INVESTIGATING HIGH SCHOOL STUDENTS' UNDERSTANDINGS OF ANGLE MEASURE

by

HAMILTON L. HARDISON

(Under the Direction of Leslie P. Steffe)

ABSTRACT

The purpose of this study was to understand how ninth-grade students quantify angularity. Angle and angle measure are critical topics in mathematics; however, students' difficulties with these topics are well documented in research literature. In contrast, little is known about how students develop propitious quantifications of angularity, and researchers have critiqued previous curricular approaches. Although some scholars have identified quantifications of angularity beneficial for the study trigonometry, these quantifications leverage multiplicative comparisons of circular quantities (e.g. arc length and circumference) and, as such, are an unlikely starting point for students' initial quantifications. Previous studies examining how students reason with angles have obfuscated students' quantifications by failing to emphasize the attribute to be measured or providing measurement tools like protractors. This dissertation research is a response to calls for studies investigating students' quantifications of angularity, particularly at the high school level.

As teacher-researcher, I taught a pair of ninth-grade students at a rural high school in the southeastern U.S. in a constructivist teaching experiment from October 2015 to

April 2016. Teaching practices included creating and posing problems involving angle models, interacting with students to understand their ways of reasoning, and reflecting on students' mathematical activities. From video records of these sessions, I constructed second-order models accounting for students' initial quantifications of angularity and modifications to these quantifications occurring throughout the teaching experiment.

From the students' activities, I abstracted three motions involved in the construction of angularity and five operations used to alter the side lengths of angle models while preserving angularity. During the teaching experiment, both students constructed quantifications of angularity involving extensive quantitative operations (e.g., segmenting, iterating); these extensive quantifications have not been discussed in previous empirical literature. Students' constructions of right-angle and full-angle templates were a significant resource for students' understandings of the degree as a standard unit of angular measure. Students developed conceptions of degrees that involved positing familiar angular templates as composite units (e.g., a right angle as a 90-unit composite or a full angle as a 360-unit composite), and their quantifications differed across rotational and non-rotational angle contexts.

INDEX WORDS: Angle, Angle Measure, Angularity, Degree Measure, Quantification, Quantity, Quantitative Reasoning, Radical Constructivism, Scheme Theory, Teaching Experiment

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When I elected to attend the University of Georgia for my undergraduate studies, I was entirely unaware the Department of Mathematics and Science Education housed nationally (and internationally) renowned programs for mathematics teacher education and research in mathematics education. I thank the mathematics education faculty who comprised the department during my years of study, and I consider myself extraordinarily fortunate to have studied under a multitude of headliners in the field. In the following paragraph, I thank some individual faculty members for a few salient memories from my years as a student of the department.

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CHAPTER 1

INTRODUCTION

In this dissertation, I describe and interpret my teaching experiment (Steffe & Thompson, 2000) involving two ninth-grade students and their ways of reasoning about angularity. Broadly, my goals in conducting the teaching experiment were to establish these students' understandings of angle measure at the onset of the study and to investigate how these understandings changed throughout the teaching experiment.

In the introductory chapter, I first discuss how I came to be interested in investigating individuals' conceptions of angle measure. During this discussion, I introduce a conceptual framework—quantitative reasoning (Thompson, 2011)—that guided this study. Then, I present a multipart argument for the importance of this study to the field of mathematics education. I close the chapter by providing an abbreviated statement of the problem, research questions, and descriptions of the contents of subsequent chapters.

Initial Motivation: Two Courses

My interest in angle measure was occasioned in part through my participation in two courses during the fall of 2014.[1] The first was a content course with a precalculus emphasis for prospective secondary mathematics teachers. The second course was a one-hour graduate seminar on quantitative reasoning.

[1] Kevin Moore was the instructor for both courses.

What Does It Mean for an Angle to have a Measure of One Degree?

The instructor's goals during the first unit for the content course included examining and enriching the prospective teachers' understandings of angle measure. In some sense, one prompt posed to the class during these initial course meetings motivated this dissertation: what does it mean for an angle to have a measure of one degree?

When this prompt was posed in the content course, I found I had two different, coherent ways an individual might reason about a one-degree angle.[2] First, if an individual drew an arbitrary circle centered at the vertex of a one-degree angle, then the minor arc intercepted by the angle would have an arc length that was 1/360 of the circle's circumference; therefore, an individual could produce a one-degree angle by partitioning an arbitrary circle into 360 equal-length arcs and selecting a central angle subtending any one of these arcs. For convenience in later paragraphs, I refer this first way of reasoning as *circle reasoning*. Second, if an individual were to iterate a one-degree angle 360 times, then the iteration would exhaust the plane; therefore, an individual could produce a one-degree angle by radially partitioning the plane into 360 equiangular parts and selecting any one of those parts. I refer to this second way of reasoning as *non-circle reasoning*. [3] Models of one-degree angles (shown in red) produced via circle reasoning and non-circle reasoning are depicted in Figures 1.1 and 1.2, respectively.[4]

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[2] Throughout this dissertation, I use "*n*-degree angle" as shorthand for "an angle with a measure of *n* degrees."

[3] The terminology I use for these two preliminary distinctions is intentionally vague for introductory purposes. As I will show throughout this dissertation and once I have introduced the necessary constructs, individuals' meanings of angularity are more nuanced.

[4] These figures push the boundaries of my current image production capabilities.

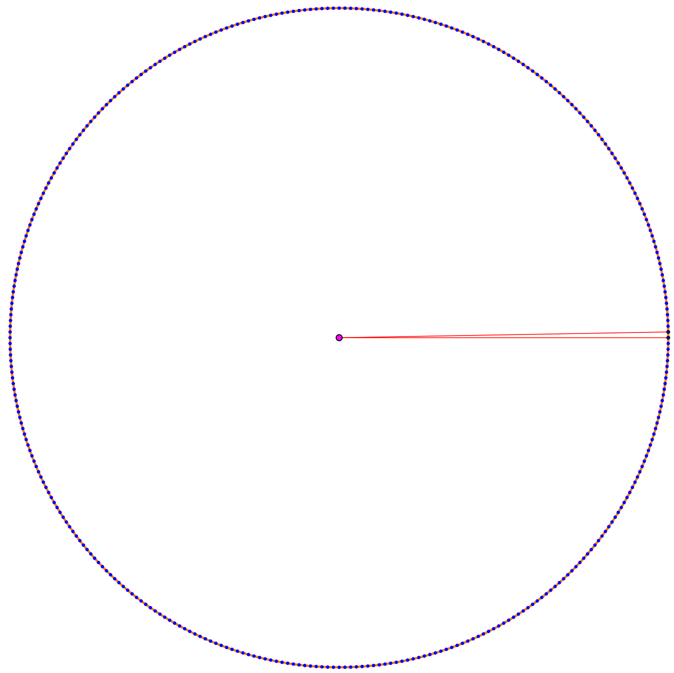

Figure 1.1. A model of a one-degree angle produced by partitioning a circle.

