

by

KENNETH SCOTT JACOBS

(Under the Direction of Robert Rumely)

ABSTRACT

Rumely recently introduced three arithmetic equivariants attached to a rational map ϕ over a non-Archimedean field. The first is a function $\mathrm{ordRes}_\phi : \mathbf{P}^1 \to \mathbb{R}$ carrying information about the resultant of various conjugates of ϕ. The second is the set of points where ordRes_ϕ is minimized; in the case that ϕ has potential good reduction, this set identifies the conjugate ϕ^γ at which ϕ attains good reduction. The third object is a measure ν_ϕ, a weighted sum of finitely many points in the hyperbolic Berkovich line; the weights are determined by local geometric properties of the map ϕ. In this dissertation, we study the asymptotic behavior of the corresponding objects attached to the iterates of ϕ.

INDEX WORDS: Arithmetic dynamics, reduction, resultant, crucial measures, equidistribution, non-Archimedean

Asymptotic Behavior of Arithmetic Equivariants in non-Archimedean Dynamics

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Dedication

To my grandmothers, Ann Garvey and Helen Jacobs.

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Finally, to my mother, who continues to give me unfailing support, to my father, who first instilled in me a sense of curiosity and wonder about the world, and to my sister, who set an example by going before me in the task of earning a PhD, I owe a debt of thanks that I will happily work for the rest of my life to repay.

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Chapter 1

Introduction

This thesis investigates the asymptotic behaviour of arithmetic equivariants attached to iterates $\phi^{(n)} = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$ of a rational map ϕ. We will be interested almost exclusively in the case that ϕ is defined over a complete, algebraically closed non-Archimedean valued field, which we denote by K; its ring of integers will be written $\mathcal{O} = \{x \in K \ : \ |x| \leq 1\}$. The Berkovich hyperbolic line over K will be written $\mathbf{H}^1$, while the Berkovich projective line will be denoted $\mathbf{P}^1$ (formal definitions of these spaces will be given in Chapter 2).

We consider three equivariants in this thesis in this thesis, each of which encodes some of the arithmetic information about the map ϕ. These objects originated in work of Rumely [24, 25], who was motivated by the following question: which $\mathrm{PGL}_2(K)$-conjugates $\phi^\gamma := \gamma^{-1} \circ \phi \circ \gamma$ of ϕ attain the 'best possible reduction'? One of Rumely's key insights was that elements of $\mathrm{PGL}_2(K)/\mathrm{PGL}_2(\mathcal{O})$ are in correspondence with (a dense subset of) the hyperbolic Berkovich line $\mathbf{H}^1$; using this correspondence, he translated the arithmetic question of 'best possible reduction' to more geometric questions about functions and measures on $\mathbf{H}^1$.

The first tool which Rumely introduced in studying this question was a function $\mathrm{ordRes}_\phi : \mathbf{H}^1 \to \mathbb{R}$ which is defined as follows: given a normalized[1] homogeneous lift $\Phi = (F, G)$ of ϕ, first define $\mathrm{ordRes}(\phi) := -\log_v |\mathrm{Res}(F, G)|_v$, where $\mathrm{Res}(F, G)$ is the resultant of F and G and $\log_v$ is a suitably normalized logarithm. Using the correspondence between points of $\mathbf{H}^1$ and elements of

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[1] A homogeneous lift $\Phi = (F, G)$ of the map ϕ is said to be normalized if the coefficients of F and G all have absolute value at most 1, and at least one coefficient has absolute value equal to 1.

$\mathrm{PGL}_2(K)$, this definition extends to a dense subset of $\mathbf{H}^1$ by $\mathrm{ordRes}_\phi(\zeta) := \mathrm{ordRes}(\phi^{\gamma_\zeta})$, where $\zeta \in \mathbf{H}^1$ corresponds to the matrix $\gamma_\zeta \in \mathrm{PGL}_2(K)$. Rumely shows in [24] Theorem 0.1 that this function extends to be continuous with respect to the strong topology and piecewise affine along segments in $\mathbf{H}^1$.

Our first main result shows that the (normalized) family of functions $\left\{\frac{1}{d^{2n}-d^n}\,\mathrm{ordRes}_{\phi^n}\right\}_{n\in\mathbb{N}}$ attached to the iterates of ϕ converges to the diagonal values of the Arakelov-Green's function:

Theorem 1.1. *The functions* $\left\{\frac{1}{d^{2n}-d^n}\,\mathrm{ordRes}_{\phi^n}\right\}_{n\in\mathbb{N}}$ *converge locally uniformly in the strong topology on* $\boldsymbol{H}^1$ *to* $g_\phi(x,x)$.

Theorem 1.1 gives an arithmetic interpretation of the Arakelov-Green's function, which is a priori defined in terms of potential-theoretic objects. In Chapter 3, we give an explicit error term for this convergence (Theorem 3.3).

Our main interest in Theorem 1.1, however, is to use it in showing the weak convergence of the *crucial measures* ν_{ϕ^n}. Rumely first defined the crucial measure ν_ϕ attached to a rational map in [25] in terms of the Berkovich Laplacian of ordRes_ϕ. He showed that it is a probability measure with finite support in $\mathbf{H}^1$ that can be realized as a weighted sum of points

$$\nu_\phi = \frac{1}{d-1}\sum_{P\in\mathbf{P}^1} w_\phi(P)\delta_P \ ;$$

here the weights $w_\phi(P)$ can be explicitly determined by looking at the reduction of ϕ at P and the behaviour of the tangent map ϕ_* (see [25] Definition 8 or Definition 1 below). If one considers now the family of measures $\{\nu_{\phi^n}\}_{n\in\mathbb{N}}$ coming from the iterates of ϕ, we find

Theorem 1.2. *The measures* ν_{ϕ^n} *converge weakly to the equilibrium measure* μ_ϕ *attached to* ϕ.

Here again, we are able to give an explicit estimate on the error term for a given dense class of test functions on $\mathbf{H}^1$ (Theorem 3.11).

Having established these two convergence results, we next turn our attention to an important subset of $\mathbf{H}^1$ determined by ordRes_ϕ, namely the collection of points where ordRes_ϕ is minimized. Rumely denotes this set $\mathrm{MinResLoc}(\phi)$, the minimial resultant locus of ϕ, and has shown that this is a non-empty subset of $\mathbf{H}^1$ which is either a point or a segment ([24], Theorem 0.1). Points in

$\mathrm{MinResLoc}(\phi)$ correspond to conjugates ϕ^γ which attain the 'best possible reduction', and as such carry important arithmetic information about the map ϕ.

An interesting fact about the sets $\mathrm{MinResLoc}(\phi)$ is that it is the barycenter of the measure ν_ϕ ([24] Theorem 7.1); the *barycenter* of a probability measure ν on the Berkovich projective line $\mathbf{P}^1$ is the collection of points which are 'balanced' with respect to ν (a precise definition is given in Definition 2 below). Given that the measures ν_{ϕ^n} converge weakly to μ_ϕ, one might expect that the barycenters $\mathrm{MinResLoc}(\phi^n)$ of ν_{ϕ^n} might converge to the barycenter $\mathrm{Bary}(\mu_\phi)$. We show

Theorem 1.3. *Suppose char$(K) = 0$. Let $\phi \in K(z)$ have degree $d \geq 2$, and let $R = \frac{2}{d-1} \operatorname{ord} \mathrm{Res}(\phi)$. Let $B_\rho(\zeta, R)$ denote the ball of radius R about ζ in the ρ-metric on $\boldsymbol{H}^1$.*

[*A*] *(Approximation) For every $\epsilon > 0$, there exists an N so that $\mathrm{MinResLoc}(\phi^n)$ is contained in the ϵ-ball around $\mathrm{Bary}(\mu_\phi)$ for every $n \geq N$.*

[*B*] *(Uniform Bounds) For every n we have $\mathrm{MinResLoc}(\phi^n) \subseteq B_\rho(\zeta_G, R)$; moreover, there is a constant M depending only on ϕ such that $\mathrm{Bary}(\mu_\phi) \subseteq B_\rho(\zeta_G, R + M)$.*

Part [A] can be viewed as 'one half' of Hausdorff convergence of closed sets; unfortunately, an concrete example shows that the other 'half' of Hausdorff convergence cannot hold in general (see Example 4 below).

The last main goal in this thesis is to prove a strengthening of Theorem 1.1, namely to show that

$$\int f d\nu_{\phi^n} \to \int f d\mu_\phi$$

when the functions f are allowed to have (logarithmic) singularities at the points of $\mathbb{P}^1(K)$. This stronger notion of convergence, which we call 'logarithmic equidistribution', plays an important role in potential theory and dynamics.

Establishing the logarithmic equidistribution of the crucial measures required a study of the geometric properties of the points in the support of ν_{ϕ^n}. In Chapter 5 we give explicit bounds for how quickly these points can approach the boundary of $\mathbf{H}^1$. Combining these estimates with the explicit equidistribution of the crucial measures, we show:

Theorem 1.4. *Suppose char$(K) = 0$. Let $\phi \in K(z)$ have degree $d \geq 2$. Fix any point $a \in \mathbb{P}^1(K)$; then*

$$\int_{\boldsymbol{P}^1} \log_v ||\cdot, a|| d\nu_{\phi^n} \to \int_{\boldsymbol{P}^1} \log_v ||\cdot, a|| d\mu_\phi \ .$$

Here, the Hsia kernel $||\cdot,\cdot||$ is an extension of the chordal metric on $\mathbb{P}^1(K)$ to $\mathbf{P}^1$; a precise definition is given in Chapter 2. The fact that $a \in \mathbb{P}^1(K)$ implies that $\log_v ||\zeta, a||$ has a singularity at $\zeta = a \in \mathbb{P}^1(K)$, but it is well-defined and finite for any other point $\zeta \in \mathbf{P}^1$. We are again able to give an explicit error bound for this convergence (Theorem 6.6); surprisingly, the error constant is independent of which point $a \in \mathbb{P}^1(K)$ we choose.

One other unexpected consequence of proving Theorem 1.4 was that the same method could be applied to extend other 'classical' equidistribution results to logarithmic equidistribution results. Favre and Rivera-Letelier have shown that pullbacks $\frac{1}{d^n}(\phi^n)^*\nu$ of a probablity measure ν which does not charge the exceptional set converge weakly to μ_ϕ ([13] Theéorème A). In Chapter 6, we extend their result to show that the pullback of probability measures ν which have *bounded potentials* satisfy a logarithmic equidistribution condition:

Theorem 1.5. *Let ν be a probability measure on $\boldsymbol{P}^1$ which has bounded potentials, and let $\nu_n = \frac{1}{d^n}(\phi^n)^*\nu$. Fix a point $a \in \mathbb{P}^1(K)$; then*

$$\int \log_v ||\cdot, a|| d\nu_n \to \int \log_v ||\cdot, a|| d\mu_\phi \ .$$

Here again, we are able to give an explicit error term (Theorem 6.9) which is independent of which point $a \in \mathbb{P}^1(K)$ is chosen.

We close this introduction with an outline of the thesis. In Chapter 2, we establish the necessary background concerning dynamics and potential theory on $\mathbf{P}^1$. Our primary reference for this task is the book by Baker and Rumely [2]. In Chapter 3, we prove Theorem 1.1 and show how it implies the weak convergence of the measures ν_{ϕ^n} against continuous test functions (Theorem 3.2). In Chapter 4 we take up the question of the asymptotic behavior of the sets MinResLoc(ϕ^n) and prove Theorem 1.3.

We then turn to the question of the logarithmic equidistribution of the measures ν_{ϕ^n}. For this, we establish certain quantitative bounds on the geometry of $\mathrm{supp}(\nu_{\phi^n})$ in Chapter 5, showing that the mass from these measures cannot accumulate too quickly at the boundary of $\mathbf{P}^1$; this is summarized in Theorem 5.1. Finally, in Chapter 6, we are able to prove the logarithmic equidistribution part of Theorem 1.2, as well as logarithmic equidistribution of the pullback of probability measures with bounded potentials.

Chapter 2

Background on non-Archimedean Dynamics and Potential Theory

Throughout this thesis, K will denote a field endowed with a non-Archimedean absolute value $|\cdot|$. Recall that an absolute value is said to be *non-Archimedean* if it satisfies a stronger form of the triangle inequality

$$|x+y| \leq \max(|x|, |y|) \ .$$

Most often, we will assume that the field K is complete with respect to this absolute value and is also algebraically closed. With the exception of Chapters 5 and 6, where we assume $\mathrm{char}(K) = 0$, we will make no assumption about the characteristic of K.

We will denote the closed disc of radius r about a point $a \in K$ by $D(a,r) := \{z \in K \ : \ |z-a| \leq r\}$; similarly, the open disc of radius r about a is $D(a,r)^- := \{z \in K \ : \ |z-a| < r\}$. While we necessarily have $D(a,r)^- \subsetneq D(a,r)$, it is important to note that both of these discs are simultaneously open and closed subsets of K in the metric topology.

The closed unit disc in K is denoted $\mathcal{O} := D(0,1)$; this geometric object is in fact a ring, and the open unit disc $D(0,1)^-$ is a maximal ideal in $\mathcal{O}$, which we denote by $\mathfrak{m}$. The quotient $k = \mathcal{O}/\mathfrak{m}$ is called the *residue field* of K. If $q_v = \mathrm{char}(k) > 0$, we normalize the absolute value so that $-\log_{q_v} |x|_v = \mathrm{ord}_{\mathfrak{m}}(x)$; otherwise, we let $\log_v(x) = \ln(x)$.

2.1 The Berkovich Projective Line

Complex dynamics considers the action of a rational map $\phi \in \mathbb{C}(z)$ on the projective line $\mathbb{P}^1(\mathbb{C})$. When ϕ is instead defined over a field K endowed with a non-Archimedean absolute value $|\cdot|$, many aspects of complex dynamics change. In large part, these changes are due to the fact that $\mathbb{P}^1(K)$ is totally disconnected in the analytic topology, and when K is algebraically closed and $|\cdot|_v$ is nontrivial, it even fails to be locally compact! The most fruitful remedy to this problem is to embed the projective line $\mathbb{P}^1(K)$ into a larger space $\mathbf{P}^1$ which is better suited to the tools of analysis.

Formally, $\mathbf{P}^1$ is the collection of all multiplicative seminorms on $K[X,Y]$ which extend the absolute value on K and which do not vanish on the ideal (X,Y), modulo an appropriate equivalence relation. This construction, in terms of seminorms, can also be carried out when $K = \mathbb{C}$, and in this case it is a consequence of the Gelfand-Mazur theorem that $\mathbf{P}^1_{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$.

There are two topologies that one can put on $\mathbf{P}^1$. The first and more natural topology is the Gelfand-Mazur topology, which is also refered to as the weak topology. It is the coarsest topology ensuring that the maps

$$x \mapsto [f]_x$$

are continuous for every $f \in K[X,Y]$. In this topology, the space $\mathbf{P}^1$ has several desirable properties: it is uniquely path connected, locally compact, and Hausdorff. However, the Gelfand-Mazur topology is not metrizable in general.

The second topology will be important in the development of potential theory on $\mathbf{P}^1$. It is called the strong topology, and will be introduced in Section 2.1.3 below. It is a metric topology which is strictly finer than the weak topology; however, it is important to note that $\mathbf{P}^1$ fails to be locally compact in this topology!

2.1.1 The Points of $\mathbf{P}^1$

The projective line $\mathbb{P}^1(K)$ embeds naturally into $\mathbf{P}^1$, and when $\mathbf{P}^1$ is endowed with the Gelfand topology, $\mathbb{P}^1(K)$ is a dense subset. We call the points of $\mathbb{P}^1(K)$ *points of type I*; in terms of seminorms, a point $\zeta = [a, b] \in \mathbb{P}^1(K)$ corresponds to an evaluation seminorm

$$f \mapsto f(a, b) =: [f]_\zeta \ , \ \forall f \in K[X, Y] \ .$$

The complement of $\mathbb{P}^1(K)$ in $\mathbf{P}^1$ is the *Berkovich hyperbolic line* and is denoted $\mathbf{H}^1$. We now describe the points in $\mathbf{H}^1$.

There are two classes of points, called *points of type II and III*, which correspond to disc seminorms: any disc $D(a, r) \subseteq K$ defines a point $\zeta = \zeta_{a,r} \in \mathbf{P}^1$ by

$$f \mapsto \sup_{z \in D(a,r)} |f(z, 1)| := [f]_{D(a,r)} \ , \ \forall f \in K[X, Y] \ .$$

We will often denote these seminorms by $[f]_\zeta$. The point ζ is a *type II point* if r lies in the value group $|K^\times|$, and a *type III point* if $r \notin |K^\times|$. Depending on the field K, there can also be *points of type IV*, which correspond to (cofinal equivalence classes of) nested discs $D(a_i, r_i) \supseteq D(a_{i+1}, r_{i+1})$ which have an empty intersection but for which $\lim_{i\to\infty} r_i > 0$; concretely, a point $\zeta \in \mathbf{P}^1$ of type IV is given by

$$f \mapsto \lim_{i\to\infty} [f]_{a_i, r_i} \ , \ \forall f \in K[X, Y]$$

for such a family of discs. These points can only exist when K is not spherically complete. Taken together, points of type I, II, III, and IV constitute the entirety of $\mathbf{P}^1$. We write $\mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\}$ for the Berkovich affine line.

2.1.2 Tree Structure

The collection of type II and type III points can be given the structure of an $\mathbb{R}$-tree [1]. An edge in this tree is defined by familes of discs $\{D(a, s)\}_{s\in[r,R]}$ for some constants $0 < r < R < \infty$. We write $[\zeta_{a,r}, \zeta_{a,R}]$ for such a segment. Points of type I or of type IV form the endpoints of this tree.

This tree is *far* from being 'well-behaved': when K is algebraically closed, every type II point $\zeta \in \mathbf{H}^1$ has an infinite number of branches away from itself (these are in one-to-one correspondence

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[1] An $\mathbb{R}$-tree is a metric space (X, d) where there is a unique arc connecting any two points $x, y \in X$. See [2] Appendix B for a detailed discussion.

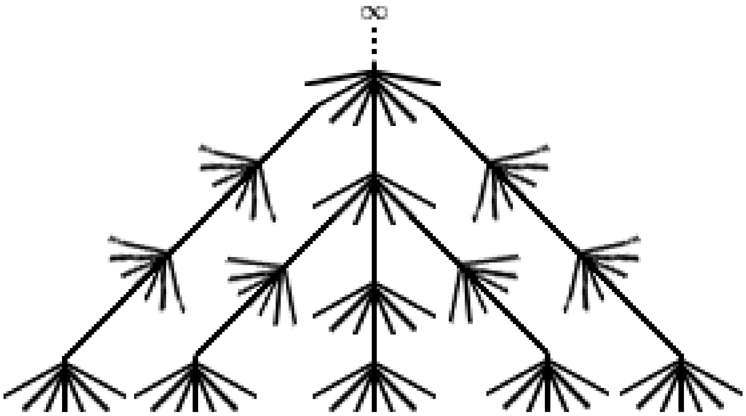

Figure 2.1: A sketch of $\mathbf{A}^1$. The points along the bottom of the tree type I points. Points with many branches are type II points, and are in fact dense along any edge in $\mathbf{H}^1$. Points of type III are non-branch points along segments in $\mathbf{H}^1$. There are no points of type IV pictured here.

with the elements of the residue field of K, which is infinite). When K is complete and algebraically closed, the value group $|K^\times|$ is dense in $\mathbb{R}$, and hence there is a dense set of points along any segment $[\zeta_{a,r}, \zeta_{a,R}]$ each having infinitely many branches away from $[\zeta_{a,r}, \zeta_{a,R}]$!

To help alleviate this situation, we often restrict our attention to *finite subtrees* $\Gamma \subseteq \mathbf{H}^1$. These are connected subsets of $\mathbf{H}^1$ formed as the union of finitely many segments $[\zeta_{a,r}, \zeta_{a,R}]$, $0 < r < R < \infty$. As such they only contain points of type II or III. In the theory of non-Archimedean analysis on $\mathbf{P}^1$, they play the role of compact subsets K in classic complex analysis.

The *tangent space* T_ζ of a point $\zeta \in \mathbf{P}^1$ is defined as the collection of equivalence classes of paths in $\mathbf{P}^1$ which originate at ζ, where two paths are equivalent if they share a common initial segment. One major point of departure from the situation over $\mathbb{C}$ is that T_ζ is *not* a vector space, and in fact the cardinality of T_ζ depends on the type of the point ζ.

An element in T_ζ is called a *direction* and is typically written as a vector $\vec{v}, \vec{w}$, etc. When ζ is a type I or a type IV point, $\#T_\zeta = 1$. For type III points ζ, we have $\#T_\zeta = 2$ while for type II points, $\#T_\zeta = \mathrm{card}(k)$.

In a similar manner, for a fixed finite subtree $\Gamma \subseteq \mathbf{H}^1$ we can define $T_\zeta(\Gamma)$ to be the equivalence classes of paths at ζ with an initial segment lying in Γ. We let $v_\Gamma(\zeta) = \#T_\zeta(\Gamma)$, which is necessarily finite for finite subtrees Γ.

The tangent directions $\vec{v} \in T_\zeta$ are in one-to-one correspondence with the connected components of $\mathbf{P}^1 \setminus \{\zeta\}$; as such, we will often denote these connected components as $B_{\vec{v}}(\zeta)^-$. The collection of all $B_{\vec{v}}(\zeta)^-$ as one ranges across all $\zeta \in \mathbf{P}^1$ and all $\vec{v} \in T_\zeta$ forms a subbasis for the weak topology on $\mathbf{P}^1$ ([2] Appendix B.6). We will denote by $B_{\vec{v}}(\zeta)$ the set $B_{\vec{v}}(\zeta)^-$ together with the point ζ.

In addition, the tangent space T_ζ at a type II point is in bijective correspondence with $\mathbb{P}^1(k)$. To see this, we suppose for simplicity that $\zeta = \zeta_{\mathrm{G}}$, which corresponds to the unit disc $D(0,1)$. The unit disc can be decomposed as

$$D(0,1) = \bigsqcup_{a \in k} D(\alpha_a, 1)^- \ ,$$

where $\alpha_a \in \mathcal{O}$ is a representative of the class $a \in k$. Paths away from ζ_{G} correspond either to families $\{D(\alpha_a, r)\}_{r \in (\epsilon, 1)}$ or $\{D(0, R)\}_{R \in (1, \epsilon)}$; the first type of families are in bijection with elements of k via the correspondence $\vec{v}_a \leftrightarrow a \in k$, while the latter family corresponds to $\vec{v}_\infty$. By appropriate changes of coördinates, a similar correspondence can be given at any type II point in $\mathbf{H}^1$.

Finally, throughout this thesis we will need to understand relationships between paths in $\mathbf{H}^1$. Fix a basepoint $z \in \mathbf{P}^1$; given any two points $\zeta, \eta \in \mathbf{P}^1$, we can consider the paths $[\zeta, z], [\eta, z]$, which we view as oriented, each ending at z. The *join of* ζ *and* η *relative to* z is the first point where the segments $[\zeta, z]$ and $[\eta, z]$ intersect. We denote this point by $\zeta \wedge_z \eta$.

2.1.3 Sizes and Distances in $\mathbf{P}^1$

There are many notions of distance that one can consider on $\mathbf{P}^1$. In this section we survey some of the main notions of size and distance in $\mathbf{P}^1$ that will be important in this thesis. The reader is directed to [2] for full details.

Perhaps the most natural notion of size on $\mathbf{P}^1$ is the *diameter* of a point $\zeta \in \mathbf{P}^1$. If $\zeta \in \mathbf{P}^1$ is a point of type I, type II or type III, we associate to it a (possibly degenerate) disc $D(a, r) \subseteq K$. In this case, we let

$$\mathrm{diam}_\infty(\zeta) := r \ .$$

If ζ is a point of type IV, it corresponds to a decreasing family $\{D(a_i, r_i)\}$ of nested discs with $\lim_{i\to\infty} r_i > 0$; in this case,

$$\mathrm{diam}_\infty(\zeta) := \lim_{i\to\infty} r_i \ .$$

This definition is well-defined and independent of the choice of family $\{D(a_i, r_i)\}$.

We can extend this idea to give a notion of distance between two points in $\mathbf{A}^1$. Given two points $\zeta, \eta \in \mathbf{A}^1$ of type I, II, or III which correspond to the discs $D(a,r), D(b,s)$ (resp.), the *Hsia kernel relative to* ∞ is given

$$\delta(\zeta,\eta)_\infty := \max(|a-b|, r, s) \ .$$

This kernel was first considered in unpublished work of L. C. Hsia, and has been generalized by Baker and Rumely in [2] Chapter 4. Geometrically, $\delta(\zeta,\eta)_\infty$ is the radius of the smallest disc in K that contains both $D(a,r)$ and $D(b,s)$. Note also that if $\zeta, \eta \in K$ are type I points, then $\delta(\zeta,\eta)_\infty = |\zeta,\eta|$; thus, $\delta(\cdot,\cdot)_\infty$ can be viewed as an extension of the standard metric on K, although it is important to note that it is *not* a metric on $\mathbf{P}^1$, since $\delta(\zeta,\zeta)_\infty > 0$ for any $\zeta \in \mathbf{H}^1$.

In a similar way, we can extend the chordal metric on $\mathbb{P}^1(K)$ to $\mathbf{P}^1$ using the Hsia kernel relative to ζ_{G}. Recall that the chordal metric on $\mathbb{P}^1(K)$ is given as follows: if $x = [x_1 : x_2], y = [y_1 : y_2] \in \mathbb{P}^1(K)$, then

$$||x,y|| = \frac{|x_1y_2 - x_2y_1|}{\max(|x_1|,|x_2|)\cdot\max(|y_1|,|y_2|)} \ .$$

Later in the thesis, we will need the following definitions:

$$B(x,r) := \{z \in \mathbb{P}^1(K) \ : \ ||x,z|| \le r\} \ , \ B(x,r)^- := \{z \in \mathbb{P}^1(K) \ : \ ||x,z|| < r\} \ .$$

We remark, however, that both $B(x,r)$ and $B(x,r)^-$ are simultaneously open and closed sets.

The Hsia kernel relative to ζ_{G} is defined

$$||\zeta,\eta|| := q_v^{-\rho(x\wedge_{\zeta_{\mathrm{G}}} y, \zeta_{\mathrm{G}})} \ .$$

In Section 2.4.1 below, we will discuss a more general construction of the Hsia kernel relative to any base point in $\mathbf{P}^1$. A detailed presentation of the Hsia kernels is given in [2] Chapter 4.

Another important notion of distance on $\mathbf{P}^1$ is the path distance metric, which is also referred to as the 'big metric'. Given two points $\zeta, \eta \in \mathbf{H}^1$, let

$$\begin{aligned}\rho(\zeta,\eta) :&= 2\log_v \operatorname{diam}_\infty(\zeta \wedge_\infty \eta) - \log_v \operatorname{diam}_\infty(\zeta) - \log_v \operatorname{diam}_\infty(\eta) \\ &= (\log_v \operatorname{diam}_\infty(\zeta \wedge_\infty \eta) - \log_v \operatorname{diam}_\infty(\zeta)) + (\log_v \operatorname{diam}_\infty(\zeta \wedge_\infty \eta) - \log_v \operatorname{diam}_\infty(\eta)) \ .\end{aligned}$$

This metric is best understood in terms of the second decomposition given above; the first two terms measure the (logarithmic) path distance between ζ and $\zeta \wedge_\infty \eta$, while the second term measures the (logarithmic) path distance between η and $\zeta \wedge_\infty \eta$. This function can be extended to all of $\mathbf{P}^1$ with the convention that $\rho(z,x)$ is ∞ for $z \neq x$ and is 0 for $z = x$.

The big metric has the nice property that it is invariant under coordinate changes ([2] Proposition 2.30): for any $\gamma \in \mathrm{PGL}_2(K)$, we have $\rho(\gamma(x), \gamma(y)) = \rho(x,y)$. The topology it defines is called the strong topology, and it is strictly finer than the weak topology. The advantage of this topology over the weak topology is that it is (evidently!) metrizable, and it remains Hausdorff and uniquely path connected. Unfortunately, though, in this topology $\mathbf{P}^1$ is no longer compact, and in fact it even fails to be locally compact! Nevertheless, it will play an important role in potential theory on $\mathbf{P}^1$.

We will at times also make use of another metric on $\mathbf{P}^1$ called the *small metric*. This metric is given

$$\mathrm{d}_{\mathbf{P}^1}(\zeta,\eta) = 2\operatorname{diam}_{\mathrm{G}}(\zeta \wedge_{\zeta_{\mathrm{G}}} \eta) - \operatorname{diam}_{\mathrm{G}}(\zeta) - \operatorname{diam}_{\mathrm{G}}(\eta) \ .$$

The small metric generates the same topology on $\mathbf{P}^1$ as does the big metric ρ, as they are locally bounded in terms of one another on $\mathbf{H}^1$. While the small metric is not coordinate invariant, it is useful from a dynamical point of view: the action of ϕ on $\mathbf{H}^1$ is Lipschitz continuous in the small metric ([2] Proposition 9.37).

2.2 Rational maps over non-Archimedean Fields

Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. In this section, we explore the interplay between the arithmetic and geometric properties of the map ϕ.

2.2.1 Geometric Action of ϕ

The action of ϕ on $\mathbb{P}^1(K)$ can be extended to all of $\mathbf{P}^1$ as follows: let $\Phi = [F, G]$ be a homogeneous lift of ϕ – a pair of homogeneous, degree d polynomials $F, G \in K[X, Y]$ having no common roots and satisfying $\phi(z) = \frac{F(z,1)}{G(z,1)}$. Then

$$[f]_{\phi(\zeta)} := [f \circ \Phi]_{\zeta} \ , \ \forall f \in K[X, Y] \ .$$

In particular, we note that the action of ϕ on $\mathbf{P}^1$ preserves the type of points ([2] Proposition 2.15).

It follows that automorphisms of $\mathbb{P}^1(K)$ extend to automorphisms of $\mathbf{P}^1$, and in fact these are the only algebraic automorphisms of $\mathbf{P}^1$ ([2] Corollary 2.13). Moreover, elements of $\mathrm{PGL}_2(K)$ act transitively on triples in $\mathbf{P}^1$ in the following sense: given triples $[a_0, A, a_1], [b_0, B, b_1]$ where $a_0, a_1, b_0, b_1 \in \mathbb{P}^1(K)$ and $A, B \in \mathbf{H}^1$ type II points lying in the segments $[a_0, a_1], [b_0, b_1]$ (resp.), there is an element of $\mathrm{PGL}_2(K)$ sending a_0 to b_0, a_1 to b_1, and A to B ([2] Corollary 2.13). The action of $\mathrm{PGL}_2(K)$ is not free, however; the stabilizer of ζ_{G} in $\mathrm{PGL}_2(K)$ is the group $\mathrm{PGL}_2(\mathcal{O})$ ([24] Proposition 1.1).

2.2.2 Arithmetic Properties of ϕ

In this section, we'll first present some arithmetic properties of ϕ; at the end of this section, we will discuss beautiful ways in which these arithmetic properties are reflected in its geometric action on ϕ on $\mathbf{P}^1$.

We will frequently work with homogeneous lifts of ϕ. These are pairs of homogeneous, degree d polynomials $F, G \in K[X, Y]$ which have no common roots over $\overline{K}$ and which satisfy $\phi(z) = \frac{F(z,1)}{G(z,1)}$. The corresponding polynomial map on K^2 will be denoted $\Phi = (F, G) : K^2 \to K^2$. We will also write $F(X, Y) = a_d X^d + ... + a_0 Y^d$ and $G(X, Y) = b_d X^d + ... + b_0 Y^d$ for the coördinate polynomials.

A homogeneous lift Φ of ϕ is said to be *normalized* if $\max(|a_i|, |b_i|) = 1$; equivalently, Φ is normalized if and only if all of the coefficients of F, G lie in the unit disc, and at least one coefficient is a unit. In this situation, one can apply the reduction map $\tilde{\cdot} : \mathcal{O} \to k$ to each of the coefficients and obtain a new polynomial map $(\tilde{F}, \tilde{G}) : k^2 \to k^2$. Unfortunately, while F and G have no common factors, it is quite possible for $\tilde{F}$ and $\tilde{G}$ to have a common factor! Letting $\tilde{A} = \mathrm{GCD}(\tilde{F}, \tilde{G})$, we factor $\tilde{F} = \tilde{A} \cdot \tilde{F}_0$, $\tilde{G} = \tilde{A} \cdot \tilde{G}_0$. The *reduction map* $\tilde{\phi} \in k(z)$ associated to ϕ is the (dehomogenization of the) polynomial map $(\tilde{F}_0, \tilde{G}_0) : k^2 \to k^2$.

Ideally, the reduction map should have dynamical properties that mirror the dynamics of ϕ. The best case is when $\deg(\phi) = \deg(\tilde{\phi})$; here, we say that ϕ has *good reduction*. If, after some change of coordinates by an element $\gamma \in \mathrm{PGL}_2(K)$, the resulting conjugate $\phi^\gamma := \gamma^{-1} \circ \phi \circ \gamma$ has good reduction, we say that the original map ϕ has *potential good reduction*. If ϕ fails to have potential good reduction, we say that ϕ has *bad reduction*.

We can also define *the reduction of ϕ at a type II point $P \in \mathbf{P}^1$*. Recall that $\mathrm{PGL}_2(K)$ acts transitively on type II points; thus, given any type II point $P \in \mathbf{H}^1$, we may write $P = \gamma(\zeta_{\mathrm{G}})$ for some $\gamma \in \mathrm{PGL}_2(K)$. The choice of γ is unique up to precomposition by an element $\tau \in \mathrm{PGL}_2(\mathcal{O})$. The *reduction of ϕ at P* is the reduction of the conjugate map ϕ^γ and we say, e.g., that ϕ is 'repelling at P' or 'indifferent at P' if ϕ^γ has the corresonding reduction type. Note that this is independent of our choice of γ, since conjugation by an element of $\mathrm{PGL}_2(\mathcal{O})$ descends to a change of coordinates on $\mathbb{P}^1(k)$, which does not affect reduction type. Similarly, we define the *degree of ϕ at P* to be $\deg_\phi(P) := \deg(\phi^\gamma)$.

For a fixed type II point $P \in \mathbf{H}^1$, we say that

- The map ϕ has *good reduction at $P = \gamma(\zeta_{\mathrm{G}})$* if $\deg(\widetilde{\phi^\gamma}) = \deg(\phi^\gamma)$.
- The map ϕ is *repelling at $P = \gamma(\zeta_{\mathrm{G}})$* if $2 \le \deg(\widetilde{\phi^\gamma}) \le \deg(\phi^\gamma)$. Note that maps of degree $d \ge 2$ with good reduction are repelling. An important consequence of Rumely's work [24, 25] is that there are at most $d-1$ repelling points in $\mathbf{H}^1$.
- The map ϕ is said to have *indifferent reduction at $P = \gamma(\zeta_{\mathrm{G}})$* if $\deg(\widetilde{\phi^\gamma}) = 1$.
- The map ϕ is said to have *constant reduction* if $\deg(\tilde{\phi}) = 0$.

Rumely has given a further stratification of the types of indifferent reduction in [25]:

- ϕ has *multiplicative indifferent reduction at P* if, after some further change of coordinates over k, the reduction map can be written $\widetilde{\phi^\gamma}(z) = \lambda z$ for some $\lambda \in k \setminus \{0, 1\}$.
- ϕ has *additively indifferent reduction at P* if, after some further change of coordinates over k, the reduction map can be written $\widetilde{\phi^\gamma}(z) = z + \lambda$ for some $\lambda \in k \setminus \{0\}$.

- ϕ has *id-indifferent reduction at* P if $\widetilde{\phi^\gamma}(z) = z$ is the identity map.

2.2.3 Interplay Between Geometry and Arithmetic

One of the beautiful parts of this theory is that there is a close interaction between the arithmetic properties of ϕ – particularly its reduction – and the geometric behaviour of ϕ.

An important related concept is the notion of the tangent map ϕ_* induced by ϕ, which can be defined as follows: fix a point $\zeta \in \mathbf{H}^1$, and a direction $\vec{v} \in T_\zeta$. For $t > 0$, let $\zeta + t\vec{v}$ denote an arbitrary point in $B_{\vec{v}}(\zeta)^-$ with $\rho(\zeta, \zeta + t\vec{v}) = t$. Baker and Rumely show ([2], Chapter 9, at the bottom of page 261) that for t sufficiently small, there is a unique direction $\vec{w} \in T_{\phi(\zeta)}$ which contains $\zeta + t\vec{v}$, regardless of which point $\zeta + t\vec{v}$ is chosen. The action of ϕ_* is then given by $\phi_*\vec{v} = \vec{w}$.

The primary connection between the arithmetic and geometric behaviour of ϕ concerns the tangent map ϕ_*. As a first example of this, we recall

Lemma 2.1 ([2] Lemma 2.17)**.** *Let $\phi(T) \in K(T)$ be non-constant. Then $\phi(T)$ has well-defined reduction $\tilde{\phi}(T) \in k(T)$ at ζ_G if and only if $\phi(\zeta_G) \notin B_{\vec{v}_\infty}(\zeta_G)^-$, and $\phi(T)$ has non-constant reduction if and only if $\phi(\zeta_G) = \zeta_G$.*

This lemma helps us understand a relation between the reduction map $\tilde{\phi}$ and the tangent map ϕ_*. Suppose that $\phi(\zeta_{\mathrm{G}}) = \zeta_{\mathrm{G}}$, so that ϕ has non-constant reduction. Labelling the tangent directions $\vec{v}_a \in T_{\zeta_{\mathrm{G}}}$ by elements of $\mathbb{P}^1(k)$ as outlined in Section 2.1.2 above, the action of the tangent map is given by the corresponding action of the reduction $\tilde{\phi}$, i.e.

$$\phi_*\vec{v}_a = \vec{v}_{\tilde{\phi}(a)} \in T_{\zeta_{\mathrm{G}}} \ .$$

If $\phi(\zeta_{\mathrm{G}}) \neq \zeta_{\mathrm{G}}$, we may choose some $\gamma \in \mathrm{PGL}_2(K)$ with $\gamma(\phi(\zeta_{\mathrm{G}})) = \zeta_{\mathrm{G}}$; then

$$\phi_*\vec{v}_a = (\gamma^{-1})_* \ \vec{v}_{\widetilde{\gamma\circ\phi}(a)} \in T_{\phi(\zeta_{\mathrm{G}})} \ .$$

We record for later use the notion of a *shearing direction.* Given a type II or type III point $\zeta \in \mathbf{H}^1$ which is fixed by ϕ, let $\vec{v} \in T_\zeta$ be a direction such that $B_{\vec{v}}(\zeta)^-$ contains at least one type I fixed point. It is reasonable to expect in this situation that $\phi_*\vec{v} = \vec{v}$, but this is not always the case. If some direction $\vec{v} \in T_\zeta$ containing a type I fixed point is *moved* by ϕ_*, so that $\phi_*\vec{v} \neq \vec{v}$, we say that $\vec{v}$ is a shearing direction. The total number of shearing directions at a point ζ is denoted by $N_{\text{Shearing}}(\zeta)$; it is necessarily a finite integer, and by the weight formula ([25], Theorem 6.2; see also Theorem 2.5 below) it will lie in the interval $[0, d-1]$.

2.2.4 Multiplicities

The reduction map also allows us to define multiplicities[2] for the map ϕ. Again for simplicity, assume $P = \zeta_{\text{G}}$, and if necessary replace ϕ by $\gamma \circ \phi$, where $\gamma \in \text{PGL}_2(K)$ satisfies $\gamma(\phi(\zeta_{\text{G}})) = \zeta_{\text{G}}$, so that $\phi(\zeta_{\text{G}}) = \zeta_{\text{G}}$. Then $\tilde{\phi}$ has non-constant reduction, and we define the *mutliplicity of* ϕ *at* $P = \zeta_G$ to be $m_\phi(P) := \deg(\tilde{\phi})$. The multiplicity at a general point P is found by pre- and post-composing ϕ by appropriate maps $\gamma, \eta \in \text{PGL}_2(K)$ so that $\gamma \circ \phi \circ \eta(\zeta_{\text{G}}) = \zeta_{\text{G}}$.

We will also use two refinements of the multiplicity of ϕ which do not appear to have direct analogs in complex dynamics. For simplicity let $P = \zeta_{\text{G}}$ and assume that $\phi(\zeta_{\text{G}}) = \zeta_{\text{G}}$. Let $\Phi = (F, G)$ be a normalized lift of ϕ, and suppose that $(\tilde{F}, \tilde{G}) = (\tilde{A} \cdot \tilde{F}_0, \tilde{A} \cdot \tilde{G}_0)$, where $\tilde{A} = \text{GCD}(\tilde{F}, \tilde{G})$. Then for any direction $\vec{v}_a \in T_{\zeta_{\text{G}}}$, which we view as can element $a \in \mathbb{P}^1(k)$, the *directional multiplicity* $m_\phi(P, \vec{v}_a)$ is the multiplicity of a as a root of $(\tilde{F}_0, \tilde{G}_0)$. The *surplus multiplicity* $s_\phi(P, \vec{v}_a)$ is the multiplicity of a as a root of $\tilde{A}$. These quantities have geometric interpretation via the following theorem:

Proposition 2.2 ([9], Proposition 3.10)**.** *Let* $\phi \in K(z)$ *be a non-constant rational function. Fix a point* $P \in \boldsymbol{H}^1$ *and a direction* $\vec{v} \in T_P$*. Then either* $\phi(B_{\vec{v}}(P)^-) = B_{\phi_*\vec{v}}(\phi(P))^-$*, or* $\phi(B_{\vec{v}}(P)^-) = \boldsymbol{P}^1$*. More precisely,*

1. *If* $\phi(B_{\vec{v}}(P)^-) = B_{\phi_*\vec{v}}(\phi(P))^-$*, then for each* $y \in B_{\phi_*\vec{v}}(\phi(P))^-$ *there are exactly* $m_\phi(P, \vec{v})$ *solutions to the equation* $\phi(x) = y$ *in* $B_{\vec{v}}(P)^-$*.*

2. *Otherwise, if* $\phi(B_{\vec{v}}(P)^-) = \boldsymbol{P}^1$*, then for each* $y \in B_{\phi_*\vec{v}}(\phi(P))^-$*, there are exactly* $m_\phi(P, \vec{v}) + s_\phi(P, \vec{v})$ *solutions to the equation* $\phi(x) = y$ *in* $B_{\vec{v}}(P)^-$*.*

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[2] Several other approaches to defining multiplicities at Berkovich points have been considered. For example, Favre and Rivera-Letelier [13] and Thuillier [29] have constructed multiplicities using the ranks of certain modules. Rivear-Letelier [23, 21] has defined multiplicities from a topological perspective by counting preimages. The fact that these notions of multiplicity agree with the one given here is discussed in [13] and [2] Section 9.1.

This theorem justifies the name ‘surplus multiplicity’, as $s_\phi(P,\vec{v})$ counts the number of ‘extra’ solutions to the equation $\phi(x) = y$ when the image $\phi(B_{\vec{v}}(P)^-)$ is all of $\mathbf{P}^1$.

Finally, we introduce two notions of multiplicities that were first considered by Rumely in [25]. The *fixed point multiplicity* of ϕ at a point $P \in \mathbf{H}^1$ in a direction $\vec{v} \in T_P$ counts the number of type I fixed points lying in that direction:

$$F_\phi(P,\vec{v}) = \#\{\text{type I fixed points in } B_{\vec{v}}(P)^-\} \ .$$

The *reduced fixed point multiplicity* of ϕ at P in the direction $\vec{v}$ is given

$$\tilde{F}_\phi(P,\vec{v}) := \text{multiplicity of } \vec{v} \text{ as a fixed point of } \phi_* \ .$$

2.3 The Resultant of a Rational Map

The main arithmetic tool studied in this thesis is the resultant of a lift $\Phi = (F, G)$ of the map ϕ. If $F(X,Y) = a_d X^d + ... + a_0 Y^d$ and $G(X,Y) = b_d X^d + ... + b_0 Y^d$, then the resultant is the determinant of the Sylvester matrix:

$$\operatorname{Res}(F,G) = \det \begin{pmatrix} a_d & a_{d-1} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & a_d & a_{d-1} & \dots & a_1 & a_0 & \dots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_d & a_{d-1} & \dots & a_1 & a_0 \\ b_d & b_{d-1} & \dots & b_1 & b_0 & 0 & \dots & 0 \\ 0 & b_d & b_{d-1} & \dots & b_1 & b_0 & \dots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & b_d & b_{d-1} & \dots & b_1 & b_0 \end{pmatrix} .$$

If Φ is replaced by $c\Phi = (cF, cG)$ for some $c \in K^\times$, the resultant transforms as $\operatorname{Res}(cF, cG) = c^{2d} \operatorname{Res}(F,G)$. Another transformation property of the resultant, which will be quite important throughout this thesis, is the following:

Lemma 2.3. *Let F, G be as above, and let $f(X,Y), g(X,Y)$ be homogeneous polynomials of degree e. Define*

$$A(X,Y) = F(f(X,Y), g(X,Y)) \ , \ B(X,Y) = G(f(X,Y), g(X,Y)) \ .$$

Then

$$\operatorname{Res}(A,B) = \operatorname{Res}(F,G)^e \operatorname{Res}(f,g)^{d^2} \ .$$

The key property of the resultant that we will rely is that $\operatorname{Res}(F,G) = 0$ if and only if F, G have a common root over $\overline{K}$. The proof of this fact can be found in [28] Proposition 2.13. This fact can also be used to study the reduction behaviour of ϕ: suppose that K is algebraically closed, and that Φ is a *normalized* lift of ϕ. We know that ϕ has good reduction if and only if $\tilde{F}, \tilde{G}$ have no common factor over k, or equivalently, $\operatorname{Res}(\tilde{F}, \tilde{G}) \neq 0$. Since the resultant is a polynomial expression in the coefficients of F and G, this is equivalent to $\widetilde{\operatorname{Res}(F,G)} \neq 0$, i.e $|\operatorname{Res}(F,G)| = 1$; in short, ϕ has good reduction (at ζ_{G}) if and only if $|\operatorname{Res}(\Phi)| = 1$ for some normalized lift Φ of ϕ. Note that $|\operatorname{Res}(F,G)|$ is independent of the normalized lift $\Phi = (F,G)$, since scaling Φ by a unit does not affect $|\operatorname{Res}(F,G)|$.

Rumely [24] first defined

$$\operatorname{ordRes}(\phi) := \operatorname{ord}(\operatorname{Res}(F,G)) = -\log_v |\operatorname{Res}(F,G)| \ ,$$

where (F,G) is a normalized lift of ϕ. Note that $\operatorname{ordRes}(\phi)$ vanishes if and only if ϕ has good reduction. Rumely extended this idea by defining a function on $\mathbf{H}^1$ as follows: given a type II point $\zeta \in \mathbf{H}^1$, write $\zeta = \gamma(\zeta_{\mathrm{G}})$; then

$$\operatorname{ordRes}_{\phi}(\zeta) := \operatorname{ordRes}(\phi^{\gamma}) \ .$$

Rumely's key insight was that this function is well-defined independent of which $\gamma \in \mathrm{PGL}_2(K)$ is used: if $\gamma(\zeta_{\mathrm{G}}) = \zeta = \hat{\gamma}(\zeta_{\mathrm{G}})$, then $\gamma = \hat{\gamma} \cdot \tau$ for some $\tau \in \mathrm{PGL}_2(\mathcal{O})$ ([24] Proposition 1.1A), and the resultant is unaffected under conjugation by an element of $\mathrm{PGL}_2(\mathcal{O})$ (this follows from

Lemma 2.3 above and the fact that the resultant of a Möbius transformation is the determinant of the corresponding matrix).

This function, which is a priori only defined for type II points in $\mathbf{H}^1$, in fact extends to a well-behaved function on all of $\mathbf{P}^1$: it is continuous and piecewise affine with respect to the ρ-metric; its slope along any segment is constrained by certain explicit congruence conditions on d, and it is convex up (see [24] Lemma 1.2).

Perhaps most importantly, $\mathrm{ordRes}_\phi(\cdot)$ attains a minimum on $\mathbf{H}^1$ ([24] Theorem 0.1), and the collection of points where ordRes_ϕ is minimized is denoted $\mathrm{MinResLoc}(\phi)$, which is called the *Minimal Resultant Locus*. Type II points $\zeta \in \mathrm{MinResLoc}(\phi)$ correspond to normalized lifts $\Phi^\gamma = (F^\gamma, G^\gamma)$ of conjugates ϕ^γ of ϕ for which $|\operatorname{Res}(F^\gamma, G^\gamma)|$ is as close to 1 as possible; n another sense, these are conjugates with the 'best possible reduction'. Indeed, by the discussion above we have that ϕ has good reduction if and only if the value of $\mathrm{ordRes}_\phi(\cdot)$ on $\mathrm{MinResLoc}(\phi)$ is identically 0.

Geometrically, $\mathrm{MinResLoc}(\phi)$ is either a point or a segment in the tree $\mathbf{H}^1$, with the former condition necessarily happening when d is even ([24] Theorem 0.1). Rumely was able to restrict the location of $\mathrm{MinResLoc}(\phi)$: if Γ_{FR} denotes the tree in $\mathbf{P}^1$ spanned by the type I fixed points of ϕ and the type II *repelling* fixed points, then $\mathrm{MinResLoc}(\phi) \subseteq \Gamma_{\mathrm{FR}}$ ([25] Proposition 4.4).

2.3.1 The Crucial Measures

To construct the crucial measure, Rumely uses the Berkovich Laplacian Δ, which wwill be defined in Section 2.4 below. For now, we will be content to define the Laplacian Δ of a 'sufficiently nice' real-valued function f on $\mathbf{P}^1$ to be a signed Borel measure $\Delta(f)$ of total mass 0 on $\mathbf{P}^1$. We can also consider a restricted Laplacian Δ_Γ associated to a subgraph $\Gamma \subseteq \mathbf{H}^1$; the resulting measure will be supported on the graph Γ.