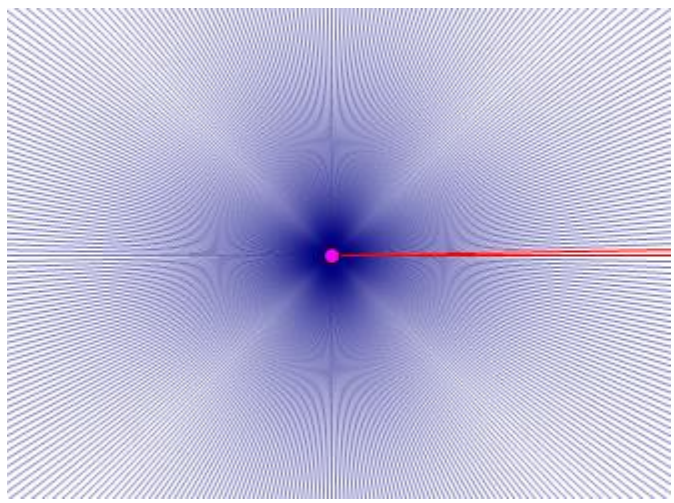

Figure 1.2. A model of a one-degree angle produced by partitioning a plane.

On one hand, these two ways of reasoning are compatible. By compatible, I mean that if an individual, *A*, produced a one-degree angle using circle reasoning and another individual, *B*, verified the measure of this angle using non-circle reasoning, both individuals would agree the angle in question was a one-degree angle. Similarly, if *B* produced a one-degree angle using non-circle reasoning and *A* verified using circle reasoning, both individuals would again agree the angle in question was a one-degree angle.

On the other hand, these two ways of reasoning about degrees as a unit of angular measure are distinct in that different geometrical objects are foregrounded. As the name suggests, circle reasoning foregrounds circles and arcs; the measure of a central angle is the determined via a multiplicative relationship between circumference and arc length.[5] In contrast, neither circles nor arc lengths are involved in non-circle reasoning. Instead, non-circle reasoning involves determining the measure of an angle via a multiplicative relationship between the interior of an angle and the plane.

The initial distinction between circle reasoning and non-circle reasoning was occasioned by my participation in the course for prospective secondary teachers. In subsequent sections, I return to, as well as refine, this initial distinction. As I mentioned at the onset of this section, my interest in angle measure was occasioned in part by two courses. The second course that piqued by interest in angle measure was a graduate seminar on quantitative reasoning (Thompson, 2011), a conceptual framework that guided this study. I describe this conceptual framework in the following section.

Quantitative Reasoning

Quantitative reasoning is grounded in an individuals' analyses of situations, often situations they experience (or have experienced) firsthand (Smith & Thompson, 2008).[6] Quantitative reasoning involves attending to quantities—measurable attributes in a situation (Thompson, 1994)—and relationships between these quantities. Quantities and

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[5] I use "multiplicative relationship" to refer to the structure of the situation as conceived by the individual and not to calculations. The arcs are inserted into the circle, and the circle is composed of arcs.
[6] Quantitative reasoning is compatible with my broader theoretical orientation, radical constructivism, which I will discuss in Chapter 2.

the relationships between them constitute a quantitative structure for the situation at hand (Thompson, 1993). Quantities and quantitative structures do not exist in an omnipresent sense; instead, they are produced by individuals (Thompson, 1994). Examples of quantities that individuals might construct include height, area, volume, time, speed, and angularity. As quantities are mental constructions, they vary both across individuals and within an individual overtime.

A quantity consists of three interrelated components, which are also mental constructions: (a) an object or situation, (b) an attribute or quality, and (c) a quantification, which is a set of operations an individual can enact on the attribute (e.g., a measurement process). Although multiple individuals may encounter the "same" object within their experiential fields from an observer's perspective, the situation from the perspective of one individual is not identical to the situation from the perspective of another. Individuals impute qualities to the situations they construct.

For example, imagine a father, his four-year-old daughter, and his newborn son visit the zoo and observe a giraffe within an enclosure.[7] If a security guard were observing the family looking toward the giraffe, all three family members would be experiencing the "same" giraffe from the security guard's perspective. Yet, each of the family members would experience the situation differently. In such a situation, the father, as an adult, would be able separate the giraffe from the non-giraffe elements within his visual field—he would see a giraffe. Furthermore, the father would likely be aware of the giraffe's height and might even consider it in relation to his own height, perhaps remarking, "that giraffe is three times as tall as I am!" In contrast, the newborn would

[7] This example was inspired by a conversation in the 2014 graduate seminar on quantitative reasoning wherein participants were debating whether height was an inherent property of giraffes.

experience visual stimuli, but he would not likely be able to isolate what the father would call "giraffe" as a single item among these visual stimuli; in other words, the newborn son likely would not see a giraffe, much less be aware of its height. The giraffe's height would be a quantity for the father, but not for newborn.

Like the father, the four-year-old would also likely identify the giraffe as a singular item within her visual field and might exclaim, "that giraffe is way taller than me!" Within this hypothetical example, the father and the daughter each demonstrated an awareness of the quantity, giraffe's height. However, their quantifications are very different. The father's hypothetical remark, "that giraffe is three times as tall as I am," suggests the father can iterate heights. Specifically, the father might have imagined exhausting the height of the giraffe by mentally uniting copies of his own height while counting these copies. It would be most unusual if the four-year-old daughter spontaneously considered the giraffe's height as composed of units of her own height. Instead, her reasoning indicates only a comparative structure. Thus, the father and daughter can subject heights to different mental processes; therefore, they have different quantifications for height in this context.

The daughter's comparative conception exhibited above is an example of *gross quantity* (Piaget, 1965; Steffe, 1991a; Thompson, 1994). An individual has constructed a gross quantity when she can compare relative extents of an attribute (Thompson, 1994), though in some cases this comparison may be available to an individual only when the object is perceptually available (Steffe, 1991a). An *extensive quantity* arises when an individual introduces units into an attribute (Piaget, 1965; Steffe, 1991a; Thompson, 1994). In the giraffe context, the father conceived the giraffe's height in terms of units of his own height; therefore, the father demonstrated an extensive quantification of height.

A major purpose of this dissertation study was to examine high school students' quantifications of angularity. I use the plural, "quantifications," because I hypothesized that individuals can quantify angularity differently, just as individuals can quantify height differently. In the previous section, I provided two different descriptions for how individuals might conceive of one-degree angles. These descriptions suggest different quantifications of angularity, with circles being crucial for one quantification but not the other. My interest in students' quantifications of angularity peaked when I discovered no extant empirical research had focused on whether individuals develop non-circular quantifications of angularity.

In this section, I have addressed two courses which occasioned my interest in studying students' quantifications of angularity. In the following section, I provide a rationale for why this study is critical for the field of mathematics education.

Rationale

I present my argument for the importance of this study in several parts. First, angle measure is pervasive within and beyond school mathematics. Second, the field knows little about students' understandings of angularity, particularly at the high school level; what we do know is that quantifying angularity is non-trivial for students. Some researchers have attributed students' difficulties in quantifying angularity to problems with curricular and pedagogical approaches to teaching angle measure in the United States. Although some researchers have convincingly argued that circular quantifications of angularity are productive for the study of trigonometry, I argue these quantifications are not accessible to all students, particularly elementary students. Although some practitioner articles have recommended activities for elementary and middle grades students targeting non-circular quantifications of angularity, circular quantifications are the only quantifications that have received attention in empirical and theoretical literature on angle. In the following sections, I elaborate on each of these reasons for the significance of the present study.

The Prevalence of Angle Within and Beyond School Mathematics

Angle and angle measure are pervasive topics in school mathematics curricula. For example, within the Common Core State Standards for Mathematics (CCSSM), students are first exposed to angles in Grade 2 when they are expected to recognize and draw shapes with a specified number of angles (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). Angle measure, as well as protractors, are introduced in the Grade 4 standards for measurement and data:

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
    a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An arc that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
    b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure. (pp. 31–32)

Other Grade 4 standards within the CCSSM suggest students should classify angles (e.g., acute, right, obtuse), recognize right triangles, draw perpendicular lines, and understand that angle measure is additive. In Grade 5, students classify two-dimensional figures into categories (e.g., square, rectangle, etc.) according to properties, including number and size of angles. In Grade 7, students construct triangles when given three angle measures and relationships between angles (e.g., supplementary, complementary, vertical, and adjacent) to find the unknown measure of an angle within a given figure. In Grade 8, students are to learn that angles are taken to angles of the same measure under rigid motions, use angles to justify similarity or congruence of two-dimensional figures, and establish properties involving the sums of measures of interior, as well as exterior, angles of triangles.

At the high school level in the CCSSM, angle and angle measure are featured prominently in the standards for geometry and trigonometry. Students investigate angle and angle measure when learning about transformations, proof, Euclidean construction, congruence, similarity, right-triangle trigonometry, and complex numbers in polar form. Students use angles when reasoning about relationships in circle contexts using arcs, chords, radii, and sectors. Students are to understand radian measure using arcs on a unit circle and extend the domain of trigonometric functions to all real numbers.

Right angles occur other contexts throughout the CCSSM. The classifications of some geometric solids (e.g., right rectangular prisms) are rooted in perpendicularity. The Pythagorean theorem and its converse involve right triangles. The Cartesian coordinate system is formed by orthogonally oriented axes. When graphing lines in the Cartesian plane, students are asked to write equations of lines perpendicular to a given line.

In the paragraphs above, I have illustrated the pervasiveness of angle and angle measure within the CCSSM. Without a doubt, there are other mathematical ideas and curricula involving angle and angle measure. For example, angle measure is critical in the construction of other coordinate systems (e.g., polar, bi-polar, cylindrical, and spherical), which do not appear in the CCSSM. Barabash (2017) outlined the role of angle and angle measure in higher mathematics and applied fields. These topics are found in vector calculus, complex analysis, astronomy, and physics. On the concept of angle, Barabash noted, "no mathematical curriculum can do without it" (p. 31).

Although angles are prevalent in school mathematics, individuals also use angles outside of traditional classroom environments. Angles are often used in carrying out practical activities like arts, crafts, and construction (Smith & Barrett, 2017). The construction of angle plays a critical role when young children come to differentiate round and polygonal shapes (Piaget & Inhelder, 1967). Individuals spontaneously use angle measure when describing the locations and motions of points in two- and three-dimensional space (Lee, 2017).

As I've shown in the preceding paragraphs, angle and angle measure are pervasive topics within and beyond school mathematics. As such, mathematics educators need to understand how students quantify angularity, which is a major goal of the present study.

Students' Challenges with Angle are Documented

Examining how students quantify angularity is important because considerably less is known about this topic than students' understandings of other quantities like volume, length, and area (Smith & Barrett, 2017). What mathematics educators do know is that students have difficulty understanding angles and their measures. As such, this study is important precisely because mathematics educators know that quantifying angularity is non-trivial. In the following sections, I synthesize findings from extant research, which has been conducted with elementary, middle, and undergraduate students. Research on high school students' quantifications of angularity is scarce, which is an additional reason why this study is merited with high-school aged students.

Findings from studies with elementary and middle grades students. In the elementary and middle grades, researchers have found students often conflate angularity with other attributes. Students often judge which of two angles is larger by comparing the lengths of the angle models' sides (Baya'a, Daher, & Mahagna, 2017; Clements, Battista, Sarama, & Swaminathan, 1996; Crompton, 2017; Devichi & Munier, 2013; Keiser, 2004; Lehrer, Jenkins, & Osana, 1998). Students also consider angle measure to refer to the linear distance between an angle model's sides (Baya'a, Daher, & Mahagna, 2017; Keiser, 2000; 2004; Lehrer, Jenkins, & Osana, 1998; Thompson, 2013). For example, when 60 elementary children enrolled in a three-year longitudinal study were asked to measure the amount of opening in various contexts (hinged wooden jaws, open door, or a bent straw), the children measured the linear distance between the ends of the objects 95% of the time (Lehrer, Jenkins, & Osana, 1998). It is common for students to believe an angle has a different measure if its orientation changes (Crompton, 2017; Devichi & Munier, 2013; Lehrer, Jenkins, & Osana, 1998), though the effects of orientation decrease substantially by Grade 5 (Lehrer, Jenkins, & Osana, 1998). Students also associate angle measure with the area of a triangle (Thompson, 2013). Finally, some students interpret angle measure as a characterization of sharpness and thus might describe an obtuse angle as having a lesser measure than an acute angle (Keiser, 2004; Owens, 1996).

Researchers have argued an awareness of rotational motion is an important development in a student's ability to estimate the measures of angles in degrees (Clements et al., 1996). The physical enactment of body or hand rotations has been shown to be productive for students learning of angle measure in degrees (Clements et al., 1996; Clements & Burns, 2000; Smith, King, & Hoyte, 2014). Over time, students' may internalize rotations, which results in a curtailment of the physical gestures as students become able to enact the motions mentally (Clements et al., 1996; Clements & Burns, 2000).

Though rotational motion may be important for students' understandings of angle measure, many studies have shown students rarely conceptualize rotational motion when an angle is presented as two sides sharing a common vertex (Clements, et al., 1996; Crompton, 2013; Lehrer, Jenkins, & Osana, 1998; Mitchelmore, 1998; Mitchelmore & White, 1995; 1998; 2000). About this difficulty, Mitchelmore concisely remarked, "…young students do not spontaneously conceptualize turning (as found in rotation and hinging situations) in terms of angles" (1998, p. 278). The recognition of a dynamic turn as a static angle model is difficult for students because it requires that students re-present an internalized mental image of a rotating ray, including the initial and final headings (Clements et al., 1996; Clements & Burns, 2000; Mitchelmore, 1998; 2009; Mitchelmore & White, 1995; 1998). Less than 10% of elementary participants in a three-year longitudinal study described or represented angular contexts in terms of turns, sweeps, or rotations (Lehrer, Jenkins, & Osana, 1998). Even when students do recognize turns as static angle models, they can still have difficulty matching the size of these static models to the size of the turn (Mitchelmore, 1998). Due to students' difficulties interpreting turns as static angle models, Mitchelmore & White recommended that defining angles via turn be excluded from students' early instruction on angle: "the definition of angle as an amount of turning between two lines has been widely promoted in the literature on the elementary mathematics curriculum…but must now be regarded as completely inappropriate" (1998, p. 25).