Unfortunately, the function $\mathrm{ordRes}_\phi(\cdot)$ is not 'sufficiently nice' so as to have a Laplacian on all of $\mathbf{P}^1$. However, its restriction to the tree Γ_{FR} spanned by the type I fixed points and type II repelling periodic points *is* sufficently nice to allow us to define its Laplacian on Γ_{FR}. In order to define the Laplacian of $\mathrm{ordRes}_\phi(\cdot)$ on this tree, we 'prune' the tree by excising segments $(Q, f]$ terminating at type I points on which the slope of ordRes_ϕ is constant; this gives a finite tree in

$\mathbf{H}^1$ (recall that there are only finitely many repelling points in $\mathbf{H}^1$) which we denote by $\Gamma_{\widehat{\mathrm{FR}}}$. With this, we have

Theorem 2.4 ([25] Corollary 6.5)**.** *Define* $\Gamma_{\widehat{FR}}$ *as above, and let* $f(\cdot)$ *be the restriction of* $\mathrm{ordRes}_\phi(\cdot)$ *to* $\Gamma_{\widehat{FR}}$*. Then*

$$\Delta_{\Gamma_{\widehat{FR}}}(f) = 2(d^2 - d)(\mu_{\Gamma_{\widehat{FR}},Br} - \nu_\phi) \ .$$

Here, $\mu_{\Gamma,\mathrm{Br}}$ is the *branching measure*[3] attached to a graph Γ from [7], and ν_ϕ is the crucial measure. The measure $\mu_{\Gamma,\mathrm{Br}}$ can be given explicitly by

$$\mu_{\Gamma,\mathrm{Br}} = \frac{1}{2}\sum_{P\in\Gamma}(2 - v_\Gamma(P))\delta_P \ .$$

Here, and throughout the thesis, δ_P is the Dirac measure which assigns measure 1 to any set A that contains P, and 0 otherwise.

2.3.2 Weight Formulae

The formulation for the cruical measure given above is somewhat inconvenient to work with in practice. Instead, Rumely's original definition of the crucial measure was given as a weighted sum of point masses:

Definition 1 ([25] Definition 8)**.** For each $P \in \mathbf{P}^1$, the weight $w_\phi(P)$ is the following non-negative integer:

1. If $P \in \mathbf{H}^1$ and P is fixed by ϕ, define

$$w_\phi(P) = \deg_\phi(P) - 1 + N_{\mathrm{Shearing}}(P) \ .$$

2. If $P \in \mathbf{H}^1$ and P is not fixed by ϕ, let $v(P)$ be the number of directions $\vec{v} \in T_P$ such that $B_P(\vec{v})^-$ contains a type I fixed point of ϕ, and define $w_\phi(P) = \max(0, v(P) - 2)$.

3. If $P \in \mathbb{P}^1(K)$, define $w_\phi(P) = 0$.

---

[3] In [7], it is called the canonical measure attached to a graph Γ; this conflicts with the usual use of 'canonical measure' in arithmetic dynamics, so we have chosen to call this the branching measure.

An amazing and beautiful theorem is the following weight formula:

Theorem 2.5 ([25] Theorem 6.2)**.** *Let* $\phi(z) \in K(z)$ *have degree* $d \geq 2$*. Then the following weight formula holds:*

$$\sum_{P \in \boldsymbol{P}^1} w_\phi(P) = (d-1) \ .$$

With this, the crucial measure ν_ϕ is the probability measure which assigns to each point of $\mathbf{P}^1$ its weight $w_\phi(\cdot)$:

$$\nu_\phi := \frac{1}{d-1} \sum_{P \in \mathbf{P}^1} w_\phi(P) \delta_P \ .$$

From this formulation, it is immediately clear that the measure ν_ϕ is a probability measure *with discrete support*, and that the mass of any point is a rational number.

The support of ν_ϕ is called *the crucial set*. Rumely has given a classification of the points P in the crucial set in terms of the reduction behaviour of ϕ at P (see [25] Proposition 6.1). We summarize this classification as follows: a point P receives weight if and only if one of the following, mutually exclusive conditions holds:

(i) P is a type II repelling fixed point for ϕ.

(ii) P is a type II multiplicatively indifferent fixed point of ϕ which is a branch point of Γ_{FR}.

(iii) P is a type II additively indifferent fixed point of ϕ which is contained in Γ_{FR}.

(iv) P is a type II branch point of Γ_{FR} which is moved by ϕ.

In any of these cases, the explicit weight of P can be calculated using the formulas given in Definition 1.

Finally, we have already seen how the crucial measures arise from the function $\mathrm{ordRes}_\phi(\cdot)$; we now recall a relationship between the crucial measures and the set $\mathrm{MinResLoc}(\phi)$:

Theorem 2.6 ([25] Theorem 7.1)**.** *Let* $\phi(z) \in K(z)$ *have degree* $d \geq 2$*. A point* $Q \in \boldsymbol{P}^1$ *belongs to* $\mathrm{MinResLoc}(\phi)$ *if and only if for each* $\vec{w} \in T_Q$*,*

$$\sum_{P \in B_{\vec{w}}(Q)^-} w_\phi(P) \leq \frac{d-1}{2} \ .$$

If d is even, MinResLoc(ϕ) *is a vertex of the tree spanned by the elements of the crucial set. If d is odd,* MinResLoc(ϕ) *may be either a vertex or an edge of this tree.*

This theorem says that the minimal resultant locus is the *barycenter* of the crucial measure ν_ϕ; the notion of barycenter in the non-Archimedean context was first introduced by Rivera-Letelier, and will be discussed at greater length in Chapter 4 where we show that the minimal resultant locus is related to the barycenter of the equilibrium measure μ_ϕ of ϕ (see Section 2.6.2 below for a definition of the equilibrium measure in this context).

2.4 Potential Theory on $\mathbf{P}^1$

Several authors have given non-Archimedean analogues of classical potential theory; here we will follow a blend of the approaches in [2] and [12, 11].

2.4.1 The Fundamental Potential Kernel

The fundamental potential kernel[4] $\langle \cdot, \cdot \rangle_z$ relative to a base point $z \in \mathbf{H}^1$ will play an important role in the development of potential theory on $\mathbf{P}^1$. Fix a base point $z \in \mathbf{H}^1$, and let $\zeta, \eta \in \mathbf{H}^1$. If $w = \zeta \wedge_z \eta$, we define

$$\langle \zeta, \eta \rangle_z := \rho(z, w) \ .$$

A more intrinsic definition of the potential kernel is that, for $z, \eta \in \mathbf{H}^1$ fixed, the function $\langle \cdot, \eta \rangle_z$ is the fundamental solution to the differential equation

$$\Delta_\Gamma \langle \cdot, \eta \rangle_z = \delta_\eta - \delta_z \ , \text{ normalized by } \langle z, \eta \rangle_z = 0$$

where Γ is the segment $[z, \eta]$ and Δ_Γ is the graph theoretic Laplacian to be discussed in Section 2.4.3 below ([2] Section 3.3).

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[4] Baker and Rumely [2] denote this kernel as $j_z(\cdot, \cdot)$.

There is an important change-of-basepoint formula for the potential kernel, when both the new and the old basepoints are in $\mathbf{H}^1$: if $z, w \in \mathbf{H}^1$, then for any point $\zeta, \eta \in \mathbf{H}^1$ we have

$$\langle \zeta, \eta \rangle_w = \langle \zeta, \eta \rangle_z - \langle \zeta, w \rangle_z - \langle \eta, w \rangle_z - \langle w, w \rangle_z \ . \tag{2.1}$$

This formula is proved using properties of the Laplacian on a metrized graph; see [2] Proposition 3.3 for the full proof.

The fundamental potential kernel $\langle \cdot, \cdot \rangle_z$ also allows us to give a simpler formulation of the Hsia kernel relative to ζ_{G}, and together with (2.1) will lead to an expression for the general Hsia kernel. First, we see that the Hsia kernel relative to ζ_{G} can be given in terms of $\langle \cdot, \cdot \rangle_{\zeta_{\mathrm{G}}}$ as

$$||\zeta, \eta|| = q_v^{-\langle \zeta, \eta \rangle_{\zeta_{\mathrm{G}}}} \ .$$

The Hsia kernel $\delta(\cdot, \cdot)_{\zeta_{\mathrm{G}}}$ is symmetric and non-negative; it is continuous in each variable separately, and as a function of two variables it is upper semicontinuous ([2] Proposition 4.7).

We define the *generalized Hsia kernel relative to any basepoint* $\zeta \in \boldsymbol{P}^1$ as

$$\delta(x, y)_\zeta = \frac{||x, y||}{||x, \zeta|| \cdot ||y, \zeta||} \ , \tag{2.2}$$

which is valid for any $x, y \in \mathbf{P}^1 \setminus \{\zeta\}$. Using the change of base formula for the potential kernel, when the base ζ is in $\mathbf{H}^1$ we can also write

$$\delta(x, y)_\zeta = C q_v^{-\langle x, y \rangle_\zeta} \ , \tag{2.3}$$

where the constant C here is given explicitly as $C = q_v^{-\langle \zeta, \zeta \rangle_{\zeta_{\mathrm{G}}}}$. Like the Hsia kernel relative to ζ_{G}, the function $\delta(x, y)_\zeta$ is symmetric and continuous as a function of each variable separately; as a function of two variables, it is only upper semicontinuous ([2] Proposition 4.10).

Finally, one can use the generalized Hsia kernel to define a generalized notion of diameter relative to a fixed base point $z \in \mathbf{H}^1$. The diameter of a point $\zeta \in \mathbf{P}^1$ relative to z is given

$$\operatorname{diam}_z(\zeta) = \delta(\zeta,\zeta)_z \ . \tag{2.4}$$

From the definition of the Hsia kernel, this is

$$\operatorname{diam}_z(\zeta) = q_v^{-\langle \zeta,\zeta\rangle_z} = q_v^{-\rho(\zeta,z)} \ .$$

An exposition of the Hsia kernel, along with a motivation for the connection between the Hsia kernel and the spherical distance, is given in [2] Chapter 4.

There is also an important relationship between the Hsia kernels $\delta(\cdot,z)_\xi$ and seminorms $[\cdot]_\xi$. Fix a type I point $a \in \mathbb{P}^1(K)$. We find that $[T-a]_\xi = \delta(a,\xi)_\infty$ ([2] Corollary 4.2). Now let $\phi \in K(T)$ be a rational map, and write its divisor as $\operatorname{div}(\phi) = \sum n_i(a_i) - \sum m_j(b_j)$ where $a_i, b_j \in \mathbb{P}^1(K)$ are the roots and poles (resp.) of ϕ, and n_i, m_i are the corresponding multiplicities. Then

$$\begin{aligned}\log_v[\phi]_\xi &= \log_v C_\phi + \sum n_i \log_v[T-a_i]_\xi - \sum m_j \log_v[T-b_j]_\xi \\ &= \log_v C_\phi + \sum n_i \log_v \delta(a_i,\xi)_\infty - \sum m_j \log_v \delta(b_j,\xi)_\infty \ ,\end{aligned}$$

for some constant C_ϕ depending on ϕ. More generally, using the change of variables formula for the Hsia kernel, we can express $\log_v[\phi]_\zeta$ in terms of any Hsia kernel as (see [2] Corollary 4.14)

$$\log_v[\phi]_\zeta = \log_v \widetilde{C}_{\phi,\eta} + \sum n_i \log_v \delta(\zeta,a_i)_\eta - \sum m_j \log_v \delta(\zeta,b_j)_\eta \ ,$$

provided η is disjoint from the support of $\operatorname{div}(\phi)$.

2.4.2 Potential Functions

Let $\mathcal{M}^+$ denote the space of all finite, positive, real-valued Borel measures on $\mathbf{P}^1$ and let $\mathcal{M}$ denote the space of $\mathbb{R}$-linear combinations of elements of $\mathcal{M}$. Each measure in $\mathcal{M}$ can be decomposed $\nu = \nu_+ + \nu_-$ as a sum of positive and negative measures, each of which has finite mass; we will therefore work almost entirely with positive measures ν. By normalizing $\nu_-, -\nu_+$ appropriately, we typically assume that $\nu \in \mathcal{M}$ is a probability measure. In later sections, we will also make use of the total variation of ν, a positive measure denoted by $|\nu| = \nu_+ - \nu_-$.

A *potential function* attached to a probability measure $\nu \in \mathcal{M}$ is a function of the form

$$u_\nu(\zeta, \xi) := -\int_{\mathbf{P}^1} \log_v \delta(w, \zeta)_\xi d\nu(w) + C \ .$$

When the basepoint ξ is in $\mathbf{H}^1$, we can also express this as

$$u_\nu(\zeta, \xi) = \int \langle w, \zeta \rangle_\xi d\nu(w) + C' \ .$$

In either case, the potential function is essentially measuring the (logarithmic) distance between the point ζ and the points in the support of ν, where distance is measured relative to the point ξ. Several remarks are in order:

- As we have defined it above, the potential function isn't always well-defined; this happens, for example if $\nu = \delta_\xi$ for some type I point $\xi \in \mathbb{P}^1(K)$; in this case, $u_\nu(\zeta, \xi) = \log_v \delta(\xi, \zeta)_\xi = -\log_v ||\xi, \xi||$ is infinite. However, if we assume either that $\xi \in \mathbf{H}^1$, or that $\xi \in \mathbb{P}^1(K) \backslash \mathrm{supp}(\nu)$, then the potential function is always well defined ([2] Proposition 6.12).

- A potential function is only defined up to an additive constant; this is because, at a more fundamental level, potential functions are solutions to a specific differential equation (to be discussed below). At times it will be convenient to choose a specific normalization; more often, though, it will be okay to assume $C = 0$.

- We will almost always view the second argument ξ as being fixed. If $\xi \in \mathbf{H}^1$, we can rewrite $u_\nu(\cdot, \xi)$ in terms of any other base point $z \in \mathbf{H}^1$ using the change of variables formula given in (2.1); more precisely,

$$u_\nu(\cdot, \xi) = u_\nu(\cdot, z) - u_\nu(\xi, z) - \nu(\mathbf{P}^1)\langle \cdot, \xi \rangle_z - \nu(\mathbf{P}^1)\langle \xi, \xi \rangle_z \ .$$

The function $u_\nu(\cdot,\xi)$ is continuous on $\mathbf{P}^1 \setminus \mathrm{supp}(\nu)$, but in general is only lower semicontinuous ([2] Proposition 6.12). The measure ν is said to have *continuous potentials* if for some $\xi \in \mathbf{H}^1$, the function $u_\nu(\cdot,\xi) : \mathbf{P}^1 \to \mathbb{R}$ is continuous; by the change-of-base formula, this holds at one $\xi \in \mathbf{H}^1$ if and only if it holds for every $\xi \in \mathbf{H}^1$. Similarly, we say that ν has *bounded potentials* if $u_\nu(\cdot,\xi) : \mathbf{P}^1 \to \mathbb{R}$ is bounded for some (and hence every) $\xi \in \mathbf{H}^1$.

2.4.3 The Laplacian

The potential functions have a more implicit definition in terms of the Laplace operator. We will first give an abstract construction, due to Favre, Jonsson, and Rivera-Letelier [12, 11]; we will then give a more explicit construction in terms of directional derivatives due to Baker and Rumely. The fact that these two constructions give the same operator (up to a sign!) is addressed in [2] Section 5.8.

Fix a base point $\xi \in \mathbf{H}^1$, and choose the normalizing constant for the potential kernel to be $C' = \nu(\mathbf{P}^1)$, so that $u_\nu(\cdot,\xi) = \int_{\mathbf{P}^1} \langle w, \cdot \rangle_\xi d\nu(w) + \nu(\mathbf{P}^1)$ ([13], Equation 2.6). The assignment

$$\nu \mapsto -u_\nu(\cdot,\xi)$$

defines a map from the space $\mathcal{M}$ to a subset $\mathcal{P}$ of functions on $\mathbf{P}^1$ which Favre, Jonsson, and Rivera-Letelier call *the space of potential functions*[5]. They show that this map is bijective ([11] Theorem 7.50), and define their Laplacian to be

$$\Delta\left(-u_\nu(\cdot,\xi)\right) := \nu - \nu(\mathbf{P}^1)\delta_\xi \ .$$

Baker and Rumely construct their Laplacian in terms of directional derivatives. For a full discussion of their construction, see [2] Chapters 3 and 5. Here, we will only present a summary.

We begin with the notion of a directional derivative of a function $g : U \to \mathbb{R}$. Given a point $P \in U$ and a direction $\vec{v} \in T_P$, choose an arbitrary point $Q \in B_{\vec{v}}(P)^-$. For each $0 < t < \rho(P,Q)$, let $P + t\vec{v}$ denote the unique point on the segment $[P,Q]$ with $\rho(P, P + t\vec{v}) = t$. The directional

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[5] The negative sign is unfortunately necessary; the Laplacian of Favre–Jonsson–Rivera-Letelier and the Laplacian of Baker –Rumely agree up to a sign. We will choose the sign convention of Baker–Rumely.

derivative (relative to Q) is defined

$$\partial_{\vec{v}}^{Q}(g)(P) := \lim_{t\to 0} \frac{g(P+t\vec{v})-g(P)}{t} \ ,$$

provided the limit exists. If the limit does exist, it is in fact independent of the choice Q used in the above construction, since for any two points $Q_1, Q_2 \in B_{\vec{v}}(P)^-$, the segments $[P, Q_1], [P, Q_2]$ share a common initial segment, and hence for t sufficiently small, $P + t\vec{v}$ is independent of whether we use Q_1 or Q_2. We define the directional derivative to be $\partial_{\vec{v}}(g)(P) := \partial_{\vec{v}}^{Q}(g)(P)$ for any $Q \in B_{\vec{v}}(P)^-$.

Baker and Rumely define their Laplacian by first introducing a Laplacian on a finite subgraph $\Gamma \subseteq \mathbf{H}^1$. A function $f : \Gamma \to \mathbb{R}$ is said to be *continuous and piecewise affine* if there exists a finite subset $S \subseteq \Gamma$, which necessarily includes the vertices of Γ, such that f is continuous on Γ and is an affine function with respect to the ρ-metric when restricted to the segments in $\Gamma \setminus S$. The space of all such functions on Γ is denoted $\mathrm{CPA}(\Gamma)$. For any element $f \in \mathrm{CPA}(\Gamma)$, the *Laplacian of f on* Γ is the signed Borel measure

$$\Delta_\Gamma(f) := -\sum_{P\in\Gamma} \sum_{\vec{v}\in T_P(\Gamma)} \partial_{\vec{v}}(f)(P)\delta_P \ .$$

It is important to note that $\Delta_\Gamma(f)(\Gamma) = 0$ ([2] Proposition 3.14E). Since $f \in \mathrm{CPA}(\Gamma)$ has only finitely many points where its slope changes, and Γ has only finitely many branch points, its Laplacian on Γ can be decomposed as a weighted sum of points $\Delta_\Gamma(f) = \sum_{P\in\Gamma} c_P \ \delta_P$.

Baker and Rumely go on to extend their Laplacian on Γ to include functions of *bounded differential variation* on Γ, a space which they denote $\mathrm{BDV}(\Gamma)$. While we will not give a construction of this space of functions (it is quite involved – see [2] Section 3.5), we remark that for functions $f \in \mathrm{BDV}(\Gamma) \setminus \mathrm{CPA}(\Gamma)$, the Laplacian on Γ will involve higher order derivatives of f (when they exist). We also note an important formula involving the Laplacian of a pair of functions in $\mathrm{BDV}(\Gamma)$ (see [2] Proposition 3.14C): if $f, g \in \mathrm{BDV}(\Gamma)$, then for each $z \in \Gamma$ we have

$$\int f(x) \ d\Delta(g)(x) = \int_\Gamma g \ d\Delta(f)(y) = \iint_{\Gamma\times\Gamma} \langle x, y\rangle_z \ d\Delta(f)(x) d\Delta(g)(y) \ .$$

Finally, Baker and Rumely extend their Laplacian to subdomains $U \subseteq \mathbf{P}^1$ by considering exhaustions of U by finite graphs Γ and taking limits of the Laplacians Δ_Γ. This is carried out in full detail in [2] Chapter 5.

2.4.4 Arakelov-Green's Functions

Let $\nu \in \mathcal{M}$ be a probability measure with continuous potentials. The *Arakelov-Green's function* associated to ν is a function of two variables $g_\nu(x, y)$ defined by

$$g_\nu(x,y) = \int_{\mathbf{P}^1} -\log_v \delta(x,y)_\zeta d\nu(\zeta) + C \ . \tag{2.5}$$

Here, the normalizing constant C is chosen so that

$$\iint g_\nu(x,y) d\nu(x) d\nu(y) = 0 \ .$$

Observe that, by the definition of the generalized Hsia kernel given in Section 2.1.3, this can be expressed in terms of the potential functions $u_\nu(\cdot, \zeta_{\mathrm{G}})$ as

$$g_\nu(x,y) = -\log_v ||x,y|| - u_\nu(x, \zeta_{\mathrm{G}}) - u_\nu(y, \zeta_{\mathrm{G}}) + C \ .$$

Since we are assuming that $u_\nu(\cdot, \zeta_{\mathrm{G}})$ is continuous, $g_\nu(x,y)$ inherits its continuity properties from $-\log_v ||x,y||$, namely, $g_\nu(x,y)$ is continuous in each variable separately, but only lower semicontinuous as a function of two variables ([2] Proposition 8.66).

2.5 Regularization of Measures

In this section, we will build on a notion of regularization of measures originally introduced by Favre and Rivera-Letelier [12] in their work on the equidistribution of points of small height. Regularization in the non-Archimedean context is, loosely speaking, the process of retracting a measure ν towards a designated point[6] $\zeta \in \mathbf{P}^1$. This is often used when a test function f has a

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[6] Favre and Rivera-Letelier worked with retractions to the point $\zeta = \infty$.

singularity: by retracting away from the singular point, it becomes easier to estimate the integral against f.

In [12], Favre and Rivera-Letelier relied on a notion of smoothing functions and measures relative to the point ∞. Baker and Rumely have given a smoothing on $\mathbf{P}^1$ relative to any point in $\mathbf{H}^1$ ([2] Section 5.7); we recall here Baker and Rumely's generalization, and apply this to give a notion of retracting a measure towards an arbitrary basepoint in $\mathbf{P}^1$. This will prove to be an important tool in Chapter 6, where we establish several logarithmic equidistribution results.

Fix $\zeta_0 \in \mathbf{P}^1$. Recall that the ζ_0-diameter of a point $z \in \mathbf{P}^1$ is the quantity $\mathrm{diam}_{\zeta_0}(z) = \delta(z,z)_{\zeta_0}$. Following [2], for every $0 \leq \delta \leq \frac{1}{||\zeta_0,\zeta_0||}$, we let

$$X(\zeta_0,\delta) := \{z \in \mathbf{H}^1 \ : \ \mathrm{diam}_{\zeta_0}(z) \geq \delta\} \ .$$

We first note that when $\zeta_0 \in \mathbf{H}^1$, these sets coincide with balls relative to the strong metric:

Lemma 2.7 (See [2] Section 5.7)**.** *Fix $\zeta_0 \in \boldsymbol{H}^1$. For every $0 < \delta < \frac{1}{||\zeta_0,\zeta_0||}$, let $R = \log\left(\frac{||\zeta_0,\zeta_0||}{\delta}\right)$. Then*

$$X(\zeta_0,\delta) = \mathcal{B}_\rho(\zeta_0,R) \ ,$$

where $\mathcal{B}_\rho(\zeta_0,R)$ is the closed ball of radius R about ζ_0 in the path distance metric ρ on $\boldsymbol{H}^1$.

Proof. By our normalization of the Hsia kernel, we have

$$\begin{aligned}
\log_v \mathrm{diam}_{\zeta_0}(z) - \log_v \mathrm{diam}_{\zeta_0}(\zeta_0) &= \log_v \delta(z,z)_{\zeta_0} - \log_v \delta(\zeta_0,\zeta_0)_{\zeta_0} \\
&= -\langle z,z\rangle_{\zeta_G} + 2\langle z,\zeta_0\rangle_{\zeta_G} - \langle \zeta_0,\zeta_0\rangle_{\zeta_G} \\
&= -\rho(z,\zeta_0) \ .
\end{aligned}$$

Thus, $\mathrm{diam}_{\zeta_0}(z) \geq \delta$ if and only if $\rho(z,\zeta_0) \leq \log_v \frac{\mathrm{diam}_{\zeta_0}(\zeta_0)}{\delta} = \log_v\left(\frac{||\zeta_0,\zeta_0||}{\delta}\right)$, which is the desired assertion. □

Our smoothing will be defined as a retraction. Generally speaking, if $U \subseteq \mathbf{P}^1$ is a connected closed subset, we can define a retraction $r_{\mathbf{P}^1,U} : \mathbf{P}^1 \to U$ as follows: if $P \in U$, we define $r_{\mathbf{P}^1,U}(P) = P$. Otherwise, choose a point $Q \in U$, and for any other point $P \in \mathbf{P}^1$, consider the segment $[P,Q]$

joining P and Q. Since $P \notin U$, there is a unique 'first point' R in $[P, Q]$ which lies in U, where we view the segment as being oriented from P to Q. We define $r_{\mathbf{P}^1,U}(P) = R$. This is independent of our choice of reference point $Q \in U$, since U is connected and $\mathbf{P}^1$ is uniquely path connected. An important fact about the retraction is that $r_{\mathbf{P}^1,U}$ is continuous in the weak topology ([2] Lemma 5.3).

Let $\epsilon \geq 0$. For $\zeta_0 \in \mathbf{P}^1$, we define $\pi_{\epsilon,\zeta_0} := r_{\mathbf{P}^1,X(\zeta_0,\epsilon)}$. We obtain the following generalization of [12] Lemma 4.7:

Lemma 2.8 ([12] Lemma 4.7)**.** *Fix $\zeta_0 \in \boldsymbol{P}^1$. For every $0 \leq \epsilon < \frac{1}{||\zeta_0,\zeta_0||}$ and every ball $B = B_\zeta(\vec{v})^- \in \boldsymbol{P}^1$, we have*

$$\pi_{\epsilon,\zeta_0}^{-1}(B) = \begin{cases} B, & B \cap X(\zeta_0,\epsilon) \neq \emptyset, \text{ but } X(\zeta_0,\epsilon) \not\subseteq B \\ \boldsymbol{P}^1, & X(\zeta_0,\epsilon) \subseteq B \\ \emptyset, & B \cap X(\zeta_0,\epsilon) = \emptyset \end{cases}$$

Proof. Let $B := B_{\vec{v}}(\zeta_1)^- \subseteq \mathbf{P}^1$ be a connected component of $\mathbf{P}^1 \setminus \{\zeta_1\}$ for some point $\zeta_1 \in \mathbf{P}^1$. If $B \cap X(\zeta_0,\epsilon) = \emptyset$, then $\pi_{\epsilon,\zeta_0}^{-1}(B) = \emptyset$ since the image of π_{ϵ,ζ_0} lies in $X(\zeta_0,\epsilon)$. If $X(\zeta_0,\epsilon) \subseteq B$, then necessarily $\pi_{\epsilon,\zeta_0}^{-1}(B) = \mathbf{P}^1$ since $\pi_{\epsilon,\zeta_0}^{-1}(X(\zeta_0,\epsilon)) = \mathbf{P}^1$.

Finally, suppose $B \cap X(\zeta_0,\epsilon) \neq \emptyset$, but $X(\zeta_0,\epsilon) \not\subseteq B = B_{\vec{v}}(\zeta_1)^-$. We claim that $\zeta_1 \in X(\zeta_0,\epsilon)$: if not, let $\vec{v} \in T_{\zeta_1}$ be the direction towards $X(\zeta_0,\epsilon)$. Since $X(\zeta_0,\epsilon) \not\subseteq B$, $\vec{v}$ is not the direction pointing into B. Thus any path from a point $y \in B$ to $X(\zeta_0,\epsilon)$ must pass through ζ_1. Since $X(\zeta_0,\epsilon)$ is (path)connected, it follows that no point $y \in B$ lies in $X(\zeta_0,\epsilon)$, contradicting that $B \cap X(\zeta_0,\epsilon) \neq \emptyset$. Hence $\zeta_1 \in X(\zeta_0,\epsilon)$.

Now choose any point $z \in \mathbf{P}^1$ with $\pi_{\epsilon,\zeta_0}(z) \in B$, and consider the segment $[z, \pi_{\epsilon,\zeta_0}(z)]$. If $z \notin B$, then $[z, \pi_{\epsilon,\zeta_0}(z)]$ must contain ζ_1. But since $\zeta_1 \in X(\zeta_0,\epsilon)$, this implies that $\pi_{\epsilon,\zeta_0}(z) = \zeta_1 \notin B = B_{\vec{v}}(\zeta_1)^-$, which contradicts that $\pi_{\epsilon,\zeta_0}(z) \in B$. We conclude that $z \in B$, and hence $\pi_{\epsilon,\zeta_0}^{-1}(B) \subseteq B$. The reverse inclusion follows from the definition of the retraction π_{ϵ,ζ_0}. □

We can use the map π_{ϵ,ζ_0} to define a regularization (retraction) of Borel measures. Fix a finite signed Borel measure ν and a point $\zeta_0 \in \mathbf{P}^1$. For any $0 \leq \epsilon < \frac{1}{||\zeta_0,\zeta_0||}$, let

$$\nu_{\epsilon,\zeta_0} := (\pi_{\epsilon,\zeta_0})_* \nu \ .$$

As an example, if $\nu = \delta_z$ is the point mass supported at $z \in \mathbf{P}^1$, then $\nu_{\epsilon,\zeta_0} = \delta_{\pi_{\epsilon,\zeta_0}(z)}$. This follows because

$$\nu_{\epsilon,\zeta_0}(B) = \nu(\pi_{\epsilon,\zeta_0}^{-1}(B)) = \begin{cases} 1, & z \in \pi_{\epsilon,\zeta_0}^{-1}(B) \\ 0, & z \notin \pi_{\epsilon,\zeta_0}^{-1}(B) \end{cases} = \begin{cases} 1, & \pi_{\epsilon,\zeta_0}(z) \in B \\ 0, & \pi_{\epsilon,\zeta_0}(z) \notin B \end{cases} .$$

The next lemma shows that the retraction operator $(\pi_{\epsilon,\zeta_0})_*$ satisfies several continuity properties with respect to the weak-* topology on the set of continuous functions $\mathcal{C}(\mathbf{P}^1)$. More generally, if ν_n is a family of measures on $\mathbf{P}^1$, we say that $\nu_n \to \nu$ if, for every continuous function $f : \mathbf{P}^1 \to \mathbb{R}$, we have

$$\int f \ d\nu_n \to \int f \ d\nu \ .$$

In the next lemma, and throughout this thesis, we will write $|F|$ for the cardinality of a finite set F.

Lemma 2.9 ([12] Lemma 4.8)**.** *Fix $\zeta_0 \in \boldsymbol{P}^1$. For a finite signed Borel measure ν, we have $\nu_{\epsilon,\zeta_0} \to \nu$ as $\epsilon \to 0$. For a sequence of finite signed Borel measures ν_n with $\nu_n \to \nu$, we have $(\nu_n)_{\epsilon,\zeta_0} \to \nu_{\epsilon,\zeta_0}$ as $n \to \infty$. Moreover, if $F \subseteq \boldsymbol{P}^1 \setminus \{\zeta_0\}$ is a finite set, and $[F] = |F|^{-1} \sum_{z \in F} [z]$, then for every $\zeta_0 \in \boldsymbol{H}^1$ and every $0 < \epsilon \leq \frac{1}{||\zeta_0, \zeta_0||}$, the measure $[F]_{\epsilon,\zeta_0}$ has continuous potentials.*

Proof. By [2] Proposition 5.4, as Γ ranges over all finite subtrees and f ranges over CPA(Γ), the functions of the form $f \circ r_{\mathbf{P}^1,\Gamma}$ are dense in the set of all continuous functions $\mathcal{C}(\mathbf{P}^1)$. Thus, we will show these convergences for functions of the form $f \circ r_{\mathbf{P}^1,\Gamma}$ for fixed Γ and f.

Fix a finite tree $\Gamma \subseteq \mathbf{H}^1$, and let $f \in$ CPA(Γ). By definition we have

$$\int f \circ r_{\mathbf{P}^1,\Gamma} \ d\nu_{\epsilon,\zeta_0} = \int (f \circ r_{\mathbf{P}^1,\Gamma}) \circ \pi_{\epsilon,\zeta_0} d\nu \ .$$

For ϵ sufficiently small (chosen so that $\Gamma \subseteq X(\zeta_0, \epsilon)$), we have $f \circ r_{\mathbf{P}^1,\Gamma} = (f \circ r_{\mathbf{P}^1,\Gamma}) \circ \pi_{\epsilon,\zeta_0}$ and hence

$$\int f \circ r_{\mathbf{P}^1,\Gamma} \ d\nu_{\epsilon,\zeta_0} = \int \circ r_{\mathbf{P}^1,\Gamma} \ d\nu \ .$$

Since such functions $f \circ r_{\mathbf{P}^1,\Gamma} \in \mathrm{CPA}(\Gamma)$ are dense in $\mathcal{C}(\mathbf{P}^1)$ (see [2], Proposition 5.4), the first assertion holds. If $\nu_n \to \nu$, then since $(f \circ r_{\mathbf{P}^1,\Gamma}) \circ \pi_{\epsilon,\zeta_0}$ is continuous for each fixed ϵ we have

$$\int f \circ r_{\mathbf{P}^1,\Gamma}\ d(\nu_n)_{\epsilon,\zeta_0} = \int (f \circ r_{\mathbf{P}^1,\Gamma}) \circ \pi_{\epsilon,\zeta_0} d\nu_n \to \int (f \circ r_{\mathbf{P}^1,\Gamma}) \circ \pi_{\epsilon,\zeta_0} d\nu = \int f \circ r_{\mathbf{P}^1,\Gamma}\ d\nu_{\epsilon,\zeta_0}\ .$$

For the final assertion, note that since $\zeta_0 \notin F$, the regularized measure $[F]_{\epsilon,\zeta_0}$ is a sum of atomic measures *supported in* $\mathbf{H}^1$; hence $[F]_{\epsilon,\zeta_0}$ has continuous potentials.

□

2.6 Arithmetic Dynamics

In this section, we review several topics at the intersection of arithmetic dynamics and potential theory over non-Archimedean fields. We will not go into depth on the non-Archimedean Fatou and Julia sets, as they are largely unnecessary for the results in this thesis. The reader who is interested in the Fatou-Julia theory of non-Archimedean rational functions is directed to the thesis of J. Rivera-Letelier ([20]) and Chapter 10 of [2].

2.6.1 Height Functions

Over $\mathbb{C}$, an important tool in studying the dynamics of ϕ is the Green's function (or escape rate function) $G_\Phi : \mathbb{C}^2 \to \mathbb{R}$ attached to a homogeneous lift of ϕ, which measures the asymptotic rate of escape of a point (x, y) under the iteration of Φ. It is defined

$$G_\Phi(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log ||\Phi^n(x,y)||\ ,$$

where $||\cdot||$ is the usual norm on $\mathbb{C}^2$. One can define a non-Archimedean analogue of this by replacing $\mathbb{C}^2$ with K^2 and the Euclidean norm $||(x,y)||$ with $\max(|x|,|y|)$; in this context, it is more commonly called the *height function attached to* Φ because of its relation to a Néron-Tate height on an elliptic curve.

For us, it will be useful to consider a dehomogenized version of this function on $\mathbb{P}^1(K)$ and its extension to $\mathbf{P}^1$. More precisely, if $\Phi = (F, G)$ is a homogeneous lift of ϕ, then for $\zeta \in \mathbf{A}^1$ we define

$$\hat{h}_{\Phi,v,(\infty)}(\zeta) := \lim_{n\to\infty} \frac{1}{d^n} \log_v \max\left([F(T,1)]_\zeta, [G(T,1)]_\zeta\right) \ . \tag{2.6}$$

The fact that this limit exists is proved using a telescoping series argument; the details can be found in [2] Section 10.1. In Chapter 3, the expressions $\frac{1}{d^n} \log_v \max\left([F^n(T,1)]_\zeta, [G^n(T,1)]_\zeta\right)$ will play an important role in establishing the convergence of the normalized functions $\frac{1}{d^{2n}-d^n} \operatorname{ordRes}_\phi(\cdot)$, and as such we will write

$$\hat{\ell}^{(n)}_{\phi,v}(\zeta) = \log_v \max\Big([F^{(n)}(T,1)]_\zeta, [G^{(n)}(T,1)]_\zeta\Big) \ , \tag{2.7}$$

so that $\hat{h}_{\phi,v,(\infty)}(\zeta) = \lim_{n\to\infty} \frac{1}{d^n}\ \hat{\ell}^{(n)}_{\phi,v}(\zeta)$.

2.6.2 The Equilibrium Measure

The height function also gives rise to the equilibrium measure μ_ϕ attached to ϕ: a ϕ-invariant probability measure on $\mathbf{P}^1$ which satisfies $\phi^*\mu_\phi = d \cdot \mu_\phi$. Concretely, the equilibrium measure is the unique probability measure on $\mathbf{P}^1$ satisfying

$$\Delta \hat{h}_{\phi,v,(\infty)} = \delta_\infty - \mu_\phi \ .$$

Over $\mathbb{C}$ this measure was first considered as the limiting measure for the pullbacks of a point mass $\delta^n_a = \frac{1}{d^n} \sum_{\phi^n(w)=a} \delta(w)$ for non-exceptional a, and is also realized as the unique measure of maximal entropy ([4], [17], [14]).

The fact that the non-Archimedean equilibrium measure is also the weak limit of the measures δ^n_a is due to Favre—Rivera-Letelier [13]. However, in this context μ_ϕ need not be the measure of maximal entropy; see [13] Section 5.2 for a discussion of this phenomenon.

Over $\mathbb{C}$, the equilibrium measure never charges points ([8] Théorème 3.7.1), and in the non-Archimedean case the same is essentially true, *except* in the case that ϕ has potential good reduction; in this case, $\mu_\phi = \delta_\zeta$, where $\zeta = \gamma(\zeta_{\mathrm{G}})$ and ϕ^γ has good reduction ([12] or [2] Corollary 10.47). In the non-Archimedean context, as well as over $\mathbb{C}$, the equilibrium measure has continuous potentials ([2] Proposition 10.7 over K, [8] Théorème 3.7.1 over $\mathbb{C}$); as such, it gives rise to an Arakelov-Green's function $g_{\mu_\phi}(x,y)$ as described in the preceeding section; for brevity we often write $g_\phi(x,y)$.

In [2] Section 10.2, Baker and Rumely give an alternative formulas for the Arakelov-Green's function that are very useful in practice and which will play a very important role in the proof of our Theorem 3.1 . They show ([2] Theorem 10.21) that for $x, y \in \mathbf{P}^1 \setminus \{\infty\}$,

$$g_{\mu_\phi}(x,y) = -\log_v \delta(x,y)_\infty + \hat{h}_{\phi,v,(\infty)}(x) + \hat{h}_{\phi,v,(\infty)}(y) + \log_v R_{\Phi,v} \ . \tag{2.8}$$

where $R_{\Phi,v} = |\operatorname{Res}(F,G)|_v^{-\frac{1}{d(d-1)}}$ (in fact they give a more general formula that holds even if $x = \infty$ or $y = \infty$). Of particular use to us will be the case that $\mathbf{H}^1 \ni x = y \neq \infty$, which gives rise to the formula

$$g_\phi(x,x) = -\log_v \delta(x,x)_\infty + 2\hat{h}_{\phi,v,(\infty)}(x) + \log_v R_{\Phi,v} \ . \tag{2.9}$$

2.6.3 The Julia and Fatou Sets in $\mathbf{P}^1$

We close this chapter with a brief discussion of the Fatou and Julia sets of a rational map ϕ defined over a non-Archimedean valued field K. The Julia set $\mathcal{J}(\phi)$ of ϕ is defined to be the support of the measure μ_ϕ introduced in the previous section; its complement is called the Fatou set, which we denote by $\mathcal{F}(\phi) := \mathbf{P}^1 \setminus \mathcal{J}(\phi)$.

The Julia and Fatou sets are both totally invariant; that is, $\mathcal{F}(\phi^n) = \mathcal{F}(\phi)$ and $\mathcal{J}(\phi^n) = \mathcal{J}(\phi)$ ([2] Lemma 10.51). There are a number of alternative defining properties that one can give for $\mathcal{F}(\phi)$ and $\mathcal{J}(\phi)$ (see [2] Section 10.5 for a detailed discussion). We remark here that the Julia set is also the closure of the repelling periodic points[7] in $\mathbf{P}^1$ ([2] Theorem 10.88). The Fatou set can be described as the set of all points $x \in \mathbf{P}^1$ for which there is a neighborhood U of x (open in the weak topology) for which $\cup_{n=0}^\infty \phi^n(U)$ omits at least three points of $\mathbb{P}^1(K)$. In particular, if an open set $U \neq \mathbf{P}^1$ satisfies $\phi(U) \subseteq U$, then necessarily $U \subseteq \mathcal{F}(\phi)$.

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[7] It is interesting to note, however, that an important open question is whether the 'classical' Julia set $\mathcal{J}(\phi) \cap \mathbb{P}^1(K)$ is the closure of type I repelling periodic points, as is the case over $\mathbb{C}$. Hsia has conjectured that this is the case; see [15] Conjecture 4.3

Chapter 3

Equidistribution of ν_{ϕ^n}

Let K be a complete, non-Archimedean valued field, and fix $\phi \in K(z)$ of degree $d \geq 2$. In this chapter, we establish the weak convergence of the measures ν_{ϕ^n} to the equilibrium measure μ_ϕ on $\mathbf{P}^1$. The convergence of the measures ν_{ϕ^n} is built on the following convergence result for the family of functions $\mathrm{ordRes}_{\phi^n}(\cdot)$:

Theorem 3.1. *The normalized functions*

$$\frac{1}{d^{2n}-d^n}\,\mathrm{ord}\,\mathrm{Res}_{\phi^n}(x)$$

converge to the diagonal values of the Arakelov-Green's function $g_\phi(x,x)$, locally uniformly in the strong topology.

The proof of this theorem is based on a decomposition of the function $\mathrm{ordRes}_{\phi^n}(\cdot)$ and the function $g_\phi(\cdot,\cdot)$; see Table 3.1 below for the explicit decomposition. In Section 3.1 below, we give an explicit estimate on the error term $\left|\frac{1}{d^{2n}-d^n}\,\mathrm{ordRes}_{\phi^n}(x) - g_\phi(x,x)\right|$; the error term is $O\left(\frac{1}{d^n}\right)$, where the implied constant depends on the map ϕ and on the point x.

The fact that Theorem 3.1 implies the weak convergence of the measures ν_{ϕ^n} can be understood from the heuristic that "$\Delta\frac{1}{d^{2n}-d^n}\,\mathrm{ordRes}_{\phi^n}(\cdot) = 2\nu_{\phi^n}$" and "$\Delta g_\phi(\cdot,\cdot) = 2\mu_\phi$". Neither of these equations is exactly correct; there are certain other measures involved in the right hand sides of these Laplacians. However, these 'extra measures' become insignificant in the limit, and we obtain

Theorem 3.2. *The measures ν_{ϕ^n} converge weakly to the canonical measure μ_ϕ.*

Here too, an explicit estimate on $\left|\int f d(\nu_{\phi^n} - \mu_\phi)\right|$ will be given (Theorem 3.3) for test functions f that are continuous and piecewise affine on fixed graphs $\Gamma \subseteq \mathbf{H}^1$ and constant on branches off of Γ. As was the case above, the error term is $O\left(\frac{1}{d^n}\right)$, where the implied constant now depends on the map ϕ, the function f and the graph Γ.

The rest of this chapter is divided into two sections. In Section 3.1 we first establish an explicit convergence result for $\mathrm{ordRes}_{\phi^n}(x)$ for a fixed point $x \in \mathbf{H}^1$ (Theorem 3.3), and use it to deduce Theorem 3.1. Here the primary tool is a parallel decomposition of $\mathrm{ord}\,\mathrm{Res}_{\phi^n}(\cdot)$ and $g_\phi(\cdot,\cdot)$ given in Table 3.1 below.

In Section 3.2 we establish the weak convergence of the family of crucial measures. For this we develop formulae for the slopes of $\mathrm{ord}\,\mathrm{Res}_{\phi^n}(x)$ similar to those given in [25], Propositions 5.2-5.4. Taken together, the formulae in [25] and those in Section 3.2 allow us to explicitly compute the Laplacian of $f_n(x) = \mathrm{ord}\,\mathrm{Res}_{\phi^n}(x) + \log\delta(x,x)_\infty$ on an arbitrary graph (Proposition 3.23). We then prove Theorem 3.11, which gives an explicit estimate on the error $\int f d(\nu_{\phi^n} - \mu_\phi)$ for test functions f which are continuous and piecewise affine on a fixed finite graph $\Gamma \subseteq \mathbf{H}^1$. An approximation theorem for arbitrary continuous functions allows this result to be extended as needed to show weak convergence.

Finally, in Section 3.3 we give two explicit examples of the family of measures ν_{ϕ^n} and demonstrate their weak convergence.

3.1 Convergence of the functions ordRes_{ϕ^n}

In this section we establish Theorem 3.1 by proving the following more quantitative result:

Theorem 3.3. *Let K be a complete, non-Archimedean valued field, and let $\phi \in K(z)$ have degree $d \geq 2$. There is a constant $C = C(\phi) > 0$ depending only on ϕ such that for any $x \in \boldsymbol{H}^1$, we have*

$$\left|\frac{1}{d^{2n} - d^n}\,\mathrm{ord}\,\mathrm{Res}_{\phi^{(n)}}(x) - g_{\mu_\phi}(x,x)\right| \leq \frac{4}{d^n - 1}\max\left(C, \rho(x,\zeta_G)\right) .$$

The idea behind the proof of Theorem 3.3 comes from the following decomposition of $\mathrm{ordRes}_\phi(\cdot)$ and $g_\phi(\cdot,\cdot)$ (see [24] Equation 8 and (2.8) above):

Table 3.1: Comparison of Decompositions. Here, $x \in \mathbf{H}^1$ has type II and $\gamma \in \mathrm{PGL}_2(K)$ satisfies $x = \gamma(\zeta_{\mathrm{G}})$.

$\operatorname{ord}\operatorname{Res}_{\phi^{(n)}}(\gamma)$	$g_{\mu_\phi}(x,x)$
$\operatorname{ord}\operatorname{Res}(F^{(n)}, G^{(n)})$	$-\frac{1}{d(d-1)}\log_v(\lvert\operatorname{Res}(F,G)\rvert)$
$(d^{2n}+d^n)\operatorname{ord}(\det(\gamma))$	$-\log_v(\delta(x,x)_\infty)$
$-2d^n\min(\operatorname{ord}((F^{(n)})^\gamma), \operatorname{ord}((G^{(n)})^\gamma))$	$2\hat{h}_{\phi,v,(\infty)}(x)$

Heuristically, we show that each term in the left column, when normalized by $\frac{1}{d^{2n}-d^n}$, converges to the corresponding term on the right side; this only works heuristically, and we will in fact need to work with Rows 2 and 3 as a single unit rather than treating them separately. The convergence of the terms in Row 1 is straightforward:

Lemma 3.4. *For every n, we have*

$$\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}\left(F^{(n)}, G^{(n)}\right) = -\frac{1}{d^2-d}\log_v\lvert\operatorname{Res}(F,G)\rvert \ .$$

Proof. Using the formula in Lemma 2.3 above, we find:

$$\operatorname{Res}(F^{(n)}, G^{(n)}) = \operatorname{Res}(F,G)^{d^{n-1}}\operatorname{Res}(F^{(n-1)}, G^{(n-1)})^{d^2} \ .$$

Applying this inductively,

$$\begin{aligned}\operatorname{Res}(F^{(n)}(X,Y), G^{(n)}(X,Y)) &= \operatorname{Res}(F,G)^{d^{n-1}+\ldots+d^{2n-2}} \\ &= \operatorname{Res}(F,G)^{d^{n-1}(1+d+\ldots+d^{n-1})} \\ &= \operatorname{Res}(F,G)^{d^{n-1}\frac{d^n-1}{d-1}} \\ &= \operatorname{Res}(F,G)^{\frac{d^{2n}-d^n}{d(d-1)}} \ .\end{aligned}$$

Taking the ord and normalizing, we obtain the result

$$\begin{aligned}\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}(F^{(n)},G^{(n)}) &= \frac{1}{d^{2n}-d^n}\operatorname{ord}\left(\operatorname{Res}(F,G)^{\frac{d^{2n}-d^n}{d^2-d}}\right)\\ &= \frac{1}{d^2-d}\operatorname{ord}\operatorname{Res}(F,G)\\ &= -\frac{1}{d^2-d}\log_v|\operatorname{Res}(F,G)|\ .\end{aligned}$$

□

We now turn to the convergence of the terms in the second and third rows of Table 3.1. The terms on the second line of Table 3.1 are related by the following lemma:

Lemma 3.5. *If x is the type II point $\zeta_{a,r} \in \boldsymbol{H}^1$, then the transformation $\gamma \in \mathrm{PGL}_2(K)$ given $\gamma(z) = bz + a$, where $|b| = r$, sends ζ_G to x, and we have*

$$\operatorname{ord}(\det(\gamma)) = -\log_v(\delta(x,x)_\infty)\ .$$

Proof. Let γ be as in the statement of the lemma; as a matrix, γ is represented by $\left[\begin{smallmatrix} b & a\\ 0 & 1\end{smallmatrix}\right]$; clearly $x = \gamma(\zeta_{\mathrm{G}})$. Since x corresponds to a disk of radius r and $\delta(x,x)_\infty = \operatorname{diam}_\infty(x) = r$, we have

$$\begin{aligned}-\log_v(\delta(x,x)_\infty) &= -\log_v|b|\\ &= \operatorname{ord}(b)\ .\end{aligned}$$

Note that $\det(\gamma) = b$, and so $\operatorname{ord}(\det(\gamma)) = \operatorname{ord}(b) = -\log_v(\delta(x,x)_\infty)$. □

Next we look to compare the terms

$$-2d^n\min\left(\operatorname{ord}\left((F^{(n)})^\gamma\right),\operatorname{ord}\left((G^{(n)})^\gamma\right)\right)$$

to $2\hat{h}_{\phi,v}$. Let $|F(X,Y)| = \max_{1\le i\le d}|a_i|$ denote the absolute value of the largest coefficient of $F(X,Y)$. We can rewrite the above expression in terms of a log max of the absolute values:

$$-2d^n \min\left(\operatorname{ord}\left((F^{(n)})^\gamma\right), \operatorname{ord}\left((G^{(n)})^\gamma\right)\right) = 2d^n \log\max\left(\left|(F^{(n)})^\gamma\right|, \left|(G^{(n)})^\gamma\right|\right) \ . \tag{3.1}$$

In the discussion below, we make two simplifying assumptions. First, we will only work with affine transformations $\gamma(z) = az + b$; these maps can be used to carry ζ_{G} to any type II point in $\mathbf{H}^1$, and so will be sufficient for our purposes. Second, rather than working with iterates $F^{(n)}, G^{(n)}$, we will state and prove our results for arbitrary $F, G \in K[X,Y]$ and then apply the results to the iterates in proving Theorem 3.1.