In the literature, there are many other documented issues related to students' recognition of angles. Students often consider two curves intersecting at a common point to form an angle (Keiser, 2004; Matos, 1999). In terms of static angle models, right angles are the prototypical angle for many students (Devichi & Munier, 2013; Matos, 1999); however, children are less likely to view obtuse angles as angles than acute angles (Matos, 1999). Students recognize convex angles as angles more often than concave angles (Matos, 1999). Finally, some students have difficulty interpreting 0°, 180°, and 360° angles in terms of static angle models because there is no visible vertex at the corner of two distinct sides (Keiser, 2000; 2004).[8]

Findings from studies with undergraduate students. At the undergraduate level, researchers have found that students rarely hold coherent meanings for different standard angular units. Preservice teachers are more comfortable with degrees than radians and often convert from radians to degrees to understand the magnitude of an angle described in radian measure (Akkoc, 2008; Fi, 2003; Topçu, Kertil, Akkoc, Kamil, & Osman, 2006). Several researchers have found that few preservice teachers define radian measure as the ratio of an intercepted arc to a circles radius (Topçu et al., 2006; Tuna, 2013). Numerical values for angle measures are often interpreted as measured in

[8] Historically, mathematicians have debated whether angles can take these measures as well, according to Keiser (2004).

degrees by default, unless the value contains the symbol, π (Akkoc, 2008). Like some elementary and middle grades students, some preservice teachers also have difficulty identifying 0°, 180°, and 360° angles due to the absence of two visible, distinct rays (Yigit, 2014). Many researchers have attributed students' difficulties in trigonometry to impoverished meanings for angle measure (e.g., Akkoc, 2008; Fi, 2003; Topçu et al., 2006); this issue is discussed in greater detail in the next section.

Problems with Angle Measure in US Curricula

Studying how students quantify angularity is important for improving mathematics pedagogy and curricula. Both Thompson (2008) and Moore (2013) have noted that trigonometry is a difficult topic for students in the U.S, and they attribute this difficulty in part to fragmented meanings for angle measure. Thompson (2008) argued triangle trigonometry is often developed without sufficient emphasis on angle measure. Such an emphasis requires that angle measures be taken as the arguments of trigonometric functions rather than triangles. The later emphasis on triangles is often enforced in US classrooms when teachers request that students directly apply the pneumonic SOH-CAH-TOA when evaluating trigonometric functions in right triangle contexts.[9] Thompson also noted that triangles, and therefore angles, are often presented as static objects in triangle trigonometry. As a result, angles and their measures lack the variation needed to develop coherent meanings for trigonometric ratios across right triangle, unit circle, and function contexts (Moore, 2013; Thompson, 2008).

In addition to considering the relationship between the development of trigonometry and angle concepts, both Moore (2013) and Thompson (2008) have argued

[9] SOH-CAH-TOA: Sine is Opposite over Hypotenuse; Cosine is Adjacent over Hypotenuse; Tangent is Opposite over Adjacent.

that the development of angle measure in earlier grade levels is also deficient. In the earlier grades, degrees are emphasized as a unit of measure without explicit attention to what quality of an angle is being measured in these units. Furthermore, little attention is given to the process by which this quality is measured. Moore (2013) argued students' initial experiences with angle measure often involve learning how to use a protractor and categorize particular angular configurations (e.g., acute, obtuse, supplementary, complementary, and vertical).[10] By relying on pre-constructed measurement tools (e.g., physical or electronic protractors), opportunities for students to quantify angularity are, at worst, absent and obscured at best. Thompson (2008) claimed when angle measure is developed in earlier grades it is often developed procedurally by asking students to convert fractions of complete rotations to equivalent fractions of 360. This failure to attend to both the measurement process and the quality being measured does not support students to quantify angle measure.

In addition to problematic treatments of angle measure in earlier grade levels, Moore (2013) posited different approaches are often taken to introduce degrees and radians as units for measuring the openness of angles. Although degrees are often introduced with instruction as to how to appropriately position and read numerical values from a protractor, radians are rarely, if ever, introduced with a protractor approach. Instead, radian measure is developed as a multiplicative comparison between an arc length subtended by a central angle and the radius of the corresponding circle. Moore claimed these different approaches to angle measure result in disconnected meanings for angle measure.

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[10] Moore's critique is consistent with the summary of CCSSM standards I presented earlier in this chapter. In particular, it is noteworthy that angle measure, degrees, and protractors are all introduced in Grade 4.

Thompson (2008) remarked that arcs in angle diagrams add little to students' quantifications of angle measure. An arc like the one indicated on the left in Figure 1.3 serves primarily as a pointer to an *object* and might as well be replaced by the pointer shown at right in Figure 1.3. Instead, Thompson argued arcs on angle diagrams should serve as pointers to the *process* by which an angle can be measured. The following section elaborates on the quantifications for which Moore (2013) and Thompson (2008) advocated.

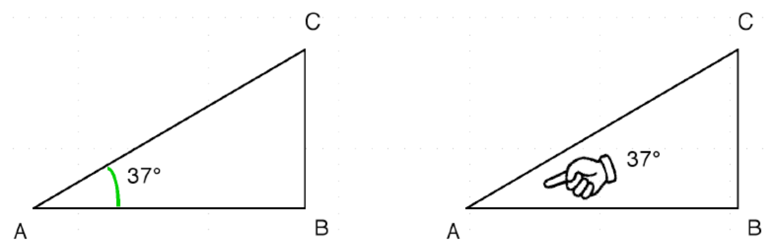

Figure 1.3. Arc as pointer (Thompson, 2008, p. 49).

A Quantitative Approach to Angle Measure

In this section, I synthesize existing literature that has approached angle measure from a quantitative reasoning perspective. I outline the circular quantification of angularity for which researchers have previously advocated, and I describe the results from one teaching experiment in angle measure, which emphasized quantification. Following this discussion, I argue that, while this circular quantification of angularity is productive for the study of trigonometry, this quantification is not appropriate for all mathematics students.

Existing approaches to quantifying angularity. Thompson (2008) recommended a circular quantification of angularity, which he framed as a process for measuring the *openness* of an angle. This measurement process consists of drawing a circle centered at the angle’s vertex, measuring the arc subtended by the angle in units that are 1/360 the circle’s circumference, and using this measure to describe the openness of the angle. Thompson argued taking an arc-length approach to developing angle measure is productive as it allows for coherent interpretations of angles measured in both degrees and radians.

Moore (2013) argued for a similar development of angle measure and has elaborated this approach.[11] If degrees and radians are intended to measure the same attribute, then teachers should strive to engender compatible meanings for these units of measure in the minds of students. Angle measures in both radians and degrees “can be thought of as conveying the fractional amount of any circle’s circumference subtended by the angle’s rays, provided that the circle is centered at the vertex of the angle” (Moore, 2013, p. 78). Thus, one degree is defined in relation to an arc whose length is 1/360 of the corresponding circle’s circumference; similarly, one radian is defined by an arc whose length is $1/(2\pi)$ of the corresponding circle’s circumference. In fact, any unit that is proportional to a circle’s circumference could be used to measure an angle (e.g., diameter or quarter-circumference). Thus, all units of angle measure that develop in conjunction with this circular quantification of angle measure ultimately describe the same quantitative relationship: the multiplicative comparison of an arc subtended by a central angle and the circumference of the corresponding circle; this multiplicative comparison is invariant across all possible circles (Figure 1.3).

[11] See also Moore (2009; 2010; 2012; 2014). Tallman (2015) took an equivalent approach in his work with a practicing secondary teacher.