The expression for the conjugate Φ^γ can be given

$$\begin{aligned} \begin{bmatrix} F^\gamma(X,Y) \\ G^\gamma(X,Y) \end{bmatrix} &= \left(\begin{bmatrix} 1 & -a \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} F \\ G \end{bmatrix}\right)\left(\begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix}\right) \\ &= \begin{bmatrix} F(bX+aY,Y) - aG(bX+aY,Y) \\ bG(bX+aY,Y) \end{bmatrix} \ . \end{aligned}$$

It is worth noting here that $[F^\gamma, G^\gamma]$ may not be normalized.

We will address the relation between the coefficients of $[F^\gamma, G^\gamma]$ and $[F,G]$ in two steps, first looking at the effect of postcomposition by $\operatorname{Adj}(\gamma)$, and then the effect of precomposition by γ.

Lemma 3.6. *Let $F(X,Y), G(X,Y)$ be a pair of homogeneous degree d polynomials in $K[X,Y]$. For $a, b \in K$, if $x = \zeta_{a,|b|} \in \boldsymbol{H}^1$, we have*

$$\Big|\log\max\left(|F(X,Y) - aG(X,Y)|, |bG(X,Y)|\right) - \log\max\left(|F(X,Y)|, |G(X,Y)|\right)\Big| \le \rho(x, \zeta_G) \ . \tag{3.2}$$

Proof. The result follows from explicit estimates on the coefficients, making precise the computations laid out in the proof of [28] Theorem 3.11.

Write $F(X,Y) = a_d X^d + a_{d-1} X^{d-1} Y + ... + a_0 Y^d$, $G(X,Y) = b_d X^d + ... + b_0 Y^d$. The coefficients of $F(X,Y) - aG(X,Y)$ are of the form $a_i - a \cdot b_i$, and likewise the coefficients of $bG(X,Y)$ are $b \cdot b_i$. By the ultrametric inequality, we obtain estimates towards the lower bound by:

$$|a_i - a \cdot b_i| \leq \max(|a_i|, |b_i|) \cdot \max(1, |a|) \ ,$$
$$|b \cdot b_i| \leq \max(|a_i|, |b_i|) \cdot \max(1, |b|) \ .$$

Hence

$$\max(|a_i - a \cdot b_i|, |b \cdot b_i|) \leq \max(|a_i|, |b_i|) \cdot \max(1, |a|, |b|) \ ,$$

or equivalently

$$\frac{\max(|a_i - a \cdot b_i|, |b \cdot b_i|)}{\max(|a_i|, |b_i|)|} \leq \max(1, |a|, |b|) \ . \tag{3.3}$$

Similarly, for the upper bound, we have

$$|a_i| = |a_i - ab_i + ab_i| \leq \max(|a_i - a \cdot b_i|, |b \cdot b_i|) \max\left(1, \frac{|a|}{|b|}\right) \ ,$$
$$|b_i| = \frac{1}{|b|}|bb_i| \leq \max(|a_i - a \cdot b_i|, |b \cdot b_i|) \max\left(1, \frac{1}{|b|}\right) \ .$$

Hence

$$\max(|a_i|, |b_i|) \leq \max(|a_i - a \cdot b_i|, |b \cdot b_i|) \cdot \max\left(1, \frac{|a|}{|b|}, \frac{1}{|b|}\right) \ ,$$

or equivalently

$$\frac{\max(|a_i|, |b_i|)}{\max(|a_i - a \cdot b_i|, |b \cdot b_i|)} \leq \max\left(1, \frac{|a|}{|b|}, \frac{1}{|b|}\right) \ . \tag{3.4}$$

Combining (3.3) and (3.4), taking logs, and doing some algebra yields:

$$\begin{aligned}
&\big|\log\max\left(|a_i - a\cdot b_i|, |b\cdot b_i|\right) - \log\max\left(|a_i|, |b_i|\right)\big| \\
&\qquad\qquad \le \log\max\left(\max\left(1, \frac{|a|}{|b|}, \frac{1}{|b|}\right), \max(1, |a|, |b|)\right) \\
&\qquad\qquad \le \log\max\left(1, \frac{|a|}{|b|}, \frac{1}{|b|}\right) + \log_v \max(1, |a|, |b|) \\
&\qquad\qquad = 2\log\max(1, |a|, |b|) - \log(|b|)\ .
\end{aligned}$$

Finally, we note that if $x = \zeta_{a,|b|}$, the smallest disc containing $D(a, |b|)$ and $D(0, 1)$ has radius $R = \max(|a|, |b|, 1)$; hence $x \wedge_\infty \zeta_{\mathrm{G}} = \zeta_{0,R}$. The above estimate now reads

$$\begin{aligned}
&\big|\log\max\left(|a_i - a\cdot b_i|, |b\cdot b_i|\right) - \log\max\left(|a_i|, |b_i|\right)\big| \\
&\qquad\qquad \le 2\log\max(1, |a|, |b|) - \log(|b|) \\
&\qquad\qquad = 2\log R - \log(|b|) \\
&\qquad\qquad = \rho(x, \zeta_{\mathrm{G}})\ .
\end{aligned}$$

□

We now have a lemma that makes explicit the effect of precomposition of $[F, G]$ by γ:

Lemma 3.7. *Let $F(X, Y), G(X, Y)$ be a pair of homogeneous degree d polynomials in $K[X, Y]$. For $a, b \in K$, if $x = \zeta_{a,|b|} \in \boldsymbol{H}^1$, we have*

$$\log\max(|F(bX + aY, Y)|, |G(bX + aY, Y)|) = \log\max([F(T, 1)]_x, [G(T, 1)]_x)\ ,$$

where $[F(T, 1)]_x$ denotes the (semi)norm corresponding to x.

Proof. First recall that the norm induced by the Gauss point is indeed the Gauss norm: $[F(T, 1)]_{\zeta_{Gauss}} = \max_{0\le i\le d}(|a_i|) = |F(T, 1)|$.

Let $T = X/Y$. We have $F(bX + aY, Y) = \frac{1}{Y^d} F(bT + a, 1)$, and since the division by Y does not affect the maximum of the coefficients, we have

$$\begin{aligned} |F(bX + aY, Y)| &= |F(bT + a, 1)| \\ &= |F(\gamma(T), 1)| \\ &= [F(\gamma(T), 1)]_{\zeta_{Gauss}} \\ &= [F(T, 1)]_{\gamma(\zeta_{Gauss})} \\ &= [F(T, 1)]_x \ . \end{aligned}$$

The similar statement holds for $G(X, Y)$, and so the result follows. □

We can combine the two preceeding lemmas to obtain a result that expresses the effect of conjugation by an affine map γ on the size of the coefficients of a pair $[F, G]$:

Lemma 3.8. *Let $F(X, Y), G(X, Y)$ be a pair of homogeneous degree d polynomials in $K[X, Y]$. Let $x \in \boldsymbol{H}^1$ be of type II, and let $\gamma(z) = bz + a$ be the affine map sending ζ_G to x. Let $\hat{\ell}^{(n)}_{\phi,v}(x)$ denote the convergent of the canonical height given in (2.7):*

$$\hat{\ell}^{(n)}_{\phi,v}(x) := \log \max\left([F^{(n)}(T, 1)]_x, [G^{(n)}(T, 1)]_x \right) \ .$$

Then

$$\left| \log \max\left(\left| \left(F^{(n)}\right)^{\gamma} \right|, \left| \left(G^{(n)}\right)^{\gamma} \right| \right) - \hat{\ell}^{(n)}_{\phi,v}(x) \right| \leq \rho(x, \zeta_G) \ .$$

Proof. We first apply the result of Lemma 3.6 to find

$$\left| \log \max\left(\left| \left(F^{(n)}\right)^{\gamma} \right|, \left| \left(G^{(n)}\right)^{\gamma} \right| \right) - \log \max\left(\left| F^{(n)}(bX + aY, Y) \right|, \left| G^{(n)}(bX + aY, Y) \right| \right) \right| \leq \rho(x, \zeta_{\mathrm{G}}) \ .$$

Now applying Lemma 3.7 we find that

$$\left| \log \max\left(\left| \left(F^{(n)}\right)^{\gamma} \right|, \left| \left(G^{(n)}\right)^{\gamma} \right| \right) - \log \max\left(\left[F^{(n)}(T, 1) \right]_x, \left[G^{(n)}(T, 1) \right]_x \right) \right| \leq \rho(x, \zeta_{\mathrm{G}}) \ .$$

Equivalently,

$$\left|\log\max\left(\left|\left(F^{(n)}\right)^{\gamma}\right|,\left|\left(G^{(n)}\right)^{\gamma}\right|\right)-\hat{\ell}^{(n)}_{\phi,v}(x)\right|\leq\rho(x,\zeta_{\mathrm{G}})\ .$$

□

The above proposition shows that the terms $\frac{1}{d^n}\log\max\left(\left|\left(F^{(n)}\right)^{\gamma}\right|,\left|\left(G^{(n)}\right)^{\gamma}\right|\right)$ behave very similarly to the convergents of $\hat{h}_{\phi,v}$ given in [2], Equation (10.9); we make this relation precise in the following proposition:

Proposition 3.9. *Let $x\in\boldsymbol{H}^1$ be given by $x=\gamma(\zeta_G)$, where $\gamma(z)=bz+a$. There exists a constant C_ϕ depending only on ϕ such that:*

$$\begin{aligned}\Bigg|-\frac{1}{d^n-1}\min\left(\mathrm{ord}\left((F^{(n)})^{\gamma}\right),\mathrm{ord}\left((G^{(n)})^{\gamma}\right)\right)-\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v(\delta(x,x)_\infty)\Bigg|&\\ \leq\frac{2}{d^n-1}\max\left(C_\phi,\ \rho(x,\zeta_G)\right)\ .&\qquad(3.5)\end{aligned}$$

Remark: There is a seemingly 'extra' term $\frac{1}{d^n-1}\log_v(\delta(x,x)_\infty)$ appearing in the left side of the inequality (3.5); this term both cleans up the proof below and facilitates the proof of Theorem 3.1.

Proof. To ease notation, let

$$\begin{aligned}\hat{k}^{(n)}_{\phi,v}(x)&:=\log\max\left(\left|\left(F^{(n)}\right)^{\gamma}\right|,\left|\left(G^{(n)}\right)^{\gamma}\right|\right)\\&=-\min\left(\mathrm{ord}\left((F^{(n)})^{\gamma}\right),\mathrm{ord}\left((G^{(n)})^{\gamma}\right)\right)\ .\end{aligned}$$

The statement of Lemma 3.8 tells us that

$$|\hat{k}^{(n)}_{\phi,v}(x)-\hat{\ell}^{(n)}_{\phi,v}(x)|\leq\rho(x,\zeta_{\mathrm{G}})\ .\qquad(3.6)$$

We find that

$$\begin{aligned}\left|\frac{1}{d^n-1}\hat{\ell}^{(n)}_{\phi,v}(x)\right. & \left. -\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v(\delta(x,x)_\infty)\right| \\ &\leq \frac{1}{d^n-1}|\hat{k}^{(n)}_{\phi,v}(x)-\hat{\ell}^{(n)}_{\phi,v}(x)|+\left|\frac{1}{d^n-1}\hat{\ell}^{(n)}_{\phi,v}(x)-\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v\delta(x,x)_\infty\right| \\ &\leq \frac{1}{d^n-1}\rho(x,\zeta_{\mathrm{G}})+\left|\frac{1}{d^n-1}\hat{\ell}^{(n)}_{\phi,v}(x)-\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v\delta(x,x)_\infty\right| \qquad (3.7)\end{aligned}$$

Using (3.6), we estimate the second term in (3.7) as (note these are real-valued functions, so we cannot use the ultrametric inequality):

$$\begin{aligned}\left|\frac{1}{d^n-1}\hat{\ell}^{(n)}_{\phi,v}(x)\right. & \left. -\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v(\delta(x,x)_\infty)\right| = \\ &= \left|\frac{d^n}{d^n-1}\frac{1}{d^n}\hat{\ell}^{(n)}_{\phi,v}(x)-\frac{d^n}{d^n-1}\hat{h}_{phi,v}(x)+\frac{1}{d^n-1}\hat{h}_{\phi,v}(x)-\frac{1}{d^n-1}\log_v\delta(x,x)_\infty\right| \\ &\leq \left|\frac{d^n}{d^n-1}\left(\frac{1}{d^n}\hat{\ell}^{(n)}_{\phi,v}(x)-\hat{h}_{\phi,v}(x)\right)\right|+\left|\frac{1}{d^n-1}\left(\hat{h}_{\phi,v}(x)-\log(\delta(x,x)_\infty)\right)\right| \ . \qquad (3.8)\end{aligned}$$

By the construction of $\hat{h}_{\phi,v}(x)$ on $\mathbf{P}^1$ (see [2], Section 10.1), there is a constant $C_1 = C_1(\phi)$ depending only on ϕ so that the first term in term (3.8) is bounded above:

$$\left|\frac{d^n}{d^n-1}\left(\frac{1}{d^n}\hat{\ell}^{(n)}_{\phi,v}(x)-\hat{h}_{\phi,v}(x)\right)\right| \leq \frac{1}{d-1}\cdot\frac{1}{d^n-1}\cdot C_1 \ . \qquad (3.9)$$

We next rewrite the second term in (3.8) as

$$\begin{aligned}\left|\frac{1}{d^n-1}\left(\hat{h}_{\phi,v}(x)-\log_v\delta(x,x)_\infty\right)\right| &\leq \left|\frac{1}{d^n-1}\left(\hat{h}_{\phi,v,(\infty)}(x)-\log_v\max(1,[T])_x)\right)\right| \qquad (3.10) \\ &+\left|\frac{1}{d^n-1}\left(\log_v\max(1,[T]_x)-\log_v\delta(x,x)_\infty\right)\right| \ . \qquad (3.11)\end{aligned}$$

Applying [2] Equation (10.11) (see also the remark after (10.7)), the expression on the right side of (3.10) is no greater than $\frac{1}{(d-1)(d^n-1)}C_1$. For the term appearing in (3.11), we have that

$$\begin{aligned}\left|\frac{1}{d^n-1}\left(|\log_v \max(1,[T]_x) - \log_v \delta(x,x)_\infty\right)\right| &= \frac{1}{d^n-1}\big(\log_v \max(1,|a|,|b|) - \log_v |b|\big)\\ &\le \frac{1}{d^n-1}\big(2\log_v \max(1,|a|,|b|) - \log_v |b|\big)\\ &= \frac{1}{d^n-1}\rho(x,\zeta_{\mathrm{G}})\ ,\end{aligned}$$

where the final equality was established in the course of the proof of Lemma 3.6. Inserting these estimates into (3.7), we find

$$\begin{aligned}&\left|\frac{1}{d^n-1}\hat{\ell}^{(n)}_{\phi,v}(x)\ -\hat{h}_{\phi,v}(x) - \frac{1}{d^n-1}\log_v(\delta(x,x)_\infty)\right|\\ &\qquad\le \frac{1}{d^n-1}\rho(x,\zeta_{\mathrm{G}}) + \frac{1}{d^n-1}\cdot\frac{2C_1}{d-1} + \frac{1}{d^n-1}\rho(x,\zeta_{\mathrm{G}})\\ &\qquad\le \frac{2}{d^n-1}\max\left(\rho(x,\zeta_{\mathrm{G}}), \frac{C_1}{d-1}\right)\ .\end{aligned}$$

Letting $C_\phi = \frac{C_1}{d-1}$ gives the asserted bound.

□

Remark: Note that Proposition 3.9 gives an effective, geometrically convergent algorithm for approximating the Berkovich canonical height $\hat{h}_{\phi,v,(\infty)}(x)$ by using the convergents $\hat{k}^{(n)}_{\phi,v}(x)$ instead of the 'classical' convergents $\hat{h}_{\phi,v}(x)$. The advantage of these new convergents is that they require only taking the maximum over the coefficients of $(F^{(n)})^\gamma$, $(G^{(n)})^\gamma$ rather than the supremum of their values on discs.

We are now ready to show the convergence of the normalized function $\dfrac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(x)$ to the function $g_{\mu_\phi}(x,x)$.

Proof of Theorem 3.3. Let $x = \gamma(\zeta_{\mathrm{G}})$, where $\gamma(z) = bz + a$. Using the decompositions in Table 3.1 above we have

$$\left|\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(x) - g_{\mu_\phi}(x,x)\right| \leq \left|\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}(F^{(n)},G^{(n)}) - \left(\frac{-1}{d(d-1)}\log|\operatorname{Res}(F,G)|\right)\right| \tag{3.12}$$

$$+\left|\frac{d^{2n}+d^n}{d^{2n}-d^n}\operatorname{ord}\det(\gamma) + \log(\delta(x,x)_\infty)\right. \tag{3.13}$$

$$\left.-\frac{2}{d^n-1}\min\left(\operatorname{ord}\left((F^{(n)})^\gamma\right), \operatorname{ord}\left((G^{(n)})^\gamma\right)\right) - 2\hat{h}_{\phi,v}(x)\right| \ . \tag{3.14}$$

By Lemma 3.4, the term (3.12) is identically zero. By Lemma 3.5, the term in (3.13) above is

$$-\frac{2}{d^n-1}\log_v(\delta(x,x)_\infty)) \ .$$

The terms in (3.13) and (3.14) are precisely (twice) the terms bounded in Proposition 3.9, and so we have

$$\left|\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^{(n)}}(x) - g_{\mu_\phi}(x,x)\right| \leq \frac{4}{d^n-1}\max\left(C_\phi, \rho(x,\zeta_{\mathrm{G}})\right) \ . \tag{3.15}$$

This establishes both pointwise convergence on type II points and uniform convergence in the sets $\mathrm{B}_\rho(\zeta_{\mathrm{G}}, R)$ for fixed $R > 0$. □

We note the following corollary to the convergence; the result in fact holds for diagonal Green's functions attached to arbitrary probability measures, as we will see in Lemma 4.3 below.

Corollary 3.10. *The function $g_{\mu_\phi}(\cdot,\cdot)$ is convex up on segments $[x,y]$ in $\boldsymbol{H}^1$ and is Lipschitz continuous with respect to the path distance metric with Lipschitz constant 1.*

Proof. Let $r_n(x) := \frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(x)$; it was shown in [24] Proposition 1.3 that this function is convex up on $\mathbf{P}^1$, and also that the r_n are each Lipschitz continuous with Lipschitz constant $1 + \frac{2}{d^n-1}$.

For the convexity of g_{μ_ϕ}, fix a segment $[x, y] \subseteq \mathbf{H}^1$. There exists a constant $R > 0$ for which $[x, y] \subseteq B_\rho(\zeta_{\mathrm{G}}, R)$, and so we may assume that $r_n \to g_{\mu_\phi}$ uniformly on $[x, y]$.

For brevity of notation, let $g(z) = g_{\mu_\phi}(z, z)$. Fix $t \in [x, y]$; we need to show

$$\frac{g(y) - g(x)}{y - x} \geq \frac{g(t) - g(x)}{t - x} \ .$$

By Theorem 3.1, we can choose n large enough so that $\left|\frac{g(y)-r_n(y)}{y-x}\right| < \frac{\epsilon}{4}$, $\left|\frac{r_n(x)-g(x)}{y-x}\right| < \frac{\epsilon}{4}$, and $\left|\frac{g(t)-r_n(t)}{t-x}\right| < \frac{\epsilon}{4}$. We have

$$\begin{aligned}
\frac{g(y) - g(x)}{y - x} &= \frac{g(y) - r_n(y) + r_n(y) - r_n(x) + r_n(x) - g(x)}{y - x} \\
&\geq -\frac{\epsilon}{4} - \frac{\epsilon}{4} + \frac{r_n(y) - r_n(x)}{y - x} \\
&\geq -\frac{\epsilon}{2} + \frac{r_n(t) - r_n(x)}{t - x} \\
&= -\frac{\epsilon}{2} + \frac{r_n(t) - g(t) + g(t) - g(x) + g(x) - r_n(x)}{t - x} \\
&\geq -\frac{\epsilon}{2} - \frac{\epsilon}{4} - \frac{\epsilon}{4} + \frac{g(t) - g(x)}{t - x} \\
&= -\epsilon + \frac{g(t) - g(x)}{t - x} \ .
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $g(t)$ is convex up on $[x, y]$.

To see that $g_{\mu_\phi}(x, x)$ is Lipschitz continuous, fix $x, y \in \mathbf{H}^1$, and let $0 < \epsilon < \frac{\rho(x,y)}{2}$. Choose n sufficiently large so that

$$|g(x) - r_n(x)| < \epsilon/2 \quad \text{and} \quad |g(y) - r_n(y)| < \epsilon/2 \ .$$

Since $|r_n(x) - r_n(y)| \leq 1 + \frac{2}{d^n - 1}$, we have

$$\begin{aligned}
|g(x) - g(y)| &= |g(x) - r_n(x) + r_n(x) - r_n(y) + r_n(y) - g(y)| \\
&\leq \left(1 + \frac{2}{d^n - 1}\right) \rho(x, y) + \epsilon \ .
\end{aligned}$$

Letting $\epsilon \to 0$ and $n \to \infty$ gives the desired result. □

Additional properties of the function $g_\nu(x,x)$ for arbitrary probability measures ν will be given in Chapter 4 below.

3.2 Weak Convergence of the Measures ν_{ϕ^n}

We now apply the results of the previous section to show the weak convergence of the measures ν_{ϕ^n} to the canonical measure μ_ϕ. The proof will come from the following more explicit theorem

Theorem 3.11. *If $\Gamma \subseteq \boldsymbol{H}^1$ is a finite connected subgraph and f is a continuous piecewise affine map on Γ, then there exist a constant $C_\phi > 0$ depending only on ϕ, and constants $R_\Gamma, D_\Gamma > 0$ depending only on Γ so that*

$$\left|\int_\Gamma f d(\mu_\phi - \nu_{\phi^n})\right| \leq \frac{4}{d^n - 1}\left(\max\left(C_\phi, R_\Gamma\right) \cdot |\Delta|(f) + \max_\Gamma |f| \cdot D_\Gamma\right) .$$

We prove Theorem 3.11 in Section 3.2.3 below, and after we give the proof of Theorem 3.2. Before doing either of these, we briefly outline the remainder of this section. In Section 3.2.1 we expand on the slope formulae for $\operatorname{ord}\operatorname{Res}_\phi(x)$ developed in [25] to include slopes at arbitrary points in $\mathbf{H}^1$. It turns out that the more convenient function to study is

$$f_n(x) = \frac{1}{d^{2n} - d^n} \operatorname{ord}\operatorname{Res}_{\phi^n}(x) + \log(\delta(x,x)_\infty) . \tag{3.16}$$

Not only does this function give cleaner slope formulae, it also has the property that

$$\frac{1}{d^{2n} - d^n} \operatorname{ord}\operatorname{Res}_{\phi^n}(x) - g_{\mu_\phi}(x,x) = f_n(x) - 2\hat{h}_{\phi,v}(x) + \log_v |\operatorname{Res}(\Phi)|^{-1/(d(d-1))} . \tag{3.17}$$

Our strategy of proof is as follows: let $\Gamma_{\mathrm{FR,\,n}}$ denote the tree spanned by the type I n-periodic points and type II repelling n-periodic points, and let $\Gamma_{\widehat{\mathrm{FR}},n}$ denote the truncation of this tree obtained by removing terminal segments $[f, Q_0)$ on which the slope of $\operatorname{ordRes}_{\phi^n}(\cdot)$ is constant. We compute the Laplacian of $f_n(x)$ on arbitrary subgraphs Γ, first by joining such a graph to $\Gamma_{\widehat{FR},n}$ and then computing the (retraction of the) Laplacian on the larger graph. We prove weak convergence for CPA test functions on an (arbitrary) fixed subgraph $\Gamma \subseteq \mathbf{H}^1$, and then using an approximation theorem for continuous functions on $\mathbf{P}^1$ extend this to show weak convergence in general.

3.2.1 Slope Formulae Revisited

Here we compute the slope of the functions $f_n(x)$ on connected subgraphs $\Gamma \in \mathbf{H}^1$ which share at most one point in common with the corresponding $\Gamma_{FR,n}$; in the following section these results will be used to give explicit formulae for $\Delta_\Gamma(f_n)$ for such Γ. The parallel result for graphs $\Gamma \subseteq \Gamma_{\widehat{FR,n}}$ is found in [25] Corollary 6.5, which will be discussed in the next section.

Lemma 3.12. *Let $\Gamma \subseteq \boldsymbol{P}^1$ be a finite tree. Let $\mu_{Br,\Gamma}$ be the branching measure attached to this tree. Then*

$$\Delta(\log(\delta(x,x)_\infty)) = -2\mu_{Br,\Gamma} + 2\delta_{r_\Gamma(\infty)}$$

where $r_\Gamma(\infty)$ is the retraction of ∞ to Γ.

Proof. This is a straightforward computation. Let $w = r_\Gamma(\infty)$. Note that $\log(\delta(x,x)_\infty)$ is the arclength parameterization for a segment of Γ; thus for all $P \in \Gamma \setminus \{w\}$ we have

$$\begin{aligned}
\sum_{\vec{v} \in T_p\Gamma} \partial_{\vec{v}}(\log(\delta(P,P)_\infty)) &= \sum_{\vec{v} \neq \vec{v}_w} \partial_{\vec{v}}(\log(\delta(P,P)_\infty)) + \partial_{\vec{v}_w}(\log(\delta(P,P)_\infty)) \\
&= \left(\sum_{\vec{v} \neq \vec{v}_w} -1 \right) + 1 \\
&= (v(P)-1)(-1) + 1 \\
&= (2 - v(P)) \ .
\end{aligned}$$

For $P = w$:

$$\begin{aligned}
\sum_{\vec{v} \in T_w\Gamma} \partial_{\vec{v}}((\log(\delta(P,P)_\infty)) &= \sum_{\vec{v} \in T_w\Gamma} -1 \\
&= -v(w) \\
&= (2 - v(w)) - 2 \ .
\end{aligned}$$

Thus

$$\begin{aligned}\Delta_\Gamma(\log(\delta(x,x)_\infty)) &= \sum_{P\in\Gamma}(v(p)-2)\delta_P + 2\delta_w \\ &= -2\mu_{Br,\Gamma} + 2\delta_{r_\Gamma(\infty)} \ .\end{aligned}$$

□

Now let Γ be a finite, connected subgraph of $\mathbf{H}^1$ that intersects $\Gamma_{FR,n}$ in at most one point. For fixed n, let w_n denote the point of Γ that is nearest to $\Gamma_{FR,n}$.

Lemma 3.13. *If $P \in \Gamma \setminus \{w_n\}$, then*

$$\sum_{\vec{v}\in T_P\Gamma} \partial_{\vec{v}} f_n = \begin{cases} \frac{2}{d^n-1}(v(P)-2), & \text{if } \phi^n(P) \neq P, \\ \frac{2}{d^n-1}v(P), & \text{if } \phi^n(P) = P \text{and } P \text{ is not id-indifferent for } \phi^n, \\ 0, & \text{if } \phi^n(P) = P \text{and } P \text{ is id-indifferent for } \phi^n. \end{cases}$$

Proof. We begin with the case of $\phi^n(P) \neq P$. Here we use the formula from Proposition 5.4 in [25], together with the fact that the term $\log(\delta(P,P)_\infty)$ is the arclength parameterization. Let $\vec{v}_w$ be the direction at P pointing towards w_n. Note that $\#F_{\phi^n}(P,\vec{v}) = 0$ for any $\vec{v} \neq \vec{v}_w$, and $\#F_{\phi^n}(P,\vec{v}_w) = d^n + 1$. Then

$$\begin{aligned}\sum_{\vec{v}\in T_p\Gamma} \partial_{\vec{v}} f_n &= \left(\sum_{\vec{v}\neq\vec{v}_w} \partial_{\vec{v}} f_n\right) + \partial_{\vec{v}_w} f_n \\ &= \left(\sum_{\vec{v}\neq\vec{v}_w}\left(\frac{d^{2n}+d^n}{d^{2n}-d^n} - \frac{2d^n\#F_{\phi^n}(P,\vec{v})}{d^{2n}-d^n} - 1\right)\right) + \left(\frac{d^{2n}+d^n}{d^{2n}-d^n} - \frac{2d^n\#F_{\phi^n}(P,\vec{v}_w)}{d^{2n}-d^n} + 1\right) \\ &= \sum_{\vec{v}\neq\vec{v}_w}\left(\frac{2}{d^n-1}\right) + \frac{d^{2n}+d^n-2d^{2n}-2d^n+d^{2n}-d^n}{d^{2n}-d^n} \\ &= \frac{2}{d^n-1}(v(P)-1) - \frac{2}{d^n-1} \\ &= \frac{2}{d^n-1}(v(P)-2) \ .\end{aligned}$$

In the case $\phi^n(P) = P$, we need to separate into the cases where P is id-indifferent for ϕ^n and where it is not.

If P is not id-indifferent for ϕ^n, then the First Identification Lemma ([25], Lemma 2.1) implies that for all directions $\vec{v} \in T_P\Gamma \setminus \{\vec{v}_w\}$, we have $s_{\phi^n}(P, \vec{v}) = 0$ and $(\phi^n)_*(\vec{v}) \neq \vec{v}$. For $\vec{v} = \vec{v}_w$, we claim that $s_{\phi^n}(P, \vec{v}) = d^n - 1$ and $(\phi^n)_*(\vec{v}) = \vec{v}$. To see this, we note that $F_\phi(P, \vec{v}_w) = d^n - 1$, as all of the fixed points lie in the direction towards w. The first identification lemma then gives that

$$d^n + 1 = s_{\phi^n}(P, \vec{v}_w) + \tilde{F}_{\phi^n}(P, \vec{v}_w) \ .$$

Since $s_{\phi^n}(P, \vec{v}_w) \leq d$, this forces $\tilde{F}_{\phi^n}(P, \vec{v}_w) \geq 1$, i.e. $(\phi^n)_*\vec{v}_w = \vec{v}_w$. Now note that P is additively indifferent for ϕ^n: by assumption it is not id-indifferent, and it cannot be repelling, as it does not lie in $\Gamma_{\mathrm{FR},n}$. If it were multiplicatively indifferent, then it would necessarily fix two directions, since $\widetilde{\phi^n}(z) = \tilde{\lambda} z$ fixes two points of $\mathbb{P}^1(k)$; but the First Identification Lemma ([25] Lemma 2.1) would then imply that there are two directions containing fixed points, which is a contradiction. So P is addtivively indifferent for ϕ^n. Since the linear map $z \mapsto z + \tilde{\lambda}$ has a unique degree 2 fixed point, we conclude that $\tilde{F}_{\phi^n}(P, \vec{v}_w) = 2$, hence $s_{\phi^n}(P, \vec{v}_w) = d^n - 1$.

Therefore, applying [25] Proposition 5.2, we find

$$\begin{aligned}
\sum_{\vec{v} \in T_P\Gamma} \partial_{\vec{v}} f_n &= \sum_{\vec{v} \neq \vec{v}_w} \left(\frac{d^{2n} - d^n}{d^{2n} - d^n} - \frac{2d^n s_{\phi^n}(P, \vec{v})}{d^{2n} - d^n} + \frac{2d^n \cdot 1}{d^{2n} - d^n} - 1 \right) + \left(\frac{d^{2n} - d^n}{d^{2n} - d^n} - \frac{2d^n(d^n - 1)}{d^{2n} - d^n} + 1 \right) \\
&= \sum_{\vec{v} \neq \vec{v}_w} \left(\frac{2}{d^n - 1} \right) + \frac{d^{2n} - d^n - 2d^{2n} + 2d^n + d^{2n} - d^n}{d^{2n} - d^n} \\
&= \frac{2}{d^n - 1}(v(p) - 1) \ .
\end{aligned}$$

The proof when P is id-indifferent is similar. We again claim that $s_{\phi^n}(P, \vec{v}_w) = d^n - 1$ in this case: to see this, we apply the Third Identification Lemma ([25] Lemma 4.5), which implies that $\vec{v}_w$ is the only direction with $s_{\phi^n}(P, \vec{v}_w) > 0$. Since $d^n = \deg_{\phi^n}(P) + \sum_{\vec{v} \in T_P} s_{\phi^n}(P, \vec{v})$ (see [9] Equation 3.1), this implies that $d^n - 1 = s_{\phi^n}(P, \vec{v}_w)$.

Using [25] Proposition 5.3 we have

$$\begin{aligned}\sum_{\vec{v}\in T_P\Gamma}\partial_{\vec{v}}f_n &= \sum_{\vec{v}\neq\vec{v}_w}\left(\frac{d^{2n}-d^n}{d^{2n}-d^n}-\frac{2d^n s_{\phi^n}(P,\vec{v})}{d^{2n}-d^n}-1\right)+\left(\frac{d^{2n}-d^n}{d^{2n}-d^n}-\frac{2d^n s_{\phi^n}(P,\vec{v}_w)}{d^{2n}-d^n}+1\right)\\ &= \sum_{\vec{v}\neq\vec{v}_w}\left(\frac{d^{2n}-d^n-d^{2n}+d^n}{d^{2n}-d^n}\right)+\frac{d^{2n}-d^n-2d^n(d^n-1)+d^{2n}-d^n}{d^{2n}-d^n}\\ &= 0\ .\end{aligned}$$

□

Note that the above formulae hold even when P is an endpoint of Γ. Thus we are left to consider the case of $P=w$. In this case, $s_{\phi^n}(w,\vec{v})$ and $\#F_{\phi^n}(w,\vec{v})$ are zero for all directions $\vec{v}$ pointing into Γ; to see this, note that these are precisely the directions which point *away* from $\Gamma_{\mathrm{FR},n}$, hence $F_{\phi^n}(w,\vec{v})=0$. If w is fixed by ϕ^n, but not id-indifferent, then the First Identification Lemma ([25] Lemma 2.1) gives that $0=s_{\phi^n}(w,\vec{v})+\tilde{F}_{\phi^n}(w,\vec{v})$, hence $s_{\phi^n}(w,\vec{v})=0$. If w is id-indifferent, then the Third Identification Lemma ([25] Lemma 4.5) implies that $s_{\phi^n}(w,\vec{v})=0$ since there are no type I fixed points in $B_{\vec{v}}(w)^-$. Likewise, if w is moved by ϕ^n, then the Second Identification Lemma ([25] Lemma 2.2) gives that $s_{\phi^n}(w,\vec{v})=0$.

Lemma 3.14. *For $P=w_n\in\Gamma$, we have*

$$\sum_{\vec{v}\in T_{w_n}\Gamma}\partial_{\vec{v}}f_n=\begin{cases}0, & w_n \textit{ is an id-indifferent fixed point,}\\ \dfrac{2}{d^n-1}v(w_n), & \textit{otherwise.}\end{cases}$$

Proof. Here again the proof splits into three cases. If w_n is not fixed, then by [25] Proposition 5.4,

$$\begin{aligned}\sum_{\vec{v}\in T_{w_n}\Gamma}\partial_{\vec{v}}f_n &= \sum_{\vec{v}\in T_{w_n}\Gamma}\frac{d^{2n}+d^n}{d^{2n}-d^n}-\frac{2d^n\#F_{\phi^n}(w_n,\vec{v})}{d^{2n}-d^n}-1\\ &= \sum_{\vec{v}\in T_{w_n}\Gamma}\frac{d^{2n}+d^n-d^{2n}+d^n}{d^{2n}-d^n}\\ &= \frac{2}{d^n-1}v(w_n)\ .\end{aligned}$$

If w_n is fixed by ϕ^n but is not id-indifferent, we argued above that $s_{\phi^n}(w_n, \vec{v}) = 0$, hence $(\phi^n)_*\vec{v} \neq \vec{v}$ for any $\vec{v}$ pointing into Γ (this is again the First Identification Lemma, [25] Lemma 2.1). Therefore, applying [25] Proposition 5.2, we find

$$\begin{aligned}\sum_{\vec{v}\in T_{w_n}\Gamma} \partial_v f_n &= \sum_{\vec{v}\in T_{w_n}\Gamma} \frac{d^{2n}-d^n}{d^{2n}-d^n} - \frac{2d^n s_{\phi^n}(w_n,\vec{v})}{d^{2n}-d^n} + \frac{2d^n \cdot 1}{d^{2n}-d^n} - 1 \\ &= \sum_{\vec{v}\in T_{w_n}\Gamma} \frac{d^{2n}-d^n+2d^n-d^{2n}+d^n}{d^{2n}-d^n} \\ &= \frac{2}{d^n-1} v(w) \ .\end{aligned}$$

Finally, if w_n is fixed by ϕ^n and is id-indifferent, applying [25] Proposition 5.3 we find

$$\begin{aligned}\sum_{\vec{v}\in T_{w_n}\Gamma} \partial_{\vec{v}} f_n &= \sum_{\vec{v}\in T_{w_n}\Gamma} \frac{d^{2n}-d^n}{d^{2n}-d^n} - \frac{2d^n s_{\phi^n}(w_n,\vec{v})}{d^{2n}-d^n} - 1 \\ &= \sum_{\vec{v}\in T_{w_n}\Gamma} \frac{d^{2n}-d^n-d^{2n}+d^n}{d^{2n}} \\ &= 0 \ .\end{aligned}$$

□

3.2.2 Applications To Laplacians

In this section we use the slope formulae described above to relate the Laplacian of f_n to both the crucial measure and the canonical measure; these relations will be key in the estimates that give weak convergence.

Recall that $\Gamma_{FR,n}$ is the tree in $\mathbf{P}^1$ spanned by the classical n-periodic points and the type II repelling n-periodic points; in [25] it was shown that this is spanned by finitely many points. Fix n, and fix also an arbitrary tree $\Gamma \subseteq \mathbf{H}^1$. Take $R_\Gamma > 0$ sufficiently large so that $\Gamma \subseteq \mathcal{B}_\rho(\zeta_{\mathrm{G}}, R_\Gamma)$.

We work with a truncated version of $\Gamma_{\mathrm{FR},n}$, which we denote by $\Gamma_{\widehat{FR},n}$. It is constructed as follows: for each classical fixed point α_i of ϕ^n, choose a point $Q_i \in \mathbf{H}^1$ sufficiently near to α_i so that $[Q_i, \alpha_i]$ contains no branch points of $\Gamma_{FR,n}$, and that the slope of $\operatorname{ord}\operatorname{Res}_{\phi^n}(\cdot)$ is constant on this segment (see [25]). We further extend these branches if necessary to ensure that none of the endpoints of $\Gamma_{\widehat{\mathrm{FR}},n}$ lie in $\mathcal{B}_\rho(\zeta_{\mathrm{G}}, R_\Gamma)$.

Our goal is to compute the Laplacian of f_n on Γ, which we will do by first computing the Laplacian of f_n on a larger tree $\Gamma^{(n)}$, and then retracting to Γ.

If $\Gamma \cap \Gamma_{\widehat{FR},n} \neq \emptyset$, let $\Gamma^{(n)} = \Gamma \cup \Gamma_{\widehat{FR},n}$. Otherwise, letting $x_n \in \Gamma_{\widehat{FR},n}$ denote the point in $\Gamma_{\widehat{FR},n}$ lying closest to Γ, and $w_n \in \Gamma$ the point lying closest to $\Gamma_{\widehat{FR},n}$, we define $\Gamma^{(n)} = \Gamma \cup [w_n, x_n]$. In either case, we let $\Gamma_0^{(n)} = \Gamma^{(n)} \cap \Gamma_{\widehat{FR},n}$, and for $i = 1, 2, ..., N = N(n, \Gamma)$, we let $\Gamma_i^{(n)}$ denote the *closures* of the various components of $\Gamma^{(n)} \setminus \Gamma_0^{(n)}$.

We begin with a lemma that shows that, in the case $\Gamma \cap \Gamma_{\widehat{FR},n} \neq \emptyset$, the number of connected components $\Gamma_i^{(n)}$, $i \geq 1$, is uniformly bounded in terms of Γ:

Lemma 3.15. *There is a constant $K(\Gamma)$ such that for any finite tree $\Gamma \subseteq \boldsymbol{H}^1$ having exactly s edges, and any connected subtree $\Gamma_0 \subseteq \Gamma$, the number of connected components of $\Gamma \setminus \Gamma_0$ is bounded by $K(\Gamma) = 2s$.*

Proof. we proceed by induction on the number of edges of Γ. If Γ has one edge, then Γ is an interval and any connected subset is again an interval, so by removing a subinterval we form at most 2 connected components. Hence $K(\Gamma) = 2$, which establishes the claim when $s = 1$.

Now let Γ be a tree with $s > 1$ edges, and let $\Gamma_0 \subseteq \Gamma$ be a connected subtree. Let e be an edge of Γ such that $\Gamma \setminus e$ is connected, and let $\hat{\Gamma}$ be the tree formed by removing e from Γ. Similarly, let $\hat{\Gamma_0}$ be the tree formed by removing from Γ_0 any part that intersects e. By induction, $\hat{\Gamma} \setminus \hat{\Gamma_0}$ has at most $K(\hat{\Gamma} \setminus \hat{\Gamma_0}) = 2(s-1)$ components. If we consider the edge e alone, then it will have at most two components in $e \setminus (e \cap \Gamma_0)$; thus by adding e back into $\hat{\Gamma}$ and $\hat{\Gamma_0}$ we will add at most two components to $\hat{\Gamma} \setminus \hat{\Gamma_0}$ (in general, only one, unless $\Gamma_0 \subseteq e$), thus there are at most $2(s-2) + 2 = 2s$ components of $\Gamma \setminus \Gamma_0$, i.e $K(\Gamma) = 2s$. ☐

In particular, the preceeding lemma shows that the number of components $\Gamma_i^{(n)}$, $i \geq 1$, is bounded above by $2s$, where s is the number of edges of Γ.

In the following two subsections, we compute the Laplacian of f_n on each of the $\Gamma_i^{(n)}$; the results are combined to give the Laplacian on $\Gamma^{(n)}$ (Lemma 3.21) and finally the Laplacian of $\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(x) - g_{\mu_\phi}(x,x)$ on Γ (Proposition 3.23).

The Laplacian of f_n on $\Gamma_0^{(n)}$

Recall that $\Gamma_0^{(n)}$ is either the intersection of Γ with $\Gamma_{\widehat{\mathrm{FR}},n}$, or is the point in $\Gamma_{\widehat{FR},n}$ lying closest to Γ; in either case, we derive a formula for $\Delta_{\Gamma_0^{(n)}}(f_n)$, which is given below in Proposition 3.18. As a first step in this direction we recall:

Corollary 3.16. (*Rumely, [25]*) *Let $\phi(z) \in K(z)$ have degree $d \geq 2$. Then*

$$\Delta_{\Gamma_{\widehat{FR}}}(\operatorname{ord}\operatorname{Res}_\phi(\cdot)) = 2(d^2-d)(\mu_{\Gamma_{\widehat{FR}},Br} - \nu_\phi) \ .$$

This is easily generalized to higher iterates, and combined with Lemma 3.12 above we obtain the result:

Lemma 3.17. *If $\Gamma_{\widehat{FR},n}$ is as defined above, then*

$$\Delta_{\Gamma_{\widehat{FR},n}}\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(\cdot) + \log(\delta(\cdot,\cdot)_\infty)\right) = -2\nu_{\phi^n} + 2\delta_{r_{\Gamma_{\widehat{FR},n}}(\infty)} \ .$$

Proof. This is a straightforward computation: by Lemma 3.12 and Corollary 3.16

$$\begin{aligned}\Delta_{\Gamma_{\widehat{FR},n}}\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(\cdot) + \log(\delta(\cdot,\cdot)_\infty)\right) &= \left(2\mu_{\Gamma_{\widehat{FR},n},Br} - 2\nu_{\phi^n}\right) + \left(-2\mu_{\Gamma_{\widehat{FR},n},Br} + 2\delta_{r_{\Gamma_{\widehat{FR},n}}(\infty)}\right) \\ &= -2\nu_{\phi^n} + 2\delta_{r_{\Gamma_{\widehat{FR},n}}(\infty)} \ .\end{aligned}$$

□

Finally we apply this to the Laplacian on $\Gamma_0^{(n)}$ by taking retractions:

Proposition 3.18. *The Laplacian of f_n on $\Gamma_0^{(n)}$ is given*

$$\Delta_{\Gamma_0^{(n)}}(f_n) = -2(r_{\Gamma_{\widehat{FR},n},\Gamma_0^{(n)}})_*\nu_{\phi^n} + 2\delta_{r_{\Gamma_0^{(n)}}(\infty)} \ .$$

Proof. Since $\Gamma_0^{(n)} \subseteq \Gamma_{\widehat{FR},n}$, we have

$$\Delta_{\Gamma_0^{(n)}}(f_n) = (r_{\Gamma_{\widehat{FR},n},\Gamma_0^{(n)}})_* \Delta_{\Gamma_{\widehat{FR},n}}(f_n) \ .$$

The result now holds by explicitly retracting the expression appearing in Lemma 3.17. □

The Laplacian of f_n on $\Gamma_i^{(n)}$ for $i \geq 1$

We now aim to compute $\Delta_\Gamma(f_n)$ on the graphs $\Gamma_i^{(n)}$ for $i \geq 1$; these graphs share exactly one point in common with $\Gamma_{\widehat{FR},n}$, hence we can apply the results of Section 3.2.1. We begin with the following:

Proposition 3.19. *Let* $\Gamma = \Gamma_i^{(n)}$ *for some fixed* $i \geq 1$. *Let* w_n^i *denote the point in* $\Gamma_i^{(n)} \cap \Gamma_{\widehat{FR},n}$. *We have*

1. *Every fixed point in* Γ *that is not id-indifferent is additively indifferent. If* A_Γ *is the number of such fixed points in* Γ*, and* E_Γ *is the number of edges of* Γ*, then*

$$A_\Gamma \leq E_\Gamma \ .$$

2. *The Laplacian of* f_n *on* Γ *is*

$$\Delta_\Gamma(f_n) = \frac{2}{d^n - 1}\left(\Theta_\Gamma + \Omega_{n,i}\delta_{w_n^i}\right) \ ,$$

where

$$\Theta_\Gamma = - \sum_{\substack{Q \in \Gamma \setminus \{w_n^i\}, \\ \phi^n(Q) = Q \\ Q \text{ is not id-indifferent}}} v_\Gamma(Q)\delta_Q - \sum_{\substack{P \in \Gamma \setminus \{w_n^i\}, \\ \phi^n(P) \neq P}} (v_\Gamma(P) - 2)\delta_P \ ,$$

and

$$\Omega_{n,i} = \begin{cases} v_\Gamma(w_n^i), & \text{if } w_n^i \text{ is not id-indifferent,} \\ 0, & \text{if } w_n^i \text{ is id-indifferent.} \end{cases} \ .$$

are measures depending only on Γ, ϕ, *and* n.

3. If $B_\Gamma = \sum_{P\in\Gamma} |v_\Gamma(P) - 2| + (E_\Gamma + 1) \cdot \max_{P\in\Gamma} v_\Gamma(P)$, then

$$|\Delta|_\Gamma(f_n) \leq \frac{2}{d^n - 1} B_\Gamma \ .$$

Proof. For (A), note that by [25], Corollary 10.6 and the definition of $\Gamma_{FR,n}$, all of the multiplicatively indifferent and repelling fixed points of ϕ^n lie in $\Gamma_{FR,n}$; hence any fixed point of ϕ^n in Γ that is not id-indifferent must be additively indifferent.

If P is an additively indifferent point, it has a unique fixed direction $\vec{v}_a$ with multiplier 1. By the Second Persistence Lemma ([25], Lemma 9.5), it follows that there is a segment (P, P_0) in $\mathrm{B}_P(\vec{v}_a)^-$ on which ϕ is id-indifferent. Hence P sits on the boundary of the locus of id-indifference; since there are finitely many components in the locus of id-indifference, there can be only finitely many additively indifferent fixed points.

Moreover, by Corollary 10.2 of [25], the closure of the component of the locus of id-indifference which P bounds must contain at least two classical fixed points. Hence there can be no other additively indifferent fixed points in the segment (P, w_n^i). The number of additively indifferent fixed points in Γ is therefore bounded by the number of edges E_Γ in Γ. This completes (1).

The proof of (2) is a straightforward application of the definition of the Laplacian on a finite subgraph along with the slope formulae derived in Lemmas 3.13 and 3.14.

For (3), the contribution from each fixed point P to $|\Delta_\Gamma|(f_n)$ is at most $v(P)$, and by parts (1) and (2) above there are at most E_Γ additively indifferent fixed points in Γ. The term for non-fixed points is evidently bounded above by $\sum_{P\in\Gamma} |v(p) - 2|$. This, together with the $v(P)$ term that bounds $\Omega_n \delta_{w_i^n}$, gives the result. □

Applying the previous proposition to each of the trees $\Gamma_i^{(n)}$ $(i \geq 1)$, we have

Lemma 3.20. *Let $\Theta_{\Gamma_i^{(n)}}, \Omega_{n,i}$ be the measure and the constant from Proposition 3.19. There is a constant D_Γ such that*

$$\left| \sum_{i=1}^{N_n} \left(\Theta_{\Gamma_i^{(n)}} + \Omega_{n,i} \delta_{w_i^n} \right) \right| < D_\Gamma \ .$$

Remark: The constant D_Γ in this Lemma depends only on Γ, and not on $\Gamma^{(n)}$ or its partitions.

Proof. Let $\Theta_{\Gamma_i^{(n)}}$ and $\Omega_{n,i}$ be as in the statement of Proposition 3.19 for the respective $\Gamma_i^{(n)}$. To obtain the constant D_Γ, note that the constant in Proposition 3.19, part 3 depends only on the maximum valence and number of edges in $\Gamma_i^{(n)}$; the valence is certainly no more than $\max_{P\in\Gamma} v(P)$, and similarly $E_{\Gamma_i^{(n)}} \leq E_\Gamma$. Therefore we can take

$$D_\Gamma = K(\Gamma)\cdot\left(\sum_{P\in\Gamma}(v(P)-2) + (E_\Gamma+1)\max_{P\in\Gamma} v(P)\right) ,$$

where $K(\Gamma)$ is the constant from Lemma 3.15. □

By our choice of decomposition of $\Gamma^{(n)}$, we have

$$\Delta_{\Gamma^{(n)}} = \Delta_{\Gamma_0^{(n)}} + \sum_{i=1}^{N(n,\Gamma)} \Delta_{\Gamma_i^{(n)}} . \tag{3.18}$$

To see this, note that while the various components $\Gamma_i^{(n)}$ may intersect $\Gamma_0^{(n)}$ at a point P, the collection $T_P\Gamma^{(n)}$ is accounted for by taking the Laplacians on *all* of the components. We can therefore compute the Laplacian of f_n on $\Gamma^{(n)}$:

Lemma 3.21. *We have that*

$$\Delta_{\Gamma^{(n)}}(f_n) = -2(r_{\boldsymbol{P}^1,\Gamma_0^{(n)}})_*\nu_{\phi^n} + 2\delta_{r_{\Gamma_0^{(n)}}(\infty)} + \frac{2}{d^n-1}\Lambda_n ,$$

where Λ_n is a measure supported on $\Gamma^{(n)}$ such that $|\Lambda_n| < D_\Gamma$.

Proof. Combine Proposition 3.18 and Lemma 3.20, together with the decomposition of the Laplacian given in (3.18). □

Finally we have

Lemma 3.22. $\Delta_\Gamma(f_n) = -2(r_{\boldsymbol{P}^1,\Gamma})_* v_{\phi^n} + 2\delta_{r_\Gamma(\infty)} + \dfrac{2}{d^n-1}(r_{\boldsymbol{P}^1,\Gamma})_*\Lambda_n$.

Proof. Note that $\Delta_\Gamma(f_n) = (r_{\Gamma^{(n)},\Gamma})_*\Delta_{\Gamma^{(n)}}(f_n)$. Note that any path connecting a point $x \in \Gamma_{\widehat{\mathrm{FR}},n}$ to a point $y \in \Gamma$ must intersect $\Gamma_0^{(n)}$; consequently,

$$(r_{\Gamma^{(n)},\Gamma})_*(r_{\Gamma_{\widehat{\mathrm{FR}},n},\Gamma_0^n})_*\nu_{\phi^n} = (r_{\Gamma_{\widehat{\mathrm{FR}},n},\Gamma})_*\nu_{\phi^n} .$$

Since the support of ν_{ϕ^n} is contained in $\Gamma_{\widehat{\mathrm{FR}},n}$, this gives

$$(r_{\Gamma^{(n)},\Gamma})_*(r_{\Gamma_{\widehat{\mathrm{FR}},n},\Gamma_0^n})_*\nu_{\phi^n} = (r_{\mathbf{P}^1,\Gamma})_*\nu_{\phi^n} \ . \tag{3.19}$$

In a similar manner, we find that

$$(r_{\Gamma^{(n)},\Gamma})_*\delta_{r_{\Gamma_0^{(n)}}(\infty)} = \delta_{r_\Gamma(\infty)} \ . \tag{3.20}$$

Finally, using the fact that the support of Λ_n lies in $\Gamma^{(n)}$, we find

$$(r_{\Gamma^{(n)},\Gamma})_*\Lambda_n = (r_{\mathbf{P}^1,\Gamma})_*\Lambda_n \ . \tag{3.21}$$

Combining (3.19), (3.20), and (3.21), and using the decomposition of $\Delta_{\Gamma^{(n)}}(f_n)$ given in Lemma 3.21 yields the asserted result. $\square$

From these results we obtain the proposition that will facilitate the weak convergence.

Proposition 3.23. *For* Γ *a fixed finite graph in* $\boldsymbol{H}^1$,

$$\begin{aligned}\Delta_\Gamma &\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(\cdot) - g_{\mu_\phi}(\cdot,\cdot)\right) \\ &\qquad = 2(r_{\boldsymbol{P}^1,\Gamma})_*(\mu_\phi - \nu_{\phi^n}) + \frac{2}{d^n-1}(r_{\boldsymbol{P}^1,\Gamma})_*\Lambda_n \ .\end{aligned}$$

Proof. Using the decomposition of $g_{\mu_\phi}(x,x) = -\log(\delta(x,x)_\infty) + 2\hat{h}_\phi(x) + M$ (where M is the constant $\log_v|\operatorname{Res}(\Phi)|^{-1/(d(d-1))}$), we can write

$$\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(x) - g_{\mu_\phi}(x,x) = f_n - 2\hat{h}_\phi(x) - M \ .$$

Taking Laplacians on Γ, we obtain

$$\begin{aligned}\Delta_\Gamma\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}(\cdot)-g_{\mu_\phi}(\cdot,\cdot)\right) &= \Delta_\Gamma(f_n-2\hat{h}_\phi-M)\\ &= \Delta_\Gamma(f_n)-2\Delta_\Gamma(\hat{h}_\phi)\\ &= \left(-2(r_{\mathbf{P}^1,\Gamma})_*\nu_{\phi^n}+2\delta_{r_\Gamma(\infty)}+\frac{2}{d^n-1}(r_{\mathbf{P}^1,\Gamma})_*\Lambda_n\right)\\ &\qquad -2(r_{\mathbf{P}^1,\Gamma})_*(\delta_\infty-\mu_\phi)\\ &= 2(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi-\nu_{\phi^n})+\frac{2}{d^n-1}(r_{\mathbf{P}^1,\Gamma})_*\Lambda_n\ .\end{aligned}$$

□

3.2.3 Proof of Convergence

We are now ready to prove Theorem 3.11, after which we readily obtain the proof of Theorem 3.2.

Proof of Theorem 3.11. Let Γ be a finite graph in $\mathbf{H}^1$, and $f\in$CPA(Γ). Let R_Γ be chosen so that $\Gamma\subseteq\mathcal{B}_\rho(\zeta_{\mathrm{G}},R_\Gamma)$. We are interested in estimating

$$\left|\int_\Gamma f d(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi-\nu_{\phi^n})\right|\ .$$

From Proposition 3.23, we can express the measure as

$$(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi-\nu_{\phi^n})=\Delta_\Gamma\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}-g_{\mu_\phi}\right)-\frac{2}{d^n-1}(r_{\mathbf{P}^1,\Gamma})_*\Lambda_n\ .$$

Thus we can decompose our integral and estimate:

$$\begin{aligned}\left|\int_\Gamma f d(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi-\nu_{\phi^n})\right| &= \left|\int_\Gamma f d\Delta_\Gamma\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}-g_{\mu_\phi}\right)-\frac{2}{d^n-1}\int_\Gamma f d(r_{\mathbf{P}^1,\Gamma})_*\Lambda_n\right|\\ &\le \left|\int\left(\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}-g_{\mu_\phi}\right)d\Delta_\Gamma(f)\right|+\frac{2}{d^n-1}\left|\int_\Gamma f d(r_{\mathbf{P}^1,\Gamma})_*\Lambda_n\right|\\ &\le \max_\Gamma\left|\frac{1}{d^{2n}-d^n}\operatorname{ord}\operatorname{Res}_{\phi^n}-g_{\mu_\phi}\right|\cdot|\Delta|(f)+\frac{2}{d^n-1}\max_\Gamma|f|\cdot D_\Gamma\ .\end{aligned}$$

Using the explicit estimate from Theorem 3.3, we find

$$\left|\int_{\Gamma} f d(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi - \nu_{\phi^n})\right| \leq \frac{4}{d^n - 1}\left(\max\left(C_\phi, R_\Gamma\right) \cdot |\Delta|\left(f\right) + \max_{\Gamma}|f| \cdot D_\Gamma\right) .$$

□

With this we are able to prove Theorem 3.2, that the measures $\{\nu_{\phi^n}\}$ converge weakly to μ_ϕ. In order to show weak convergence, we will need that for all choices of $F \in \mathcal{C}(\mathbf{P}^1)$, we have

$$\int_{\mathbf{P}^1} F d(\mu_\phi - \nu_{\phi^n}) \to 0$$

as $n \to \infty$. Here is the proof.

Proof of Theorem 3.2. Let $\epsilon > 0$. Choose $F \in \mathcal{C}(\mathbf{P}^1)$. By Proposition 5.4 in [2], we know that there exists a finite graph Γ and a function $f \in \mathrm{CPA}(\Gamma)$ such that

$$\sup_{\mathbf{P}^1} |F(x) - f \circ r_{\mathbf{P}^1,\Gamma}(x)| < \frac{\epsilon}{4} .$$

Since both μ_ϕ and ν_{ϕ^n} are both probability measures, we have that

$$\begin{aligned}\left|\int_{\mathbf{P}^1} F d(\mu_\phi - \nu_{\phi^n})\right| &= \left|\int_{\mathbf{P}^1} \left(F - f \circ r_{\mathbf{P}^1,\Gamma}\right) d(\mu_\phi - \nu_{\phi^n}) + \int_{\mathbf{P}^1} f \circ r_{\mathbf{P}^1,\Gamma} d(\mu_\phi - \nu_{\phi^n})\right| \\ &\leq \frac{\epsilon}{2} + \left|\int_{\mathbf{P}^1} f \circ r_{\mathbf{P}^1,\Gamma} d(\mu\phi - \nu_{\phi^n})\right| \\ &= \frac{\epsilon}{2} + \left|\int_{\Gamma} f d(r_{\mathbf{P}^1,\Gamma})_*(\mu_\phi - \nu_{\phi^n})\right| .\end{aligned}$$

Since f and Γ are fixed, Theorem 3.11 tells us that for n sufficiently large, the remaining integral term can be made smaller than $\frac{\epsilon}{2}$. This establishes weak convergence. □

3.3 Examples

In this section, we give explicit examples showing the weak convergence of the crucial measures.

Example 1. Let $K = \mathbb{C}_p$ for some prime $p \geq 3$, and let $\phi(z) = \dfrac{z^p - z}{p}$. It is known (see [2], Example 10.120) that the invariant measure attached to ϕ is the Haar measure on $\mathbb{Z}_p := \lim_{\leftarrow} \mathcal{O}_K/(\mathfrak{m}_K^n)$. The classical fixed points of ϕ are ∞ and points $\zeta_1, ..., \zeta_p$, where the ζ_i lie in the different cosets of $\mathbb{Z}_p/p\mathbb{Z}_p$; we have that Γ_{Fix} is the tree spanned by the ζ_i and ∞. The Gauss point ζ_{G} is a non-fixed branch point of Γ_{Fix}, with valence $p+1$; hence $w_\phi(\zeta_{\text{G}}) = p + 1 - 2 = p - 1 = \deg(\phi) - 1$. By the weight formula, it is the only weighted point.

We now look to preimages of ζ_{G} under ϕ. The set $\phi^{-1}(\zeta_{\text{G}})$ is a collection of disjoint discs $D(a_1, r_1), ..., D(a_p, r_p)$ where a_i lie in the various directions towards fixed points, and the r_i can all be taken to be $1/p$. To see this, note that the preimages of zero are the points satisfying $a_i^p - a_i = 0$; in the reduction modulo $\mathfrak{m}_K$ these are the same as the classical fixed points ζ_i above. For the radii, one checks that $|\phi(a_i + p) - \phi(a_i)| = 1$, which establishes that $r_i = |p|$ for each i. It follows from these two facts that the discs are disjoint.

More generally, we claim that a point in the n^{th} preimage of ζ_{G} corresponds to a disc

$$D(a_{i_n i_{n-1} \cdots i_2 i_1}, p^{-n})$$

where the points $\{a_{i_n i_{n-1} \cdots i_2 i_1}\}$ are the successive preimages of 0 indexed in such a way that $\phi(a_{i_n i_{n-1} \cdots i_2 i_1}) = a_{i_{n-1} \cdots i_2 i_1}$, $\phi^{(2)}(a_{i_n i_{n-1} \cdots i_2 i_1}) = a_{i_{n-2} \cdots i_2 i_1}$, ... and finally $\phi^{(n)}(a_{i_n i_{n-1} \cdots i_2 i_1}) = 0$. Note that we make no assertion as to whether the points $a_{i_n \cdots i_1}$ are distinct, though in the end we will deduce that in fact they are:

Claim: The points $a_{i_n \cdots i_1}$ lie in distinct coset classes modulo p^n.

Note that we have already seen this for $n = 1$, where the preimages of 0 are $a_1 = \zeta_1, ..., a_p = \zeta_p$. We proceed now by induction. Suppose $a_{i_n \cdots i_1} \equiv a_{j_n \cdots j_1} \mod p^n$. By the relation $\phi(a_{i_n \cdots i_1}) = a_{i_{n-1} \cdots i_1}$, we find that

$$a_{i_n \cdots i_1} = a_{i_n \cdots i_1}^p - p \cdot a_{i_{n-1} \cdots i_1} \ ,$$

and likewise

$$a_{j_n \cdots j_1} = a_{j_n \cdots j_1}^p - p \cdot a_{j_{n-1} \cdots j_1} \ .$$

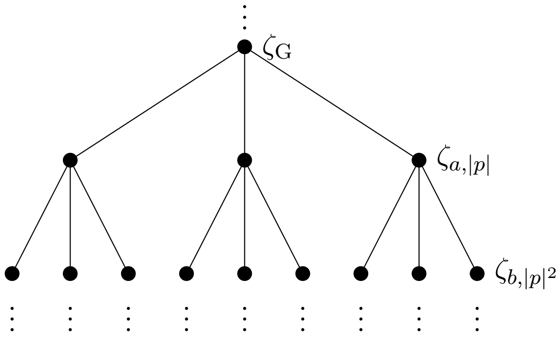

Figure 3.1: An example when $p = 3$ and $n = 3$. The bold vertices are points in the support of ν_{ϕ^3}. Each has valence 4 in the tree $\Gamma_{\mathrm{Fix},3}$ spanned by the classical fixed points, and each of these points is moved by ϕ; hence these points each have weight $\frac{1}{13}$.

The congruence $a_{i_n\cdots i_1} \equiv a_{j_n\cdots j_1} \mod p^n$ implies that for some $r \in \mathbb{Z}$ we have

$$p^n r = a^p_{i_n\cdots i_1} - a^p_{j_n\cdots j_1} + p(a_{j_{n-1}\cdots j_1} - a_{i_{n-1}\cdots i_1}) \; .$$

By our induction hypothesis, $p^{n-1} \nmid a_{j_{n-1}\cdots j_1} - a_{i_{n-1}\cdots i_1}$, and therefore we conclude that $p^n \nmid a^p_{i_n\cdots i_1} - a^p_{j_n\cdots j_1}$. But this contradicts that $a_{i_n\cdots i_1} \equiv a_{j_n\cdots j_1} \mod p^n$, and thus establishes the claim.

In particular, the above claim implies that the preimages $\phi^{-n}(0)$ are in one-to-one correspondence with the coests of $\mathbb{Z}_p/p^n\mathbb{Z}_p$. The fact that the radii corresponding to the n^{th} preimages of ζ_{G} are p^{-n} can be seen by the fact that $a_{i_n\cdots i_1} + p^n$ maps to a point lying at distance p^{n-1} from $a_{i_{n-1}\cdots i_1}$.

We next claim that each point $\zeta \in \bigcup_{i=0}^{n-1} \phi^{-i}(\zeta_{\mathrm{G}})$ is a branch point of $\Gamma_{\mathrm{FR},n}$ with valence at least $p+1$. To see this, note that $D(a_{i_k\cdots i_1}, p^{-k}) \subseteq D(0,1)$, and that ϕ^k maps $D(a_{i_k\cdots i_1}, p^{-k})$ onto $D(0,1)$. Then since $\phi^{k+1}(D(a_{i_{k+1}i_k\cdots i_1}, p^{-(k+1)})) = D(0,1)$, it must contain a fixed point of ϕ^{k+1}, which necessarily lies in $D(a_{i_k\cdots i_1}, p^{-k})$. There are p such discs $D(a_{i_{k+1}\cdots i_1}, p^{-(k+1)})$ lying in $D(a_{i_k\cdots i_1}, p^{-k})$ (corresponding to the cosets of $\mathbb{Z}_p/p^{k+1}\mathbb{Z}_p$ in $D(a_{i_k\cdots i_1}, p^{-k})$). This, together with the direction towards ∞ implies that $\zeta_{a_{i_k\cdots i_1}, p^{-k}}$ has valence at least $p+1$ in $\Gamma_{\mathrm{FR},n}$ for each $k = 1, ..., n-1$.

Finally, note that each point $\zeta \in \bigcup_{i=0}^{n-1} \phi^{-i}(\zeta_{\mathrm{G}})$ is moved, hence $w_{\phi^{(n)}}(\zeta) = v_{\Gamma_{\mathrm{FR},n}}(\zeta) - 2 \geq p-1$. There are $\sum_{i=0}^{n-1} p^i = \frac{p^n-1}{p-1}$ such points, and by summing the corresponding weights, we find a total weight of at least $p^n - 1$. As this is equal to $\deg(\phi^n) - 1$, these are the only points which bear weight and their weights must be eactly equal to $p-1$.

It also follows from the above remarks that the points $\phi^{-n}(\zeta_{\mathrm{G}})$ distribute themselves equally among the representatives of $\mathcal{O}_K/(\mathfrak{m}_K^n)$, and so as n tends to infinity these points converge to the points of $\mathbb{Z}_p$. For each fixed k, we know $\mu_{\mathrm{Haar}}(D(a, p^{-k})) = p^{-k}$ for any center $a \in \mathbb{Z}_p$. The corresponding ν_{ϕ^n} measure can be computed by considering the convex hull of $D(a, p^{-k})$ in $\mathbf{P}^1$, which we will denote by $\mathcal{D}(a, p^{-k})$ (concretely, this is $\mathbf{P}^1 \setminus B_{\vec{v}_\infty}(\zeta_{a,p^{-k}})^-$. The set $\mathcal{D}(a, p^{-k})$ will only receive ν_{ϕ^n} weight when $n \geq k$, and in this case, each point in $\mathcal{D}(a, p^{-k})$ that receives ν_{ϕ^n}-mass will have weight $\frac{p-1}{p^n-1}$. We need only count how many such points there are in a given $\mathcal{D}(a, p^{-k})$.

It will suffice to assume our center is of the form $a_{i_k i_{k-1} \cdots i_1}$, since the discs centered at these points of radius p^{-k} form a partition of $\mathbb{Z}_p$. From the arguments above, $\mathcal{D}(a_{i_k i_{k-1} \cdots i_1}, p^{-k})$ contains $1 + p + p^2 + ... + p^{(n-1)-k} = \frac{p^{n-k}-1}{p-1}$ points which receive ν_{ϕ^n}-mass, corresponding to the preimages $\phi^{-k}(\zeta_{\mathrm{G}}), \phi^{-(k+1)}(\zeta_{\mathrm{G}}), \ldots, \phi^{-(n-1)}(\zeta_{\mathrm{G}})$ which lie in $\mathcal{D}(a_{i_k i_{k-1} \cdots i_1})$ (there are $1, p, p^2, ... p^{(n-1)-k}$ such points, resp.). Therefore,

$$\nu_{\phi^n}(\mathcal{D}(a_{i_k i_{k-1} \cdots i_1}, p^{-k})) = \frac{p-1}{p^n-1} \cdot \frac{p^{n-k}-1}{p-1} \to \frac{1}{p^k} = \mu_{\mathrm{Haar}}(\mathcal{D}(a_{i_k i_{k-1} \cdots i_1}, p^{-k})) \ ,$$

which completes the proof.

Example 2 (Lattès Maps)**.** Let $K = \mathbb{C}_p$ be the p-adic complex numbers, and fix $q \in K$ satisfying $0 < |q| < 1$. A Tate curve E/K is an elliptic curve which is isomorphic (as a group) to the quotient $K^\times/q^{\mathbb{Z}}$ (see, e.g., [27] Appendix C). In particular, the multiplication-by-m map $[m] : E \to E$ is given by the quotient of the map $z \mapsto z^m$ on $K^\times$. Viewing E as an affine plane curve $y^2 = x^3 + Ax + B$ for $A, B \in K$, we let $\pi : E \to \mathbb{P}^1(K)$ be the map $(x, y) \mapsto x$. The *Lattès map* corresponding to multiplication by m is the rational map ϕ_m of degree m^2 on $\mathbb{P}^1(K)$ which completes the diagram

$$\begin{tikzcd}
E \arrow[r, "{[m]}"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\
\mathbb{P}^1(K) \arrow[r, "\phi_m"] & \mathbb{P}^1(K)
\end{tikzcd}$$

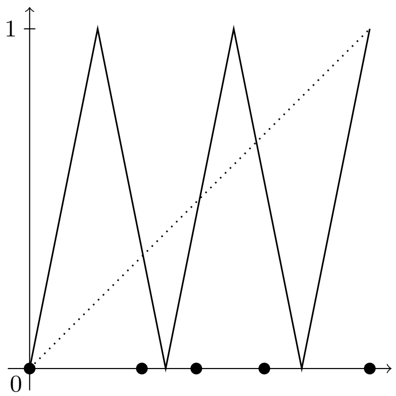

Figure 3.2: The action of ϕ_5 on $\mathcal{J}$ is represented by the sawtooth graph to the left. The bold points along the x-axis are the type II fixed points. Those points corresponding to edges where the graph is decreasing have 2 shearing directions, while those corresponding to edges where the graph is increasing have no shearing.

The map ϕ^m can be extended to the Berkovich line $\mathbf{P}^1$ over K in a natural way. Choosing suitable coördinates on $\mathbb{P}^1(K)$, the Julia set of ϕ_m is the segment $\mathcal{J} = [\zeta_{\mathrm{G}}, \zeta_{0,|q|^{-1/2}}]$, and the equilibrium measure is the uniform measure on this segment ([12], Proposition 5.1). For $i = 0, 1, ..., m-1$, let $\mathcal{I}_i = [\zeta_{0,|q|^{-i/2m}}, \zeta_{0,|q|^{-(i+1)/2m}}]$, so that $\mathcal{J} = \bigcup_{i=0}^{m-1} \mathcal{I}_i$. The restriction of ϕ_m to $\mathcal{I}_i$ is an affine map (with respect to the metric ρ), sending $\mathcal{I}_i$ bijectively onto $\mathcal{J}$. Along each segment $\mathcal{I}_i$, the map ϕ_m has slope $(-1)^i m$ ([12], Proposition 5.1).

Each interval $\mathcal{I}_i$ contains a unique type II fixed point of ϕ_m which we denote by ζ_i, for $i = 0, 1, ..., m-1$. Since ϕ_m has slope $(-1)^i m$, the rate of repulsion $r_{\phi_m}(\zeta_i, \vec{v}) = m$ for any direction $\vec{v} \in T_{\zeta_i}$ pointing into $\mathcal{J}$. Hence the points ζ_i are repelling fixed points with $\deg_{\phi_m}(\zeta_i) \geq m$.

The type I fixed points of ϕ_m all lie in branches off of the points ζ_i. For $i = 1, 2, ..., m-2$, the points ζ_i each have two direction $\vec{v}_0^{(i)}, \vec{v}_\infty^{(i)} \in T_{\zeta_i}$ pointing into $\mathcal{J}$ which contain classical fixed points. If i is odd (so that the slope of ϕ_m on $\mathcal{I}_i$ is $-m < 0$), these directions are flipped by $(\phi_m)_*$; hence ζ_i has at least two shearing directions. If i is even, $(\phi_m)_*$ fixes these directions at the respective ζ_i.

At the endpoints, ζ_0 has a unique direction $\vec{v}_\infty^{(0)} \in T_{\zeta_0}$ pointing into $\mathcal{J}$ which contains a type I fixed point, and it is fixed by $(\phi_m)_*$. The point ζ_{m-1} also has only one direction $\vec{v}_0^{(m-1)}$ containing a type I fixed point; if m is even, this direction is mapped to $\vec{v}_\infty \in T_{\zeta_{m-1}}$ by $(\phi_m)_*$ (since the slope of ϕ_m on $\mathcal{I}_{m-1}$ is negative when m is even). If m is odd, then the direction $\vec{v}_0^{(m-1)}$ is fixed by $(\phi_m)_*$. Therefore, for m even we have

$$w_{\phi_m}(\zeta_i) = \deg_{\phi_m}(\zeta_i) - 1 + N_{\text{Shearing}}(\zeta_i) \geq \begin{cases} m+1, & i = 1, 3, 5, ...m-3 \\ m, & i = m-1 \\ m-1, & i = 0, 2, 4, ..., m-2 \end{cases} . \tag{3.22}$$

Summing the lower bounds over each of the points ζ_i gives

$$\begin{aligned} \sum_{i=0}^{m-1} w_{\phi_m}(\zeta_i) &\geq (m+1)\left(\frac{m-2}{2}\right) + m + (m-1)\left(\frac{m}{2}\right) \\ &= m^2 - 1 \ . \end{aligned}$$

Since the sum of the weights is always equal to $\deg(\phi_m) - 1 = m^2 - 1$, the lower bounds given in (3.22) must be equalities.

Similarly, if m is odd we have

$$w_{\phi_m}(\zeta_i) = \deg_{\phi_m}(\zeta_i) - 1 + N_{\text{Shearing}}(\zeta_i) \geq \begin{cases} m+1, & i = 1, 3, 5, ...m-2 \\ m-1, & i = 0, 2, 4, ..., m-1 \end{cases} . \tag{3.23}$$

Again summing the lower bounds over each of the ζ_i gives

$$\begin{aligned} \sum_{i=0}^{m-1} w_{\phi_m}(\zeta_i) &\geq (m+1)\left(\frac{m-1}{2}\right) + (m-1)\left(\frac{m+1}{2}\right) \\ &= m^2 - 1 \ . \end{aligned}$$

Since the sum of the weights is $m^2 - 1$, we again conclude that the lower bounds in (3.23) are equalities.

Since the iterates of ϕ_m satisfy $\phi_m^{(n)} = \phi_{m^n}$, we see that for fixed m, the points ζ_i distribute themselves uniformly (with respect to the weight functions) along the interval $[\zeta_{\text{G}}, \zeta_{0,|q|^{-1/2}}]$ as $n \to \infty$.

Chapter 4

Results on $\mathrm{MinResLoc}(\phi^n)$

In this chapter, we use the convergence results in Chapter 3 to study the sets $\mathrm{MinResLoc}(\phi^n)$. Recall that $\mathrm{MinResLoc}(\phi)$ is the set of points in $\mathbf{H}^1$ where $\mathrm{ordRes}_\phi(\cdot)$ is minimized.

Rumely has already established important properties of $\mathrm{MinResLoc}(\phi)$. He shows that this set is always either a point or a segment in $\mathbf{H}^1$, that it lies in the ρ-ball of radius $\frac{2}{d-1}\,\mathrm{ordRes}_\phi(\zeta_{\mathrm{G}})$ about ζ_{G}, and more precisely that it lies in the tree spanned by the type I fixed points and the Berkovich repelling points of ϕ. ([24] Theorems 0.1 and 3.8). Rumely went on in a later paper [25] to show that $\mathrm{MinResLoc}(\phi)$ is the *barycenter*[1] of the measure ν_ϕ (Theorem 7.1), and that points in $\mathrm{MinResLoc}(\phi)$ correspond to conjugates having (semi)-stable reduction (in the sense of GIT).

In this chapter, we pursue some of the geometric questions answered by Rumely, now for the sets $\mathrm{MinResLoc}(\phi^n)$ attached to the *iterates* of ϕ. To study the asymptotic behaviour of these sets, one is naturally led to study the barycenter $\mathrm{Bary}(\mu_\phi)$ of the canonical measure. This subset of $\mathbf{H}^1$ has many properties in common with the sets $\mathrm{MinResLoc}(\phi^n)$:

Table 4.1: Similarities between $\mathrm{MinResLoc}(\phi)$ and $\mathrm{Bary}(\mu_\phi)$

$\mathrm{MinResLoc}(\phi^n)$	$\mathrm{Bary}(\mu_\phi)$
Either a point or a segment in $\mathbf{H}^1$	Either a point or a segment in $\mathbf{H}^1$
Barycenter of the measure ν_{ϕ^n}	Barycenter of the measure μ_ϕ
Set of points which minimize $\mathrm{ordRes}_{\phi^n}(\cdot)$	Set of points which minimize $g_\phi(\cdot,\cdot)$

---

[1] The formal definition of the Barycenter will be given in Section 4.1 below.

Our first result concerning the sets $\mathrm{MinResLoc}(\phi^n)$ relates them to the set $\mathrm{Bary}(\mu_\phi)$. Heuristically, one hopes that the measure-theoretic convergence $\nu_{\phi^n} \to \mu_\phi$ implies the geometric convergence $\mathrm{MinResLoc}(\phi^n) = \mathrm{Bary}(\nu_{\phi^n}) \to \mathrm{Bary}(\mu_\phi)$. It is somewhat disappointing, then, that the latter convergence does not always hold under the Hausdorff topology; one way to understand the obstruction is that, for some maps, $\mathrm{MinResLoc}(\phi^n)$ is always equal to a point $\{\zeta_n\}$ while the set $\mathrm{Bary}(\mu_\phi)$ is a segment. This is shown explicitly in Example 4 below.

However, there is one 'half' of Hausdorff convergence that always does hold:

Proposition 4.1. *Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. For any $\epsilon > 0$, there exists an N such that for every $n \geq N$ we have*

$$\mathrm{MinResLoc}(\phi^n) \subseteq \mathcal{B}_\epsilon(\mathrm{Bary}(\mu_\phi)) \ ,$$

where $\mathcal{B}_\epsilon(A) = \{\zeta \in \boldsymbol{H}^1 \ : \ \min_{x \in A} \rho(x, \zeta) < \epsilon\}$.

Apart from studying the convergence of the sets $\mathrm{MinResLoc}(\phi^n)$, one could also ask a weaker question: are the sets $\mathrm{MinResLoc}(\phi^n)$ 'bounded' in $\mathbf{H}^1$ in some sense? Rumely has shown ([24], Theorem 0.1) that the set $\mathrm{MinResLoc}(\phi)$ lies in the set $\{z \in \mathbf{H}^1 \ : \ \rho(z, \zeta_{\mathrm{G}}) \leq \frac{2}{d-1} \mathrm{ordRes}_\phi(\zeta_{\mathrm{G}})\}$. Applying his bound to successive iterates gives an upper bound on the distance from $\mathrm{MinResLoc}(\phi^n)$ to ζ_{G} that grows geometrically in d (for this we also use Theorem 3.1 above). This sort of asymptotic behaviour was not considered by Rumely in [24], and it turns out that one can do much better:

Theorem 4.2. *Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$, and let $R = \frac{2}{d-1} \mathrm{ordRes}(\phi)$. Then for each n,*

$$\mathrm{Bary}(\nu_{\phi^n}) = \mathrm{MinResLoc}(\phi^n) \subseteq \mathcal{B}_\rho(\zeta_G, R) \ .$$

If $m_0 = \min_{x \in \boldsymbol{P}^1} g_\phi(x, x)$ and μ_ϕ denotes the equilibrium measure of ϕ, then

$$\mathrm{Bary}(\mu_\phi) \subseteq \mathcal{B}_\rho(\zeta_G, R + m_0 - g_\phi(\zeta_G, \zeta_G)) \ .$$

4.1 Some Preliminary Results about Arakelov-Green's Functions

In this section we establish results relating the sets $\mathrm{MinResLoc}(\phi^n)$ to the barycenter of the set μ_ϕ. We first recall the definition of the barycenter of a measure in $\mathbf{P}^1$:

Definition 2. (Rivera-Letelier) Let ν be a finite, positive Radon measure on $\mathbf{P}^1$. The barycenter of ν, denoted $\mathrm{Bary}(\nu)$, is the collection of points $Q \in \mathbf{P}^1$ such that $\nu(\mathrm{B}_Q(\vec{v})^-) \leq \frac{1}{2}\nu(\mathbf{P}^1)$ for each $\vec{v} \in T_Q$.

Example 3. Here, we give several examples of barycenters for various probability measures on $\mathbf{P}^1$.

1. Let p be an odd prime and let $K = \mathbb{C}_p$, and let μ denote the Haar measure on $\mathbb{Z}_p$. Here the barycenter is $\{\zeta_{\mathrm{G}}\}$. To see this, note that each coset $k + p\mathbb{Z}_p$ has $\mu(k + p\mathbb{Z}_p) = \frac{1}{p}$. Letting $\vec{v}_1, ..., \vec{v}_p$ denote the directions at ζ_{G} corresponding to these cosets, we have $\mu(B_{\zeta_{\mathrm{G}}}(\vec{v}_i)^-) = \frac{1}{p}$. In particular, if $Q \neq \zeta_{\mathrm{G}}$ and $\vec{v}_G \in T_Q$ is the direction towards ζ_{G}, then $\mu(B_Q(\vec{v}_G)^-) \geq (p-1)\frac{1}{p} > \frac{1}{2}$, where the final inequality holds because p is odd. Thus $Q \notin \mathrm{Bary}(\mu)$.

2. Let $\nu = \frac{1}{2}\delta_A + \frac{1}{2}\delta_B$. Then the barycenter of ν is precisely the segment $[A, B]$. Let $\vec{v}_B \in T_A$ be the direction pointing towards B. This example also shows that $\nu(\cup_{\vec{v} \in T_A \setminus \{\vec{v}_B\}} B_A(\vec{v})^-)$ need not equal $\frac{1}{2}$.

3. Let $K = \mathbb{C}_p$. The barycenter of the canonical measure attached to $\phi(T) = \frac{T^2-1}{p}$ is the interval $[\zeta_{D(1,\frac{1}{p})}, \zeta_{D(-1,\frac{1}{p})}]$; a proof will be given in Example 4 below. This example is due to Rob Benedetto (personal communication).

4. Let $\nu = \delta_\zeta$ be a point mass at some point $\zeta \in \mathbf{H}^1$. Then the barycenter for ν is ζ itself.

We will restrict our attention to probability measures ν on $\mathbf{P}^1$. There are several important facts about the barycenter proved in Section 4.1 below, all of which are due to Rivera-Letelier but which have yet to appear. As they are essential to the proof of Proposition 4.1, we include our own proofs here. More explicitly, we will show (i) the barycenter of a probability measure is always non-empty (Proposition 4.4), (ii) it is always a point or a segment (Proposition 4.4), and (iii) the associated Arakelov-Green's function $g_\nu(x, x)$ attains its minimum precisely on $\mathrm{Bary}(\nu)$ (Proposition 4.4).

Having established these preliminary properties of the Arakelov-Green's functions, we use them to establish results relating the sets $\mathrm{MinResLoc}(\phi^n)$ to $\mathrm{Bary}(\mu_\phi)$.

We begin with several results about the Arakelov-Green's function $g_\nu(x, y)$ attached to a probability measure ν on $\mathbf{P}^1$. In particular we are interested in the values of this function on the diagonal of $\mathbf{P}^1$; we will let $g_\nu(x) = g_\nu(x, x)$.

Lemma 4.3. *Let ν be a probability measure on $\boldsymbol{P}^1$.*

1. *The function $g_\nu(\cdot)$ is convex up along segments in $\boldsymbol{P}^1$.*
2. *If $Q \in \boldsymbol{H}^1$, and $\vec{v}$ is any direction in T_Q, we have*

$$\partial_v(g_\nu)(Q) = 1 - 2\nu(B_Q(\vec{v})^-) \ .$$

3. *If Q is a type I point with $c = \nu(\{Q\})$, then for any $\epsilon > 0$ there exists a type II point Q_0 sufficiently close to Q such that if $\vec{v} \in T_{Q_0}$ denotes the direction towards Q, we have*

$$|(1 - 2c) - \partial_{\vec{v}}(g_\nu)(Q_0)| < \epsilon \ .$$

In particular, if ν does not charge Q, then

$$|1 - \partial_{\vec{v}}(g_\nu)(Q_0)| < \epsilon \ .$$

Proof. Fix $Q \in \mathbf{H}^1$. The integral representation of $g_\nu(x)$ gives us

$$g_\nu(x) = \rho(Q, x) - 2\int \langle x, \zeta\rangle_Q \ d\nu(\zeta) + C_Q \ . \tag{4.1}$$

Here, the term C_Q differs from the constant given in (2.5), but only by a fixed amount determined by Q and ν. Explicitly, we have

$$C_Q = C - \rho(Q, \zeta_{\mathrm{G}}) + 2\int \langle x, \zeta\rangle_Q \ d\nu(\zeta) = g_\nu(Q) \ .$$

To prove (A), fix a segment $[P, Q] \subseteq \mathbf{P}^1$, and let R_t be the unique point in $[P, Q]$ satisfying $\rho(R_t, Q) = t\rho(P, Q)$, for any $0 \leq t \leq 1$. Convexity along $[P, Q]$ amounts to showing

$$g_\nu(R_t) \leq t g_\nu(P) + (1-t) g_\nu(Q) \ . \tag{4.2}$$

Using the decomposition in (4.1), the left side of (4.2) is

$$g_\nu(R_t) = t\rho(P, Q) - 2\int \langle R_t, \zeta \rangle_Q \ d\nu(\zeta) + g_\nu(Q) \ , \tag{4.3}$$

and the right side of (4.2) is

$$\begin{aligned} t g_\nu(P) + (1-t) g_\nu(Q) &= \left(t\rho(P, Q) - 2t \int \langle P, \zeta \rangle_Q \ d\nu(\zeta) + t g_\nu(Q) \right) + (1-t) g_\nu(Q) \\ &= t\rho(P, Q) - 2\int \langle P, \zeta \rangle_Q \ d\nu(\zeta) + g_\nu(Q) \ . \end{aligned} \tag{4.4}$$

Comparing (4.3) and (4.4), it is enough to show that for every $0 \leq t \leq 1$, we have

$$\int \langle P, \zeta \rangle_Q \ d\nu(\zeta) \geq \int \langle R_t, \zeta \rangle_Q \ d\nu(\zeta) \ . \tag{4.5}$$

In fact, we claim that more is true: for any point $R \in [P, Q]$, and any $\zeta \in \mathbf{P}^1$, we have $\langle P, \zeta \rangle_Q \geq \langle R, \zeta \rangle_Q$. To see this, let $w \in [P, Q]$ be the point $\zeta \wedge_Q P$ be the first point where the segment $[P, Q]$ and $[\zeta, Q]$ intersect; then $\langle P, \zeta \rangle_Q = \rho(w, Q)$. We consider two cases:

1. If $w \in [R, Q]$, then $R \wedge_Q \zeta = w$ as well, and $\langle R, \zeta \rangle_Q = \rho(w, Q) = \langle P, \zeta \rangle_Q$, which establishes the claim in this case.

2. If $w \notin [R, Q]$, then $R \wedge_Q \zeta = R$, so that $\langle R, \zeta \rangle_Q = \rho(R, Q) \leq \rho(w, Q) = \langle P, \zeta \rangle_Q$, which proves the claim in this case.

Thus, in either case, the inequality (4.5) holds for the integrand at each fixed ζ, and hence the integral inequality follows. This finishes the proof of convexity.

We next prove (B). Fix $\vec{v} \in T_Q$. We can evaluate the integral in (4.1) for a point[2] $Q + t\vec{v}$ by restricting to the segment $[Q, Q+t\vec{v}]$. Let $\nu_{[Q,Q+t\vec{v}]} = (r_{\mathbf{P}^1,[Q,Q+t\vec{v}]})_*\nu$ denote the retraction of the measure[3]; then (4.1) becomes

$$g_\nu(Q+t\vec{v}) = \rho(Q, Q+t\vec{v}) - 2\int_Q^{Q+t\vec{v}} \rho(Q,s)d\nu_{[Q,Q+t\vec{v}]}(s) + C_Q \ .$$

We will explicitly estimate the quantity

$$\frac{g_\nu(Q+t_0\vec{v}) - g_\nu(Q)}{t_0}$$

for small values of t_0. Let $c_1 = 1 - \nu(B_Q(\vec{v})^-)$; then the retraction measure $\nu_{[Q,Q+t_0\vec{v}]}$ decomposes as $\nu_{[Q,Q+t_0\vec{v}]} = c_1\delta_Q + \nu_{(Q,Q+t_0\vec{v})} + c_2(t_0)\delta_{Q+t_0\vec{v}}$, where $c_2(t_0) \to \nu(B_Q(\vec{v})^-)$ as $t_0 \to 0$; the existence of $c_2(t_0)$ follows from the regularity[4] of ν. Inserting this decomposition into Equation (4.1), and using the fact that $C_Q = g_\nu(Q)$, we find

$$\begin{aligned} g_\nu(Q+t_0\vec{v}) - g_\nu(Q) &= \rho(Q, Q+t_0\vec{v}) - 2\int_Q^{Q+t_0\vec{v}} \rho(Q,t)d\nu_{[Q,Q+t_0\vec{v}]}(t) \\ &= t_0 - 2\int_Q^{Q+t_0\vec{v}} \rho(Q,t)d\nu_{(Q,Q+t_0\vec{v})} - 2c_2(t_0)\cdot t_0 \ . \end{aligned} \tag{4.6}$$

Observe that $\nu_{(Q,Q+t_0\vec{v})}(Q, Q+t_0\vec{v}) \leq 1 - c_1 - c_2(t_0)$, hence we can estimate the above integral as

$$0 \leq \int_Q^{Q+t_0\vec{v}} \rho(Q,t)d\nu_{(Q,Q+t_0\vec{v})} \leq (1 - c_1 - c_2(t_0))\cdot t_0$$

And so for t_0 sufficiently small,

$$(2c_1 - 1) \leq \frac{g_\nu(Q+t_0\vec{v}) - g_\nu(Q)}{t_0} \leq 1 - 2c_2(t_0) \ .$$

---

[2] Recall that $Q + t\vec{v}$ is actually an arbitrary point in $B_{\vec{v}}(Q)^-$ satisfying $\rho(Q+t\vec{v}, Q) = t$, and the limit of the difference quotient is unaffected by which point we choose. See the discussion in Section 2.4.3.

[3] More generally, if $\mathcal{I} \subseteq \mathbf{H}^1$ is an interval, we will let $\nu_{\mathcal{I}} = (r_{\mathbf{P}^1,\mathcal{I}})_*\nu$ denote the retraction of ν to $\mathcal{I}$.

[4] Explicitly, $c_2(t_0)$ can be given as $c_2(t_0) := 1 - \nu(B_{Q+t_0\vec{v}}(\vec{v}_Q)^-)$, the complementary mass of the ball at $Q + t_0\vec{v}$ pointing towards Q.

Rewriting the left side with the explicit value of c_1 we have

$$1 - 2\nu(B_Q(\vec{v})^-) \leq \frac{g_\nu(Q + t_0\vec{v}) - g_\nu(Q)}{t_0} \leq 1 - 2c_2(t_0) \ .$$

Letting $t_0 \to 0$, we have

$$\partial_v(g_\nu)(Q) = \lim_{t_0 \to 0} \frac{g_\nu(Q + t_0\vec{v}) - g_\nu(Q)}{t_0} = 1 - 2\nu(B_Q(\vec{v})^-)$$

as asserted.

We now show part (C). Let Q be a type I point, set $c = \nu(\{Q\})$, and fix $\epsilon > 0$. By the regularity of ν, we can find a type II point Q_0 so that, if $\vec{v} \in T_{Q_0}$ is the direction towards Q, then $c - \frac{\epsilon}{2} \leq \nu(B_{Q_0}(\vec{v})^-) < c + \frac{\epsilon}{2}$. Thus

$$|1 - 2c - \partial_{\vec{v}}(g_\nu)(Q_0)| = 2|\nu(B_{Q_0}(\vec{v})^-) - c| < \epsilon \ .$$

If $Q \notin \mathrm{supp}(\nu)$ this reduces to

$$|1 - \partial_{\vec{v}}(g_\nu)(Q_0)| < \epsilon$$

as asserted.

$\square$

With the above lemma, we can prove the following result about the geometry of the barycenter of a probability measure. This result is due originally to Rivera-Letelier, but the proof given here is our own:

Proposition 4.4. *(Rivera-Letelier) Let ν be a probability measure on $\boldsymbol{P}^1$ with continuous potentials. Let $g_\nu(x) := g_\nu(x,x)$ be the diagonal values of the Arakelov-Green's function.*

1. *The function $g_\nu(x)$ is minimized precisely on* $\mathrm{Bary}(\nu)$.
2. $\mathrm{Bary}(\nu)$ *is a nonempty subset of $\boldsymbol{H}^1$, and it either consists of a single point or is a closed segment.*

Proof. Since ν has continuous potentials, the function $g_\nu(x)$ is lower semi-continuous on $\mathbf{P}^1$. Also, ν does not charge any type I points: if $\nu(\{Q\}) > 0$ for some type I point Q, then we can decompose the potential function as

$$u_\nu(z, \zeta_{\mathrm{G}}) = -\nu(\{Q\}) \log_v \delta(z, Q)_{\zeta_{\mathrm{G}}} + \int_{\mathbf{P}^1 \setminus \{Q\}} -\log_v \delta(z, w)_{\zeta_{\mathrm{G}}} d\nu(w) \ .$$

But then $\lim_{z \to Q} u_\nu(z, \zeta_{\mathrm{G}}) = -\infty$, contradicting that $u_\nu(z, \zeta_{\mathrm{G}})$ is continuous as a function to $\mathbb{R}$.

Since $\mathbf{P}^1$ is compact in the weak topology and $g_\nu(x)$ is lower semicontinuous, it must assume a minimum. Moreover, the points at which $g_\nu(x)$ attains its minimum lie in $\mathbf{H}^1$: if it contained a type I point Q, then necessarily $\nu(\{Q\}) = 0$ and by Lemma 4.3 there exists a type II point Q_0 sufficiently near Q such that if $\vec{v} \in T_{Q_0}$ is the direction towards Q, then $\partial_{\vec{v}}(g_\nu)(Q_0) > \frac{1}{2}$. In particular, $g_\nu(Q_0) < g_\nu(Q)$, contradicting that Q is a minimum value of $g_\nu(x)$.

If Q is a point at which $g_\nu(x)$ is minimized, then $\partial_{\vec{v}}(g_\nu)(Q) \geq 0$ for every $\vec{v} \in T_Q$. In particular, it follows from Lemma 4.3 that $\nu(B_Q(\vec{v})^-) \leq \frac{1}{2}$ for every $\vec{v} \in T_Q$; thus Q is in the barycenter of ν. It follows that $\mathrm{Bary}(\nu)$ is nonempty. Conversely, suppose that $Q \in \mathrm{Bary}(\nu)$; then $\nu(B_Q(\vec{v})^-) \leq \frac{1}{2}$ for every $\vec{v} \in T_Q$, and by Lemma 4.3, we find that $\partial_{\vec{v}}(g_\nu)(Q) \geq 0$ for every $\vec{v} \in T_Q$. Now choose any point $P \in \mathbf{H}^1$ at which $g_\nu(x)$ is minimized, and consider the segment $[P, Q]$. Since $\partial_{\vec{v}_P}(g_\nu)(Q) \geq 0$, it follows that $g_\nu(Q) \leq g_\nu(P)$; since g_ν is minimized at P, we find that $g_\nu(Q) = g_\nu(P)$, i.e. g_ν is minimized at Q as well. This finishes the proof that g_ν is minimized precisely on $\mathrm{Bary}(\nu)$.

We now show that $\mathrm{Bary}(\nu)$ is connected. Suppose there are two points P, Q in the barcyenter of ν, and let $R \in [P, Q]$. Then R is also in the barycenter of ν, since for any $\vec{v} \in T_R$ we have either $B_R(\vec{v})^- \subseteq \mathrm{B}_P(\vec{v}_R)^-$ or $B_R(\vec{v})^- \subseteq B_Q(\vec{v}_R)^-$, where $\vec{v}_R$ is the direction towards R originating at P or Q, as is appropriate. Thus $\nu(B_R(\vec{v})^-) \leq \frac{1}{2}$ for each $\vec{v} \in T_R$. In particular, the barycenter of ν is connected.

Next we show that $\mathrm{Bary}(\nu)$ is a segment. Let $P \in \mathrm{Bary}(\nu)$, and suppose that Q is any other point in the barycenter. Let $\vec{v}_Q \in T_P$ be the direction towards Q; then $\nu(B_P(\vec{v}_Q)^-) \leq \frac{1}{2}$, hence

$$\nu\left(\cup_{\vec{v} \in T_P \setminus \{\vec{v}_Q\}} B_P(\vec{v})^-\right) + \nu(\{P\}) \geq \frac{1}{2} \ .$$

In a similar way let $\vec{w}_P \in T_Q$ be the direction at Q towards P. We have that $\nu(B_Q(\vec{w}_P)^-) \leq \frac{1}{2}$ and moreover

$$\frac{1}{2} \leq \nu\left(\cup_{\vec{v} \in T_P \setminus \{\vec{v}_Q\}} B_P(\vec{v})^-\right) + \nu(\{P\}) \leq \nu(B_Q(\vec{w}_P)^-) \leq \frac{1}{2} \ . \tag{4.7}$$

Hence $\frac{1}{2} = \nu(\left(\cup_{\vec{v} \in T_P \setminus \{\vec{v}_Q\}} B_P(\vec{v})^-\right) + \nu(\{P\})$. Thus if Γ is any connected subgraph of $\mathrm{Bary}(\nu)$, it can have at most two endpoints, i.e. Γ must be a segment. Since finite graphs exhaust $\mathrm{Bary}(\nu)$, it follows that if $\mathrm{Bary}(\nu)$ has more than one point, then it must be a segment. This segment must be closed, as it is the collection of points where g_ν is minimized.