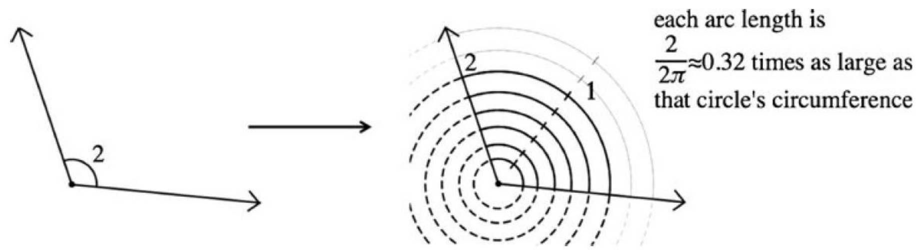

Figure 1.4. An image of angle measure that involves an equivalence class of arcs (Moore, 2013, p. 228).

Empirical results from Moore. In his 2013 report, Moore presented results from a teaching experiment with two college students wherein he tried to foster the circular quantification of angularity described above. Moore characterized the development of the students' quantifications across four sessions: a pre-interview session, two teaching sessions, and a post-interview session. In the pre-interviews, the students' ways of reasoning about angle measure were dominated by thinking about geometrical objects in terms of degrees (e.g., a circle has 360°, a line has 180°, perpendicular lines have 90°). Additionally, the students labeled angles using arcs, but the arcs were merely indicators of an angle to be considered and were not associated with a measurement process. Moore noted the students "appeared to have no discernible scheme of what one measures when measuring an angle, nor did their angle measure understandings include a quantitative meaning for the units used to measure the openness of an angle" (p. 241). Although the students were able to perform appropriate calculations in given situations by dividing a subtended arc length by a circle's circumference, the students did not interpret the result of the division quantitatively as the fraction of the circle's circumference subtended by the angle.

During the teaching sessions, the students engaged in tasks designed to engender an understanding of angle measure as a multiplicative comparison between the arc length subtended by a central angle and a circle's circumference. Tasks posed to the students included creating a protractor for measuring angles in nonstandard units (e.g., where 8 units or 15 units were required to rotate a circle), investigating the number of times a radius length is contained in a circle's circumference, and creating central angles that subtended arcs of a specified number of radius lengths. Tasks in the post-interviews involved determining arc-lengths subtended by a 35° central angle across three concentric circles with known radius lengths and developing a formula to relate arc length, radius length, and angle measure.

Over the course of the teaching sessions, Moore's students developed an understanding that measurements of an angle in various units convey the same information: the proportion of an arbitrary circle's circumference that is subtended by an angle with the given measure. The students' explanations of numerical expressions after the pre-interviews suggested that they were able to interpret their numerical operations within the quantitative structure of the given situation. By the end of the study, the students did not need to carry out numerical operations in order to provide quantitative interpretations; instead, the students were able to reason with the unspecified quantities of arc length, radius length, and angle measure in order to write equations relating the unspecified quantities.

Circular quantifications of angularity are not appropriate for all students. Moore's (2013) study indicates some undergraduate students can develop powerful circular quantifications of angularity. I agree with Moore and Thompson that circular quantifications of angularity are especially productive for trigonometry, precisely because these quantifications result in coherent interpretations across various units. However, these circular quantifications of angularity are quite sophisticated and rely upon generalizing multiplicative comparisons of circular lengths. Mathematics educators cannot take for granted that all students can make these kinds of multiplicative comparisons, much less generalize them as holding across all possible circles.

As I showed in my synopsis of CCSSM standards involving angles, students are formally introduced to angle measure in Grade 4. According to estimates from Steffe (2017), 75% of fifth graders would not yet be able to enact the sophisticated multiplicative coordinations required to establish a circular quantification of angularity.[12] However, the only process for measuring an angle outlined in the CCSSM relies on such a quantification. The present study is crucial because I examine both circular and non-circular quantifications of angularity; as such, the results of this study have significant implications for early instruction on angle measure. My hypothesis is that non-circular quantifications of angularity necessarily precede circular quantifications. However, no empirical research to date has examined students' non-circular quantifications of angularity.

Critique of Existing Literature from a Quantification Perspective

In the preceding section, I argued that circular quantifications of angularity are not appropriate for all mathematics students. In this section, I argue that researchers, save Moore, Thompson, and Tallman, have not sufficiently attended to students' quantifications of angularity in previous studies on angles. Here, I critique existing

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[12]Specifically, Steffe estimated that only 25% of fifth-grade students could coordinate three levels of units in assimilation. Units coordination will be discussed in more detail in Chapter 2.

literature focused on children's understandings of angle by considering the three elements involving in Thompsons (2011) characterization of a quantity—object, attribute, and measurement process (i.e., quantification). Additionally, I consider non-empirical practitioner articles which are relevant for the present study.

Studies focused primarily on angle as an object. The majority of extant angle literature has focused primarily on students' conceptions of angle as an object with little attention to (a) an attribute to be measured or (b) a process by which to measure this attribute (Keiser, 2000; 2004; Keiser, Klee, & Fitch 2003; Mitchelmore, 1997; 1998; Mitchelmore & White, 1995; 1998; 2000; Prescott, Mitchelmore, & White, 2002; Silfverberg & Joutsenlahti, 2014). Keiser and colleagues have focused primarily on students' definitions of angles (Keiser, 2000; Keiser, Klee, & Fitch 2003) and whether students think an angle is formed when two curves share a common vertex (Keiser, 2004). Silfverberg and Joutsenlahti (2014) investigated whether individuals viewed points within the interior of an angle as part of the angle itself.

Elementary and middle grades students' conceptions of angles in physical situations have been the focus of many studies by Mitchelmore & colleagues (Mitchelmore, 1997; 1998; Mitchelmore & White, 1995; 1998; 2000; Prescott, Mitchelmore, & White, 2002). For example, Mitchelmore & White (2000) interviewed 194 participants enrolled in grades 2, 4, 6, and 8 to investigate students' recognition of angles in various contexts (e.g., wheel, door, scissors, etc.; see Figure 1.4). During the interviews, students were provided with a bendable straw and asked to use the bendable straw to represent each contextualized model (e.g., "how the wheel turns" or "how the hill slopes" (p. 223).) Apart from examining whether students could establish angular congruence for different models, Mitchelmore and White (2000) did not investigate students' quantifications of angularity, which would have entailed examining how students might measure the relevant angular attribute for each of the angle models used in the study.

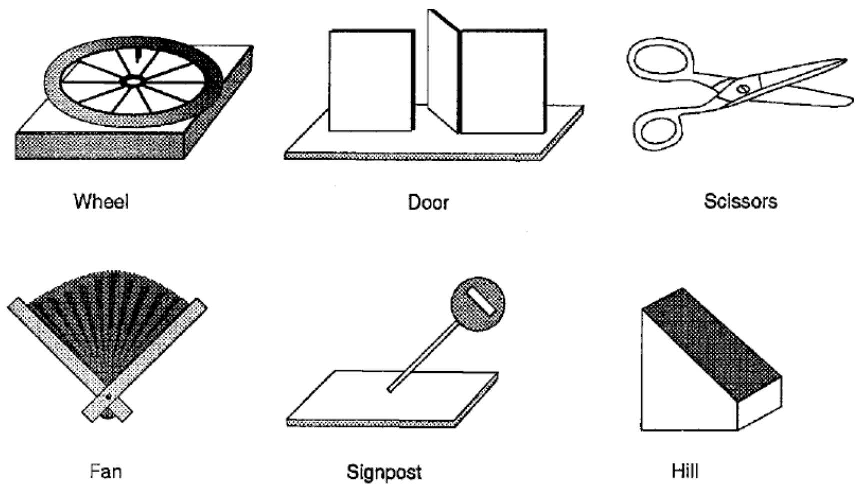

Figure 1.5. Examples of angle models used by Mitchelmore & White (2000, p. 221).