□

Though it will not be needed for the present work, we make note of the following minimization statement for Arakelov-Green's functions of two variables attached to measures with finite support in $\mathbf{H}^1$:

Proposition 4.5. *Let $\nu = \sum_{n=1}^N c_n \delta_{P_n}$ be a probability measure on $\boldsymbol{P}^1$ supported at finitely many points $P_i \in \boldsymbol{H}^1$. Then there exists a pair (P_i, P_j) such that*

$$\min_{x,y \in \boldsymbol{P}^1} g_\nu(x,y) = g_\nu(P_i, P_j) \ .$$

Proof. Let Γ denote the tree spanned by the points $\{P_i\}$. Recall that the Arakelov-Green's function can be decomposed as

$$\begin{aligned} g_\nu(x,y) &= -\sum_{i=1}^N c_{P_i} \log_v \delta(x,y)_{P_i} + C \\ &= \sum_{i=1}^N \left(c_{P_i} \cdot \rho(x \wedge_{P_i} y, P_i) + c_{P_i} \cdot \log_v ||P_i, P_i||\right) + C \ , \end{aligned}$$

where C is the normalizing constant appearing in (2.5). Let (x_0, y_0) be a point in $\mathbf{P}^1 \times \mathbf{P}^1$ where $g_\nu(x,y)$ is minimized. Suppose x_0 does not lie in the tree spanned by the P_i; then for each i, retracting x_0 to this tree will either leave $\rho(x_0 \wedge_{P_i} y_0, P_i)$ constant or cause this term to decrease.

Hence we may assume that x_0 lies in the tree spanned by the P_i; reversing the roles of x_0 and y_0, we may also assume that y_0 lies in the tree spanned by the P_i.

Fix $y \in \Gamma$. As a function of x each summand $-\log_v \delta(x,y)_{P_i}$ is linear along the segment $[P_i, y]$, attaining its minimum at $x = P_i$ and its maximum at $x = y$. On branches off of $[P_i, y]$, the function is constant. It follows that the minimum of $g_\nu(x,y)$ on the tree Γ is attained for $x = P_i$ for some weighted point $P_i = P_i(y)$ depending on y. Conversely, if we fix x then the minimum is attained at some weighted point $y = Q_j$ where $Q_j = Q_j(x)$ depends on x.

Thus for any fixed pair $(s,t) \in \Gamma \times \Gamma$, we find

$$\begin{aligned}
g_\nu(s,t) &\geq g_\nu(P_i(t), t) \\
&\geq \min_{P_i} \min_{y \in \Gamma} g_\nu(P_i, y) \\
&= \min_{P_i} g_\nu(P_i, Q_j(P_i)) \\
&\geq \min_{P,Q \in \{P_1,\ldots,P_N\}} g_\nu(P,Q) \\
&\geq \min_{x,y \in \Gamma \times \Gamma} g_\nu(x,y) \ .
\end{aligned}$$

Choosing (s,t) to be the point at which $g_\nu(x,y)$ is minimized, we find

$$\min_{x,y \in \Gamma} g_\nu(x,y) = g_\nu(s,t) \geq \min_{P,Q \in \{P_1,\ldots,P_N\}} g_\nu(P,Q) \geq \min_{x,y \in \Gamma} g_\nu(x,y) \ .$$

In particular,

$$\min_{P,Q \in \{P_1,\ldots,P_N\}} g_\nu(P,Q) = \min_{x,y \in \Gamma} g_\nu(x,y) \ .$$

The statement of the proposition now follows. □

4.2 Asymptotics of MinResLoc(ϕ^n)

We now apply the results of the previous section to the functions $g_{\mu_\phi}(x,x)$ and the sets MinResLoc(ϕ^n). Note that the invariant measure μ_ϕ has continuous potentials ([2] Proposition 10.7), hence the results of Proposition 4.4 apply. Several of the proofs in this section will rely on the following definition:

Definition 3. Let ν be a probability measure on $\mathbf{P}^1$ that does not charge type I or type IV points. If $Q \in \mathrm{Bary}(\nu)$, the set of directions at Q which point away from $\mathrm{Bary}(\nu)$ is denoted

$$T_Q^* := \{\vec{v} \in T_Q \ : \ \text{For } t \text{ sufficiently small, } Q + t\vec{v} \notin \mathrm{Bary}(\nu)\} \ .$$

If $Q \notin \mathrm{Bary}(\nu)$, then the directions which do not contain $\mathrm{Bary}(\nu)$ are similarly denoted

$$T_Q^* := \{\vec{v} \in T_Q \ : \ \mathrm{Bary}(\nu) \not\subseteq B_Q(\vec{v})^-\} \ .$$

We will be most interested in the case that $\nu = \mu_\phi$. Our first main result in this section is:

Proposition 4.6. *Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. Then there exists an $N = N(\phi)$ such that, for every $n \geq N$, we have*

$$\mathrm{Bary}(\mu_\phi) \subseteq \Gamma_{\widehat{FR},n} \ .$$

Proof. If ϕ has potential good reduction, then $\mathrm{Bary}(\mu_\phi) = \mathrm{MinResLoc}(\phi^n)$ for every n and there is nothing to prove. So we suppose that ϕ has bad reduction. In particular, μ_ϕ does not charge points (see [12], Théorème E).

We first prove the result when $\mathrm{Bary}(\mu_\phi)$ is a single point. Let $\mathrm{Bary}(\mu_\phi) = \{A\}$. Necessarily we can find two directions $\vec{v}, \vec{w} \in T_A$ so that $\mu_\phi(B_A(\vec{v})^-), \mu_\phi(B_A(\vec{w})^-) > 0$; let

$$\epsilon = \frac{1}{2}\min(\mu_\phi(B_A(\vec{v})^-), \mu_\phi(B_A(\vec{w})^-)) \ .$$

Note that the discs $B_A(\vec{v})^-, B_A(\vec{w})^-$ have the common boundary $\partial(B_A(\vec{v})^-) = \partial(B_A(\vec{w})^-) = \{A\}$. Since μ_ϕ does not charge points, and since the measures ν_{ϕ^n} converge weakly to μ_ϕ, we may apply the Portmanteau theorem ([2], Theorem A.13) to find N sufficiently large so that $\nu_{\phi^n}(B_A(\vec{v})^-), \nu_{\phi^n}(B_A(\vec{w})^-) > \frac{\epsilon}{2} > 0$ for $n \geq N$. In particular, there is a point of $\Gamma_{\widehat{FR},n}$ in each of

$B_A(\vec{v})^-, B_A(\vec{w})^-$ for $n \geq N$, and since $\Gamma_{\widehat{FR},n}$ is connected, it follows that $\mathrm{Bary}(\mu_\phi) = \{A\} \subseteq \Gamma_{\widehat{FR},n}$ whenever $n \geq N$.

A similar argument will address the case that $\mathrm{Bary}(\mu_\phi)$ is a segment. Let A, B be the endpoints of segment, and choose $\vec{v} \in T_A^*, \vec{w} \in T_B^*$ with $\mu_\phi(B_A(\vec{v})^-), \mu_\phi(B_B(\vec{w})^-) > 0$. Again we let

$$\epsilon = \frac{1}{2}\min(\mu_\phi(B_A(\vec{v})^-), \mu_\phi(B_B(\vec{w})^-)) \ .$$

The same argument as above ensures that there is an N so that, for $n \geq N$, we have

$$\nu_{\phi^n}(B_A(\vec{v})^-), \nu_{\phi^n}(B_B(\vec{w})^-) > \frac{\epsilon}{2} > 0 \ .$$

Thus there is a point of $\Gamma_{\widehat{FR},n}$ in each of $B_A(\vec{v})^-, B_B(\vec{w})^-$ for $n \geq N$. By connectedness, it follows that $\mathrm{Bary}(\mu_\phi) = [A, B] \subseteq \Gamma_{\widehat{FR},n}$.

□

Lemma 4.7. *For every $\epsilon > 0$, there exists $\delta = \delta(\phi, \epsilon) < \frac{1}{2}$ so that for every $x \in \boldsymbol{H}^1$ with $\rho(x, \mathrm{Bary}(\mu_\phi)) = \epsilon$ and every $\vec{v} \in T_x$ pointing away from* $\mathrm{Bary}(\mu_\phi)$*, we have*

$$\mu_\phi(B_x(\vec{v})^-) < \delta \ .$$

Proof. In the case that $\mathrm{Bary}(\mu_\phi)$ is a segment $[A, B]$, for any $Q \in (A, B)$ and any direction $\vec{v} \in T_Q^*$ pointing away from $\mathrm{Bary}(\mu_\phi)$, it follows from (4.7) that $\mu_\phi(B_Q(\vec{v})^-) = 0$. Thus, it suffices to prove the assertion for the end point(s) of $\mathrm{Bary}(\mu_\phi)$.

Let A be an endpoint of $\mathrm{Bary}(\mu_\phi)$, and let $x \in \mathbf{H}^1$ be any point with $\rho(x, A) = \epsilon$ for which A is the point in $\mathrm{Bary}(\mu_\phi)$ that is nearest to x. We consider two cases:

(i) First, suppose that there are no directions $\vec{v} \in T_A^*$ with $\mu_\phi(B_A(\vec{v})^-) = \frac{1}{2}$. Necessarily we have

$$\sum_{\vec{v} \in T_A^*} \mu_\phi(B_A(\vec{v})^-) \leq 1 \ , \tag{4.8}$$

and our hypothesis that $\mu_\phi(B_A(\vec{v})^-) < \frac{1}{2}$ for all $\vec{v} \in T_A^*$ ensures that

$$s_A = \sup_{\vec{v} \in T_A^*} \mu_\phi(B_A(\vec{v})^-) < \frac{1}{2} \ .$$

Indeed, if $s_A = \frac{1}{2}$, then there would be an infinite sequence of directions $\vec{v}_1, \vec{v}_2, ... \in T_A$ with $\mu_\phi(B_A(\vec{v}_i)^-) > \frac{1}{2} - \epsilon > 0$; but this would contradict (4.8). Then for any $\vec{v} \in T_A^*$ and any $x \in B_A(\vec{v})^-$ with $\rho(x, A) = \epsilon$, we have

$$\mu_\phi(B_x(\vec{w})^-) \leq \mu_\phi(B_A(\vec{v})^-) \leq s_A < \frac{1}{2}$$

for each $\vec{w} \in T_x \setminus \{\vec{v}_A\}$.

(ii) Now suppose that, for some $\vec{v} \in T_A^*$, we have $\mu_\phi(B_A(\vec{v})^-) = \frac{1}{2}$. Let x_ϵ denote a generic point in $B_A(\vec{v})^-$ with $\rho(x_\epsilon, A) = \epsilon$ We have that

$$\sum_{x_\epsilon} \sum_{\vec{v} \in T_{x_\epsilon}^*} \mu_\phi(B_{x_\epsilon}(\vec{v})^-) \leq \frac{1}{2} \ . \tag{4.9}$$

In particular, at most countably many x_ϵ have directions $\vec{v} \in T_{x_\epsilon}^*$ that carry mass. Let

$$\delta_A = \sup_{x_\epsilon, \vec{v} \in T_{x_\epsilon}^*} \mu_\phi(B_{x_\epsilon}(\vec{v})^-) \leq \frac{1}{2} \ .$$

Note that it is impossible to have some x_ϵ and a direction $\vec{v} \in T_{x_\epsilon}^*$ with $\mu_\phi(B_{x_\epsilon}(\vec{v})^-) = \frac{1}{2}$; if this were the case, then any $y \in [A, x_\epsilon]$ would also be in the barycenter, contradicting A is an endpoint of $\mathrm{Bary}(\mu_\phi)$. This, together with (4.9), implies that $\delta_A < \frac{1}{2}$, and so we have $\mu_\phi(B_{x_\epsilon}(\vec{v})^-) < \delta_A$ for every $\vec{v} \in T_{x_\epsilon}^*$.

If $\mathrm{Bary}(\mu_\phi)$ is a single point A, then δ_A is the constant asserted in the lemma. Otherwise, if $\mathrm{Bary}(\mu_\phi) = [A, B]$, it suffices to take $\delta = \min(\delta_A, \delta_B)$. □

Proof of Proposition 4.1. Fix $0 < \epsilon < 1$ and let $\delta = \delta(\phi, \epsilon/2)$ be the constant arising from Lemma 4.7. Note that here we are using the constant attached to $\frac{\epsilon}{2}$ rather than the one attached to ϵ.

Observe that we can interpret the conclusion of Lemma 4.7 as a statement about the slope of $g_\phi(x,x)$; namely, if $x \in \mathbf{H}^1$ with $\rho(x, \mathrm{Bary}(\mu_\phi)) = \frac{\epsilon}{2}$ and $\vec{v} \in T_x^*$, then

$$\partial_{\vec{v}}(g_\phi)(x) = 1 - 2\mu_\phi(B_x(\vec{v})^-) > 1 - 2\delta > 0 \ .$$

Fix R large enough so that $B_\epsilon(\mathrm{Bary}(\mu_\phi)) \subseteq B_\rho(\zeta_{\mathrm{G}}, R)$. Choose any y with $\rho(y, \mathrm{Bary}(\mu_\phi)) = \epsilon$, and let x be the unique point on the path joining y to $\mathrm{Bary}(\mu_\phi)$ satisfying $\rho(x, \mathrm{Bary}(\mu_\phi)) = \frac{\epsilon}{2}$. Set

$$s = (1 - 2\delta) \cdot \frac{\epsilon}{4} > 0 \ .$$

Since $g_\phi(x,x)$ is convex along segments (see Corollary 3.10), we have

$$\begin{aligned} g_\phi(y,y) - g_\phi(x,x) &\geq \partial_{\vec{v}}(g_\phi)(x) \cdot \frac{\epsilon}{2} \\ &> (1-2\delta)\frac{\epsilon}{2} = 2s \ . \end{aligned}$$

Equivalently,

$$g_\phi(x,x) + s < g_\phi(y,y) - s \ .$$

By Theorem 3.1 above, we may choose N so that for $n \geq N$, we have

$$\left| \frac{1}{d^{2n} - d^n} \operatorname{ord} \mathrm{Res}_{\phi^n}(z) - g_\phi(z,z) \right| < s$$

for every $z \in B_\rho(\zeta_{\mathrm{G}}, R)$. In particular,

$$\begin{aligned} \frac{1}{d^{2n} - d^n} \operatorname{ord} \mathrm{Res}_{\phi^n}(x) &\leq g_\phi(x,x) + s \\ &< g_\phi(y,y) - s \\ &\leq \frac{1}{d^{2n} - d^n} \operatorname{ord} \mathrm{Res}_{\phi^n}(y) \ . \end{aligned}$$

Thus for $n \geq N$, the function $\frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x)$ is increasing as one moves from points at distance $\frac{\epsilon}{2}$ from $\operatorname{Bary}(\mu_\phi)$ to points at distance ϵ from $\operatorname{Bary}(\mu_\phi)$. Since $\operatorname{ordRes}_\phi(\cdot)$ is convex up along segments, it must attain its minimum on $B_{\frac{\epsilon}{2}}(\operatorname{Bary}(\mu_\phi)) \subseteq B_\epsilon(\operatorname{Bary}(\mu_\phi))$. $\square$

As a consequence, we have a result that gives an interpretation of the minimal value that $g_\phi(x,x)$ takes on $\mathbf{P}^1$:

Corollary 4.8. *Let* $m_n = \min_{x \in \boldsymbol{P}^1} \frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x)$ *be the value that* $\frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x)$ *takes on* $\operatorname{MinResLoc}(\phi^n)$*. Then*

$$\min_{x \in \boldsymbol{P}^1} g_\phi(x,x) = \lim_{n \to \infty} m_n \ .$$

Proof. For $n \geq 1$, let $x_n \in \operatorname{MinResLoc}(\phi^n)$, and set $m_n = \frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x_n)$. Let $m_0 = \min_{x \in \mathbf{P}^1} g_\phi(x,x)$.

Fix $\epsilon > 0$. By Proposition 4.1, we may choose N_1 sufficiently large so that for $n \geq N_1$, $\rho(x_n, \operatorname{Bary}(\mu_\phi)) < \frac{\epsilon}{2}$; by the (Lipschitz) continuity of $g_\phi(x,x)$ with respect to ρ (Corollary 3.10), this implies

$$|m_0 - g_\phi(x_n, x_n)| < \frac{\epsilon}{2} \ . \tag{4.10}$$

Further, since $\operatorname{Bary}(\mu_\phi)$ is bounded, by Theorem 3.3 we may choose N_2 so that for $n \geq N_2$, we have

$$\left| \frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x_n) - g_\phi(x_n, x_n) \right| < \frac{\epsilon}{2} \ . \tag{4.11}$$

Taking $n \geq \max(N_1, N_2)$, Equations (4.10) and (4.11) give

$$\left| m_0 - \frac{1}{d^{2n}-d^n} \operatorname{ord} \operatorname{Res}_{\phi^n}(x_n) \right| < \epsilon$$

$\square$

4.3 Uniform Bounds on $\mathrm{MinResLoc}(\phi^n)$ and $\mathrm{Bary}(\mu_\phi)$

In this section, we study the distance between points in $\mathrm{MinResLoc}(\phi^n)$ and the Gauss point, and also the distance between points in $\mathrm{Bary}(\mu_\phi)$ and ζ_{G}. The main lemma used in this task is the following estimate on the growth of certain coefficients of Φ^n:

Lemma 4.9. *Let Φ be a normalized lift of ϕ. Let $\Phi^n = [F, G]$ be a normalized lift for the nth iterate of ϕ, where $F(X,Y) = \alpha_D X^D + ... + \alpha_0 Y^D$, $G(X,Y) = \beta_D X^D + ... + \beta_0 Y^D$ and $D = d^n$. Then*

$$\max(|\alpha_0|, |\beta_0|) \geq |\operatorname{Res}(\Phi)|^{\frac{d^n-1}{d-1}}$$
$$\max(|\alpha_{d^n}|, |\beta_{d^n}|) \geq |\operatorname{Res}(\Phi)|^{\frac{d^n-1}{d-1}}\ .$$

Proof. We observe that $|\alpha_0| = |F(0,1)|, |\alpha_D| = |F(1,0)|$ and $|\beta_0| = |G(0,1)|, |\beta_D| = |G(1,0)|$. For a pair (x,y), let $||(x,y)|| = \max(|x|,|y|)$. Then by [2] Lemma 10.1, we have

$$\begin{aligned}\max(|\alpha_0|, |\beta_0|) = ||\Phi^n(0,1)|| &\geq ||\Phi^{n-1}(0,1)||^d \cdot |\operatorname{Res}(\Phi)| \\ &\geq ||\Phi^{n-2}(0,1)||^{d^2} \cdot |\operatorname{Res}(\Phi)|^{1+d} \\ &\cdots \\ &\geq ||(0,1)|| \cdot |\operatorname{Res}(\Phi)|^{1+d+...+d^{n-1}} = |\operatorname{Res}(\Phi)|^{\frac{d^n-1}{d-1}}\ .\end{aligned}$$

A similar argument holds for $\max(|\alpha_{d^n}|, |\beta_{d^n}|)$.

□

Lemma 4.9 above gives us a bound on the size of leading and constant coefficients of the polynomials that form a normalized lift of ϕ^n. Similar bounds appeared in the proof of [24] Proposition 1.8, which gave a bound for the set $\mathrm{MinResLoc}(\phi)$. We can use the previous lemma to strengthen this bound for iterates:

Proposition 4.10. *Let $d \geq 2$ and let $R = \frac{2}{d-1}\operatorname{ordRes}(\phi)$. Fix a point $x \in \mathbb{P}^1(K)$. For any point $\zeta \in [\zeta_G, x]$, the function $\operatorname{ordRes}_{\phi^n}$ satisfies*

$$\frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta) \geq \rho(\zeta_G,\zeta) + \frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta_G) - R\ . \tag{4.12}$$

Let ξ be the unique point in $[\zeta_G, x]$ such that $\rho(\zeta_G,\xi) = \frac{2}{d-1}\operatorname{ordRes}(\phi)$. Then for each n, the function $\operatorname{ordRes}_{\phi^n}(\cdot)$ is increasing along $[\xi, x]$ as one moves away from ξ.

Proof. The proof follows [24] Proposition 1.8 closely. After a change of coördinates by some $\gamma \in \mathrm{GL}_2(\mathcal{O})$, we can assume that $x = 0$. Let $\Phi^n = [F, G]$ be a normalized lift of ϕ^n, where $D = D^n$, $F(X,Y) = a_D X^D + ... + a_0 Y^D$, $G(X,Y) = b_D X^D + ... + b_0 Y^D$, where $a_i, b_j \in \mathcal{O}$ and at least one coefficient is a unit.

Given $A \in K^\times$, let $\tau_A(z) = Az$. In [24] Proposition 1.8, Rumely shows that

$$\begin{aligned}\operatorname{ordRes}_{\phi^n}(\zeta_{0,|A|}) - {}&\operatorname{ordRes}_{\phi^n}(\zeta_{\mathrm{G}}) \\ &\geq \max\big(-2D\operatorname{ord}(a_0) + (D^2+D)\operatorname{ord}(A), -2D\operatorname{ord}(b_0) + (D^2-D)\operatorname{ord}(A), \\ &\quad -2D\operatorname{ord}(a_D) + (D-D^2)\operatorname{ord}(A), -2D\operatorname{ord}(b_D) + (-D-D^2)\operatorname{ord}(A)\big)\ .\end{aligned}$$

Using the bounds in Lemma 4.9, this gives that

$$\begin{aligned}\operatorname{ordRes}_{\phi^n}(\zeta_{0,|A|}) - {}&\operatorname{ordRes}_{\phi^n}(\zeta_{\mathrm{G}}) \\ &\geq -2D\frac{d^n-1}{d-1}\operatorname{ordRes}(\phi) + \max\big((D^2-D)\operatorname{ord}(A), (D-D^2)\operatorname{ord}(A)\big)\ .\end{aligned} \tag{4.13}$$

Restricting ourselves to $\operatorname{ord}(A) > 0$, the right side of (4.13) is

$$-2\frac{d^{2n}-d^n}{d-1}\operatorname{ordRes}(\phi) + (d^{2n}-d^n)\operatorname{ord}(A)\ ,$$

which establishes the first claim:

$$\frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta_{0,|A|}) \geq \operatorname{ord}(A) - \frac{2}{d-1}\operatorname{ordRes}(\phi) + \frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta_{\mathrm{G}})\ .$$

When $\mathrm{ord}(A) = 0$, the left hand side of (4.13) is exactly equal to 0. Thus, if $\mathrm{ord}(A)$ is chosen large enough so that the right hand side of (4.13) is positive, the function $\mathrm{ordRes}_{\phi^n}(\cdot)$ must be increasing for all larger values of $\mathrm{ord}(A)$. This is attained for

$$(D^2 - D)\,\mathrm{ord}(A) \geq \frac{2D(d^n - 1)}{d-1}\,\mathrm{ordRes}(\phi) \ ,$$

or equivalently, inserting the definition of $D = d^n$,

$$\mathrm{ord}(A) \geq \frac{2}{d-1}\,\mathrm{ordRes}(\phi) \ .$$

□

Corollary 4.11. *Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. Let $R = \frac{2}{d-1}\,\mathrm{ordRes}(\phi)$. Then for each n,*

$$\mathrm{MinResLoc}(\phi^n) \subseteq B_\rho(\zeta_G, R) \ .$$

In particular, $\mathrm{diam}(\mathrm{MinResLoc}(\phi^n)) \leq \frac{4}{d-1}\,\mathrm{ordRes}(\phi)$.

Note that this proposition and its corollary imply that the bound in Lemma 4.9 is as sharp as one would expect in general. In particular, if the bound grew more slowly, say exponentially of order n rather than order d^n, we could find a sequence of radii $R_n \to 0$ with $\mathrm{MinResLoc}(\phi^n) \subseteq B_\rho(\zeta_{\mathrm{G}}, R_n)$, which isn't true in general. Proposition 4.10 can also be used to give a lower bound for the Arakelov-Green's function:

Lemma 4.12. *Let $R = \frac{2}{d-1}\,\mathrm{ordRes}(\phi)$. Fix any type I point x. For any point $\zeta \in [\zeta_G, x]$, we have*

$$g_\phi(\zeta,\zeta) \geq \rho(\zeta_G,\zeta) + g_\phi(\zeta_G,\zeta_G) - R \ .$$

Proof. We use the convergence of the functions $\frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(x)$ given in Theorem 3.1. Let $\epsilon > 0$, and fix $\zeta \in [\zeta_{\mathrm{G}}, x]$. We may choose n large enough so that

$$\left|\frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta) - g_\phi(\zeta,\zeta)\right| < \epsilon$$
$$\left|\frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta_{\mathrm{G}}) - g_\phi(\zeta_{\mathrm{G}},\zeta_{\mathrm{G}})\right| < \epsilon \ .$$

Combining this with (4.12), we find

$$\begin{aligned} g_\phi(\zeta,\zeta) + \epsilon &\geq \frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta) \\ &\geq \rho(\zeta_{\mathrm{G}},\zeta) - R + \frac{1}{d^{2n}-d^n}\operatorname{ordRes}_{\phi^n}(\zeta_{\mathrm{G}}) \\ &\geq \rho(\zeta_{\mathrm{G}},\zeta) - R + g_\phi(\zeta_{\mathrm{G}},\zeta_{\mathrm{G}}) - \epsilon \ . \end{aligned}$$

Letting $\epsilon \to 0$ gives the result.

$\square$

We can apply this to obtain a bound on the distance of $\operatorname{Bary}(\mu_\phi)$ to ζ_{G}:

Proposition 4.13. *Let $R = \frac{2}{d-1}\operatorname{ordRes}(\phi)$ and $m_0 = \min_{x\in\boldsymbol{P}^1} g_\phi(x,x)$. Then*

$$\operatorname{Bary}(\mu_\phi) \subseteq B_\rho(\zeta_G, R + m_0 - g_\phi(\zeta_G)) \ .$$

We further have

$$\operatorname{diam}^\rho(\operatorname{Bary}(\mu_\phi)) \leq 2(R + m_0 - g_\phi(\zeta_G)) \ ,$$

where diam^ρ is the diameter in the ρ-metric. In particular, if we choose a coördinate system so that $\zeta_G \in \operatorname{Bary}(\mu_\phi)$, then

$$\operatorname{Bary}(\mu_\phi) \subseteq B_\rho(\zeta_G, R)$$

and

$$\operatorname{diam}^\rho(\operatorname{Bary}(\mu_\phi)) \leq 2R \ .$$

Proof. Let $R = \frac{2}{d-1}\operatorname{ordRes}(\phi)$, and fix $\epsilon > 0$. Let $\mathrm{Bary}(\mu_\phi)$ be the segment $[\zeta_1, \zeta_2]$, and without loss of generality assume $\rho(\zeta_{\mathrm{G}}, \zeta_2) \geq \rho(\zeta_{\mathrm{G}}, \zeta_1)$. By the preceeding lemma,

$$\rho(\zeta_{\mathrm{G}}, \zeta_2) \leq g_\phi(\zeta_2, \zeta_2) + R - g_\phi(\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}}) \ .$$

Since $\zeta_2 \in \mathrm{Bary}(\mu_\phi)$ and $g_\phi(x, x)$ is minimized on $\mathrm{Bary}(\mu_\phi)$, this gives

$$\rho(\zeta_{\mathrm{G}}, \zeta_2) \leq R + m_0 - g_\phi(\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}}) \ .$$

The last assertion follows from the fact that $g_\phi(\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}}) = m_0$ if $\zeta_{\mathrm{G}} \in \mathrm{Bary}(\mu_\phi)$. The statements about the diameters are immediate.

□

4.4 An Example: The Failure of Hausdorff Convergence

Several results in this chapter suggest that we may be able to say something stronger than the conclusion of Proposition 4.1, namely, that the sets $\mathrm{MinResLoc}(\phi^n)$ converge in the Hausdorff metric to $\mathrm{Bary}(\mu_\phi)$. However, the following example, suggested by Rob Benedetto, shows that this cannot be the case in general:

Example 4. Let $K = \mathbb{C}_p$ for some prime $p \neq 2$. Take $\phi(T) = \dfrac{T^2 - 1}{p}$. Since $\phi^{(n)}$ has even degree, $\mathrm{MinResLoc}(\phi^n)$ will always be a single type II point ([24], Theorem 0.1). However, we can show that the barycenter of μ_ϕ is a segment.

First we claim that ϕ has bad reduction. If ϕ had potential good reduction, then there would be a repelling fixed point $\zeta \in \mathbf{H}^1$, which would necessarily carry ν_ϕ-weight. However, we will show that the only point carrying ν_ϕ-weight is a non-fixed point: The classical fixed points of ϕ satisfy $T^2 - pT - 1 = 0$; by the theory of Newton polygons they both have absolute value 1, and by looking at the reduction we see that they lie off of two different directions at ζ_{G}. Thus, the tree Γ_{Fix} spanned by the classical fixed points is the union of $[\gamma_1, \infty]$ and $[\gamma_2, \infty]$, where γ_1, γ_2 are the type I finite fixed points. These two segments meet at ζ_{G}, hence ζ_{G} is a branch point of Γ_{Fix}. Moreover, because

ϕ has constant reduction we know that $\phi(\zeta_{\mathrm{G}}) \neq \zeta_{\mathrm{G}}$, and therefore $w(\zeta_{\mathrm{G}}) = v_{\Gamma_{\mathrm{Fix}}}(\zeta_{\mathrm{G}}) - 2 = 1$. Since $\deg(\phi) = 2$, this is the only weighted point.

We now turn to finding $\mathrm{Bary}(\mu_\phi)$. For this, note that the preimages of $D(0,1)$ in K are the discs $D\left(1, \frac{1}{p}\right)$ and $D\left(-1, \frac{1}{p}\right)$. Arguing inductively, we claim $\phi^{-j}(D(0,1))$ is a disjoint union of 2^j discs $D(a,r)$ with $a \equiv \pm 1 \mod \mathfrak{m}_K$, and $r = \dfrac{1}{p^j}$. To see this, suppose that $\phi^{j-1}(D(b,s)) = D(0,1)$, and consider the preimage of $D(b,s)$. The points $w \in \phi^{-1}(b)$ satisfy

$$w^2 = 1 + pb \ .$$

Hence $\phi^{-1}(D(b,s))$ are discs centered at points w with $w \equiv 1$ or $w \equiv -1$ in $\tilde{k}$. For the radius, we note that

$$\begin{aligned} \left|\phi(w+p^j) - \phi(w)\right| = \left|\frac{(w+p^j)^2 - 1}{p} - \frac{w^2 - 1}{p}\right| &= \left|\frac{2wp^j + p^{2j}}{p}\right| \\ &= \left|p^{j-1}\right| \cdot \left|2w + p^j\right| \\ &= \frac{1}{p^{j-1}} \ . \end{aligned}$$

Thus the point $w + p^j$ lies on the boundary of $\phi^{-1}(D(b,s))$, and so $\phi^{-1}(D(b,s)) = D\left(w, \dfrac{1}{p^j}\right) \sqcup D\left(w', \dfrac{1}{p^j}\right)$. Since $|w - w'| = 1$, these two discs are disjoint. Moreover, they are disjoint from any other disc in $\phi^{-j}(D(0,1))$: suppose $D(c,t) \neq D(b,s)$ is another disc with $\phi^{j-1}(D(c,t)) = D(0,1)$ whose preimages are $D(v,p^{-j}), D(v',p^{-j})$. If $D(v,p^{-j}) \cap D(w,p^{-j}) \neq \emptyset$, then these discs would necessarily be equal since they have the same diameter. In particular, $D(c,t) = \phi(D(v,p^{-j})) = \phi(D(w,p^{-j})) = D(b,s)$, which contradicts that $D(c,t) \neq D(b,s)$.

Let $\zeta_1 = \zeta_{1,p^{-1}}, \zeta_2 = \zeta_{-1,p^{-1}}$. From the above arguments, we see that $\phi^{-j}(\zeta_{\mathrm{G}})$ consists of 2^n points, half of which lie in $\mathbf{P}^1 \setminus B_{\vec{v}_\infty}(\zeta_1)$ (these are the preimages $\phi^{-j}(D(0,1)) = D(a,p^{-j})$ with $a \equiv 1 \mod \mathfrak{m}_K$) and the other half of which lie in $\mathbf{P}^1 \setminus B_{\vec{v}_\infty}(\zeta_2)^-$. Since the measures $\frac{1}{2^n}(\phi^n)^*\delta_{\zeta_{\mathrm{G}}}$ converge weakly to μ_ϕ ([13] Théorème A), it follows that $\mu_\phi(B_{\vec{v}}(\zeta_1)^-) \leq \frac{1}{2}$ for every $\vec{v} \in T_{\zeta_1} \setminus \{\vec{v}_{\zeta_{\mathrm{G}}}\}$

and $\mu_\phi(B_{\vec{v}}(\zeta_2)^-) \leq \frac{1}{2}$ for every $\vec{v} \in T_{\zeta_2} \setminus \{\vec{v}_{\zeta_{\mathrm{G}}}\}$. Hence the segment $[\zeta_1, \zeta_2]$ is contained in the barycenter. Moreover, the barycenter cannot be any larger.

The conclusion of Proposition 4.1 shows that the sets $\mathrm{MinResLoc}(\phi^n)$ approach $\mathrm{Bary}(\mu_\phi)$ in some sense, though as the above example shows we do not have Hausdorff convergence. A natural question, then, is whether the sets $\mathrm{MinResLoc}(\phi^n)$ converge to some *subset* of $\mathrm{Bary}(\mu_\phi)$. If this happens, the natural follow-up question is what dynamical significance this limit set has; the writer of this thesis does not yet have a good answer to either of these questions.

Chapter 5

Geometry of the Crucial Set

This chapter is devoted to a (coarse) geometric analysis of the asymptotic behaviour of the support of the crucial measures ν_{ϕ^n}. It is centered around proving the following theorem:

Theorem 5.1. *Assume that K has characteristic 0 and residue characteristic $p \geq 0$. Let $\phi \in K(z)$ have degree $d \geq 2$, and let $\mathcal{L}_\phi$ denote the Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric. There exists a constant $N_0 = N_0(\phi)$ depending only on ϕ so that if $n \geq N_0$ and $P \in \boldsymbol{H}^1$ is a point with $w_{\phi^n}(P) > 0$, we have*

$$\rho(P, \zeta_G) \leq 3n \log_v \mathcal{L}_\phi \ .$$

Geometrically, this theorem bounds the rate at which the crucial measures accumulate mass near type I points; in Chapter 6 it will be used to give the quantitative logarithmic equidistribution of the crucial measures.

The proof of Theorem 5.1 is based on the interplay between the action of ϕ on $\mathbf{P}^1$ and the action of ϕ on the underlying discs $D(a, r) \subseteq \mathbb{P}^1(K)$. In the first section, we will establish several preliminary lemmas. The first set of lemmas bound the distance between the root and pole of a rational map ϕ, and are built on the work of Rumely and Winburn [26]. The second set of lemmas give a quantitative, non-Archimedean generalization of a lemma of Przytycki [19], which bounds the distance between critical points and (Julia) fixed points. With these lemmas, we establish

Theorem 5.1 for each point P in $\mathrm{supp}(\nu_{\phi^n})$ by considering the various reduction types of ϕ at P according to the classification in [25] Proposition 6.1 (A).

5.1 Bounds For Roots, Poles, Critical Points, and Fixed Points

The Lipschitz constant $\mathcal{L}_\phi$ for the action of ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric $||\cdot,\cdot||$ will play an important role in this chapter. We recall that

$$\mathcal{L}_\phi := \sup_{\substack{x,y\in\mathbb{P}^1(K)\\ x\neq y}} \frac{||\phi(x),\phi(y)||}{||x,y||} \ .$$

Note that the Lipschitz constant is always at least 1: choose two points $x \neq y$ with $\phi(x) = 0, \phi(y) = \infty$. Hence $||\phi(x),\phi(y)|| = 2$, and $0 < ||x,y|| \leq 2$. In particular, $\mathcal{L}_\phi \geq \frac{||\phi(x),\phi(y)||}{||x,y||} = \frac{2}{||x,y||} \geq 1$. An upper bound for $\mathcal{L}_\phi$ is considered in work of Rumely and Winburn (see [26]), which will be discussed again below.

We also remark that the Lipschitz constant $\mathcal{L}_\phi$ is $\mathrm{GL}_2(\mathcal{O})$-invariant, in the sense that $\mathcal{L}_\phi = \mathcal{L}_{\phi^\tau}$ for any $\tau \in \mathrm{GL}_2(\mathcal{O})$. This follows from the fact that the chordal metric is $\mathrm{GL}_2(\mathcal{O})$-invariant.

Lemma 5.2. *Fix $\zeta_0 \in \boldsymbol{H}^1$ of type II, and let $\zeta_1, ...\zeta_k$ satisfy $\phi^n(\zeta_i) = \zeta_0$. Choose some $\gamma \in \mathrm{PGL}_2(K)$ with $\gamma(\zeta_G) = \zeta_0$, and let $\mathcal{L}_{\phi^\gamma}$ be the Lipschitz constant for the action of ϕ^γ on $\mathbb{P}^1(K)$ with respect to the chordal metric. Then for each $i = 1, ..., k$, we have*

$$\rho(\zeta_i, \zeta_0) \leq n \log_v \mathcal{L}_{\phi^\gamma} \ .$$

Proof. We may conjugate ϕ by an element $\gamma \in \mathrm{PGL}_2(K)$ that satisfies $\gamma(\zeta_{\mathrm{G}}) = \zeta_0$. While such a γ is not unique, it is uniquely determined up to precomposition by an element $\mathrm{PGL}_2(\mathcal{O})$. Fix an index i, and let $r_i = \rho(\zeta_0, \zeta_i)$. We can find $\tau_i \in \mathrm{PGL}_2(\mathcal{O})$ so that $\gamma \circ \tau_i(\zeta_{0,r_i}) = \zeta_i$. Replacing ϕ by $\phi^{\gamma\circ\tau_i}$, we may assume $\zeta_0 = \zeta_{\mathrm{G}}$, and $\zeta_i = \zeta_{0,r}$ for some r. By the $\mathrm{PGL}_2(K)$-invariance of ρ, it is enough to estimate $\rho(\zeta_i, \zeta_0) = \rho(\zeta_{0,r}, \zeta_{\mathrm{G}}) = -\log_v r$.

We use a description of the action of ϕ on $\mathbf{P}^1$ given in [2] Proposition 2.18. More precisely, we can find $a_1, a_2, ..., a_k \in D(0,r)$ and $b_1, ..., b_s \in D(0,1)$ so that the image in $\mathbb{P}^1(K)$ of the closed affinoid

$D(0,r) \setminus \cup D(a_i,r)^-$ under ϕ^n is the closed affinoid $D(0,1) \setminus \cup D(b_k,1)^-$. Choose two points $x, y \in D(0,1) \setminus \cup D(b_k,1)$ with $||x,y|| = 1$, and write $x = \phi^n(z), y = \phi^n(w)$ for $z, w \in D(0,r) \setminus \cup D(a_i,r)$. We find

$$\begin{aligned} 1 = |x-y| = ||x,y|| &\leq \mathcal{L}_\phi ||\phi^{n-1}(z), \phi^{n-1}(w)|| \\ &\leq \dots \\ &\leq \mathcal{L}_\phi^n ||z,w|| \\ &\leq \mathcal{L}_\phi^n \cdot r \ . \end{aligned}$$

Thus we have the lower bound

$$\mathcal{L}_\phi^{-n} \leq r \ .$$

In particular,

$$\rho(\zeta_{\mathrm{G}}, \zeta_{0,r}) = \log_v \left(\frac{1}{r}\right) \leq \log_v \mathcal{L}_\phi^n = n \log_v \mathcal{L}_\phi \ .$$

Translating back to the original map ϕ and the original point ζ_0 gives the assertion in the lemma. $\square$

We obtain an important corollary:

Corollary 5.3. *Let $\phi \in K(z)$ of degree $d \geq 2$ Then ϕ has good reduction at $\zeta = \gamma(\zeta_G)$ if and only if $\mathcal{L}_{\phi^\gamma} = 1$.*

Proof. Without loss of generality, after replaicing ϕ with ϕ^γ we can assume that $\zeta = \zeta_{\mathrm{G}}$. In [28] Theorem 2.14, Silverman shows[1] that

$$\frac{||\phi(x), \phi(y)||}{||x,y||} \leq |\operatorname{Res}(\Phi)|^{-2}$$

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[1] Rumely and Winburn [26] have given a detailed study of the Lipschitz constant of a rational map; among other things, they strengthen Silverman's bound to obtain

$$\frac{||\phi(x), \phi(y)||}{||x,y||} \leq |\operatorname{Res}(\Phi)|^{-1} \ .$$

for any normalized lift Φ and for any pair of points $x, y \in \mathbb{P}^1(K)$. In particular, we find that

$$\begin{aligned} 1 \leq \mathcal{L}_\phi &= \sup_{x,y \in \mathbb{P}^1(K)} \frac{||\phi(x), \phi(y)||}{||x, y||} \\ &\leq |\operatorname{Res}(\Phi)|^{-2} = 1 \ , \end{aligned}$$

where the final equality follows because ϕ has good reduction; hence $\mathcal{L}_\phi = 1$.

Conversely, suppose that $\mathcal{L}_\phi = 1$. Then by the preceeding lemma, if ζ_i is any preimage of ζ_{G} under ϕ^n, we have

$$\rho(\zeta_i, \zeta_{\mathrm{G}}) \leq n \cdot \log_v \mathcal{L}_\phi = 0 \ .$$

In particular, ζ_{G} is a totally ramified point which, by [2] Proposition 10.45 (A) implies that $\mu_\phi = \delta_{\zeta_{\mathrm{G}}}$ which, by [2] Proposition 10.5, implies that ϕ has good reduction. $\square$

In their work on Lipschitz constants for ϕ, Rumely and Winburn [26] define the following constants:

- The root-pole number of ϕ is given

$$\mathrm{RP}(\phi) = \min\{||\alpha, \beta|| \ : \ \alpha, \beta \in \mathbb{P}^1(K), \ \phi(\alpha) = 0, \ \phi(\beta) = \infty\} \ .$$

- The Gauss preimage radius of ϕ is

$$\mathrm{GPR}(\phi) = \min\{\mathrm{diam}_{\zeta_{\mathrm{G}}}(x) \ : \ x \in \mathbf{P}^1, \phi(x) = \zeta_{\mathrm{G}}\} \ .$$

- The Ball-mapping radius of ϕ is

$$S_0(\phi) = \sup\{0 < r \leq 1 \ : \ \text{for all } a \in \mathbb{P}^1(K), \ \phi(B(a, r)^-) \neq \mathbb{P}^1(K)\} \ .$$

They further establish the following relationship between these quantities:

Proposition 5.4. *(Rumely-Winburn, [26]) Let $\phi \in K(z)$ have degree $d \geq 1$. Then*

$$0 < \mathrm{GPR}(\phi) \leq \mathrm{RP}(\phi) \leq S_0(\phi) \leq 1 \ .$$

This, together with Lemma 5.2 gives

Lemma 5.5. *Let $\phi \in K(z)$ have degree $d \geq 1$, and let $\mathcal{L}_\phi$ denote the Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric. Then*

$$\mathcal{L}_\phi^{-n} \leq \mathrm{GPR}(\phi^n) \leq \mathrm{RP}(\phi^n) \ .$$

Proof. The conclusion of Lemma 5.2 gives that, for each ζ_i satisfying $\phi^n(\zeta_i) = \zeta_{\mathrm{G}}$, we have

$$\rho(\zeta_i, \zeta_{\mathrm{G}}) \leq n \log_v \mathcal{L}_\phi \ .$$

Inserting this into the definition of $\mathrm{diam}_{\zeta_{\mathrm{G}}}(x)$ and taking the minimum now gives the result. □

An important corollary of this that will be used many times below is that if, for some $r < 1$, the closed disc $D(0, r)$ contains a root α and a pole β for ϕ, then we can bound r below by

$$\mathcal{L}_\phi^{-n} \leq ||\alpha, \beta|| = |\alpha - \beta| \leq r \ .$$

Remark: Lemma 5.2 can also be proven using a result of Rumely and Winburn, who show ([26], Theorem 0.3) that $\mathrm{GPR}(\phi) = \mathcal{L}_\phi^{-1}$; the Lemma follows by noting that $\mathcal{L}_{\phi^n}^{-1} \geq \mathcal{L}_\phi^{-n}$.

We close this section by using $\mathcal{L}_\phi$ to estimate the expansion of ϕ with respect to the Hsia kernel $||\cdot, \cdot||$:

Lemma 5.6. *Let $\eta, \zeta \in \boldsymbol{P}^1$ be either type I or type II (though they need not have the same type). Then*

$$||\phi(\zeta), \phi(\eta)|| \leq \mathcal{L}_\phi ||\zeta, \eta|| \ .$$

Proof. The statement is true by definition if ζ, η are both type I points, so we proceed to the case when at least one of ζ, η lies in $\mathbf{H}^1$. As a matter of notation, for fixed $\xi \in \mathbf{H}^1$ and any point $\zeta_0 \in \mathbf{P}^1 \setminus \{\xi\}$, let $\vec{v}_{\xi,\zeta_0} \in T_\xi$ denote the direction at ξ with $\zeta_0 \in B_{\vec{v}_{\xi,\zeta_0}}(\xi)^-$.

Suppose first that $\zeta \in \mathbf{H}^1$ is a type II point and $\eta \in \mathbb{P}^1(K)$ is a type I point. Choose a direction

$$\vec{w} \in T_{\phi(\zeta)} \setminus \{\vec{v}_{\phi(\zeta),\zeta_{\mathrm{G}}},\ \vec{v}_{\phi(\zeta),\phi(\eta)},\ \phi_*\vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \phi_*\vec{v}_{\zeta,\eta}\}\ ,$$

as well as a direction

$$\vec{v} \in T_\zeta \setminus \{\vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \vec{v}_{\zeta,\eta}\}$$

with $\phi_*\vec{v} = \vec{w}$. Then for any type I point $x \in B_{\vec{v}}(\zeta)^-$, we have $x \wedge_{\zeta_{\mathrm{G}}} \eta = \zeta \wedge_{\zeta_{\mathrm{G}}} \eta$, hence $||\zeta, \eta|| = ||x, \eta||$; in a similar way, by our choice of $\vec{w}$ we have that $\phi(x) \wedge_{\zeta_{\mathrm{G}}} \phi(\eta) = \phi(\zeta) \wedge_{\zeta_{\mathrm{G}}} \phi(\eta)$, so that $||\phi(\zeta), \phi(\eta)|| = ||\phi(x), \phi(\eta)||$. In particular, since both x and η are type I points, we have

$$||\phi(\zeta), \phi(\eta)|| = ||\phi(x), \phi(\eta)|| \le \mathcal{L}_\phi \cdot ||x, \eta|| = \mathcal{L}_\phi \cdot ||\zeta, \eta||$$

as asserted.

Suppose now that both ζ and η are type II points. Here, we consider first the case that $\zeta = \eta$. Choose distinct directions $\vec{v}_0, \vec{v}_1 \in T_\zeta \setminus \{\vec{v}_{\zeta,\infty}, \vec{v}_{\zeta,\zeta_{\mathrm{G}}}\}$ whose images under the tangent map ϕ_* are distinct. Then for any pair of type I points $x \in B_{\vec{v}_0}(\zeta)^-, y \in B_{\vec{v}_1}(\zeta)^-$, we have that $x \wedge_{\zeta_{\mathrm{G}}} y = \zeta = \eta$, so that $||\zeta, \eta|| = ||x, y||$. Moreover, $\phi(x), \phi(y)$ lie in distinct directions away from $\phi(\zeta) = \phi(\eta)$, so that $\phi(x) \wedge_{\zeta_{\mathrm{G}}} \phi(y) = \phi(\zeta) = \phi(\eta)$, and hence $||\phi(\zeta), \phi(\eta)|| = ||\phi(x), \phi(y)||$. Therefore,

$$||\phi(\zeta), \phi(\eta)|| = ||\phi(x), \phi(y)|| \le \mathcal{L}_\phi ||x, y|| = \mathcal{L}_\phi ||\zeta, \eta||$$

as desired.