Studies primarily focused on salient attributes. Other studies have focused primarily on the attributes students find salient when attending to angle models (Devichi, & Munier, 2013; Lehrer, Jenkins, & Osana, 1998; Noss, 1987). For instance, several tasks used in a three-year longitudinal study reported by Lehrer, Jenkins, and Osana (1998) fall into this category. I discuss three such tasks here.

In a one task, students were presented with four triads of bent pipe cleaners and were asked to select the pair that was most similar within each triad. The authors explain the triads were designed to explore students' conceptions related to categories of angles (acute, right, obtuse), variations in length, and variations in orientation. The four triads are shown in Figure 1.5.

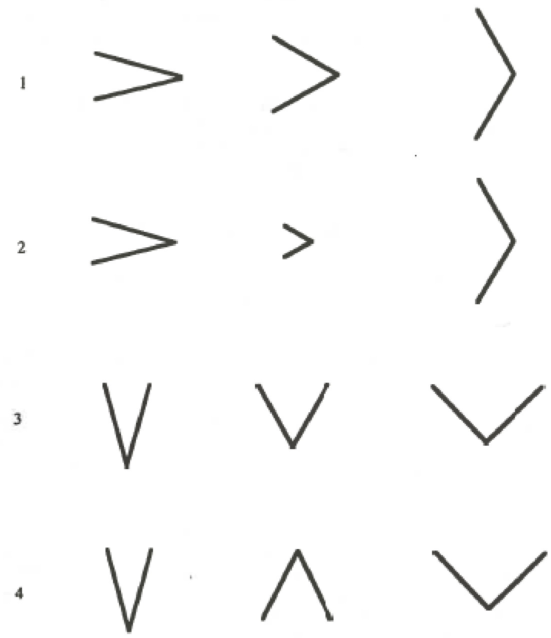

Figure 1.6. Triads of angle models (Lehrer, Jenkins, & Osana, 1998, p. 147).

Students' justifications for the most similar pair within each triad were coded as belonging to one of six categories: (a) orientation, (b) length of segments, (c) distance between legs, (d) sweep, arc, or turn, (e) overall appearance (e.g., these two look the same), or (f) conventional angle measure.

In a second task, students were asked to draw three different angles and explain how each was different from the other; this task was coded in the same way as the responses for previous task. Variations of this task were used by Noss (1987) and Devichi & Munier (2013). In a third task, students were asked to measure the opening (or bending) of four situations: two hinged pieces of wood, an open door, a bent pipe cleaner, and a bent straw. Students' responses were coded based on the part of the angle that students measured: ray length, distance between endpoints, rotation, conventional angle measure, or other.

Studies like the one reported by Lehrer, Jenkins, and Osana (1998) offer important insights regarding the attributes of angle models that are salient for students. These studies offer little information regarding students' quantifications of angularity as researchers have primarily tracked the attributes students *would* measure (e.g., length or rotation) without attending to the mental operations undergirding these measurement processes.

Studies focused on measurement, but not quantification. Finally, some studies have focused on students' abilities to measure angles in degrees (Crompton, 2013; 2017; Clements, Battista, Sarama, & Swaminathan, 1996; Clements & Burns, 2000; Simmons & Cope, 1990; Smith, King, & Hoyte, 2014). However, these studies have provided students with protractors, either through LOGO (Clements, Battista, Sarama, & Swaminathan, 1996; Clements & Burns, 2000; Simmons & Cope, 1990) or other dynamic geometry software (Crompton, 2013; 2017; Smith, King, & Hoyte, 2014). For example, Clements & Burns (2000) examined two fourth-grade students use of body rotations to establish an understanding of angle as turn. The students engaged in a set of activities within a LOGO computer environment that featured a turtle whose movement was directed by distance commands and turn commands. For example, the command sequence "fd 50 rt 30 fd 40" would direct the turtle to move in the direction of its current heading 50 units, to turn 30 degrees to the right of its current heading, and then move 40 units in the direction of its new heading. An additional tool within the computer environment showed an electronic protractor in 30° increments (Figure 1.6).

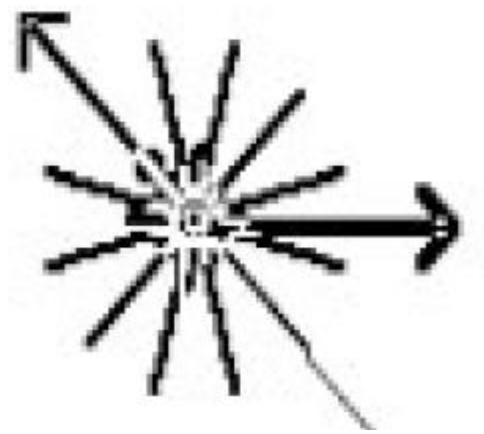

Figure 1.7. An electronic protractor in 30° increments (Clements & Burns, 2000, p. 33).

Studies in this category are limited in examining students' quantifications of angularity precisely because researchers have provided students with standard tools for measuring angles during the study. Providing students with a protractor impedes an investigation of how students would quantify angularity in a given context.

Recommendations for practitioners. Non-empirical publications directed toward practicing elementary and middle grades mathematics teachers have recommended classroom activities pointing toward non-circular quantifications of angularity (Browning, Garza-Kling, & Sundling, 2007; Host, Baynham, & McMaster, 2014; Millsaps, 2012; Sherman, 2017; Wilson, 1990; Wilson & Adams, 1992). In each of these articles, the authors recommend tasks involving measuring one angle using another angle (see Figure 1.7); in these tasks, the angle taken as a unit for measuring is often referred to as a "wedge."

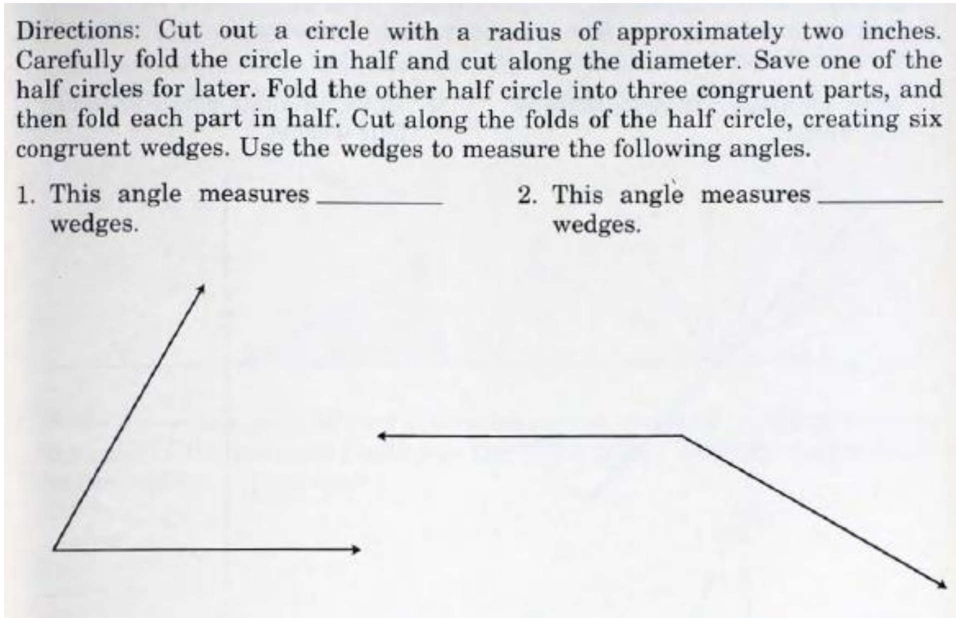

Figure 1.8. A task targeting non-circular quantification (Wilson, 1990, p. 297).