Now suppose that $\zeta \neq \eta$, but that $\phi(\zeta) = \phi(\eta)$. Choose two distinct directions

$$\vec{w}_0, \vec{w}_1 \in T_{\phi(\zeta)} \setminus \{\vec{v}_{\phi(\zeta),\zeta_{\mathrm{G}}},\ \vec{v}_{\phi(\zeta),\infty},\ \phi_*\vec{v}_{\zeta,\infty},\ \phi_*\vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \phi_*\vec{v}_{\zeta,\eta},\ \phi_*\vec{v}_{\eta,\infty},\ \phi_*\vec{v}_{\eta,\zeta_{\mathrm{G}}},\ \phi_*\vec{v}_{\eta,\zeta}\}\ ;$$

we can therefore choose directions

$$\vec{v}_0 \in T_\zeta \setminus \{\vec{v}_{\zeta,\infty},\ \vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \vec{v}_{\zeta,\eta}\}$$
$$\vec{v}_1 \in T_\eta \setminus \{\vec{v}_{\eta,\infty},\ \vec{v}_{\eta,\zeta_{\mathrm{G}}},\ \vec{v}_{\eta,\zeta}\}$$

with $\phi_*\vec{v}_0 = \vec{w}_0$ and $\phi_*\vec{v}_1 = \vec{w}_1$. Then if we choose $x \in B_{\vec{v}_0}(\zeta)^-$ and $y \in B_{\vec{v}_1}(\eta)^-$, we have $x \wedge_{\zeta_{\mathrm{G}}} y = \zeta \wedge_{\zeta_{\mathrm{G}}} \eta$, hence $||\zeta,\eta|| = ||x,y||$; similarly, since $\vec{w}_0, \vec{w}_1$ are distinct, and $\phi_*\vec{v}_i = \vec{w}_i$ for $i = 0,1$, we have that $\phi(x) \wedge_{\zeta_{\mathrm{G}}} \phi(y) = \phi(\zeta) = \phi(\eta)$, and so $||\phi(\zeta),\phi(\eta)|| = ||\phi(x),\phi(y)||$; in particular, we find

$$||\phi(\zeta),\phi(\eta)|| = ||\phi(x),\phi(y)|| \le \mathcal{L}_\phi ||x,y|| = \mathcal{L}_\phi ||\zeta,\eta|| \ .$$

The final case to consider is $\zeta \neq \eta$ and $\phi(\zeta) \neq \phi(\eta)$. The method of proof is similar to the one given above, though it is a little more technical. We begin by choosing directions

$$\vec{w}_0 \in T_{\phi(\zeta)} \setminus \{\vec{v}_{\phi(\zeta),\infty},\ \vec{v}_{\phi(\zeta),\zeta_{\mathrm{G}}},\ \vec{v}_{\phi(\zeta),\phi(\eta)},\ \phi_*\vec{v}_{\zeta,\infty},\ \phi_*\vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \phi_*\vec{v}_{\zeta,\eta}\}$$
$$\vec{w}_1 \in T_{\phi(\eta)} \setminus \{\vec{v}_{\phi(\eta),\infty},\ \vec{v}_{\phi(\eta),\zeta_{\mathrm{G}}},\ \vec{v}_{\phi(\eta),\phi(\zeta)},\ \phi_*\vec{v}_{\eta,\infty},\ \phi_*\vec{v}_{\eta,\zeta_{\mathrm{G}}},\ \phi_*\vec{v}_{\eta,\zeta}\} \ ,$$

and corresponding directions

$$\vec{v}_0 \in T_\zeta \setminus \{\vec{v}_{\zeta,\infty},\ \vec{v}_{\zeta,\zeta_{\mathrm{G}}},\ \vec{v}_{\zeta,\eta}\}$$
$$\vec{v}_1 \in T_\eta \setminus \{\vec{v}_{\eta,\infty},\ \vec{v}_{\eta,\zeta_{\mathrm{G}}},\ \vec{v}_{\eta,\zeta}\} \ .$$

with $\phi_*\vec{v}_i = \vec{w}_i$ for $i = 0,1$. Choosing type I points $x \in B_{\vec{v}_0}(\zeta)^-, y \in B_{\vec{v}_1}(\eta)^-$, we find that $x \wedge_{\zeta_{\mathrm{G}}} y = \zeta \wedge_{\zeta_{\mathrm{G}}} \eta$ and $\phi(x) \wedge_{\zeta_{\mathrm{G}}} \phi(y) = \phi(\zeta) \wedge_{\zeta_{\mathrm{G}}} \phi(\eta)$, whereby

$$||\phi(\zeta),\phi(\eta)|| = ||\phi(x),\phi(y)|| \le \mathcal{L}_\phi ||x,y|| = \mathcal{L}_\phi ||\zeta,\eta|| \ .$$

□

5.1.1 Bounds Concerning Critical Points and Periodic Points

We now prove two technical lemmas pertaining to the proximity of critical points and n-periodic points. The first is an adaptation of a lemma of Przytycki to $\mathbf{P}^1$ (see [19] Lemma 1). Fix a point $a \in \mathbb{P}^1(K)$ and a radius $r < 1$. Recall that we denote by $B(a, r)$ the closed disc of radius r around a with respect to the chordal metric, and the corresponding generalization to $\mathbf{P}^1$ is

$$\mathcal{B}(a,r)^- = \{x \in \mathbf{P}^1 \ : \ ||x, a|| < r\} \ ,$$

where $||\cdot, \cdot||$ here is the Hsia kernel relative to ζ_{G}. In a similar way, we define

$$\mathcal{D}(a,r) = \{x \in \mathbf{P}^1 \ : \ \delta(x,a)_\infty < r\} \ .$$

Note that, when $[T]_a, [T]_x \leq 1$, we have $\delta(x,a)_\infty = ||x,a||$, and so in this case we have $\mathcal{B}(a,r) = \mathcal{D}(a,r)$ (recall $r < 1$).

Lemma 5.7. *Let $\mathcal{L}_\phi$ denote a Lipschitz constant for ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric. There exists a constant $0 < A_\phi < 1$ depending only on ϕ such that the following holds: for any critical point $c \in \mathbb{P}^1(K)$ of ϕ and any $n > 0$, if $\epsilon < A_\phi \cdot \mathcal{L}_\phi^{-(n-1)}$ and $\phi^n(\mathcal{B}(c,\epsilon)^-) \cap \mathcal{B}(c,\epsilon)^- \neq \emptyset$ for some n, then $\phi^n(\mathcal{B}(c,\epsilon)^-) \subseteq \mathcal{B}(c,\epsilon)^-$.*

Proof. We claim that we can conjugate ϕ by an element of $\mathrm{PGL}_2(\mathcal{O})$ so as to assume that $c = 0$ and $|\phi(c)| \leq 1$. To see that this is possible, we consider two cases. If $c, \phi(c)$ lie in the same connected component $B_{\vec{v}_a}(\zeta_{\mathrm{G}})^- \subseteq \mathbf{P}^1 \setminus \{\zeta_{\mathrm{G}}\}$, we can lift a Möbius transformation $\widetilde{\gamma} \in \mathrm{PGL}_2(k)$ with $\gamma(a) = 0$ to a map $\gamma \in \mathrm{PGL}_2(\mathcal{O})$ which sends $B_{\vec{v}_a}(\zeta_{\mathrm{G}})^-$ to $B_{\vec{v}_0}(\zeta_{\mathrm{G}})^-$. Conjugating by γ gives the desired configuration. If $c, \phi(c)$ lie in different connected components of $\mathbf{P}^1 \setminus \{\zeta_{\mathrm{G}}\}$, then we can find an element of $\gamma \in \mathrm{PGL}_2(K)$ sending the triple $(c, \zeta_{\mathrm{G}}, \phi(c))$ to the triple $(0, \zeta_{\mathrm{G}}, 1)$ (see [2] Corollary 2.13); necessarily γ fixes ζ_{G}, hence $\gamma \in \mathrm{PGL}_2(\mathcal{O})$, and conjugation by γ achieves the desired configuration.

Writing ϕ as a Taylor series about $c = 0$, we have

$$\phi(z) = a_0 + a_2 z^2 + ... \ .$$

Let $k \geq 2$ be chosen such that a_k is the first non-zero term in this expansion. We can find $\tilde{A}_\phi(c) < 1$ depending only on ϕ, c so that, for $\epsilon < \tilde{A}_\phi(c)$ and $|z| < \epsilon < 1$, we have

$$|\phi(z) - a_0| = |a_k| \cdot |z|^k < 1 \ .$$

This 'extends' to $\mathbf{H}^1$ as

$$[\phi(T) - a_0]_z = |a_k| \cdot [T]_z^k \ , \tag{5.1}$$

for all $z \in \mathcal{D}(0,\epsilon)^- = \mathcal{B}(0,\epsilon)^-$. Since $|a_0| = |\phi(0)| \leq 1$, the inequality in (5.1) implies that $[\phi(T)]_z \leq 1$ for all $z \in \mathcal{B}(0,\epsilon)^-$; using the relation $\delta(z,a)_\infty = [T-a]_z$ (see [2] Corollary 4.2), we find

$$\begin{aligned} ||\phi(z), a_0|| = \delta(\phi(z), a_0)_\infty &= [\phi(T) - a_0]_z \\ &= |a_k| \cdot [T]_z^k \\ &\leq |a_k| \cdot [T]_z^2 \ . \end{aligned}$$

This, along with the fact that $z \in \mathcal{B}(0,\epsilon)^- = \mathcal{D}(0,\epsilon)^-$, gives

$$\begin{aligned} ||\phi(z), a_0|| &\leq |a_k| \cdot [T]_z^2 \\ &< |a_k|\epsilon^2 \ . \end{aligned}$$

Now let $z, w \in \mathcal{B}(0,\epsilon)^- = \mathcal{D}(0,\epsilon)^-$ be any type II points; by Lemma 5.6 we find

$$\begin{aligned} ||\phi^n(z), \phi^n(w)|| &\leq \mathcal{L}_\phi ||\phi^{n-1}(z), \phi^{n-1}(w)|| \\ &\leq \dots \\ &\leq \mathcal{L}_\phi^{n-1} ||\phi(z), \phi(w)|| \\ &\leq \mathcal{L}_\phi^{n-1} \cdot |a_k|\epsilon^2 \ . \end{aligned}$$

In particular, if $A_\phi(c) := \min(\tilde{A}_\phi(c), \frac{1}{|a_k|}) < 1$, then for $\epsilon < A_\phi(c) \cdot \mathcal{L}_\phi^{-(n-1)}$ we have

$$||\phi^n(z), \phi^n(w)|| \leq \mathcal{L}_\phi^{n-1} \cdot |a_k|\epsilon^2 < \epsilon \ . \tag{5.2}$$

We next claim that $\phi^n(\mathcal{B}(0,\epsilon)^-) \subseteq \mathcal{B}(0,\epsilon)^-$. Let $\phi^n(z) \in \phi^n(\mathcal{B}(c,\epsilon)^-)$ be any point in the image of $\mathcal{B}(0,\epsilon)$. If $\phi^n(z)$ is a type I point, then (5.2) immediately implies that

$$||\phi^n(z), \phi^n(0)|| < \epsilon \ ,$$

i.e. $\phi^n(z) \in \mathcal{B}(0,\epsilon)^-$.

If, on the other hand, $\phi^n(z) \in \mathbf{H}^1$, we may choose a type II point $\hat{\eta} = \phi^n(\eta) \in \phi^n(\mathcal{B}(c,\epsilon)^-$ with $||\phi^n(z), \phi^n(\eta)|| < \epsilon$; this can always be done since $\mathcal{B}(0,\epsilon)^-$ is open in the strong topology, and they type II points are dense in $\mathbf{H}^1$ under the strong topology. Further, since $\phi^n(\mathcal{B}(0,\epsilon)^-) \cap \mathcal{B}(0,\epsilon)^- \neq \emptyset$, we can choose a type II point $\hat{w} = \phi^n(w)$ lying in this intersection.

Since $\phi^n(w) \in \phi^n(\mathcal{B}(c,\epsilon)^-)$, (5.2) implies $||\phi^n(z), \phi^n(w)|| < \epsilon$, and because $\phi^n(w) \in \mathcal{B}(c,\epsilon)^-$, we find $||\phi^n(w), 0|| < \epsilon$. By the ultrametric inequality, we have

$$\begin{aligned} ||\phi^n(z), c|| &\leq \max(||\phi^n(z), \phi^n(\eta)||, ||\phi^n(\eta), \phi^n(w)||, ||\phi^n(w), 0||) \\ &< \max(\epsilon,\ \epsilon,\ \epsilon) = \epsilon \ , \end{aligned}$$

i.e. $\phi^n(z) \in \mathcal{B}(0,\epsilon)^-$. This completes the proof. □

The preceeding lemma can be thought of as a quantitative expression of the fact that if a critical point c of ϕ is very close to an n-periodic point, then both must lie in the Fatou set. The next lemma will give a similar relationship for a critical point *of the n-th iterate* ϕ^n and an n-periodic point. The idea is that if $B(c,r)$ contains a critical point c of ϕ^n *and* an n-periodic point f, then there must be some $0 \leq j \leq n-1$ such that $\phi^j(c)$ is a critical point of c, and the corresponding image $\phi^j(B(c,r))$ will contain the critical point $\phi^j(c)$ of ϕ along with the n-periodic point $\phi^j(f)$ of ϕ. We then translate the quantitative results from the preceeding lemma to $\phi^j(\mathcal{B}(c,r))$.

Lemma 5.8. *Let* $\mathcal{L}_\phi$ *denote the Lipschitz constant for the action of* ϕ *on* $\mathbb{P}^1(K)$ *with respect to the chordal metric, and let* A_ϕ *denote the constant in Lemma 5.7. Let* $n \geq 1$*. Suppose that* $B(0,r)^-$ *contains an* n*-periodic point of* ϕ *and a critical point of* ϕ^n*. If* $r < A_\phi \cdot \mathcal{L}_\phi^{-2(n-1)}$*, then* $\mathcal{B}(0,r)^- \subseteq \mathcal{F}(\phi)$*.*

Proof. Let $f \in B(0,r)^-$ denote an n-periodic point of ϕ, and consider the sets

$$\mathcal{B}(0,r)^-, \phi(\mathcal{B}(0,r)^-), \phi^2(\mathcal{B}(0,r)^-), ..., \phi^{n-1}(\mathcal{B}(0,r)^-) \ .$$

Since ϕ^n has a critical point in $B(0,r)^-$, the map ϕ must have a critical point in some $\phi^j(\mathcal{B}(0,r)^-)$, where $j = 0, 1, ..., n-1$.

We claim that $\phi^j(\mathcal{B}(0,r)^-) \subseteq \mathcal{B}(\phi^j(0), \epsilon)^-$, where $\epsilon = \mathcal{L}_\phi^j \cdot r$. To see this, choose $\phi^j(\zeta) \in \phi^j(\mathcal{B}(0,r)^-)$; if $\phi^j(\zeta)$ is a type I point, then necessarily we have

$$\begin{aligned} ||\phi^j(\zeta), \phi^j(0)|| &\leq \mathcal{L}_\phi ||\phi^{j-1}(\zeta), \phi^{j-1}(0)|| \\ &\vdots \\ &\leq \mathcal{L}_\phi^j ||\zeta, 0|| \\ &< \mathcal{L}_\phi^j \cdot r \ , \end{aligned}$$

where the final inequality follows from the fact that $\zeta \in \mathcal{B}(0,r)^-$; from this, we see $\phi^j(\zeta) \in \mathcal{B}(\phi^j(0), \epsilon)^-$. Suppose, on the other hand, that $\phi^j(\zeta) \in \mathbf{H}^1$; by the density of type II points, we can choose a type II point $\hat{\eta} \in \phi^j(\mathcal{B}(0,r)^-)$ with $||\phi^j(\zeta), \hat{\eta}|| < r$. Writing $\hat{\eta} = \phi^j(\eta)$ for some $\eta \in \mathbf{H}^1$, we have

$$\begin{aligned} ||\phi^j(\zeta), \phi^j(0)|| &\leq \max(||\phi^j(\zeta), \phi^j(\eta)||, ||\phi^j(\eta), \phi^j(0)||) \\ &\leq \max(r, ||\phi^j(\eta), \phi^j(0)||) \ . \end{aligned}$$

Applying Lemma 5.6 to the term $||\phi^j(\eta), \phi^j(0)||$ gives

$$\begin{aligned} ||\phi^j(\zeta), \phi^j(0)|| &\leq \max(r, ||\phi^j(\eta), \phi^j(0)||) \\ &\leq \max(r, \mathcal{L}_\phi^j ||\eta, 0||) \\ &< \mathcal{L}_\phi^j \cdot r \ . \end{aligned}$$

In particular, $\phi^j(\zeta) \in \phi^j(\mathcal{B}(0,r)^-)$, and we conclude that in general, $\phi^j(\mathcal{B}(0,r))^- \subseteq \mathcal{B}(\phi^j(0), \epsilon)^-$ where $\epsilon = \mathcal{L}_\phi^j \cdot r$. We note that $\epsilon < 1$: the constant r was chosen so that $r < A_\phi \cdot \mathcal{L}_\phi^{-2(n-1)}$; since $\mathcal{L}_\phi \geq 1$ and $A_\phi < 1$ (see the proof of Lemma 5.7), this implies $r < \mathcal{L}_\phi^{-2(n-1)} < \mathcal{L}_\phi^{-j}$ for each $j = 0, 1, ..., n-1$. Hence $\epsilon = \mathcal{L}_\phi^j r < 1$.

In particular, $\mathcal{B}(\phi^j(0), \epsilon)^-$ contains a critical point of ϕ and, since it contains the n-periodic point $\phi^j(f)$, it satisfies $\phi^n(\mathcal{B}(\phi^j(0), \epsilon)^-) \cap \mathcal{B}(\phi^j(0), \epsilon)^- \neq \emptyset$. Applying Lemma 5.7, if r satisfies

$$\mathcal{L}_\phi^j \cdot r < A_\phi \cdot \mathcal{L}_\phi^{-(n-1)} \ ,$$

we have $\phi^n(\mathcal{B}(\phi^j(0), \epsilon)^-) \subseteq \mathcal{B}(\phi^j(0), \epsilon)^-$. It follows that $\mathcal{B}(\phi^j(0), \epsilon)^- \subseteq \mathcal{F}(\phi^n)$, and since $\mathcal{F}(\phi^n) = \mathcal{F}(\phi)$, we find $\mathcal{B}(\phi^j(0), \epsilon)^- \subseteq \mathcal{F}(\phi)$. Now recall that $\phi^j(\mathcal{B}(0,r)^-) \subseteq \mathcal{B}(\phi^j(0), \epsilon)^-$; since $\mathcal{F}(\phi)$ is both forwards and backwards invariant, this implies

$$\mathcal{B}(0,r)^- \subseteq \phi^{-j}(\mathcal{B}(\phi^j(0), \epsilon)^- \subseteq \mathcal{F}(\phi) \ .$$

□

5.2 Bounds for Weighted Points

In this section, we establish the following theorem:

Theorem 5.9. *Let K be a complete, algebraically closed non-Archimedean valued field of characteristic 0. Let $\phi \in K(z)$ have degree $d \geq 2$.*

(A) Suppose ϕ has potential good reduction, and let $P \in \boldsymbol{H}^1$ be the point at which ϕ attains good reduction. Let Φ be a normalized lift of ϕ at ζ_G. Then

$$\rho(P, \zeta_G) \leq \frac{2}{d-1} \log_v |\operatorname{Res}(\Phi)|^{-1} \ .$$

(B) Suppose ϕ does not have potential good reduction, and let $\mathcal{L}_\phi$ denote the Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric. Then there exists a constant $1 > A_\phi > 0$ depending only on ϕ such that the following holds: Suppose that for some $n \geq 1$ and some $P \in \boldsymbol{H}^1$, we have $w_{\phi^n}(P) > 0$. Then

$$\rho(P, \zeta_G) \leq \max\left(n \log_v \mathcal{L}_\phi, 2(n-1)\log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1}\right) \ . \tag{5.3}$$

We will establish this theorem by considering separately the different types of weighted points. Our first step is to address those maps that have good reduction; this is essentially a restatement of [24] Theorem 0.1:

Proposition 5.10. *Let Φ be a normalized lift of ϕ. If ϕ has potential good reduction, and $P = \zeta$ is the point at which ϕ attains good reduction, then*

$$\rho(P, \zeta_G) \leq \frac{2}{d-1} \log_v |\operatorname{Res}(\Phi)|^{-1} \ .$$

Proof. If ϕ has good reduction, so too does ϕ^n for all n (see, e.g., [2] Theorem 10.17). In this case, the crucial set consists of a single point ζ, which is also the Minimal Resultant Locus of ϕ. In [24] Theorem 0.1, Rumely established that the Minimal Resultant Locus lies in the ball $B_\rho(\zeta_{\mathrm{G}}, \frac{2}{d-1} \operatorname{ordRes}(\phi)) = B_\rho(\zeta_{\mathrm{G}}, -\frac{2}{d-1} \log_v |\operatorname{Res}(\Phi)|)$, where Φ is a normalized lift of ϕ. Hence the asserted bound holds. $\square$

We now consider the case when ϕ does not have potential good reduction. Here, we proceed by obtaining bounds for the different types of points appearing in the crucial set.

Proposition 5.11. *[Focused Repelling Points] Let* Φ *be a normalized lift of* ϕ*, and let* P *be a focused repelling fixed point for some iterate* ϕ^n*. Then*

$$\rho(P, \zeta_G) \leq n \log_v \mathcal{L}_\phi \ .$$

Proof. If $P = \zeta_{\mathrm{G}}$, the assertion is clear, and so we assume $P \neq \zeta_{\mathrm{G}}$. Let $\vec{v}_a \in T_{\zeta_{\mathrm{G}}}$ be the direction pointing towards P, and let $\vec{v}_b \in T_P$ be any direction pointing away from ζ_{G}. Choose a type I point $S \in B_{\vec{v}_b}(P)^-$ and a map $\gamma \in \mathrm{PGL}_2(\mathcal{O})$ with $\gamma(S) = 0$; then $\gamma(B_{\vec{v}_a}(\zeta_{\mathrm{G}})^-) = B_{\vec{v}_0}(\zeta_{\mathrm{G}})^-$, and $P = \gamma^{-1}(\zeta_{0,r})$, where $\rho(\zeta_{\mathrm{G}}, P) = -\log_v r$. Replacing ϕ by ϕ^γ, we may assume $P = \zeta_{0,r}$. It suffices to find an upper bound on $-\log_v r$, which we do by considering two cases:

Case 1: Suppose that the direction $\vec{v}_\infty \in T_P$ is the direction pointing into $\Gamma_{\mathrm{FR},n}$. Denote by $\Gamma^n_{\text{Fix, b}}$ the tree in $\mathbf{H}^1$ spanned by the n-periodic points of ϕ and the points in $\phi^{-n}(b)$. By Rumely's Tree Intersection theorem ([25], Theorem 4.2) we have that

$$\Gamma_{\mathrm{FR},n} = \bigcap_{b \in \mathbb{P}^1(K)} \Gamma^n_{\text{Fix, b}} \ .$$

Hence $P \in \Gamma^n_{\text{Fix, }0} \cap \Gamma^n_{\text{Fix, }\infty}$. But since P is a focused repelling periodic point, it does not lie in Γ^n_{Fix}, and therefore there must be both a pole β and a root α of ϕ^n in $\mathbf{P}^1 \setminus B_{\vec{v}_\infty}(P)^-$, hence in $D(0, r)$. In particular,

$$||\alpha, \beta|| = |\alpha - \beta| \leq r \ .$$

Lemma 5.5 now gives

$$r \geq \mathrm{RP}(\phi^n) \geq \mathcal{L}_\phi^{-n} \ .$$

Case 2: Suppose that some finite direction $\vec{v}_a \in T_P \setminus \{\vec{v}_\infty\}$ is the direction pointing into $\Gamma_{\text{Fix, Repel}}$. By [25] Proposition 3.1(B), we know $s_{\phi^n}(P, \vec{v}_a) > 0$, and hence $\phi^n(B_{\vec{v}_a}(P)^-) = \mathbf{P}^1$. In particular, ϕ^n has a root α and a pole β in $B_{\vec{v}_a}(P)$. So $\alpha, \beta \in D(a, r)$, and

$$||\alpha, \beta|| = |\alpha - \beta| \leq r \ ,$$

and arguing as in the previous case gives

$$r \geq \mathcal{L}_\phi^{-n} \ .$$

Thus in both cases, $r \geq \mathcal{L}_\phi^{-n}$; after taking logarithms of both sides, this gives the bound asserted in the statement of the proposition.

$\square$

Proposition 5.12. *[Fixed points that are not focused repelling points] Assume that K is a complete, algebraically closed non-Archimedean valued field with characteristic 0. Let $\mathcal{L}_\phi$ denote the Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ with respect to the chordal metric. Let P be a point with $w_{\phi^n}(P) > 0$ that is fixed by ϕ^n and which is not a focused repelling periodic point. Let A_ϕ be the constant in Lemma 5.8. Then*

$$\rho(P, \zeta_G) \leq \max\left(n \log_v \mathcal{L}_\phi, 2(n-1)\log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1}\right) \ . \tag{5.4}$$

If P has a shearing direction, then $\rho(P, \zeta_G) \leq n \log_v \mathcal{L}_\phi$.

Proof. If $P = \zeta_{\mathrm{G}}$ then the assertion is clear, so assume $P \neq \zeta_{\mathrm{G}}$. Since P is not a focused repelling periodic point, it must belong to Γ^n_{Fix}, hence we can find two distinct directions $\vec{v}_a, \vec{v}_b \in T_P(\Gamma^n_{\mathrm{Fix}})$ containing type I n-periodic points a, b (resp.). Without loss of generality we can assume $\vec{v}_a \neq \vec{v}_{\zeta_{\mathrm{G}}}$.

Let $\vec{v}_c \in T_{\zeta_{\mathrm{G}}}$ be chosen so that $P \in B_{\vec{v}_c}(\zeta_{\mathrm{G}})^-$. Choose a type I n-periodic point f in $B_{\vec{v}_a}(P)$ and another type I point e in $\mathbf{P}^1 \backslash B_{\vec{v}_c}(\zeta_{\mathrm{G}})^-$. By [2], we can find $\gamma \in \mathrm{PGL}_2(\mathcal{K})$ sending the triple $[f, \zeta_{\mathrm{G}}, e]$ to $[0, \zeta_{\mathrm{G}}, \infty]$; since γ fixes ζ_{G}, we can ensure that $\gamma \in \mathrm{PGL}_2(\mathcal{O})$ (see [24] Proposition 1.1). Replacing ϕ by ϕ^γ, the $\mathrm{PGL}_2(K)$-invariance of ρ implies that it suffices to estimate $\rho(\zeta_{\mathrm{G}}, \zeta_{0,r}) = -\log_v r$. We will further assume that $r < \gamma_p^{-1}$, where $\gamma_p = |p|^{-1/(p-1)}$, for otherwise $r \geq \gamma_p^{-1}$ implies $\rho(\zeta_{\mathrm{G}}, \zeta_{0,r}) = -\log_v r \leq \frac{1}{p-1}$, which is no larger than the second term in the maximum appearing in (5.4).

Since $w_{\phi^n}(P) > 0$, P is not id-indifferent. Thus for every $\vec{v} \in T_P(\Gamma^n_{\mathrm{Fix}})$, we have (see [25] Lemma 2.1)

$$\#F_{\phi^n}(P, \vec{v}) = s_{\phi^n}(P, \vec{v}) + \#\tilde{F}_{\phi^n}(P, \vec{v}) \ . \tag{5.5}$$

We now consider two cases:

Case 1: Suppose that $s_{\phi^n}(P,\vec{v}) > 0$ for some $\vec{v} \in T_P(\Gamma^n_{\text{Fix}}) \setminus \{\vec{v}_\infty\}$. Then $\phi^n(B_{\vec{v}}(P)^-) = \mathbf{P}^1$, and hence $B_{\vec{v}}(P)^-$ contains both a pole β and a root α of ϕ^n. Arguing as in the proof of Proposition 5.11, we have

$$\mathcal{L}_\phi^{-n} \leq r \ .$$

Case 2: Otherwise, $s_{\phi^n}(P,\vec{v}_a) = 0$ for all $\vec{v}_a \in T_P(\Gamma^n_{\text{Fix}}) \setminus \{\vec{v}_\infty\}$. By (5.5) we find that $(\phi^n)_*\vec{v}_a = \vec{v}_a$ for all $\vec{v}_a \in T_P(\Gamma^n_{\text{Fix}}) \setminus \{\vec{v}_\infty\}$. Since P is not a point of good reduction, Faber's theorem ([9] Lemma 3.17) implies that *some* direction has $s_{\phi^n}(P,\vec{v}) > 0$, and we conclude $s_{\phi^n}(P,\vec{v}_\infty) > 0$. Again by (5.5) this implies that $B_{\vec{v}_\infty}(P)^-$ contains a type I fixed point of ϕ^n. We now consider two subcases:

Case 2A: We claim that, if $(\phi^n)_*\vec{v}_\infty \neq \vec{v}_\infty$, then there is a pole of ϕ^n in $D(0,r)$: the map $(\phi^n)_* : T_P \to T_P$ is surjective, and so if $(\phi^n)_*\vec{v}_\infty \neq \vec{v}_\infty$, then there is a finite direction $\vec{v}_a$ with $(\phi^n)_*\vec{v}_a = \vec{v}_\infty$. We necessarily have that $\phi^n(B_{\vec{v}_a}(P)) \supseteq B_{\vec{v}_\infty}(P)$, whereby $B_{\vec{v}_a}(P)$ contains a pole β of ϕ^n. In particular,

$$||\beta, 0|| = |\beta - 0| \leq r \ .$$

Since 0 is a root and β is a pole of ϕ^n, we argue as above to find

$$\mathcal{L}_\phi^{-n} \leq r \ .$$

Taken together, Cases 1 and 2A imply that if P has a shearing direction, then $\mathcal{L}_\phi^{-n} \leq r$, which is the final assertion of the lemma.

Case 2B: We are thus left to the case that $(\phi^n)_*\vec{v} = \vec{v}$ for each direction $\vec{v} \in T_P(\Gamma^n_{\text{Fix}})$, and $s_{\phi^n}(P,\vec{v}_\infty) > 0$ while $s_{\phi^n}(P,\vec{v}_a) = 0$ for all $\vec{v}_a \in T_P(\Gamma^n_{\text{Fix}}) \setminus \{\vec{v}_\infty\}$. Note that such a P cannot be an additively indifferent or multiplicatively indifferent point: such points have degree 1, and so by the weight formulae (see Definition 1) they must have a shearing direction in order to receive weight. Thus, in this case:

P is a *repelling* n-periodic point. $(*)$

We claim that, after conjugating by an element $\gamma \in \mathrm{PGL}_2(\mathcal{O})$, we can assume that (1) $\phi^n(0) = 0$, (2) for $\vec{v}_0 \in T_P$, we have $(\phi^n)_*\vec{v}_0 = \vec{v}_0$, and (3) there exists some $\vec{v}_a \in T_P \setminus \{\vec{v}_\infty, \vec{v}_0\}$ with $(\phi^n)_*\vec{v}_a = \vec{v}_0$. Note that condition (1) is satisfied by our initial conjugation, and (2) is therefore satisfied since we are assuming that $s_{\phi^n}(P, \vec{v}_0) = 0$ (see (5.5)). It remains to show that (3) can be obtained in a way that preserves (1) and (2).

Note that since ϕ^n has no poles in $D(0, r)$, the only direction in T_P satisfying $(\phi^n)_*\vec{w} = \vec{v}_\infty$ is the direction $\vec{w} = \vec{v}_\infty$. Hence the reduction $\widetilde{\phi^n}$ at P is a polynomial map. If condition (3) fails, then the only preimage of 0 under $\widetilde{\phi^n}$ is again zero, hence $\widetilde{\phi^n} = z^{\tilde{d}}$ where $\tilde{d} = \deg_{\phi^n}(P)$. This polynomial has non-trivial finite fixed points $\{a_1, a_2, ..., a_\ell\} \in \mathbb{P}^1(k)$, which correspond to directions $\vec{v}_{a_i} \in T_P$ with $\widetilde{\#F}_{\phi^n}(P, \vec{v}_{a_i}) > 0$. Moreover, for each $a_i \neq 0$ we can find at least one $b \in \mathbb{P}^1(k) \setminus \{a_1, ..., a_\ell\}$ with $\widetilde{\phi^n}(b) = a_i$.

Since $\#\tilde{F}_{\phi^n}(P, \vec{v}_{a_i}) > 0$, Equation (5.5) implies that $B_{\vec{v}_{a_i}}(P)^-$ contains a type I n-periodic point f_i; further, $(\phi^n)_*\vec{v}_b = \vec{v}_{a_i}$ for b chosen as above. Conjugating ϕ^n by $\gamma(z) = z + f_i$ for any fixed $i \in \{1, 2, .., k\}$ will give a map satisfying (1), (2) and (3).

With this conjugation, we find that $\phi^n(0) = 0$ and that there is some non-zero, finite direction $\vec{w} = (\gamma^{-1})_*(\vec{v}_{a_i})$ with $\phi^n(B_{\vec{w}}(P)^-) = B_{\vec{v}_0}(P)^-$ (recall that $\vec{v}_\infty$ is the only direction with surplus multiplicity, so the image of $B_{\vec{w}}(P)^-$ is not all of $\mathbf{P}^1$). In particular, $B_{\vec{w}}(P)^-$ contains a non-zero root of ϕ^n. By the non-Archimedean Rolle's theorem ([10] Application 1), there is a critical point of ϕ^n in $D(0, r \cdot \gamma_p)$, where $\gamma_p = |p|^{-1/(p-1)} > 1$ (here we are using that K has characteristic 0).

Fix $\epsilon > 0$ such that $r\gamma_p + \epsilon < 1$, and let $P' = \zeta_{0, r\cdot\gamma_p}$, and $\vec{v}_0' \in T_{P'}$ the direction towards 0. The disc $D(0, r \cdot \gamma_p + \epsilon) = B(0, r \cdot \gamma_p + \epsilon)$ contains the n-periodic point 0 of ϕ and a critical point of ϕ^n. If $r \cdot \gamma_p + \epsilon < A_\phi \mathcal{L}_\phi^{-2(n-1)}$, then by Lemma 5.8 we find $\mathcal{B}(0, r \cdot \gamma_p + \epsilon) \subseteq \mathcal{F}(\phi)$. Since $B_{\vec{v}_0'}(P')^- \subseteq \mathcal{B}(0, r\gamma_p + \epsilon)$, we have that $P \in B_{\vec{v}_0'}(P')^- \subseteq \mathcal{F}(\phi)$; but by $(*)$, $P \in \mathrm{B}_{\vec{v}_0'}(P')^-$ is a repelling periodic point, and thus lies in the Julia set, which is a contradiction. So $r \cdot \gamma_p + \epsilon \geq A_\phi \mathcal{L}_\phi^{-2(n-1)}$. Letting $\epsilon \to 0$, moving the γ_p to the other side of the inequality and taking logarithms gives the asserted bound.

$\square$

Finally, we bound the distance from ζ_{G} to weighted points that are branch points of Γ^n_{Fix} which are moved by ϕ^n:

Proposition 5.13. *[Branch points which are moved] Assume that K is a complete, algebraically closed non-Archimedean valued field with characteristic 0. Let A_ϕ be the constant from Lemma 5.8, and let $\gamma_p = |p|^{-1/(p-1)}$. Let P be a point with $w_{\phi^n}(P) > 0$ that is moved by ϕ^n. Then P is necessarily a branch point of Γ^n_{Fix}, and*

$$\rho(P, \zeta_G) \leq \max\left(n \log_v \mathcal{L}_\phi, 2(n-1)\log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1}\right) .$$

Proof. If $P = \zeta_{\mathrm{G}}$, the result is clear, and so we assume $P \neq \zeta_{\mathrm{G}}$. By Rumely's classification of points with $w_{\phi^n}(P) > 0$ ([25] Proposition 6.1), P must be a branch point of Γ^n_{Fix} which is moved by ϕ^n.

We normalize ϕ as in the proof of the preceeding proposition, so that 0 is an n-periodic point and $P = \zeta_{0,r}$ for some $r < 1$. We can further assume that $r < \gamma_p^{-1}$, where $\gamma_p = |p|^{-1/(p-1)}$.

If ϕ^n has a pole β in $D(0,r)$, then $|\beta| \leq r$; since 0 is a root of ϕ^n, we may argue using the root-pole number as in the previous proposition to find

$$\mathcal{L}_\phi^{-n} \leq ||0, \beta|| = |\beta| \leq r .$$

Suppose instead that ϕ^n has no poles in $D(0,r)$. Then for each finite direction $\vec{v}_a \in T_P \setminus \{\vec{v}_\infty\}$, we have $\infty \notin \phi^n(B_{\vec{v}_a}(P)^-)$. In particular, $\phi^n(B_{\vec{v}_a}(P)^-) \neq \mathbf{P}^1$, and so $\phi^n(B_{\vec{v}_a}(P)^-)$ must be a generalized Berkovich disc $B_{(\phi^n)_*\vec{v}_a}(\phi^n(P))^-$ (see [22] Lemma 2.1, or also [2] Proposition 9.41); moreover, this also implies that $s_{\phi^n}(P, \vec{v}_a) = 0$ for each finite direction $\vec{v}_a \in T_P \setminus \{\vec{v}_\infty\}$.

Let $Q = \phi^n(P)$. We first claim that $Q \in (P, \infty]$. If not, let $\vec{w}_{QP} \in T_Q$ be the direction at Q pointing towards P. Then $\infty \in B_{\vec{w}_{QP}}(Q)^-$. For each finite direction $\vec{v}_a \in T_P(\Gamma^n_{\mathrm{Fix}})$ containing a fixed point, [25] Lemma 2.2 implies that either $Q \in B_{\vec{v}_a}(P)^-$ or $P \in B_{(\phi^n)_*\vec{v}_a}(Q)^-$. The first condition can hold for at most one $\vec{v}_a \in T_P(\Gamma^n_{\mathrm{Fix}})$, and since P is a branch point in Γ^n_{Fix} there must be some finite direction $\vec{v}_a \in T_P(\Gamma^n_{\mathrm{Fix}})$ with $P \in B_{(\phi^n)_*\vec{v}_a}(Q)^- = \phi^n(B_{\vec{v}_a}(P)^-)$ (again, recall that $s_{\phi^n}(P, \vec{v}_a) = 0$, so the image is not all of $\mathbf{P}^1$). This implies that $(\phi^n)_*\vec{v}_a = \vec{w}_{QP}$, and so

$\infty \in \phi^n(B_{\vec{v}_a}(P)^-)$. This contradicts that ϕ^n does not have a pole in any finite direction at P, and so we conclude that $Q \in (P, \infty]$. Write $Q = \zeta_{0,s}$, with $s > r$.

We next claim that for any finite direction $\vec{v}_a \in T_P(\Gamma^n_{\mathrm{Fix}}) \setminus \{\vec{v}_\infty\}$, we have $(\phi^n)_*\vec{v}_a = \vec{w}_0$, where $\vec{w}_0 \in T_Q$ is the direction towards 0. If $f_a \in B_{\vec{v}_a}(P)^-$ is a type I n-periodic point, then $f_a \in \phi^n(B_{\vec{v}_a}(P)^-) = B_{(\phi^n)_*\vec{v}_a}(Q)^-$. Since $|f_a| \leq r < s = \mathrm{diam}_{\zeta_{\mathrm{G}}}(Q)$, we must have $(\phi^n)_*\vec{v}_a = \vec{w}_0$, where $\vec{w}_0 \in T_Q$ is the direction towards 0.

As above, let $\vec{v}_a \in T_P(\Gamma^n_{\mathrm{Fix}}) \setminus \{\vec{v}_\infty\}$, and let $U_a = B_{\vec{v}_a}(P)^-$. Then $\phi^n(U_a) = B_{\vec{w}_0}(Q)^-$, and hence $\overline{U_a} \subseteq \phi^n(U_a)$. The repelling fixed point critieria ([23] Proposition 9.3, see also [2] Theorem 10.83) implies that each U_a contains some repelling n-periodic point (of type I or of type II). In particular, $U_0 \cap \mathcal{J}(\phi) \neq \emptyset$.

Let $\vec{v}_b \in T_P(\Gamma^n_{\mathrm{Fix}}) \setminus \{\vec{v}_\infty, \vec{v}_0\}$. The fact that $0 \notin U_b$ and $0 \in \phi^n(U_b) = B_{\vec{v}_0}(Q)^-$ implies that ϕ^n has a non-zero root in $B_{\vec{v}_b}(P)^-$. By the non-Archimedean Rolle's Theorem (see [10] Application 1), ϕ^n has a critical point in the disc $D(0, r \cdot \gamma_p) = B(0, r \cdot \gamma_p)$, where $\gamma_p = |p|^{-1/(p-1)}$ (here we are using that $\mathrm{char}(K) = 0$).

Now fix $\epsilon > 0$ so that $r \cdot \gamma_p + \epsilon < 1$. Since $\vec{v}_0 \in T_P(\Gamma^n_{\mathrm{Fix}})$, we have $B_{\vec{v}_0}(P)^- \subseteq \mathcal{B}(0, r\gamma_p + \epsilon)^-$. If r satisfies

$$r\gamma_p + \epsilon < A_\phi \cdot \mathcal{L}_\phi^{-2(n-1)} ,$$

then by Lemma 5.8 we find $U_0 = B_{\vec{v}_0}(P)^- \mathcal{B}(0, r\gamma_p + \epsilon)^- \subseteq \mathcal{F}(\phi)$; but this contradicts $B_{\vec{v}_0}(P)^- \cap \mathcal{J}(\phi) = U_0 \cap \mathcal{J}(\phi) \neq \emptyset$, and so we conclude

$$r\gamma_p + \epsilon \geq A_\phi \mathcal{L}_\phi^{-2(n-1)} ;$$

letting $\epsilon \to 0$, moving the γ_p to the other side, and taking logarithms gives the asserted bound. □

Proof of Theorem 5.9. If ϕ has potential good reduction and P is the point where ϕ attains good reduction, then the first assertion of the theorem follows immediately from Proposition 5.10. If ϕ does not have potential good reduction, then for each point P in the crucial set one of the following holds: either

1. P is a focused repelling periodic point, or P is fixed and has a shearing direction. Then by Propositions 5.11 and 5.12 we have

$$\rho(P, \zeta_{\mathrm{G}}) \leq n \log_v \mathcal{L}_\phi \ .$$

2. P is fixed by ϕ^n, but has no shearing and is not a focused repelling point, or P is moved by ϕ. Then by Propositions 5.11 and 5.13 we have

$$\rho(P, \zeta_{\mathrm{G}}) \leq 2(n-1) \log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1} \ .$$

By taking maxima, the theorem follows.

□

With Theorem 5.9, we can readily establish Theorem 5.1:

Proof of Theorem 5.1. If ϕ has good reduction, then $P = \zeta_{\mathrm{G}}$ is the only weighted point of ϕ^n for each n, and $\mathcal{L}_\phi = 1$ (see Corollary 5.3). Thus $\rho(P, \zeta_{\mathrm{G}}) = 0 = 3n \log_v \mathcal{L}_\phi$ for all n, which establishes the theorem in this case.

If ϕ attains good reduction at a point $P \neq \zeta_{\mathrm{G}}$ then $\mathcal{L}_\phi > 1$ (Corollary 5.3). So, for any normalized lift Φ, it is enough to choose N_0 so that

$$\frac{2}{d-1} \log_v |\operatorname{Res}(\Phi)|^{-1} \leq 3n \log_v \mathcal{L}_\phi$$

for all $n \geq N_0$, since Proposition 5.10 implies that

$$\rho(P, \zeta_{\mathrm{G}}) \leq \frac{2}{d-1} \log_v |\operatorname{Res}(\Phi)|^{-1} \leq 3n \log_v \mathcal{L}_\phi \ .$$

Finally, if ϕ does not have potential good reduction, we know that $\mathcal{L}_\phi > 1$ (Corollary 5.3). Here we may choose N_0 sufficiently large so that

$$\max\left(n \log_v \mathcal{L}_\phi, 2(n-1) \log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1} \right) < 3n \log_v \mathcal{L}_\phi$$

for $n \geq N_0$, where A_ϕ is the constant from Theorem 5.9. This, together with Theorem 5.9, establishes the asserted bound. $\square$

Chapter 6

Logarithmic Equidistribution on $\mathbf{P}^1$

This chapter was motivated by the following question: given a family of probability measures $\{\nu_n\}$ on $\mathbf{P}^1$, for which points $a \in \mathbb{P}^1(K)$ (if any!) does the convergence

$$\int \log_v \delta(\cdot, a)_{\zeta_{\mathrm{G}}} d\nu_n \to \int \log_v \delta(\cdot, a)_{\zeta_{\mathrm{G}}} d\mu_\phi \tag{6.1}$$

hold? We observe that because $a \in \mathbb{P}^1(K)$, the kernel $\log_v \delta(\cdot, a)_\zeta$ is no longer continuous: it has a singularity at the point $x = a$ in the same way that the function $\log x$ on the real line has a singularity at $x = 0$ (see Figure 6.1 below). We will say that a family of measures $\{\nu_n\}$ satisfies a logarithmic equidistribution condition if (6.1) holds.

The property in (6.1) has several applications: in potential theory, (6.1) says that the potential functions $u_{\nu_n}(\cdot, \zeta_{\mathrm{G}})$ converge pointwise to $u_{\mu_\phi}(\cdot, \zeta_{\mathrm{G}})$. In dynamics, if (6.1) holds for each critical point $c \in \mathbb{P}^1(K)$, a result of Okuyama implies that the Lyapunov exponent $L(\phi)$ of ϕ can be approximated in terms of the measures ν_n ([18], Lemma 3.1).

In this chapter, we discuss a technique for extending weak convergence results for families of measures – such as the equidistribution of the crucial measures given in Theorem 3.11 – to logarithmic equidistribution results. We will apply this technique to two families of measures, in each case giving explicit estimates on the error terms. In particular, we show that (6.1) holds for the crucial measures ν_{ϕ^n} (Theorem 6.1) and for the pullbacks $\frac{1}{d^n}(\phi^n)^*\nu$ of measures having bounded potentials (Theorem 6.2).

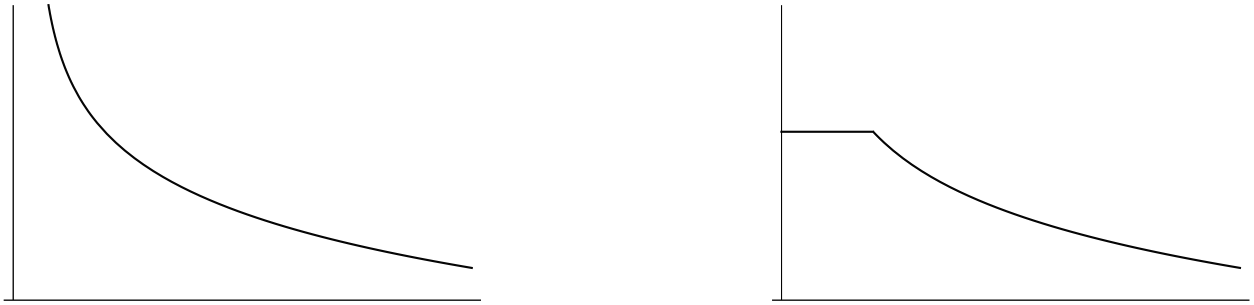

Figure 6.1: The graph on the left shows the behaviour of $\log_v \delta(\cdot, a)_{\zeta_{\mathrm{G}}}$ on any path terminating at a (the origin). The graph on the right shows the behaviour of the truncation $\log_v \delta(\cdot, a_\epsilon)_{\zeta_{\mathrm{G}}}$.

The primary tool used in this chapter is a type of truncation: for any $a \in \mathbb{P}^1(K)$ (or more generally, any $a \in \mathbf{P}^1$) and any $\epsilon \in [\mathrm{diam}_{\mathrm{G}}(a), 1]$, let a_ϵ be the point on the segment $[a, \zeta_{\mathrm{G}}]$ with $\mathrm{diam}_{\mathrm{G}}(a_\epsilon) = \epsilon$. Then the function $\log_v \delta(\cdot, a_\epsilon)_{\zeta_{\mathrm{G}}}$ agrees with $\log_v \delta(\cdot, a)_{\zeta_{\mathrm{G}}}$ off of the segment $[a, a_\epsilon]$, and on $[a, \zeta_{\mathrm{G}}]$, $\log_v \delta(\cdot, a_\epsilon)_{\zeta_{\mathrm{G}}}$ is the second function shown in Figure 6.1.

Using the truncation, we decompose the expression in (6.1) as

$$\int \log_v \delta(z, a)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)(z) = \int \log_v \delta(z, a)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, a_\epsilon)_{\zeta_{\mathrm{G}}} d\nu_n(z) \tag{6.2}$$

$$+ \int \log_v \delta(z, a_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)(z) \tag{6.3}$$

$$+ \int \log_v \delta(z, a_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, a)_{\zeta_{\mathrm{G}}} d\mu_\phi(z)\ . \tag{6.4}$$

We control each of these pieces as follows: if ϵ, n are chosen appropriately – so that the ϵ-ball around a has no ν_n-mass – then the integral in (6.2) is 0. Next, since the truncation function $\log_v \delta(z, a_\epsilon)_{\zeta_{\mathrm{G}}}$ is continuous on $\mathbf{P}^1$, classical equidistribution results for the measures ν_n can be used to estimate (6.3). Finally, using an explicit estimate on the μ_ϕ-measure of the ϵ-ball around a, we show that (6.4) is essentially bounded above by ϵ (see Proposition 6.4 for the precise statement).

We first apply this technique to show that the crucial measures satisfy the logarithmic equidistribution condition given in (6.1). Recall that a function $f : \mathbf{P}^1 \to \mathbb{R}$ is said to be Hölder continuous if there exist constants $0 < M$ and $0 < \alpha$ such that

$$|f(z) - f(w)| \leq M \mathrm{d}_{\mathbf{P}^1}(z, w)^{\alpha} \ .$$

The constants M and α are called the Hölder constant and exponent (resp.).

Theorem 6.1. *Assume that K has characteristic 0 and residue characteristic $p \geq 0$, and let $\phi \in K(Z)$ have degree $d \geq 2$. Let $\mathcal{L}_{\phi}$ denote the Lipschitz constant of ϕ, and let α denote the Hölder exponent of $u_{\phi}(\cdot, \zeta_G)$. Then for n sufficiently large (depending only on ϕ and K), and for any $\zeta \in \boldsymbol{P}^1$, the measures ν_{ϕ^n} and μ_{ϕ} satisfy*

$$\left| \int \log_v \delta(z, \zeta)_{\zeta_G} d \left(\nu_{\phi^n} - \mu_{\phi} \right)(z) \right| = O\left(\frac{n}{d^n} + \frac{1}{(\mathcal{L}_{\phi}^{\alpha})^n} \right) \ .$$

Here, the big-O constant is independent of ζ.

The proof of this theorem depends heavily on the results of Chapter 5. A quantitative version of this result is given in Theorem 6.6 below, where the error constant is given in terms of a constant C_{ϕ} depending only on ϕ, the Lipschitz constant $\mathcal{L}_{\phi}$, and the Hölder constant and exponent of the potential function $u_{\phi}(\cdot, \zeta_{\mathrm{G}})$.