The activities recommended in these practitioner articles suggest non-circular quantifications of angularity may be an alternative starting point for early angle measure instruction; however, there are no extant empirical or theoretical publications with a particular focus on angle measure supporting these activities. Therefore, the present theoretically-grounded and empirical study is significant.

Summary of the Problem and Research Questions

Angle measure is a critical topic in school mathematics; yet, relative to other quantities, few researchers have examined how students' quantifications of angle measure, and research at the high-school level is remarkably scarce. In fact, Moore (2013) explicitly called for research on high school students' quantifications of angularity. What we do know from existing literature is that quantifying angularity is non-trivial for students. Although previous researchers have found circular quantifications of angularity to be productive for the study of trigonometry; such quantifications are not an appropriate starting point for all students.

Circular quantifications aside, extant research has predominantly focused on student's conceptions of angles as objects or the attributes students find salient when considering angle models. Students' quantifications of angularity have received considerably less attention. Studies examining students' ability to measure angles in degrees have provided students with protractors, which obscures investigating students' quantifications of angularity. Although activities in practitioner literature point toward non-circular quantifications of angularity, there is no empirical research on angle supporting these recommendations. For all these reasons, this study is critical for the field of mathematical education.

Research Questions

The overarching purpose of this study was to examine how high school students understood angle measure. In this dissertation, I address the following three specific research questions:

1. What motions and operations serve in students' constructions of angularity?
2. What are students' quantifications of angularity at the onset of the teaching experiment, and how do these quantifications change throughout the teaching experiment?
3. How do students' quantifications of angularity compare in rotational and non-rotational contexts?

The overall goal of the teaching experiment is foregrounded in the second research question. The first and third research questions were motivated in part by claims from other researchers. As I explained in my review of research on angle, researchers have argued that an awareness of rotational motion is critical for students' quantifications of angularity; I developed the first research question to examine whether students might find other motions relevant when considering angles. Additionally, researchers have argued that students are less likely to recognize rotational contexts as involving angles than non-rotational contexts; therefore, I developed the third research question to examine whether students' quantifications in these contexts also differed.

Preview of Subsequent Chapters

In Chapter 2, I discuss additional theoretical elements that are relevant for this study. Chapter 3 is devoted to the research methods, and there I outline the teaching experiment methodology and specific characteristics of my teaching experiment. In Chapter 4, I describe the results from participants' initial interviews. In Chapter 5, I discuss the results of the study. In Chapter 6, present conclusions and implications from this study.

CHAPTER 2

THEORECTICAL ELEMENTS

In this chapter, I present theoretical elements relevant to the present study. I begin with an explication of the basic principles of radical constructivism. Thereafter, I outline von Glasersfeld's (1995) scheme theory, and I describe some specific mental operations I will use to model students' mathematics in subsequent chapters. I close with a brief initial conceptual analysis of angle measure, which informed my thinking when conceiving the study.

Radical Constructivism

A theoretical orientation is essential for conducting research as a primary purpose of research is to answer questions about our worlds. Therefore, assumptions about the nature of reality and how these realities can be known must be made explicit in order to determine the kinds of research questions that are sensible to ask, as well as how these questions can be investigated and answered. I have adopted the radical constructivist perspective of knowing and learning, which was developed by Ernst von Glasersfeld. Radical constructivism has two major principles:

1. Knowledge is not passively received but built up by the cognizing subject;
2. The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality. (von Glasersfeld, 1995, p. 18)

The first principle of radical constructivism highlights that knowledge is neither innate nor can an individual receive knowledge from an "exterior" source, like another individual. Instead, an individual actively constructs knowledge from her experiences and her previous constructions. The second principle of radical constructivism asserts the purpose of cognition is to organize an individual’s experiential reality, which is the only reality to which she has access; this second principle is the reason for *radical* in radical constructivism.

An individual is always constrained to her own ways of perceiving and reasoning; it is impossible for an individual to experience a reality independent from herself. Instead of characterizing knowledge as true, which suggests knowledge as a reflection of an observer-independent reality, radical constructivists characterize knowledge in terms of viability. The knowledge an individual constructs is viable precisely because this knowledge has fit the particular needs of her experience (von Glasersfeld & Cobb, 1989).

From the radical constructivist perspective, mathematics does not exist readily made somewhere in the world for individuals to discover. Instead, mathematics, like all knowledge, is a product of human intelligence and exists in the minds of individuals (Steffe & Thompson, 2000; Steffe & Wiegel, 1992). My organization of my own mathematical knowledge constitutes first-order knowledge. As a constructivist teacher and researcher, I acknowledge students have constructed mathematical concepts, which are different from my own mathematical concepts. In this sense, students create their own mathematical realities.

Steffe and Thompson (2000) use the phrase “students’ mathematics” to refer to the mathematical realities students have constructed. My goal as a constructivist teacher and researcher is to try to understand students’ mathematics, to use my understandings of students’ mathematics to shape my interactions with students, and to engender changes in students’ ways of reasoning. Students’ mathematics exists in the minds of students and, therefore, is not directly available to observers. Thus, teachers and researchers must rely on students' language and actions to make inferences about students' mathematics; Steffe and Thompson (2000) use the phrase "mathematics of students" to refer to these second-order models teachers make of students' mathematics.

In this dissertation, my goal is to create second-order models of a particular subset of students' mathematics, namely students' quantifications of angularity. In the next section, I discuss scheme theory, which is a set of analytical tools radical constructivist researchers have used to construct second-order models.

Scheme Theory

To describe and explain students' quantifications and changes in students' quantifications, I draw on von Glasersfeld's (1995) scheme theory. *Schemes* refer to characteristic ways of operating that an individual has established from her prior experiences. Schemes entail a three-part structure: (a) an individual's recognition of a particular situation; (b) an activity associated with that situation; and (c) a result that is beneficial or expected. von Glasersfeld (1989) illustrates an example of a scheme involving an infant and a rattle. An infant may develop a rattle-shaking scheme out of her experience by noticing a pleasing noise (from the infant's perspective) is produced whenever she shakes a graspable item with a round shape at one end. In this case, the first part of the scheme consists of the infant's recognition of a rattle—a graspable item with a round shape at one end. The activity of the scheme is the action of shaking the rattle. The result of the scheme is the infant's perception of the pleasing noise generated by the shaking action.

The first part of a scheme involves an *assimilation*—an individual's recognition of a previous experience in a present experience. Assimilation occurs based on the available structures of the cognizing individual; thus, situations different from an observer's perspective may be assimilated by an individual to the same scheme. If an infant, having developed a rattle-shaking scheme, were to grasp and begin shaking another item with a round shape at one end, an observer might notice the infant has grasped a spoon. However, from the infant's perspective at that point, the graspable item *is* a rattle; from an observer's perspective, the infant has yet to differentiate between the rattle and the spoon. The infant is only attending to rattle-like features of the graspable item and treats the item as an appropriate situation for the rattle-shaking scheme.