Second, we are able to give a logarithmic equidistribution result for pullbacks of probability measures ν on $\mathbf{P}^1$ with bounded potentials:

Theorem 6.2. *Let K be a complete, algebraically closed non-Archimedean valued field, and let $\phi \in K(z)$ have degree $d \geq 2$. Fix $\zeta \in \boldsymbol{P}^1$, and let ν be any probability measure on $\boldsymbol{P}^1$ with bounded potentials. Define $\nu_n = \frac{1}{d^n}(\phi^n)^* \nu$. Then*

$$\left| \int \log_v \delta(z, \zeta)_{\zeta_G} d(\nu_n - \mu_{\phi}) \right| = O\left(\frac{1}{d^n} \right) \ .$$

Here, the big-O constant depends only on the Lipschitz constant of ϕ and the potential function $u_{\nu}(\cdot, \zeta_G)$ of the measure ν, and is independent of the point $\zeta \in \boldsymbol{P}^1$.

We will again give a quantitative version of this result in Proposition 6.8 below. The classical equidistribution of pullbacks of non-exceptional type I points over number fields is a consequence of the equidistribution of points of small heights given in [3], [12], and [1, 6]. Favre and Rivera-Letelier [13] have also shown that the normalized pullbacks of a probability measure ν on $\mathbf{P}^1$ also converge weakly to μ_ϕ, provided that ν does not charge the exceptional set of ϕ. Our proof of Theorem 6.2 builds on their approach, first showing the result when $\nu = \delta_\xi$ for a fixed $\xi \in \mathbf{H}^1$, and then deriving the general case.

6.1 General Preparatory Results

We begin with two preparatory results that will play a role in proving each of the logarithmic equidistribution results stated in the introduction of this chapter. We begin with a lemma modeled on a result of Favre and Rivera-Letelier ([13] Proposition 3.3). Recall that $q_v > 1$ is the normalizing constant for the logarithm $\log_v$, so that $\log_v(q_v) = 1$.

Lemma 6.3. *Let ν be a positive Borel measure with Hölder continuous potentials (with respect to the small metric $d_{\boldsymbol{P}^1}$), and let M, α denote the Hölder constant and exponent (resp.) for $u_\nu(z, \zeta_G)$. Given $\zeta \in \boldsymbol{P}^1$ with $\mathrm{diam_G}(\zeta) \in \left(0, \frac{1}{q_v}\right)$, let $\vec{v}_{\zeta_G} \in T_\zeta$ denote the direction towards ζ_G. Then for any $\vec{v} \in T_\zeta \setminus \{\vec{v}_{\zeta_G}\}$, we have*

$$\nu(B_{\vec{v}}(\zeta)^-) \le M(q_v - 1)^\alpha \,\mathrm{diam_G}(\zeta)^\alpha \ .$$

In particular, ν does not charge type I points.

Proof. By our assumption $\mathrm{diam_G}(\zeta) \in \left(0, \frac{1}{q_v}\right)$, we have that

$$1 = -\log_v\left(\frac{1}{q_v}\right) < -\log_v \mathrm{diam_G}(\zeta) = \rho(\zeta, \zeta_\mathrm{G}) \ ,$$

i.e. $1 < \rho(\zeta, \zeta_\mathrm{G})$. Let $\hat{\zeta}$ be the unique point in $[\zeta, \zeta_\mathrm{G}]$ with

$$\rho(\hat{\zeta}, \zeta_\mathrm{G}) = \rho(\zeta, \zeta_\mathrm{G}) - 1 \ .$$

We record for later use that this translates into a statement about the diameters of $\hat{\zeta}$ and ζ, namely

$$\mathrm{diam_G}(\hat{\zeta}) = q_v \cdot \mathrm{diam_G}(\zeta) \ . \tag{6.5}$$

Let $\chi(z) := -\log_v \delta(z,\zeta)_{\hat{\zeta}} - \log_v ||\hat{\zeta},\hat{\zeta}||$; this function is constant on branches off of $[\zeta,\hat{\zeta}]$, and in particular

- $\chi(z) = 1$ for all $z \in \mathbf{P}^1 \setminus B_{\vec{v}_G}(\zeta)^-$. To see this, we compute

$$\begin{aligned}
\chi(\zeta) &= -\log_v \delta(\zeta,\zeta)_{\hat{\zeta}} - \log_v ||\hat{\zeta},\hat{\zeta}|| \\
&= -\log_v \frac{||\zeta,\zeta||}{||\zeta,\hat{\zeta}||^2} - \log_v ||\hat{\zeta},\hat{\zeta}|| \\
&= -\log_v ||\zeta,\zeta|| + 2\log_v ||\hat{\zeta},\hat{\zeta}|| - \log_v ||\hat{\zeta},\hat{\zeta}|| \qquad (\text{using } \hat{\zeta} \in [\zeta,\zeta_{\mathrm{G}}]) \\
&= \rho(\zeta,\zeta_{\mathrm{G}}) - \rho(\hat{\zeta},\zeta_{\mathrm{G}}) = 1 \ .
\end{aligned}$$

- $\chi(z) = 0$ on $\mathbf{P}^1 \setminus B_{\vec{v}_\zeta}(\hat{\zeta})^-$, where $\vec{v}_\zeta \in T_{\hat{\zeta}}$ is the direction towards ζ. To see this, we note that

$$\chi(\hat{\zeta}) = -\log_v \delta(\hat{\zeta},\zeta)_{\hat{\zeta}} - \log_v ||\hat{\zeta},\hat{\zeta}|| = \log_v ||\hat{\zeta},\hat{\zeta}|| - \log_v ||\hat{\zeta},\hat{\zeta}|| = 0 \ .$$

  In particular, this implies $\chi(\zeta_{\mathrm{G}}) = 0$.

We now estimate the ν-mass in $B_{\vec{v}}(\zeta)^-$ for $\vec{v} \in T_\zeta \setminus \{\vec{v}_G\}$. Recall that $\Delta(\xi) = \delta_\zeta - \delta_{\hat{\zeta}}$, hence

$$\begin{aligned}
\nu(B_{\vec{v}}(\zeta)^-) \leq \int \chi d\nu &= \int \chi d(\nu - \delta_{\zeta_{\mathrm{G}}}) + \chi(\zeta_{\mathrm{G}}) \\
&= \int \chi d\Delta(-u_\nu(\cdot,\zeta_{\mathrm{G}})) \\
&= \int -u_\nu(\cdot,\zeta_{\mathrm{G}}) d\Delta(\chi) \\
&= -u_\nu(\zeta,\zeta_{\mathrm{G}}) + u_\nu(\hat{\zeta},\zeta_{\mathrm{G}}) \\
&\leq M \mathrm{d}_{\mathbf{P}^1}(\zeta,\hat{\zeta})^\alpha \ .
\end{aligned} \tag{6.6}$$

We are left to estimate $\mathrm{d}_{\mathbf{P}^1}(\zeta, \hat{\zeta})$. For this, we recall from (6.5) that $\mathrm{diam}_{\mathrm{G}}(\hat{\zeta}) = q_v \,\mathrm{diam}_{\mathrm{G}}(\zeta)$, thus

$$\begin{aligned} \mathrm{d}_{\mathbf{P}^1}(\zeta, \hat{\zeta}) &= 2\,\mathrm{diam}_{\mathrm{G}}(\zeta \wedge_{\mathrm{G}} \hat{\zeta}) - \mathrm{diam}_{\mathrm{G}}(\zeta) - \mathrm{diam}_{\mathrm{G}}(\hat{\zeta}) \\ &= \mathrm{diam}_{\mathrm{G}}(\hat{\zeta}) - \mathrm{diam}_{\mathrm{G}}(\zeta) \\ &= (q_v - 1)\,\mathrm{diam}_{\mathrm{G}}(\zeta) \ . \end{aligned}$$

Inserting this into (6.6) gives the asserted inequality. ☐

The following Proposition is, in some sense, an integral version of the preceeding lemma.

Proposition 6.4. *Let ν denote a positive Borel measure with Hölder continuous potentials (with respect to the small metric $\boldsymbol{d}_{\boldsymbol{P}^1}$), and let M, α denote the Hölder constant and exponent (resp.) for $u_\nu(z, \zeta_G)$.*

Fix $\zeta \in \boldsymbol{P}^1$ with $\mathrm{diam}_{\zeta_G}(\zeta) \in [0, \frac{1}{q_v})$. Let $\epsilon \in \left[\mathrm{diam}_{\zeta_G}(\zeta), \frac{1}{q_v}\right)$, and let ζ_ϵ denote the unique point on $[\zeta, \zeta_G]$ with $\mathrm{diam}_{\mathrm{G}}(\zeta_\epsilon) = \epsilon$. Then for each $\zeta \in \boldsymbol{P}^1$,

$$\left| \int \log_v \delta(z, \zeta_\epsilon)_{\zeta_G} - \log_v \delta(z, \zeta)_{\zeta_G} d\nu(z) \right| \leq \frac{M(q_v - 1)^\alpha}{\alpha |\ln(q_v)|} (\epsilon^\alpha - \mathrm{diam}_{\mathrm{G}}(\zeta)^\alpha) \ . \tag{6.7}$$

Proof. In the case that $\epsilon = \mathrm{diam}_{\mathrm{G}}(\zeta)$, the integrand in (6.7) is 0, as is the upper bound. We may therefore assume that $\mathrm{diam}_{\mathrm{G}}(\zeta) < \epsilon$.

Let $\vec{v} \in T_{\zeta_\epsilon}$ be the direction towards ζ, and note that the integrand is zero on $U = \mathbf{P}^1 \setminus B_{\vec{v}}(\zeta_\epsilon)^-$, so we must estimate $\left| \int_U \log_v \delta(z, \zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, \zeta)_{\zeta_{\mathrm{G}}} d\nu(z) \right|$. Let $\epsilon = \epsilon_0 > \epsilon_1 > \epsilon_2 > ... > \epsilon_N = \mathrm{diam}_{\mathrm{G}}(\zeta)$ be a partition of the interval $[\mathrm{diam}_{\mathrm{G}}(\zeta), \epsilon]$. For each $k = 0, 1, ..., N-1$, let ζ_k denote the point on the segment $[\zeta, \zeta_\epsilon]$ with $\mathrm{diam}_{\mathrm{G}}(\zeta_k) = \epsilon_k$. Let $\vec{v}_k \in T_{\zeta_k}$ denote the unique direction towards ζ_{k+1}. We will sometimes also write $\mathrm{diam}_{\mathrm{G}}(\zeta) = \epsilon_\zeta$.

Since $\zeta_\epsilon = \zeta_0$ and $\zeta = \zeta_N$, we can rewrite the integral $\int_U \log_v \delta(z, \zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, \zeta)_{\zeta_{\mathrm{G}}} d\nu(z)$ as a telescoping sum

$$S(\{\epsilon_k\}) := \sum_{k=0}^{N-1} \int_U \log_v \delta(z, \zeta_k)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, \zeta_{k+1})_{\zeta_{\mathrm{G}}} d\nu(z) \ .$$

Each integrand is bounded above by $\log_v \epsilon_k - \log_v \epsilon_{k+1}$ on the ball $B_{\vec{v}_k}(\zeta_k)^-$, and is zero off of this ball. In particular, we have

$$S(\{\epsilon_k\}) \leq \sum_{k=0}^{N-1} (\log_v \epsilon_k - \log_v \epsilon_{k+1}) \nu(B_{\vec{v}_k}(\zeta_k)^-) \ . \tag{6.8}$$

By the preceeding lemma,

$$\nu(B_{\vec{v}_k}(\zeta_k)^-) \leq M(q_v - 1)^\alpha \epsilon_k^\alpha$$

where M, α are the Hölder constant and exponent for ϕ with respect to the small metric $\mathbf{d_{P^1}}$. Inserting this into (6.8) gives

$$S(\{\epsilon_k\}) \leq M(q_v - 1)^\alpha \sum_{k=0}^{N-1} (\log_v \epsilon_k - \log_v \epsilon_{k+1}) \epsilon_k^\alpha \ .$$

Making a change of variables $\eta_k = \epsilon_k^\alpha$, we find

$$S(\{\epsilon_k\}) \leq \frac{M(q_v - 1)^\alpha}{\alpha} \sum_{k=0}^{N-1} (\log_v \eta_k - \log_v \eta_{k+1}) \eta_k \ . \tag{6.9}$$

Applying summation by parts to the (negative of the) above sum gives

$$\sum_{k=0}^{N-1} (\log_v \eta_k - \log_v \eta_{k+1}) \eta_k = (\log_v \eta_0)\eta_0 - (\log_v \eta_N)\eta_N - \sum_{k=0}^{N-1} \log_v \eta_{k+1} (\eta_k - \eta_{k+1}) \ . \tag{6.10}$$

Suppose now that $\mathrm{diam}_{\mathrm{G}}(\zeta) > 0$. Let $||\epsilon_k|| = \sup_{k=1,\ldots,N} |\epsilon_k - \epsilon_{k+1}|$ denote the mesh of the partition $\{\epsilon_k\}$. Taking the limit as $||\epsilon_k|| \to 0$, the expression in (6.10) becomes a definite integral:

$$\begin{aligned}
\lim_{||\eta_k|| \to 0} \sum_{k=0}^{N-1} (\log_v \eta_{k+1} - \log_v \eta_k) \eta_k &= (\alpha \cdot \log_v \epsilon_N)\epsilon_N^\alpha - (\alpha \cdot \log_v \epsilon_0)\epsilon_0^\alpha - \int_{\epsilon_0^\alpha}^{\epsilon_N^\alpha} \log_v x \ dx \\
&= \alpha((\log_v \epsilon_N)\epsilon_N - (\log_v \epsilon_0)\epsilon_0) - \left(x \log_v x - \frac{1}{\ln(q_v)} x \right) \Bigg|_{\epsilon_0^\alpha}^{\epsilon_N^\alpha} \\
&= \frac{1}{\ln(q_v)} (\epsilon_0^\alpha - \epsilon_N^\alpha) \ .
\end{aligned} \tag{6.11}$$

Using the fact that $||\epsilon_k|| \to 0$ if and only if $||\eta_k|| \to 0$, the estimates in (6.9), (6.10) and (6.11) give

$$\begin{aligned}\left|\int_U \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\nu(z)\right| &= \lim_{||\epsilon_k||\to 0} |S(\{\epsilon_k\})| \\ &\leq \left|\frac{M(q_v-1)^\alpha}{\alpha} \lim_{||\eta_k||\to 0} \sum_{k=0}^{N-1} (\log_v \eta_k - \log_v \eta_{k+1})\eta_k\right| \\ &= \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}(\epsilon_0^\alpha - \epsilon_N^\alpha)\ .\end{aligned}$$

In the case that $\mathrm{diam}_{\mathrm{G}}(\zeta) = 0$, for every $\delta > 0$ let ζ_δ be the unique point on $[\zeta, \zeta_\epsilon]$ with $\mathrm{diam}_{\mathrm{G}}(\zeta_\delta) = \delta$. If we take partitions $\{\epsilon_k\}$ of the smaller interval $[\delta, \epsilon]$, the above estimates imply

$$\begin{aligned}\left|\int_U \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\delta)_{\zeta_{\mathrm{G}}} d\nu(z)\right| &= \lim_{||\epsilon_k||\to 0} |S(\{\epsilon_k\})| \\ &\leq \left|\frac{M(q_v-1)^\alpha}{\alpha} \lim_{||\eta_k||\to 0} \sum_{k=0}^{N-1} (\log_v \eta_{k+1} - \log_v \eta_k)\eta_k\right| \\ &= \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}(\epsilon^\alpha - \delta^\alpha)\ . \qquad (6.12)\end{aligned}$$

The integrand $\log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\delta)_{\zeta_{\mathrm{G}}}$ is non-negative on U and is non-decreasing as $\delta \to 0$. By the monotone convergence theorem, taking the limit as $\delta \to 0 = \mathrm{diam}_{\mathrm{G}}(\zeta)$ and applying (6.12) gives

$$\begin{aligned}\left|\int_U \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\nu(z)\right| &= \lim_{\delta\to 0}\left|\int_U \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\delta)_{\zeta_{\mathrm{G}}} d\nu(z)\right| \\ &\leq \lim_{\delta\to 0} \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|} \cdot (\epsilon^\alpha - \delta^\alpha) \\ &= \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}\epsilon^\alpha\end{aligned}$$

which is the asserted bound in the case $\mathrm{diam}_{\mathrm{G}}(\zeta) = 0$. □

6.2 Logarithmic Equidistribution of the Crucial Measures

In this section, we prove the logarithmic equidistribution of the crucial measures. We begin with the following lemma:

Lemma 6.5. *Fix $\zeta \in \boldsymbol{H}^1$. There exists a constant C_ϕ depending only on ϕ so that for each $n \geq 1$, we have*

$$\left| \int \log_v \delta(z,\zeta)_{\zeta_G} d(\nu_{\phi^n} - \mu_\phi)(z) \right| \leq \frac{8C_\phi + 56\rho(\zeta,\zeta_G)}{d^n - 1} .$$

Proof. Note that the integrand $f(z) := \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}}$ is in $\mathrm{CPA}(\Gamma)$ for $\Gamma = [\zeta, \zeta_{\mathrm{G}}]$. We further observe that

- $|\Delta|(f) = |\delta_\zeta| + |\delta_{\zeta_{\mathrm{G}}}| \leq 2$.
- $\max_\Gamma |f| = \max_{z\in[\zeta,\zeta_{\mathrm{G}}]} |\log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}}| = \rho(\zeta,\zeta_{\mathrm{G}})$.
- $R_\Gamma = \rho(\zeta,\zeta_{\mathrm{G}})$, the radius of a ball for which $\Gamma \subseteq B(\zeta_{\mathrm{G}}, R_\Gamma)$.
- $D_\Gamma = 12$. Recall that D_Γ was computed in Lemma 3.20 as

  $$D_\Gamma = K(\Gamma) \cdot \left(\sum_{P\in\Gamma} |v(P) - 2| + (E_\Gamma + 1) \max_{P\in\Gamma} v(P) \right) .$$

  Here, $v(P) = 2$ for each interior point of $\Gamma = [\zeta, \zeta_{\mathrm{G}}]$ and $v(P) = 1$ for each endpoint. The constant E_Γ counts the number of edges in Γ (introduced in Proposition 3.19), which in our case is 1. Finally, $K(\Gamma)$ counts the number of connected components that can arise by removing a connected subgraph $\Gamma_0 \subseteq \Gamma$, which for a segment can be taken as $K(\Gamma) = 2$ (see Lemma 3.15). Taking this together, we find that $D_\Gamma = 12$.

Inserting these estimates into the bound in Theorem 3.11, and using the estimate

$$\max(C_\phi, \rho(\zeta,\zeta_{\mathrm{G}})) \leq C_\phi + \rho(\zeta,\zeta_{\mathrm{G}})$$

gives

$$\left| \int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d(\nu_{\phi^n} - \mu_\phi)(z) \right| \leq \frac{8C_\phi + 56\rho(\zeta,\zeta_{\mathrm{G}})}{d^n - 1} .$$

□

We are now ready to give a quantitative version of the logarithmic equidistribution of the crucial measures, which immediately implies Theorem 6.1 in the introduction. Note that the error bound is independent of the point $\zeta \in \mathbb{P}^1(K)$:

Theorem 6.6. *Let K be a complete, algebraically closed non-Archimedean valued field of characteristic 0. Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. Let $\mathcal{L}_\phi$ denote a Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ in the chordal metric and let C_ϕ be the constant from Lemma 6.5. Let M, α be the Hölder constant and exponent (resp.) for the potential function $u_\phi(z, \zeta_G)$ with respect to the metric $\boldsymbol{d}_{\boldsymbol{P}^1}$.*

For n sufficiently large (depending only on ϕ and K) and for any $\zeta \in \boldsymbol{P}^1$, we have

$$\left| \int \log_v \delta(z,\zeta)_{\zeta_G} d(\nu_{\phi^n} - \mu_\phi) \right| \leq \frac{168 n \log_v \mathcal{L}_\phi + 8C_\phi}{d^n - 1} + \frac{M(q_v - 1)^\alpha}{\alpha |\ln(q_v)|} \mathcal{L}_\phi^{-n\alpha} \ . \tag{6.13}$$

Proof. If ϕ has potential good reduction, then $\nu_{\phi^n} = \mu_\phi$, and the bound in (6.13) trivially holds. We therefore assume that ϕ has bad reduction, and in particular, by Corollary 5.3, we have that $\mathcal{L}_\phi > 1$.

By Theorem 5.9, we find that a point P with $w_{\phi^n}(P) > 0$ for some n must satisfy

$$\rho(P, \zeta_{\mathrm{G}}) \leq \max \left(n \log_v \mathcal{L}_\phi, 2(n-1)\log_v \mathcal{L}_\phi - \log_v A_\phi + \frac{1}{p-1} \right) \ .$$

We can find a constant $N_0 = N_0(\phi)$ such that, for $n \geq N_0$ and $w_{\phi^n}(P) > 0$, we have

$$\rho(P, \zeta_{\mathrm{G}}) \leq 3n \log_v \mathcal{L}_\phi \ .$$

Increasing N_0 if necessary, we can also assume that $\mathcal{L}_\phi^{-3n} < \frac{1}{q_v}$ for $n \geq N_0$; note that this additional constraint depends only on ϕ and K.

Fix $\zeta \in \mathbf{H}^1$ and $n \geq N_0$. If $\rho(\zeta, \zeta_{\mathrm{G}}) \leq 3n \log_v \mathcal{L}_\phi$, then we may apply Lemma 6.5 to find

$$\left| \int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d(\nu_{\phi^n} - \mu_\phi) \right| \leq \frac{8C_\phi + 168 n \log_v \mathcal{L}_\phi}{d^n - 1} \ , \tag{6.14}$$

which is stronger than the bound in (6.13).

If $\rho(\zeta, \zeta_{\mathrm{G}}) > 3n \log_v \mathcal{L}_\phi$, then let ζ_ϵ denote the point on the path $[\zeta, \zeta_{\mathrm{G}}]$ with $\rho(\zeta_\epsilon, \zeta_{\mathrm{G}}) = 3n \log_v \mathcal{L}_\phi$. More explicitly, $\epsilon := \mathrm{diam}_{\mathrm{G}}(\zeta_\epsilon) = \mathcal{L}_\phi^{-3n}$. Recalling the decomposition of (6.1) given in the introduction, we rewrite our integral as

$$\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d(\nu_{\phi^n} - \mu_\phi) = \int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d\nu_{\phi^n} \tag{6.2}$$

$$+ \int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_{\phi^n} - \mu_\phi) \tag{6.3}$$

$$+ \int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\mu_\phi\ . \tag{6.4}$$

Since $\rho(\zeta,\zeta_{\mathrm{G}}), \rho(\zeta_\epsilon,\zeta_{\mathrm{G}}) \geq 3n\log_v \mathcal{L}_\phi$, Theorem 5.9 guarantees that ν_{ϕ^n} does not charge the segment $[\zeta,\zeta_\epsilon]$, hence (6.2) is zero. Applying Lemma 6.5 to (6.3), we find that

$$\left|\int \log_v \delta(z,\zeta_\epsilon) d(\nu_{\phi^n} - \mu_\phi)\right| \leq \frac{8C_\phi + 168n\log_v \mathcal{L}_\phi}{d^n - 1}\ .$$

Finally, by Proposition 6.4 the term (6.4) can be bounded as

$$\left|\int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\mu_\phi\right| \leq \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}(\mathcal{L}_\phi^{-n\alpha} - \mathrm{diam}_{\mathrm{G}}(\zeta)^\alpha)\ .$$

Combining these gives

$$\left|\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d(\nu_{\phi^n} - \mu_\phi)\right| \leq \frac{168n\log_v \mathcal{L}_\phi + 8C_\phi}{d^n - 1} + \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}(\mathcal{L}_\phi^{-n\alpha} - \mathrm{diam}_{\mathrm{G}}(\zeta)^\alpha) \tag{6.15}$$

The inequalities in (6.14) and (6.15) imply the bound asserted in the statement of the theorem.

□

6.3 Logarithmic Equidistribution of Pullbacks of Point Masses and Measures in $\mathbf{H}^1$

In this section, we will give a logarithmic equidistribution result for pullbacks of point masses δ_ζ, where $\zeta \in \mathbf{H}^1$, and extend this to establish logarithmic equidistribution for pullbacks of arbitrary measures with bounded potentials.

We will need the following technical lemma, which gives an explicit bound for the supremum of the Arakelov-Green's function attached to the measure $\frac{1}{d}\phi^*\delta_{\zeta_0}$:

Lemma 6.7. *Fix a type II point* $\zeta_0 \in \boldsymbol{H}^1$. *Choose* $\gamma \in \mathrm{PGL}_2(K)$ *with* $\gamma(\zeta_G) = \zeta_0$, *and let* $\mathcal{L}_{\phi^\gamma}$ *denote the Lipschitz constant for the action of* ϕ^γ *on* $\mathbb{P}^1(K)$ *in the spherical metric. Let* $g_1(\cdot, \zeta_0) = \frac{1}{d}\sum_{\phi(\zeta_i)=\zeta_0} \langle z, \zeta_i\rangle_{\zeta_0}$, *where each preimage is counted with multiplicity. Then*

$$\sup_{z \in \boldsymbol{P}^1} |g_1(z, \zeta_0)| \leq \log_v \mathcal{L}_{\phi^\gamma} \ .$$

Proof. As above, write $g_1(z, \zeta_0) = \frac{1}{d}\sum_{\phi(\zeta_i)=\zeta_0} \langle z, \zeta_i\rangle_{\zeta_0}$. Note that as a function of z, the function $g_1(z, \zeta_0)$ is constant off of the tree spanned by ζ_0 and the points ζ_i. So it will suffice to determine $\sup_{z \in \mathbf{P}^1} |\langle z, \zeta_i\rangle_{\zeta_0}|$ for each $\zeta_i \in \phi^{-1}(\zeta_0)$; this supremum occurs when $z = \zeta_i$ for some i, and the upper bound in Lemma 5.2 gives that $|g_1(z, \zeta_0)| \leq \log_v \mathcal{L}_{\phi^\gamma}$ as asserted.

□

We are now ready to show the logarithmic equidistribution of the pullback of a point mass δ_ξ where $\xi \in \mathbf{H}^1$; it is worth noting that the error term depends only on the point ξ and the map ϕ; in particular, it does *not* depend on the choice of point $\zeta \in \mathbf{P}^1$ used to define the integrand:

Proposition 6.8. *Let* K *be a complete, algebraically closed non-Archimedean valued field, and let* $\phi \in K(z)$. *Fix a point* $\xi \in \boldsymbol{H}^1$, *and let* $\gamma \in \mathrm{PGL}_2(K)$ *be such that* $\gamma(\zeta_G) = \xi$. *Denote by* $\mathcal{L}_\phi$ *the Lipschitz constant for the action of* ϕ *on* $\mathbb{P}^1(K)$ *in the spherical metric.*

Let $\nu_n = \frac{1}{d^n}(\phi^n)^*\delta_\xi$, *and fix a point* $\zeta \in \boldsymbol{P}^1$. *We have*

$$\left| \int_{\boldsymbol{P}^1} \log_v \delta(\cdot, \zeta)_{\zeta_G} d(\nu_n - \mu_\phi) \right| \leq \frac{2d \log_v \mathcal{L}_{\phi^\gamma}}{d-1} \cdot \frac{1}{d^n} \ .$$

Proof. We begin by decomposing our integral. For any $\epsilon > 0$, and let ζ_ϵ be the point on the segment $[\zeta, \zeta_{\mathrm{G}}]$ with $\mathrm{diam}_{\mathrm{G}}(\zeta_\epsilon) = \epsilon$. Then

$$\left| \int \log_v \delta(z, \zeta)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)(z) \right| \leq \left| \int \log_v \delta(z, \zeta)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, \zeta_\epsilon)_{\zeta_{\mathrm{G}}} d\nu_n(z) \right| \tag{6.16}$$

$$+ \left| \int \log_v \delta(z, \zeta_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)(z) \right| \tag{6.17}$$

$$+ \left| \int \log_v \delta(z, \zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z, \zeta)_{\zeta_{\mathrm{G}}} d\mu_\phi(z) \right| \ . \tag{6.18}$$

For (6.16), note that the measure ν_n is supported at finitely many points in $\mathbf{H}^1$, and the integrand $\log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}}$ is zero off of the connected component $B_{\vec{v}_\zeta}(\zeta_\epsilon)^-$ of $\mathbf{P}^1 \setminus \{\zeta_\epsilon\}$ containing ζ. Thus, if ϵ is taken sufficiently small relative to n, so that $B_{\vec{v}_\zeta}(\zeta_\epsilon)^-$ is disjoint from the support of ν_n, then (6.16) is zero. We will specify a suitable ϵ at the end of the proof.

For (6.17), we follow closely the proof of Proposition-Définition 3.1 of [13]. Fix a point $\xi \in \mathbf{H}^1$, and let g_1 be a potential for the measure $\nu_1 = \frac{1}{d}\phi^*\delta_\xi$, so that $\Delta g_1 = \frac{1}{d}\phi^*\delta_\xi - \delta_\xi$. Let

$$g_n(\cdot,\xi) = \sum_{k=0}^{n-1} \frac{g_1(\phi^k(\cdot),\xi)}{d^k} \ ,$$

whereby $\Delta g_n(\cdot,\xi) = \frac{1}{d^n}(\phi^n)^*\delta_\xi - \delta_\xi = \nu_n - \delta_\xi$. Since g_1 is bounded on $\mathbf{P}^1$, the functions $g_n(\cdot,\xi)$ converge uniformly to a function

$$g_\infty(\cdot,\xi) = \sum_{k=0}^{\infty} \frac{g_1(\phi^k(\cdot),\xi)}{d^k} \ ,$$

and the measures $\Delta g_n(\cdot,\xi)$ converge weakly to the measure $\mu_\phi - \delta_\xi$.

Applying this to (6.17), we have that

$$\begin{aligned}
\left| \int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi) \right| &= \left| \int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d\Delta(g_n(\cdot,\xi) - g_\infty(\cdot,\xi)) \right| \\
&= \left| \int g_n(\cdot,\xi) - g_\infty(\cdot,\xi) \ d\Delta \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} \right| \\
&= \left| \int g_n(\cdot,\xi) - g_\infty(\cdot,\xi) \ d\,(\delta_{\zeta_\epsilon} - \delta_{\zeta_{\mathrm{G}}}) \right| \\
&\le |g_n(\zeta_\epsilon,\xi) - g_\infty(\zeta_\epsilon,\xi)| + |g_n(\zeta_{\mathrm{G}},\xi) - g_\infty(\zeta_{\mathrm{G}},\xi)| \\
&\le 2\sum_{k=n}^{\infty} \frac{\sup|g_1(\cdot,\xi)|}{d^k} = \frac{2d\sup|g_1(\cdot,\xi)|}{d-1} \cdot \frac{1}{d^n} \ .
\end{aligned}$$

Let $\gamma \in \mathrm{PGL}_2(K)$ be such that $\gamma(\zeta_{\mathrm{G}}) = \xi$. Estimating $\sup|g_1(\cdot,\xi)|$ as in Lemma 6.7, we obtain

$$\left| \int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi) \right| \le \frac{2d\log_v \mathcal{L}_{\phi^\gamma}}{d-1} \cdot \frac{1}{d^n} \ . \tag{6.19}$$

It is important to note that this bound is independent of ϵ. Finally, in Proposition 6.4 it was shown that the integral in (6.18) can be bounded by

$$\begin{aligned}\left|\int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\mu_\phi(z)\right| &\leq \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}(\epsilon^\alpha - \mathrm{diam}_{\mathrm{G}}(\zeta)^\alpha) \\ &\leq \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}\epsilon^\alpha \ , \qquad (6.20)\end{aligned}$$

where M, α are the Hölder constant and exponent (resp.) for the potential function $u_{\mu_\phi}(\cdot,\zeta_{\mathrm{G}})$.

We are now ready to prove convergence of the integrals $\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\nu_n$. Fix n, and let $\epsilon > 0$ be chosen so that $B_{\vec{v}_\zeta}(\zeta_\epsilon)^-$ is disjoint from the support of ν_n; note that, once we find a particular ϵ that satisfies this property, any smaller choice of ϵ would also suffice.

Inserting the bounds in (6.19) and (6.20) into our initial decomposition, and using the fact that (6.16) is zero for our choice of ϵ gives

$$\begin{aligned}\left|\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)\right| &\leq \left|\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d\nu_n\right| \\ &\quad + \left|\int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} d(\nu_n - \mu_\phi)\right| \\ &\quad + \left|\int \log_v \delta(z,\zeta_\epsilon)_{\zeta_{\mathrm{G}}} - \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\mu_\phi\right| \\ &\leq \frac{2d \log_v \mathcal{L}_{\phi^\gamma}}{d-1}\frac{1}{d^n} + \frac{M(q_v-1)^\alpha}{\alpha|\ln(q_v)|}\epsilon^\alpha \ .\end{aligned}$$

Now taking the limit as $\epsilon \to 0$, we obtain the asserted inequality:

$$\left|\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} d\left(\nu_n - \mu_\phi\right)\right| \leq \frac{2d \log_v \mathcal{L}_{\phi^\gamma}}{d-1}\frac{1}{d^n} \ . \qquad \square$$

We now extend this to pullbacks of measures ν having bounded potentials:

Theorem 6.9. *Let K be a complete, algebraically closed non-Archimedean valued field, and let $\phi \in K(z)$. Denote by $\mathcal{L}_\phi$ the Lipschitz constant for the action of ϕ on $\mathbb{P}^1(K)$ in the spherical metric.*

Let ν be any probability measure on $\boldsymbol{P}^1$ with bounded potentials, and define $\nu_n = \frac{1}{d^n}(\phi^n)^*\nu$. Fix a point $\zeta \in \boldsymbol{P}^1$. We have

$$\left|\int_{\boldsymbol{P}^1} \log_v \delta(\cdot,\zeta)_{\zeta_G} d(\nu_n - \mu_\phi)\right| \leq \frac{2}{d^n}\left(\frac{d}{d-1}\log_v \mathcal{L}_\phi + 2\sup_{z\in\boldsymbol{P}^1} |u_\nu(z,\zeta_G)|\right) .$$

Proof. By definition, $\nu = \Delta u_\nu(\cdot,\zeta_{\mathrm{G}}) + \delta_{\zeta_{\mathrm{G}}}$. Taking the normalized pullback of each side, and using the transformation formula for the Laplacian given in [2] Proposition 9.56, we have

$$\nu_n = \frac{1}{d^n}\Delta u_\nu(\phi^n(\cdot),\zeta_{\mathrm{G}}) + \frac{1}{d^n}(\phi^n)^*\delta_{\zeta_{\mathrm{G}}} .$$

By Proposition 6.8,

$$\left|\int \log_v \delta(z,\zeta)_{\zeta_{\mathrm{G}}} \frac{1}{d^n}(\phi^n)^*\delta_{\zeta_{\mathrm{G}}} - \mu_\phi\right| \leq \frac{2d\log_v \mathcal{L}_\phi}{d-1}\cdot\frac{1}{d^n} . \tag{6.21}$$

It suffices then to estimate

$$\int_{\mathbf{P}^1} \log_v \delta(\cdot,\zeta)_{\zeta_{\mathrm{G}}}\ d\Delta\frac{1}{d^n}u_\nu(\phi^n(\cdot),\zeta_{\mathrm{G}}) .$$

By moving the Laplacian to the integrand, and using the fact that $\Delta\log_v \delta(\cdot,\zeta)_{\zeta_{\mathrm{G}}}$, we find

$$\left|\int_{\mathbf{P}^1} \frac{1}{d^n}u_\nu(\phi^n(\cdot),\zeta_{\mathrm{G}})d(\delta_\zeta - \delta_{\zeta_{\mathrm{G}}})\right| \leq \frac{2}{d^n}\sup_{z\in\mathbf{P}^1}|u_\nu(\cdot,\zeta_{\mathrm{G}})| . \tag{6.22}$$

Combining (6.21) and (6.22) gives the desired result. $\square$

Chapter 7

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Appendix A

Quantitative Equidistribution of Points with Small Height

A.1 Context and Background

Throughout this Appendix, L will denote a number field, while K will denote an arbitrary complete, algebraically closed non-Archimedean valued field. Favre and Rivera-Letelier have given a quantitative equidistribution theorem ([12] Théorème 7) for points $F \subseteq \mathbb{P}^1(\overline{L})$ of 'small adelic height' (we will make this definition precise in Section A.5 below); at the non-Archimedean places of L, their proof relied on a notion of retracting a measure on $\mathbf{P}^1$ towards the point ∞. Baker and Rumely have generalized this idea by giving a notion of retraction towards an arbitrary base point $\zeta \in \mathbf{P}^1$ ([2] Section 5.7). In this appendix, we apply the generalized retraction of Baker and Rumely to re-derive the equidistribution of Favre and Rivera-Letelier ([12] Théorème 7). A precise statement is given in Theorem A.14 below.

We first fix some notation. Recall that a number field $L/\mathbb{Q}$ is a finite extension of $\mathbb{Q}$; it is endowed with a collection of equivalence classes of absolute values, which we denote by M_L; by a theorem of Ostrowski (see, e.g., [5] Chapter 2), M_L consists of Archimedean absolute values arising from the various embeddings $L \hookrightarrow \mathbb{C}$ and the $\mathfrak{p}$-adic absolute values arising from the prime ideals $\mathfrak{p}$

in the ring of algebraic integers $\mathcal{O}_L$. The former absolute values are referred to as 'infinite' absolute values while the latter are referred to as 'finite' absolute values.

Each absolute value v on L 'divides' an absolute value on $\mathbb{Q}$; more precisely, if $v \in M_L$ is an infinite absolute value, then it is realized as an extension of the usual absolute value on $\mathbb{Q}$, and we write $v \mid \infty$. Otherwise, if $v \in M_L$ is infinite, then it is an extension of some p-adic absolute value on $\mathbb{Q}$, where $p \in \mathbb{Z}$ is a prime number; here, we write $v \mid p$.

The completions of L with respect to an absolute value $v \in M_L$ will be denoted by L_v. We write $N_v = [L_v : \mathbb{Q}_v]/[L : \mathbb{Q}]$, where the corresponding absolute value on $\mathbb{Q}$ is the one whose extension to L is v. The completion of the algebraic closure of L_v is denoted $\mathbb{C}_v$; it is both complete and algebraically closed ([16] Theorem 13).

Let ν, ν' be two signed Borel measures on $\mathbf{P}^1$. Let Diag denote the diagonal in $\mathbb{P}^1(L) \times \mathbb{P}^1(L)$. At non-Archimedean places v of L, they define the energy between ν and ν' as

$$(\nu, \nu')_{v,\infty} = -\iint_{(\mathbf{A}^1_{\mathbb{C}_v}) \times (\mathbf{A}^1_{\mathbb{C}_v}))\setminus \mathrm{Diag}} \log_v \delta(x,y)_\infty \; d(\nu \times \nu')(x,y) \ . \tag{A.1}$$

At the infinite places v, they define

$$(\nu, \nu') := -\iint_{(\mathbb{C}\times\mathbb{C})\setminus \mathrm{Diag}} \log|x-y| d(\nu \times \nu')(x,y) \ . \tag{A.2}$$

A priori, these energy pairings are relative to the point ∞: the integrals occur over $\mathbf{A}^1_{\mathbb{C}_v} = \mathbf{P}^1_{\mathbb{C}_v} \setminus \{\infty\}$ and the integrand is given in terms of either the Hsia kernel relative to ∞ (when $v \nmid \infty$) or the usual absolute value (when $v \mid \infty$).

Over an arbitrary non-Archimedean field K, one can define new energy pairing at non-Archimedean places relative to arbitrary basepoints $\zeta \in \mathbf{P}^1_K$. More precisely, for a fixed finite place v and a fixed point $\zeta_0 \in \mathbf{P}^1_K$, we define

$$(\nu, \nu')_{K,\zeta_0} = \begin{cases} -\iint_{((\mathbf{P}^1_K \setminus \{\zeta_0\}) \times (\mathbf{P}^1_K \setminus \{\zeta_0\}))\setminus \mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0} \; d(\nu \times \nu')(x,y), & \zeta_0 \in \mathbb{P}^1(K) \\ -\iint_{(\mathbf{P}^1_K \times \mathbf{P}^1_K)\setminus \mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0} \; d(\nu \times \nu')(x,y) & \zeta_0 \in \mathbf{H}^1_K \end{cases} . \tag{A.3}$$

If $K = \mathbb{C}_v$, we will abbreviate this to $(\nu, \nu')_{v,\zeta_0}$ or, if the place v is clear from context, we will simply write $(\nu, \nu')_{\zeta_0}$. We will also write $(\nu, \nu')_{\zeta_0}$ when working over an arbitrary field K if K is clear from the context.

We now set out to re-derive [12] Théorème 7 using the generalized energy pairing at the non-Archimedean places. Most of the lemmas in this Appendix, as well as their proofs, have direct analogues in [12], and we have given reference to these accordingly.

A.2 Key Lemmas Involving $(\cdot, \cdot)_{K,\zeta_0}$

In this section, we work with arbitrary complete algebraically closed non-Archimedean valued fields K. A priori, the ζ_0-energy is not finite; however, we have the following:

Lemma A.1 ([12] Lemma 4.3)**.** *Let ν be a finite signed Borel measure on $\boldsymbol{P}^1_K$ such that $|\nu|$ has continuous potentials, and let ν' be a finite signed Borel measure on $\boldsymbol{P}^1_K$. Fix $\zeta_0 \in \boldsymbol{P}^1_K$. If either ν' has finite support in $\boldsymbol{P}^1_K \backslash \{\zeta_0\}$, or if $|\nu'|$ has continuous potentials, then $\log_v \delta(x,y)_{\zeta_0} \in L^1(|\nu| \times |\nu'|)$, hence $(\nu, \nu')_{\zeta_0}$ is well-defined.*

Proof. Using the definition of the generalized Hsia kernel in (2.2), we have

$$\log_v \delta(x,y)_{\zeta_0} = -\langle x, y\rangle_{\zeta_{\mathrm{G}}} + \langle x, \zeta_0\rangle_{\zeta_{\mathrm{G}}} + \langle y, \zeta_0\rangle_{\zeta_{\mathrm{G}}} \ . \tag{A.4}$$

It will suffice to show that each of the summands in (A.4) is integrable.

Note that $\langle x, y\rangle_{\zeta_{\mathrm{G}}}$ is positive for all $x, y \in \mathbf{P}^1$, and hence we may apply Tonelli's theorem to find

$$\begin{aligned}\left|\int \langle x, y\rangle_{\zeta_{\mathrm{G}}} \ d(\nu \times \nu')(x,y)\right| &= \left|\iint \langle x, y\rangle_{\zeta_{\mathrm{G}}} d\nu(x) d\nu'(y)\right| \\ &\leq \iint \langle x, y\rangle_{\zeta_{\mathrm{G}}} d|\nu|(x) d|\nu'|(y) \\ &= \int u_{|\nu|}(y, \zeta_{\mathrm{G}}) d|\nu'|(y) \ .\end{aligned}$$

Since $|\nu|$ has continuous potentials and $\mathbf{P}^1_K$ is compact, the integrand $u_{|\nu|}(y, \zeta_{\mathrm{G}})$ is bounded. Since ν' has finite mass, we have $\langle x, y\rangle_{\zeta_{\mathrm{G}}} \in L^1(|\nu| \times |\nu'|)$.

For the second summand, we may argue as above using Tonelli's theorem to find

$$\left|\iint \langle x, \zeta_0\rangle_{\zeta_{\mathrm{G}}} d\nu \times \nu'(x,y)\right| \le \int u_{|\nu|}(\zeta_0, \zeta_{\mathrm{G}}) d|\nu'|(y) \ ,$$

and again using that ν' has finite mass we find that $\langle x, \zeta_0\rangle_{\zeta_{\mathrm{G}}} \in L^1(|\nu| \times |\nu'|)$.

Finally, for the term $\langle y, \zeta_0\rangle_{\zeta_{\mathrm{G}}}$, we find

$$\left|\iint \langle y, \zeta_0\rangle_{\zeta_{\mathrm{G}}} d\nu \times \nu'(x,y)\right| \le |\nu|(\mathbf{P}^1_K) \cdot u_{|\nu'|}(\zeta_0, \zeta_{\mathrm{G}}) \ .$$

By assumption, $|\nu|(\mathbf{P}^1_K) < \infty$. In the case that $|\nu'|$ has continuous potentials, $u_{|\nu'|}(\zeta_0, \zeta_{\mathrm{G}})$ is also bounded; similarly, if ν' charges only finitely many points in $\mathbf{P}^1_K \setminus \{\zeta_0\}$, say $\mathrm{supp}(\nu') = F$, then

$$u_{|\nu'|}(\zeta_0, \zeta_{\mathrm{G}}) = -\sum_{z \in F} \log_v \delta(z, \zeta_0)_{\zeta_{\mathrm{G}}} \ . \tag{A.5}$$

Since $z \neq \zeta_0$ for each $z \in F$, the right side of (A.5) is bounded, and we conclude that $\langle y, \zeta_0\rangle_{\zeta_{\mathrm{G}}} \in L^1(|\nu| \times |\nu'|)$. This completes the proof of the lemma. □

Lemma A.2 ([12] Lemma 4.4)**.** *Fix $\zeta_0 \in \boldsymbol{P}^1_K$, and let ν, ν' be two finite signed Borel measures with* $\log_v \delta(x,y)_{\zeta_0} \in L^1(|\nu| \times |\nu'|)$. *Then $u_\nu(z, \zeta_0)$ is integrable with respect to ν', and*

$$(\nu, \nu')_{\zeta_0} = \int u_\nu(y, \zeta_0) d\nu'(y) \ .$$

Proof. The assumption that $\log_v \delta(\cdot,\cdot)_{\zeta_0} \in L^1(|\nu| \times |\nu'|)$ implies that $|\nu| \times |\nu'|$ does not charge Diag; further, if $\zeta_0 \in \mathbb{P}^1(K)$ then $|\nu| \times |\nu'|$ does not charge $\{\zeta_0\} \times \mathbf{P}^1_K$ or $\mathbf{P}^1_K \times \{\zeta_0\}$. By Fubini's theorem,

$$\begin{aligned}(\nu, \nu')_{\zeta_0} &= -\iint_{((\mathbf{P}^1_K \setminus \{\zeta_0\}) \times (\mathbf{P}^1_K \setminus \{\zeta_0\})) \setminus \mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0} d(\nu \times \nu')(x,y) \\ &= -\iint_{\mathbf{P}^1_K \times \mathbf{P}^1_K} \log_v \delta(x,y)_{\zeta_0} d\nu(x) d\nu'(y) \\ &= \int_{\mathbf{P}^1_K} u_\nu(y, \zeta_0) d\nu'(y) \ .\end{aligned}$$

□

A.3 The Projection-Measure Function

For a fixed point $\zeta_1 \in \mathbf{H}^1_K$, we can put a partial ordering on $\mathbf{H}^1_K$ as follows: two points $x, y \in \mathbf{H}^1_K$ satisfy $x \leq_{\zeta_1} y$ if and only if $x \in [\zeta_1, y]$.

Let ν be a finite signed Borel measure. Following Favre and Rivera-Letelier, we will derive a formula for the energy integral $(\nu, \nu)_{\zeta_0}$ in terms of the *projection-measure function*:

$$f_{\nu,\zeta_1}(x) = \nu(\{y \ : \ x \leq_{\zeta_1} y\}) \ .$$

This is *exactly* the same function given in [12] Section 4.5; we are only making explicit its dependence on ζ_1.

Throughout this section, we will also make use of the *Lebesgue measure* $d\lambda$ on $\mathbf{P}^1_K$: each segment $I \subseteq \mathbf{H}^1_K$ is isometric (under ρ) to a segment in $\mathbb{R}$; the measure $d\lambda$ on I is the transport of $|dx|$ along any such isometry.

We will relate the function $f_{\nu,\zeta_1}(\cdot)$ function to the energy pairing $(\nu, \nu')_{\zeta_0}$ in Proposition A.4 below. First, we recall

Lemma A.3 ([12] Lemma 4.6)**.** *Let ν be a positive measure such that $|\nu|$ has continuous potentials. Fix $\zeta_1 \in \boldsymbol{H}^1_K$. Let*

$$\tilde{u}_\nu(y, \zeta_1) = \int_{\zeta_1}^{y} f_{\nu,\zeta_1} d\lambda \ .$$

Then[1] $\Delta \tilde{u}_\nu(\cdot, \zeta_1) = \nu - \nu(\boldsymbol{P}^1_K)\delta_{\zeta_1}$*, and*

$$\int_{\boldsymbol{H}^1_K} f^2_{\nu,\zeta_1} d\lambda = \int_{\boldsymbol{P}^1_K} \tilde{u}_\nu(y, \zeta_1) d\nu(y) \ . \tag{A.6}$$

More generally, this equation is valid for all finite signed Borel measures with $f_{\nu,\zeta_1} \in L^2(\nu)$.

Proof. The proof is exactly the same as in [12] Lemma 4.6. □

We now use this to prove a fundamental proposition on the ζ_0-energy of a measure ν paired with itself:

---

[1] Remember that Favre–Rivera-Letelier's Laplacian is the negative of the Laplacian we have been using in this thesis!

Proposition A.4 ([12] Proposition 4.5)**.** *Fix $\zeta_0 \in \boldsymbol{P}^1_K$ for the energy pairing and $\zeta_1 \in \boldsymbol{H}^1_K$. Let ν be a finite signed Radon measure on $\boldsymbol{P}^1_K$ with $\nu(\boldsymbol{P}^1_K) = 0$, and such that $|\nu|$ has continuous potentials. Then $(\nu, \nu)_{\zeta_0}$ is well-defined, $f_{\nu,\zeta_1} \in L^1(\lambda)$, and*

$$(\nu, \nu)_{\zeta_0} = \int_{\boldsymbol{H}^1_K} f^2_{\nu,\zeta_1} d\lambda \geq 0 \ .$$

Moreover, $(\nu, \nu)_{\zeta_0} = 0$ if and only if $\nu = 0$.

Proof. The previous lemma implies that $\tilde{u}_{|\nu|}(\cdot, \zeta_1)$ is a potential for $|\nu|$, hence it is continuous. In particular,

$$\int_{\mathbf{P}^1} \tilde{u}_{|\nu|}(x, \zeta_1) d|\nu|(x) < \infty \ .$$

This, together with (A.6) and $|f_{\nu,\zeta_1}| \leq f_{|\nu|,\zeta_1}$ implies that $f_{\nu,\zeta_1} \in L^2(\lambda)$.

Note that $\Delta u_\nu(\cdot, \zeta_0) = \nu - \nu(\mathbf{P}^1_K)\delta_{\zeta_0} = \nu = \Delta \tilde{u}_\nu(\cdot, \zeta_1)$. Thus $u_\nu(\cdot, \zeta_0)$ differs from $\tilde{u}_\nu(\cdot, \zeta_1)$ by a constant C. In particular, since $\nu(\mathbf{P}^1_K) = 0$, we find

$$(\nu, \nu)_{\zeta_0} = \int_{\mathbf{P}^1_K} u_\nu(y, \zeta_0) d\nu(y) = \int_{\mathbf{P}^1_K} \tilde{u}_\nu(y, \zeta_1) d\nu(y) = \int_{\mathbf{H}^1_K} f^2_{\nu,\zeta_1}(y) d\lambda(y) \geq 0 \ .$$

We now show that $(\nu, \nu)_{\zeta_0} = 0$ if and only if $\nu = 0$. Certainly if $\nu = 0$, then $(\nu, \nu)_{\zeta_0} = 0$. For the converse, suppose that $(\nu, \nu)_{\zeta_0} = 0$. Then $f_{\nu,\zeta_1} = 0$ for λ-a.e. point in $\mathbf{H}^1_K$. Since ν is Radon, the function f_{ν,ζ_1} is upper semi-continuous on segments in $\mathbf{H}^1_K$. So along any segment, f_{ν,ζ_1} is both upper semi-continuous and identically 0 λ-a.e.; hence it must be identically 0 on $\mathbf{H}^1_K$.

In particular, $\nu(B_{\vec{v}_1}(\zeta_1)^-) = 0$ for every direction $\vec{v} \in T_{\zeta_1}$. Note, however, that our choice of $\zeta_1 \in \mathbf{H}^1_K$ was arbitrary in the sense that for any other choice of $\zeta_* \in \mathbf{H}^1_K$, we still have $(\nu, \nu)_{\zeta_0} = \int f^2_{\nu,\zeta_*} d\lambda = 0$, and hence $\nu(B_{\vec{v}}(\zeta)^-) = 0$ for every $\zeta \in \mathbf{H}^1_K$ and every $\vec{v} \in T_\zeta$. These sets form a basis for the open sets in $\mathbf{P}^1_K$, and since ν is Borel we conclude that $\nu = 0$. □

A.4 Key Estimates Involving $(\cdot, \cdot)_{v,\zeta_0}$

Fix a point $\zeta_0 \in \mathbf{H}^1_K$ and $\epsilon \in \left[0, \frac{1}{||\zeta_0, \zeta_0||}\right]$. Recall that the function $\pi_{\epsilon,\zeta_0} : \mathbf{P}^1_K \to X(\zeta_0, \epsilon)$ is the retraction of $\mathbf{P}^1_K$ to $X(\zeta_0, \epsilon) = \{z \in \mathbf{P}^1_K \ : \ \mathrm{diam}_{\zeta_0}(z) \geq \epsilon\}$. This induces a retraction on Borel measures ν on $\mathbf{P}^1_K$ given

$$\nu_{\epsilon,\zeta_0} := (\pi_{\epsilon,\zeta_0})_* \nu \ .$$

Lemma A.5 ([12] Lemma 4.10)**.** *Let ν be a probability measure on $\boldsymbol{P}^1_K$ with continuous potentials, and fix any $\zeta_0 \in \boldsymbol{P}^1_K$. Let η_{ζ_G} be a modulus of continuity for $u_\nu(z, \zeta_G)$ with respect to the small metric $d_{\boldsymbol{P}^1_K}$. Then for every $\epsilon \in (0,1)$ and for all finite sets $F \subseteq \boldsymbol{P}^1_K \setminus \{\zeta_0\}$, we have*

$$|([F], \nu)_{\zeta_0} - ([F]_{\epsilon,\zeta_0}, \nu)_{\zeta_0}| \leq \eta_{\zeta_G}(\epsilon) \ .$$

Proof. It is enough to show the result when $[F] = [z]$. Let $z(\epsilon, \zeta_0)$ be the support of $(\pi_{\epsilon,\zeta_0})_*[z]$. By our choice of ϵ, we are guaranteed that $\zeta_{\mathrm{G}} \in X(\zeta_0, \epsilon)$: to see this we simply note that

$$\mathrm{diam}_{\zeta_0}(\zeta_{\mathrm{G}}) = \delta(\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}})_{\zeta_0} = ||\zeta_0, \zeta_{\mathrm{G}}||^{-2} = 1 > \epsilon \ .$$

Now note that

$$\begin{aligned} ([z], \nu)_{\zeta_0} &= u_\nu(z, \zeta_0) \\ ([z]_{\epsilon,\zeta_0}, \nu)_{\zeta_0} &= u_\nu(z(\epsilon, \zeta_0), \zeta_0) \ . \end{aligned}$$

Then

$$\begin{aligned} |([z], \nu)_{\zeta_0} - ([z]_{\epsilon,\zeta_0}, \nu)_{\zeta_0}| &= |u_\nu(z, \zeta_0) - u_\nu(z(\epsilon, \zeta_0), \zeta_0)| \\ &= |u_\nu(z, \zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z, \zeta_0) + u_{[\zeta_{\mathrm{G}}]}(z, \zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z(\epsilon, \zeta_0), \zeta_0) \\ &\quad + u_{[\zeta_{\mathrm{G}}]}(z(\epsilon, \zeta_0), \zeta_0) - u_\nu(z(\epsilon, \zeta_0), \zeta_0)| \ . \qquad \text{(A.7)} \end{aligned}$$

Since $\Delta\left(u_\nu(z, \zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z, \zeta_0)\right) = [\zeta_{\mathrm{G}}] - \nu = \Delta u_\nu(z, \zeta_{\mathrm{G}})$, there is a constant C_{ζ_0,ζ_G} such that

$$u_\nu(z, \zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z, \zeta_0) = u_\nu(z, \zeta_{\mathrm{G}}) + C_{\zeta_0,\zeta_{\mathrm{G}}} \ .$$

Grouping the first two and the last two terms of (A.7) together, the constant $C_{\zeta_0,\zeta_{\mathrm{G}}}$ cancels out; all together we find

$$\begin{aligned}|([z],\nu)_{\zeta_0} - ([z]_{\epsilon,\zeta_0},\nu)_{\zeta_0}| &\leq |u_\nu(z,\zeta_{\mathrm{G}}) - u_\nu(z(\epsilon,\zeta_0),\zeta_{\mathrm{G}})| + |u_{[\zeta_{\mathrm{G}}]}(z,\zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z(\epsilon,\zeta_0),\zeta_0)| \\ &\leq \eta_{\zeta_{\mathrm{G}}}(d_{\mathbf{P}^1}(z,z(\epsilon,\zeta_0))) + |u_{[\zeta_{\mathrm{G}}]}(z,\zeta_0) - u_{[\zeta_{\mathrm{G}}]}(z(\epsilon,\zeta_0),\zeta_0)| \ . \qquad \text{(A.8)}\end{aligned}$$

Here we make two remarks: first, by our choice of ϵ, we have that $\zeta_{\mathrm{G}} \in X(\zeta_0,\epsilon)$ and therefore $\mathrm{diam}_{\mathrm{G}}(z(\epsilon,\zeta_0) \wedge_{\zeta_{\mathrm{G}}} z) = \mathrm{diam}_{\mathrm{G}}(z(\epsilon,\zeta_0))$. In particular,

$$\begin{aligned}\mathrm{d}_{\mathbf{P}^1}(z,z(\epsilon,\zeta_0)) &= \mathrm{diam}_{\zeta_{\mathrm{G}}}(z(\epsilon,\zeta_0)) - \mathrm{diam}_{\zeta_{\mathrm{G}}}(z) \\ &\leq \mathrm{diam}_{\zeta_{\mathrm{G}}}(z(\epsilon,\zeta_0)) \\ &= \mathrm{diam}_{\zeta_0}(z(\epsilon,\zeta_0)) \cdot ||z(\epsilon,\zeta_0),\zeta_0||^2 \\ &\leq \epsilon \ ,\end{aligned}$$

where the second equality arises from the change of variables formula for the Hsia kernel given in Equation (2.2) and the definition of $\mathrm{diam}_{\zeta_0}(\cdot)$ given in Equation (2.4).

Second, since $u_{[\zeta_{\mathrm{G}}]}(z,\zeta_0) = \langle z,\zeta_0\rangle_{\zeta_{\mathrm{G}}} + C$, we can rewrite the inequality in (A.8) in terms of $\langle\cdot,\cdot\rangle_{\zeta_G}$ instead of potential functions (the constant will cancel out). Therefore we obtain

$$|([z],\nu)_{\zeta_0} - ([z]_{\epsilon,\zeta_0},\nu)_{\zeta_0}| \leq \eta_{\zeta_{\mathrm{G}}}(\epsilon) + |\langle z,\zeta_0\rangle_{\zeta_{\mathrm{G}}} - \langle z(\epsilon,\zeta_0),\zeta_0\rangle_{\zeta_{\mathrm{G}}}| \ .$$

But again using the fact that $\zeta_{\mathrm{G}} \in X(\zeta_0,\epsilon)$, we know that $z, z(\epsilon,\zeta_0)$ retract to the same point on $[\zeta_{\mathrm{G}},\zeta_0]$. Thus the latter term in the above inequality is necessarily 0, and we have

$$([z],\nu)_{\zeta_0} - ([z]_{\epsilon,\zeta_0},\nu)_{\zeta_0}| = |u_\nu(z,\zeta_0) - u_\nu z(\epsilon,\zeta_0),\zeta_0)| \leq \eta_{\zeta_{\mathrm{G}}}(\epsilon) \ .$$

□

Lemma A.6 ([12] Lemma 4.11)**.** *Fix* $\zeta_0 \in \boldsymbol{P}^1_K$ *and a finite set* $F \subseteq \boldsymbol{P}^1_K \setminus \{\zeta_0\}$*. If* $\epsilon \in (0,1)$*, then*

$$([F]_{\epsilon,\zeta_0}, [F]_{\epsilon,\zeta_0})_{\zeta_0} \leq ([F],[F])_{\zeta_0} + |F|^{-1}\log_v(\epsilon^{-1}) \ .$$

Proof. We have that

$$\langle z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0)\rangle_{\zeta_0} \leq \langle z, z'\rangle_{\zeta_0} \ ,$$

hence

$$\begin{aligned}\delta(z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0))_{\zeta_0} &= \frac{||z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0)||}{||z(\epsilon,\zeta_0),\zeta_0|| \cdot ||z'(\epsilon,\zeta_0),\zeta_0||} \\ &\geq ||z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0)|| \\ &= q_v^{-\langle z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0)\rangle_{\zeta_0}} \\ &\geq q_v^{-\langle z,z'\rangle_{\zeta_0}} \\ &= \delta(z,z')_{\zeta_0} \ .\end{aligned}$$

Therefore,

$$\begin{aligned}([F]_{\epsilon,\zeta_0}, [F]_{\epsilon,\zeta_0})_{\zeta_0} &= -\iint \log_v \delta(z,z')_{\zeta_0} d\left([F]_{\epsilon,\zeta_0} \times d[F]_{\epsilon,\zeta_0}\right)(z,z') \\ &= -\frac{1}{|F|^2}\sum_{\substack{z,z'\in F\\ z\neq z'}} \log_v \delta(z(\epsilon,\zeta_0), z'(\epsilon,\zeta_0))_{\zeta_0} - \frac{1}{|F|^2}\sum_{z\in F}\log_v \delta(z(\epsilon,\zeta_0), z(\epsilon,\zeta_0))_{\zeta_0} \\ &\leq -\frac{1}{|F|^2}\sum_{\substack{z,z'\in F\\ z\neq z'}} \log_v \delta(z,z')_{\zeta_0} - \frac{1}{|F|}\log_v \epsilon \\ &= ([F],[F])_{\zeta_0} + \frac{1}{|F|}\log_v \epsilon^{-1} \ .\end{aligned}$$

□

Proposition A.7 ([12] Proposition 4.9)**.** *Let* ν *be a probability measure that has continuous potentials, and let* η_{ζ_G} *be a modulus of continuity for* $u_\nu(z,\zeta_G)$*. Fix* $\zeta_0 \in \boldsymbol{P}^1_K$*. Then for every* $\epsilon \in (0,1)$*,* *and every finite set* $F \subseteq \boldsymbol{P}^1_K \setminus \{\zeta_0\}$, *we have*

$$([F]-\nu,[F]-\nu)_{\zeta_0} \geq ([F]_{\epsilon,\zeta_0}-\nu,[F]_{\epsilon,\zeta_0}-\nu)_{\zeta_0} - 2\eta_{\zeta_G}(\epsilon) - |F|^{-1}\log_v \epsilon^{-1} \ .$$

Proof. By bilinearity, we have

$$([F]-\nu,[F]-\nu)_{\zeta_0} = ([F],[F])_{\zeta_0} - 2([F],\nu)_{\zeta_0} + (\nu,\nu)_{\zeta_0} \ .$$

Applying the estimates in Lemma A.5 and A.6 we have

$$\begin{aligned}([F]-\nu,[F]-\nu)_{\zeta_0} &= ([F],[F])_{\zeta_0} - 2([F],\nu)_{\zeta_0} + (\nu,\nu)_{\zeta_0} \\ &\geq \big(([F]_{\epsilon,\zeta_0},[F]_{\epsilon,\zeta_0})_{\zeta_0} - |F|^{-1}\log_v \epsilon^{-1}\big) \\ &- 2(([F]_{\epsilon,\zeta_0},\nu)_{\zeta_0} + \eta_{\zeta_{\mathrm{G}}}(\epsilon)) + (\nu,\nu)_{\zeta_0} \\ &= ([F]_{\epsilon,\zeta_0}-\nu,[F]_{\epsilon,\zeta_0}-\nu)_{\zeta_0} - |F|^{-1}\log_v \epsilon^{-1} - 2\eta_{\zeta_{\mathrm{G}}}(\epsilon) \ .\end{aligned}$$

□

A.5 Centered Adelic Heights

We now return to the special case of working over number fields. In their work on points of small height, Favre and Rivera-Letelier introduce *adelic measures* $\rho = \{\rho_v\}_{v \in M_L}$ which are families of measures, one for each place $v \in M_L$, satisfying certain properties ([12] Section 5.2). In order to allow for arbitrary base points at the non-Archimedean places, we introduce *centered adelic measures*; to do this, we first define λ_v to be the measure supported uniformly on the unit circle of $\mathbb{C}$ for $v \mid \infty$, and $\lambda_v = [\zeta_{\mathrm{G}}]$ for $v \nmid \infty$.

Definition 4. Let L be a number field. A *centered adelic measure* is a family of pairs $\rho = \{(\rho_v, \zeta_v)\}_{v \in M_L}$, where ρ_v is a Radon measure on $\mathbf{P}^1_{\mathbb{C}_v}$ and ζ_v is a marked point in $\mathbf{P}^1_{\mathbb{C}_v}$, subject to the condition that $\zeta_v = \infty$ for all places $v \mid \infty$ and $(\rho_v, \zeta_v) = (\lambda_v, \zeta_{\mathrm{G}})$ for all but finitely many places $v \in M_L$.

A.5.1 Energy and Heights

Fix a place $v \in M_L$. The space of all Borel measures on $\mathbf{P}^1_{\mathbb{C}_v}$ will be denoted by $\mathcal{M}_v$. The measure $\lambda_v \in \mathcal{M}_v$ will denote the measure supported uniformly on the unit circle if v is infinite, and will denote the point-mass supported at ζ_{G} if v is finite.

The energy pairings $(\cdot,\cdot)_v$ (for v infinite) and $(\cdot,\cdot)_{v,\zeta_0}$ (for v finite) are defined as in the previous sections. To ease notations, we will usually write $(\cdot,\cdot)_{v,\zeta_0}$ for *every* place, with the understanding that the ζ_0 is superfluous in the case that v is infinite.

Fix a place v and a point $\alpha \in \mathbb{C}_v$. For v infinite, we have

$$\log^+ |\alpha|_v = -\left([\alpha], \lambda_v\right)_{v,\zeta_0}$$

while for v finite and any point $\zeta_0 \in \mathbf{P}^1$, we have

$$\log_v \delta(\zeta_{\mathrm{G}}, \alpha)_{\zeta_0} = -\left([\alpha], \lambda_v\right)_{v,\zeta_0} \ .$$

In particular, if v is finite and $\zeta_0 = \infty$, then

$$\log_v^+ |\alpha|_v = -\left([\alpha], \lambda_v\right)_{v,\infty} \ .$$

A.5.2 Centered Adelic Measures and Heights

Centered adelic measures give rise to height functions on $\mathbb{P}^1(\overline{L})$ as follows: first, let $F \subseteq \mathbb{P}^1(\overline{L})$ be a finite, $\mathrm{Gal}(\overline{L}/L)$-invariant subset, and set

$$h_\rho(F) := \frac{1}{2} \sum_{v \in M_L} N_v \left([F] - \rho_v, [F] - \rho_v\right)_{v,\zeta_v} \ . \tag{A.9}$$

The sum appearing in the definition of h_ρ is over an indexing set of infinite cardinality; to see that h_ρ is well-defined, we have

Lemma A.8. *Let $F \subseteq \mathbb{P}^1(L)$. For all but finitely many $v \in M_L$,*

$$([F] - \rho_v, [F] - \rho_v)_{v,\zeta_v} = 0 \ .$$

Proof. We can restrict to the cases that v is finite, $\rho_v = \delta_{\zeta_{\mathrm{G}}}$, and $\zeta_v = \zeta_{\mathrm{G}}$. Then

$$([F] - \rho_v, [F] - \rho_v)_{v,\zeta_{\mathrm{G}}} = - \sum_{\substack{z,w \in F \\ z \neq w}} \log_v ||z, w|| \ .$$

For all but finitely many of these places, we have

$$||z, w|| = 1$$

for all pairs $z, w \in F$. ☐

We also record for later use

Lemma A.9. *([12] Lemme 5.4) Fix a finite place $v \in M_L$, and a point $\zeta_0 \in \boldsymbol{P}^1_{\mathbb{C}_v}$. For all finite subsets $F \subseteq \boldsymbol{P}^1_{\mathbb{C}_v}$, we have*

$$([F] - \lambda_v, [F] - \lambda_v)_{v,\zeta_0} \geq 0 \ .$$

Proof. Write

$$([F] - \lambda_v, [F] - \lambda_v)_{v,\zeta_0} = ([F], [F])_{v,\zeta_0} - 2([F], \lambda_v)_{v,\zeta_0} + (\lambda_v, \lambda_v)_{v,\zeta_0} \ .$$

Using the change of variables formula for the Hsia kernel in (2.2), we find

$$\begin{aligned} (\lambda_v, \lambda_v)_{v,\zeta_0} &= - \log_v \delta(\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}})_{\zeta_0} \\ &= - \log_v \frac{||\zeta_{\mathrm{G}}, \zeta_{\mathrm{G}}||}{||\zeta_{\mathrm{G}}, \zeta_0|| \cdot ||\zeta_{\mathrm{G}}, \zeta_0||} = 0 \ . \end{aligned} \tag{A.10}$$

For brevity, if $\zeta_0 \in \mathbb{P}^1(\mathbb{C}_v) \cap F$, write $F_0 = F \setminus \{\zeta_0\}$; otherwise, let $F_0 = F$. The change of variables formula for the Hsia kernel allows us to write

$$\begin{aligned}
([F],[F])_{v,\zeta_0} &= -\frac{1}{|F|^2} \sum_{(x,y)\in F_0\times F_0\setminus\mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0} \\
&= -\frac{1}{|F|^2}\left[\sum_{(x,y)\in F_0\times F_0\setminus\mathrm{Diag}} \log_v ||x,y||\right] + \frac{2|F_0|}{|F|^2}\sum_{x\in F_0}\log_v ||x,\zeta_0|| \qquad \text{(A.11)}
\end{aligned}$$

and

$$\begin{aligned}
-2([F],\lambda_v)_{v,\zeta_0} &= \frac{2}{|F|}\sum_{x\in F_0}\log_v \delta(x,\zeta_G)_{\zeta_0} \\
&= \frac{2}{|F|}\sum_{x\in F_0}\log_v ||x,\zeta_{\mathrm{G}}|| - \log_v ||x,\zeta_0|| - \log_v ||\zeta_{\mathrm{G}},\zeta_0|| \\
&= -\frac{2}{|F|}\sum_{x\in F_0}\log_v ||x,\zeta_0|| \;. \qquad \text{(A.12)}
\end{aligned}$$

Combining (A.10), (A.11), (A.12) gives

$$\begin{aligned}
([F]-\lambda_v,[F]-\lambda_v)_{v,\zeta_0} = &-\frac{1}{|F|^2}\sum_{(x,y)\in F_0\times F_0\setminus\mathrm{Diag}} \log_v ||x,y|| \\
&+ \frac{2(|F_0|-|F|)}{|F|^2}\sum_{x\in F_0}\log_v ||x,\zeta_0|| \;.
\end{aligned}$$

Recall that $||\cdot,\cdot|| \le 1$; therefore, since $|F_0| - |F| \le 0$, we find $([F]-\lambda_v,[F]-\lambda_v)_{v,\zeta_0} \ge 0$ as asserted.

□

To define h_ρ as a function on $\mathbb{P}^1(\overline{L})$, for any $\alpha \in \mathbb{P}^1(\overline{L})$ let F_α denote the $\mathrm{Gal}(\overline{L}/L)$-orbit of α, and set

$$h_\rho(\alpha) := h_\rho(F_\alpha) \;.$$

We have called h_ρ a 'height function' and in Proposition A.10 below we will show that h_ρ is indeed a height function in the sense of Weil. Recall that the naïve Weil height on $\mathbb{P}^1(\overline{L})$ is defined

$$h_{\mathrm{nv}}(F) = \frac{1}{|F|} \sum_{\alpha \in F} \sum_{v \in M_L} N_v \log^+ |\alpha| \ .$$

Equivalently, we can express this in terms of energy pairings by

$$h_{\mathrm{nv}}(F) = \sum_{v \in M_L} - ([F], \lambda_v)_{v,\infty} \ . \tag{A.13}$$

This can be adapted to give the usual function on $\mathbb{P}^1(\overline{L})$ as above: $h_{\mathrm{nv}}(\alpha) := h_{\mathrm{nv}}(F_\alpha)$.

Our next goal is to prove the following proposition:

Proposition A.10. *([12], Proposition 5.2) Let $\rho = (\rho_v, \zeta_v)$ be a centered adelic measure such that $\lambda_v - \rho_v$ has bounded potentials for each place v. Then the function h_ρ is a Weil height function; that is, $h_\rho - h_{nv}$ is uniformly bounded on $\overline{L}$ by a constant depending only on ρ.*

The proof of this proposition will depend on three technical lemmas:

Lemma A.11. *Let $\rho = (\rho_v, \zeta_v)$ be a centered adelic measure, and let $F \subseteq \overline{L}$ be a finite set that does not contain ζ_i for any ζ_i that lie in L. For infinite places v, we have*

$$([F], [F])_{v,\zeta_v} = -\frac{1}{|F|^2} \log_v \prod_{\substack{\alpha,\beta \in F \\ \alpha \neq \beta}} |\alpha - \beta| \ ,$$

while for finite v we have

$$([F], [F])_{v,\zeta_v} = -\frac{1}{|F|^2} \log_v \prod_{\substack{\alpha,\beta \in F \\ \alpha \neq \beta}} |\alpha - \beta| + \frac{2}{|F|} \sum_{\alpha \in F} \log_v \delta(\alpha, \zeta_v)_\infty + 2 \log_v ||\zeta_v, \infty|| \ .$$

Proof. For v infinite, the definition given in (A.2) of the energy pairing gives us

$$([F], [F])_{v,\zeta_v} = ([F], [F])_{v,\infty} = -\frac{1}{|F|^2} \sum_{\substack{\alpha,\beta \in F \\ \alpha \neq \beta}} \log_v |\alpha - \beta|$$

which is the asserted equality in this case. For v finite, our choice of F implies that $\zeta_v \notin F$, hence

$$([F],[F])_{v,\zeta_v} = -\frac{1}{|F|^2} \sum_{\substack{\alpha,\beta\in F \\ \alpha\neq\beta}} \log_v \delta(\alpha,\beta)_{\zeta_v} \ .$$

Using the change of variables formula for the Hsia kernel given in [2] Equation 4.29, we have

$$([F],[F])_{v,\zeta_v} = -\frac{1}{|F|^2} \sum_{\substack{\alpha,\beta\in F \\ \alpha\neq\beta}} \log_v \delta(\alpha,\beta)_{\infty} + \frac{2}{|F|} \sum_{\alpha\in F} \log_v \delta(\alpha,\zeta_v)_\infty + 2\log_v ||\zeta_v,\infty|| \ .$$

Since $\log_v \delta(\alpha,\beta)_\infty = \log_v |\alpha-\beta|$ for $\alpha,\beta \in \overline{L}$, this gives the desired equality. □

In the following lemma, we give expressions for the difference of local energy pairings in terms of potential functions:

Lemma A.12. *Let $\rho = \{(\rho_v,\zeta_v)\}_{v\in M_K}$ be a centered adelic measure, and let $F \subseteq \overline{L}$. For v infinite we have*

$$([F],\lambda_v)_v - ([F],\rho_v)_v = \frac{1}{|F|} \sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha) \ ,$$

while for v finite we have

$$\begin{aligned} ([F],\lambda_v)_{v,\infty} - ([F],\rho_v)_{v,\zeta_v} &= \frac{1}{|F|} \left[\sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha,\infty) \right] + u_{\rho_v}(\zeta_v,\infty) \\ &\quad - \frac{1}{|F|} \left[\sum_{\alpha\in F} \log_v \delta(\alpha,\zeta_v) \right] - 2\log_v ||\zeta_v,\infty|| \ . \end{aligned}$$

Proof. For v infinite, we have

$$\begin{aligned} ([F],\lambda_v)_v - ([F],\rho_v)_v &= -\frac{1}{|F|} \sum_{\alpha\in F} \int \log_v |z-\alpha| d(\rho_v-\lambda_v)(z) \\ &= \frac{1}{|F|} \sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha) \ . \end{aligned}$$

For v finite, we have

$$([F],\lambda_v)_{v,\infty}-([F],\rho_v)_{v,\zeta_v}=-\frac{1}{|F|}\sum_{\alpha\in F}\left(\int\log_v\delta(z,\alpha)_\infty d\lambda_v-\int\log_v\delta(z,\alpha)_{\zeta_v}d\rho_v(z)\right)\ .$$

Applying the change of variables formula for the Hsia kernel in [2] Equation 4.29 twice, we obtain

$$\begin{aligned}([F],\lambda_v)_{v,\infty}-([F],\rho_v)_{v,\zeta_v}&=-\frac{1}{|F|}\sum_{\alpha\in F}\left(\int\log_v\delta(z,\alpha)_\infty d\lambda_v-\int\log_v\delta(z,\alpha)_{\zeta_v}d\rho_v(z)\right)\\&=\frac{1}{|F|}\sum_{\alpha\in F}\Big(-\int\log_v\delta(z,\alpha)_\infty d\lambda_v+\int\log_v\delta(z,\alpha)_\infty d\rho_v(z)\\&\qquad-\int\log_v\delta(z,\zeta_v)_\infty d\rho_v(z)-\int\log_v\delta(\alpha,\zeta_v)_\infty d\rho_v(z)\Big)+2\log_v||\zeta_v,\infty||\\&=\frac{1}{|F|}\sum_{\alpha\in F}u_{\lambda_v-\rho_v}(\alpha,\infty)+u_{\rho_v}(\zeta_v,\infty)-\frac{1}{|F|}\sum_{\alpha\in F}\log_v\delta(\alpha,\zeta_v)_\infty+2\log_v||\zeta_v,\infty||\ ,\end{aligned}$$

which is the asserted expression.

□

Finally, we have

Lemma A.13. *Let* $\rho=\{(\rho_v,\zeta_v\}_{v\in M_L}$ *be a centered adelic measure. For infinite places* v*, we have*

$$(\rho_v,\rho_v)_v=([\infty]-\rho_v,[\infty]-\rho_v)_v\ ,$$

while for finite places v *we have*

$$(\rho_v,\rho_v)_{v,\zeta_v}=([\infty]-\rho_v,[\infty]-\rho_v)_{v,\zeta_v}-2u_{\rho_v}(\zeta_v,\infty)+4\log_v||\zeta_v,\infty||\ .$$

Proof. For v infinite, the definition of $(\cdot,\cdot)_v$ is as an integral on $\mathbb{C}\times\mathbb{C}\setminus\text{Diag}$; in particular, expanding $([\infty]-\rho_v,[\infty]-\rho_v)_v$ by bilinearity, the terms involving $[\infty]$ are zero; more precisely,

$$\begin{aligned}([\infty]-\rho_v,[\infty]-\rho_v)_v &= ([\infty],[\infty])_v - 2([\infty],\rho_v)_v + (\rho_v,\rho_v)_v \\ &= (\rho_v,\rho_v)_v \ .\end{aligned}$$

For finite places v, we again expand using the bilinearity of the energy pairing:

$$([\infty]-\rho_v,[\infty]-\rho_v)_{v,\zeta_0} = ([\infty],[\infty])_{v,\zeta_0} - 2([\infty],\rho_v)_{v,\zeta_0} + (\rho_v,\rho_v)_{v,\zeta_0} \ .$$

Note that $([\infty],[\infty])_{v,\zeta_0} = 0$, since the energy integral does not include the type I diagonal. By Lemma A.2, the second energy pairing can be expressed in terms of the potential u_{ρ_v}, so that

$$\begin{aligned}([\infty]-\rho_v,[\infty]-\rho_v)_{v,\zeta_0} &= -2\int u_{\rho_v}(z,\zeta_0)d[\infty] + (\rho_v,\rho_v)_{v,\zeta_0} \\ &= -2u_{\rho_v}(\infty,\zeta_0) + (\rho_v,\rho_v)_{v,\zeta_0} \ . \end{aligned} \tag{A.14}$$

The potential function u_{ρ_v} is not symmetric, but it satisfies $u_{\rho_v}(\infty,\zeta_0) = -u_{\rho_v}(\zeta_0,\infty)+2\log_v||\zeta_0,\infty||$. Inserting this into (A.14) and doing some algebra yields

$$(\rho_v,\rho_v)_{v,\zeta_0} = ([\infty]-\rho_v,[\infty]-\rho_v)_{v,\zeta_0} - 2u_{\rho_v}(\zeta_0,\infty) + 4\log_v||\zeta_0,\infty|| \ .$$

□

We are now ready to prove Propostion A.10:

Proof of Proposition A.10. We are trying to show that $|h_\rho - h_{\text{nv}}|$ is uniformly bounded on $\overline{L}$. Both of these heights admit local decompositions (see (A.9) and (A.13) above); working locally, we are interested in estimating

$$\begin{aligned}\frac{1}{2}([F]-\rho_v,[F]-\rho_v)_{v,\zeta_0}+([F],\lambda_v)_{v,\infty} &= \frac{1}{2}([F],[F])_{v,\zeta_0}-([F],\rho_v)_{v,\zeta_0}+\frac{1}{2}(\rho_v,\rho_v)_{v,\zeta_0}+([F],\lambda_v)_{v,\infty}\\ &= \frac{1}{2}([F],[F])_{v,\zeta_0}+\frac{1}{2}(\rho_v,\rho_v)_{v,\zeta_0}+(([F],\lambda_v)_{v,\zeta_0}-([F],\rho_v)_{v,\zeta_0})\ .\end{aligned} \tag{A.15}$$

We begin first with places $v \mid \infty$; applying the expressions in Lemmas A.11, A.12, and A.13 to the three terms appearing in (A.15), we have

$$\begin{aligned}\frac{1}{2}([F]-\rho_v,[F]-\rho_v)_{v,\zeta_0}+([F],\lambda_v)_{v,\infty} = -\frac{1}{2|F|^2}\log_v \prod_{\substack{\alpha,\beta\in F\\ \alpha\neq\beta}}|\alpha-\beta| + \frac{1}{2}([\infty]-\rho_v,[\infty]-\rho_v)_v\\ + \frac{1}{|F|}\sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha)\ .\end{aligned} \tag{A.16}$$

At finite places v, we make the same substitutions into (A.15) and obtain a *much* larger expression:

$$\begin{aligned}\frac{1}{2}([F]-\rho_v,[F]-\rho_v)_{v,\zeta_0}&+([F],\lambda_v)_{v,\infty}\\ &= -\frac{1}{2|F|^2}\log_v \prod_{\substack{\alpha,\beta\in F\\ \alpha\neq\beta}}|\alpha-\beta| + \frac{1}{|F|}\sum_{\alpha\in F}\log_v \delta(\alpha,\zeta_v)_\infty + \log_v ||\zeta_v,\infty||\\ &+\frac{1}{2}([\infty]-\rho_v,[\infty]-\rho_v)_{v,\zeta_v} - u_{\rho_v}(\zeta_v,\infty)+2\log_v||\zeta_v,\infty||\\ &+\frac{1}{|F|}\left[\sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha,\infty)\right]+u_{\rho_v}(\zeta_v,\infty)\\ &-\frac{1}{|F|}\left[\sum_{\alpha\in F}\log_v\delta(\alpha,\zeta_v)\right]-2\log_v||\zeta_v,\infty||\ .\end{aligned}$$

After no small amount of cancellation, this reduces to

$$\begin{aligned}\frac{1}{2}([F]-\rho_v,[F]-\rho_v)_{v,\zeta_0}+([F],\lambda_v)_{v,\infty} = -\frac{1}{2|F|^2}\log_v \prod_{\substack{\alpha,\beta\in F\\ \alpha\neq\beta}}|\alpha-\beta| + \frac{1}{2}([\infty]-\rho_v,[\infty]-\rho_v)_v\\ + \frac{1}{|F|}\sum_{\alpha\in F} u_{\lambda_v-\rho_v}(\alpha)+\log_v||\zeta_v,\infty||\ ,\end{aligned} \tag{A.17}$$

which matches the case when v is infinite up to the additional term $\log_v ||\zeta_v, \infty||$. Note that for all but finitely many v, we have $\log_v ||\zeta_v, \infty|| = \log_v ||\zeta_{\mathrm{G}}, \infty|| = 0$.

Let $M_L^\circ \subseteq M_L$ denote the non-Archimedean places of L. Normalizing and summing the expressions in (A.16) and (A.17), the $\mathrm{Gal}(\overline{L}/L)$-invariance of F and the product formula yield

$$h_\rho(F) - h_{\mathrm{nv}}(F) = h_\rho(\infty) + \sum_{v \in M_L} \frac{1}{|F|} \sum_{\alpha \in F} u_{\lambda_v - \rho_v}(\alpha, \infty) + \sum_{v \in M_L^\circ} N_v \log_v ||\zeta_v, \infty|| \tag{A.18}$$

Note that since $\zeta_v = \zeta_{\mathrm{G}}$ for all but finitely many places, so that $\log_v ||\zeta_v, \infty|| = 0$; thus the latter term in (A.18) is actually a finite sum depending only on the measure ρ. Let $D_\rho = \sum_{v \in M_L^\circ} N_v \log_v ||\zeta_v, \infty||$.

We also remark that for all but finitely many places, $\rho_v = \lambda_v$, and hence $u_{\lambda_v - \rho_v}(z, \infty) \equiv 0$. Let S be the collection of places where $\rho_v \neq \lambda_v$; at each of these places, ρ_v has continuous potentials by assumption, hence $\sup_{z \in \mathbf{P}^1} |u_{\rho_v - \lambda_v}(z, \infty)| \leq C_v$ for each place $v \in S$. Thus, we may bound the expression in (A.18) as

$$|h_\rho(F) - h_{\mathrm{nv}}(F)| \leq |h_\rho(\infty)| + \left[\sum_{v \in S} C_v \right] + D_\rho \ , \tag{A.19}$$

where the upper bound depends only on the measure ρ. □

A.6 The Dirichlet Pairing

Baker and Rumely have shown ([2] Proposition 5.41) that, given a function f of bounded differential variation for which $|\Delta(f)|$ has continuous potentials, the Dirichlet pairing can be expressed

$$\langle f, f \rangle_{\mathrm{Dir}} = -\iint_{\mathbf{P}^1_{\mathbb{C}_v} \times \mathbf{P}^1_{\mathbb{C}_v}} \log_v \delta(z, w)_{\zeta_0} d\Delta(f)(z) \times d\Delta(f)(w) \ . \tag{A.20}$$

The integral in this expression appears very closely related to the integrals appearing in the definition of the energy pairing $(\Delta(f), \Delta(f))_{v,\zeta_0}$, which we recall here:

$$(\nu, \nu')_{K,\zeta_0} = \begin{cases} -\iint_{((\mathbf{P}^1_{\mathbb{C}_v}\setminus\{\zeta_0\})\times(\mathbf{P}^1_{\mathbb{C}_v}\setminus\{\zeta_0\}))\setminus\mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0}\ d(\nu\times\nu')(x,y), & \zeta_0 \in \mathbb{P}^1(\mathbb{C}_v) \\ -\iint_{(\mathbf{P}^1_{\mathbb{C}_v}\times\mathbf{P}^1_{\mathbb{C}_v})\setminus\mathrm{Diag}} \log_v \delta(x,y)_{\zeta_0}\ d(\nu\times\nu')(x,y) & \zeta_0 \in \mathbf{H}^1_{\mathbb{C}_v} \end{cases} .$$

The difference between the integrals here and the one in (A.20) is in the domain of integration; our assumption that $|\Delta(f)|$ has continuous potentials ensurese that $\Delta(f)$ does not charge any type I points $\zeta_0 \in \mathbb{P}^1(\mathbb{C}_v)$, and consequently it does not charge the diagonal. Thus,

$$\langle f, f\rangle_{\mathrm{Dir}} = (\Delta(f), \Delta(f))_{v,\zeta_0} \tag{A.21}$$

for any function f of bounded differential variation for which $|\Delta(f)|$ has continuous potentials.

Now let ρ be a signed Borel measure on $\mathbf{P}^1_{\mathbb{C}_v}$ of total mass 0 for which $|\rho|$ has continuous potentials, and let $u_\rho(\cdot, \zeta_{\mathrm{G}})$ be a potential for ρ. Recall that an alternative characterization of the Dirichlet pairing can be given for continuous functions of bounded differential variation ([2] Corollar 5.39):

$$\left|\int_{\mathbf{P}^1_{\mathbb{C}_v}} f d\rho\right| = \left|\int_{\mathbf{P}^1_{\mathbb{C}_v}} f\ d\Delta u_\rho(\cdot, \zeta_{\mathrm{G}})\right| = \left|\langle f, u_\rho\rangle_{\mathrm{Dir}}\right| \ .$$

Applying the Cauchy-Schwarz inequality, and using the relation in A.21 above, we find

$$\begin{aligned} \left|\int_{\mathbf{P}^1_{\mathbb{C}_v}} f d\rho\right| &= \left|\int_{\mathbf{P}^1_{\mathbb{C}_v}} f\ d\Delta u_\rho(\cdot, \zeta_{\mathrm{G}})\right| \\ &= |\langle f, u_\rho\rangle_{\mathrm{Dir}}| \\ &\le \langle f, f\rangle_{\mathrm{Dir}}^{1/2} \cdot \langle u_\rho, u_\rho\rangle_{\mathrm{Dir}}^{1/2} \\ &= \langle f, f\rangle_{\mathrm{Dir}}^{1/2} \cdot (\rho, \rho)_{v,\zeta_0}^{1/2} \ . \end{aligned}$$

The analogous inequality is known already to hold over $\mathbb{C}$: if ρ is any signed Borel measure of total mass 0 on $\mathbb{P}^1(\mathbb{C}) = \mathbf{P}^1_{\mathbb{C}}$, and if f is $\mathcal{C}^2$ (in the usual sense over $\mathbb{C}$), then

$$\left|\int_{\mathbf{P}^1} f\ d\rho\right| \le \langle f, f\rangle_{\mathrm{Dir}}^{1/2} \cdot (\rho, \rho)^{1/2} \ .$$

A.7 Quantitative Equidistribution

In this section, we use the general energy pairing to re-derive Favre and Rivera-Levelier's quantitative equidistribution; our proof is the same as the one given by Favre and Rivera-Letelier, differing only in the fact that we're using the generalized energy pairing at the non-Archimedean places.

Let L be a number field and let $(\rho_v, \zeta_v)_{v \in M_L}$ be a centered adelic measure for which the potential functions $u_{\rho_v - \lambda_v}(\cdot)$ (for $v \mid \infty$) or $u_{\rho_v - \lambda_v}(\cdot, \zeta_{\mathrm{G}})$ (for $v \leq \infty$) have κ-Hölder continuous potentials[2] with respect to either the chordal metric on $\mathbb{P}^1(\mathbb{C}_v)$ (for $v \mid \infty$) or the small metric $\mathbf{d}_{\mathbf{P}^1}$ on $\mathbf{P}^1_{\mathbb{C}_v}$ (for $v \leq \infty$).

The statement which we will prove in this section is

Theorem A.14 (Favre and Rivera-Letelier, [12] Théorème 7)**.** *Let L be a number field, and let $(\rho_v, \zeta_v)_{v \in M_L}$ be a centered adelic measure such that the potential functions for $\rho_v - \lambda_v$ have κ-Hölder continuous potentials at each place $v \in M_L$. Let $F \subseteq \mathbb{P}^1(L)$ be a finite,* $\mathrm{Gal}(\overline{L}/L)$*-invariant subset with $|F| > 1$. Fix a place $v \in M_L$ and a function $f : \boldsymbol{P}^1_{\mathbb{C}_v} \to \mathbb{R}$ which is either $\mathcal{C}^1$ (if $v \mid \infty$) or of bounded differential variation (if $v \leq \infty$). We have*

$$\left| \int_{\boldsymbol{P}^1_{\mathbb{C}_{v_0}}} f \; d([F] - \rho_{v_0}) \right| \leq \frac{Lip(f) + \langle f, f \rangle_{Dir}^{1/2}}{|F|^{1/(2\kappa)}} + \langle f, f \rangle_{Dir}^{1/2} \cdot \left(2[L : \mathbb{Q}] h_\rho(F) + C' \frac{\log_2 |F|}{|F|} \right)^{1/2} \quad .$$

Proof. We begin by estimating

$$\left| \int_{\mathbf{P}^1_{\mathbb{C}_v}} f d([F] - \rho_v) \right| \leq \left| \int_{\mathbf{P}^1_{\mathbb{C}_v}} f d([F] - [F]_{\epsilon, \zeta_v}) \right| + \left| \int_{\mathbf{P}^1_{\mathbb{C}_v}} f d([F]_{\epsilon, \zeta_v} - \rho_v) \right| \quad . \tag{A.22}$$

For the definition of $[F]_{\epsilon, \zeta_v} = [F]_\epsilon$ in the Archimedean context, and its basic properties, see [12] Section 2.6. The first summand appearing on the right side of (A.22) can be bounded in terms of the Lipschitz constant (in the chordal metric over $\mathbb{C}$, or in the metric $\mathrm{d}_{\mathbf{P}^1}$ over non-Archimedean fields) of f as

$$\left| \int_{\mathbf{P}^1_{\mathbb{C}_v}} f d([F] - [F]_\epsilon) \right| \leq \mathrm{Lip}(f) \cdot \epsilon \; . \tag{A.23}$$

---

[2] Here, we mean that the Hölder exponent κ is independent of the place v, though since $\rho_v - \lambda_v = 0$ for all but finitely many places, we *could* a priori begin with different κ_v for each place and then choose the smallest of the non-zero κ_v; it will be more convenient to assume that we have already chosen a κ that works at all places.

For the second summand appearing on the right side of (A.22), note that $|[F]_{\epsilon,\zeta_v} - \rho_v| = [F]_{\epsilon,\zeta_v} + \rho_v$ has continuous potentials (the fact that $[F]_\epsilon$ has continuous potentials at Archimedean places is [12] Lemme 2.7). The discussion in the previous section gives

$$\left| \int_{\mathbf{P}^1_{\mathbb{C}_v}} f d([F]_\epsilon - \rho_v) \right| \leq \langle f, f \rangle_{\mathrm{Dir}}^{1/2} \cdot ([F]_\epsilon - \rho_v, [F]_\epsilon - \rho_v)_{v,\zeta_v}^{1/2} \ . \tag{A.24}$$

It therefore suffices to estimate $([F]_\epsilon - \rho_v, [F]_\epsilon - \rho_v)_{v,\zeta_v}$.

Recall that by [12] Proposition 2.8 and Proposition A.7 above, there exists a constant $C > 0$ depending only on ρ such that

$$\begin{aligned} 0 &\leq ([F]_{\epsilon,\zeta_v} - \rho_v, [F]_{\epsilon,\zeta_v} - \rho_v)_{v,\zeta_v} \\ &\leq ([F] - \rho_v, [F] - \rho_v)_{v,\zeta_v} + 2(\hat{\eta}_v(\epsilon) + \epsilon) + |F|^{-1}(C + \log_v \epsilon^{-1}) \ , \end{aligned} \tag{A.25}$$

where the positivity follows from [12] Proposition 2.6 and Proposition A.4 above (note that $[F]_{\epsilon,\zeta_v} - \rho_v$ has continuous potentials). Here, $\hat{\eta}_v$ is a modulus of continuity for u_{ρ_v} when $v \mid \infty$ and is a modulus of continuity for $u_{\rho_v}(\cdot, \zeta_{\mathrm{G}})$ when $v \nmid \infty$ (the extra '$+\epsilon$' with $\hat{\eta}_v(\cdot)$ arises from the case $v \mid \infty$, as does the constant C; see [12] Proposition 2.8).

Fix a place $v_0 \in M_L$, and let N denote the collection of infinite places together with the finite places v satisfying $\rho_v \neq \lambda_v$ and $\zeta_v \neq \zeta_{\mathrm{G}}$. Let $\hat{\eta}(\cdot)$ denote a modulus of continuity common to all of the potentials $u_{\rho_v}(\cdot)$ (for $v \mid \infty$) and $u_{\rho_v}(\cdot, \zeta_{\mathrm{G}})$ (for $v \nmid \infty$); since u_{ρ_v} is constant for all v outside of N, the quantity $\hat{\eta}(\cdot)$ is well-defined, and by assumption, $\hat{\eta}(\epsilon) \lesssim \epsilon^\kappa$. Let $\eta(\epsilon) := \hat{\eta}(\epsilon) + \epsilon$.

Applying (A.25), we find

$$\begin{aligned} N_{v_0}([F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0}, [F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0})_{v_0,\zeta_{v_0}} &\leq \sum_{v \in N \cup \{v_0\}} N_v([F]_{\epsilon,\zeta_v} - \rho_v, [F]_{\epsilon,\zeta_v} - \rho_v)_{v,\zeta_v} \\ &\leq \left[\sum_{v \in N \cup \{v_0\}} N_v([F] - \rho_v, [F] - \rho_v)_{v,\zeta_v} \right] \\ &\quad + M\left(2\eta(\epsilon) + |F|^{-1}(C + \log_2 \epsilon^{-1})\right) \ , \end{aligned} \tag{A.26}$$

where $M = (|N|+1)\max_{v\in N} N_v$. Note that in the last line, the logarithm is taken base 2, reflecting the fact that for all $\epsilon \in (0,1)$, we have $\log_v(\epsilon^{-1}) \leq \log_2 \epsilon^{-1}$, which is in turn due to the fact that $2 \leq q_v$ for *all* places $v \in M_L$ (when $v \mid \infty$, we take $q_v = e$).

By Lemma A.9, we may add the remaining terms $N_v([F]-\rho_v, [F]-\rho_v)_{v,\zeta_v}$ for $v \notin N \cup \{v_0\}$ to the upper bound in (A.26), whereby we obtain

$$N_{v_0}([F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0}, [F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0})_{v_0,\zeta_{v_0}} \leq 2h_\rho(F) + M(2\eta(\epsilon) + |F|^{-1}(C + \log_2 \epsilon^{-1}))\ .$$

Recall that $\eta(\epsilon) = \hat{\eta}(\epsilon) + \epsilon$, and that $\hat{\eta}(\epsilon) \lesssim \epsilon^\kappa$. Choosing $\epsilon = |F|^{-1/\kappa}$, this yields

$$\begin{aligned}([F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0}, [F]_{\epsilon,\zeta_{v_0}} - \rho_{v_0})_{v_0,\zeta_{v_0}} &\leq \frac{1}{N_v}\left(2h_\rho(F) + \frac{2M}{|F|} + \frac{1}{|F|^{1/\kappa}} + \frac{C}{|F|} + \frac{\log_2|F|}{\kappa|F|}\right) \\ &\leq [L:\mathbb{Q}]\left(2h_\rho(F) + C'\frac{\log_2|F|}{|F|} + \frac{1}{|F|^{1/\kappa}}\right)\end{aligned}$$

for some constant C' depending only on ρ. Combining this with (A.23), we obtain

$$\left|\int_{\mathbf{P}^1_{\mathbb{C}_{v_0}}} f\ d([F]-\rho_{v_0})\right| \leq \frac{\mathrm{Lip}(f)}{|F|^{1/\kappa}} + \langle f,f\rangle_{\mathrm{Dir}}^{1/2}\cdot[L:\mathbb{Q}]^{1/2}\cdot\left(2h_\rho(F) + C'\frac{\log_2|F|}{|F|} + \frac{1}{|F|^{1/\kappa}}\right)^{1/2}\ .$$

□