Upon shaking the spoon, the infant may be disappointed that the spoon does not produce the expected rattling sound. The disappointment experienced by the infant when the shaking activity does not lead to the expected result is an example of a *perturbation*. Upon experiencing this perturbation, the infant may focus on the item in-hand that was originally assimilated to the rattle-shaking scheme and may come to differentiate the item as a non-rattle. If the infant were to subsequently differentiate between spoons and rattles by assimilating the later to the rattle-shaking scheme and not the former, then an observer could say that the infant made an *accommodation*—a lasting modification to a scheme.

Mental Operations

The rattle-shaking scheme described above is an example of a scheme wherein the activity of scheme involves physical action, namely the rattle shaking. Schemes can involve physical actions as well as mental operations. Mental operations involve conceptual activity rather than physical activity.[13] In the following sections, I describe some mental operations I use in Chapters 4 and 5 to model students' quantifications of angularity. In my description of these mental operations, I focus on operations that serve in the generation of extensive quantity; that is, operations that produce units. Because angularity is a continuous quantity, I present a synopsis of Steffe's (2013) explication of the operations that serve in the construction of length, which informed my conceptual analysis of angularity.

The unitizing operation. The context of the giraffe in an enclosure that I presented in Chapter 1 provides a context for discussing von Glasersfeld's (1995) unitizing operation, which is in a sense a basic operation because individuals construct this operation early in life. I claimed that the father and the four-year-old would be able to isolate the giraffe from the rest of the perceptual material in the enclosure while the newborn son would not. Units—individual items abstracted from experience—do not appear readily made in the world around us. Experiential units must be produced by a cognizing subject through isolating particular sensory data (e.g., the giraffe) and distinguishing those from other sensory data (e.g., the rest of the enclosure). von Glasersfeld's model of the unitizing operating relies on attentional pulses—moments focused on particular sensory signals. According to the model, an experiential unit item constructed from sensory input begins with an unfocused moment of attention, a sequence of focused moments, and finally a terminal unfocused moment of attention which bounds the experiential unit item.[14]

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[13] Mental operations may be coupled with observable physical actions, especially if the operations cannot be enacted entirely in re-presentation (Les Steffe, Personal Communication, March 26, 2018).
[14] See Steffe & Cobb (1988) for a full discussion of the different kinds of units that can be produced.

Re-presentation. Re-presentation is the capability to bring forth a mental image of a prior experience in the absence of perceptual input. For example, if the four-year-old were to imagine the giraffe from the zoo on the drive home, the four-year-old re-presented a giraffe. von Glasersfeld (1995) insisted on a hyphen in "re-presentation" to emphasize the "repetition of something that was present in a subject's experiential world at some other time" (p. 95). The subject an individual re-presents need not be limited to a single sensory channel. For instance, re-presenting the act of swimming might entail an individual recalling the motion of their limbs, as well as the feeling and smell of the water.

The construction of length. According to Steffe (2013), the construction of length involves bounded motion (e.g., walking along a path or visually scanning up a giraffe) of some kind. If an individual is aware of the duration of the motion from beginning to end as well as the visual records of this motion, then an individual has constructed an *awareness of experiential length*; that is to say, an individual has constructed an experiential continuous unit item. An individual has constructed an *awareness of figurative length* if she can re-present the visual records, the bounded motion, and the trace of this motion in the absence of the original perceptual input. If an individual having constructed an awareness of figurative length interiorizes the object (i.e., mentally strips the object of its particulars), then the individual has constructed an *awareness of length* of the interiorized object concept

Operations for making units from units. Throughout their discussion of children's fractional knowledge, Steffe and Olive (2010) specify operations that produce units from other units. In this section, I describe operations I use in modeling quantifications of angularity.

Uniting is the application of the unitizing operation to two or more unitary items. When the unitizing operation is applied, the result is a composite unit. If additional unit items are united with an already formed composite unit an individual may engage in *progressive integration*. If a child has established a particular collection of marbles has numerosity eight and the child is given two more marbles, a child might continue counting from eight saying, “nine, ten,” progressively integrating the two additional units to form a new composite unit.

When a unit item is iterable, a composite unit can be formed though *iteration*—mentally uniting copies of the unit item. When an individual forms composite units from iterable units, the constituent units can be simultaneously viewed as part of and not part of the composite unit. That is to say, an individual can *disembed* a part from the whole without destroying the whole. Furthermore, when a unit of one is iterable a number word like “ninety” refers to a unit “one” that could be iterated ninety times to create the composite unit. When an individual can take a composite unit as input for the iteration operation, then the individual has constructed an *iterable composite unit*. Having constructed 10, for example, as an iterable composite unit, an individual might view 90 as being composed of 9 tens where each ten is composed of 10 ones.

The operations outlined in the paragraph above build up units from other units. Units can also be produced by breaking a given unit into parts. When there is no restriction of the size of the parts, Steffe & Olive (2010) use *fragmenting* to refer to simultaneous breaking a whole into parts and *segmenting* to refer to the sequential production of parts from a whole. *Equipartitioning* is a form of fragmenting wherein a composite unit is projected into a whole and the constituent units of the composite unit are iterable. The equipartitioning of a continuous unit item requires a *simultaneous projection* of same-sized parts where any one of the parts could be used in iteration to constitute the whole (Figure 2.1). In contrast, *equisegmenting* refers to the *sequential production* of equal-sized parts intended to exhaust the whole. The sequential production of equal-sized parts is illustrated in Figure 2.2. Due to the sequential production of parts within the whole, equisegmenters tend to produce less accurate estimates when physically sharing items than equipartitioners (Ulrich, 2016b).

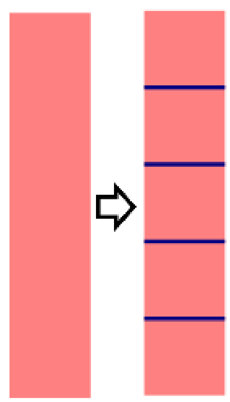

Figure 2.1. The simultaneity of the equipartitioning operation

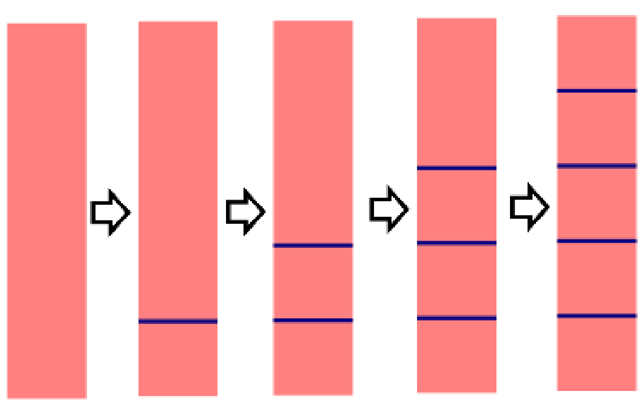

Figure 2.2. The sequentiality of equisegmenting operation

If individual comes to understand partitioning and iterating as inverse operations, he may unite these two operations into a single mental operation. The *splitting operation* refers to the simultaneous implementation of partitioning and iterating operations.

Splitting. The splitting operation is indicated if an individual uses partitioning to solve a task stated in terms of iteration (Hackenberg, 2007), such as the task shown in Figure 2.3.

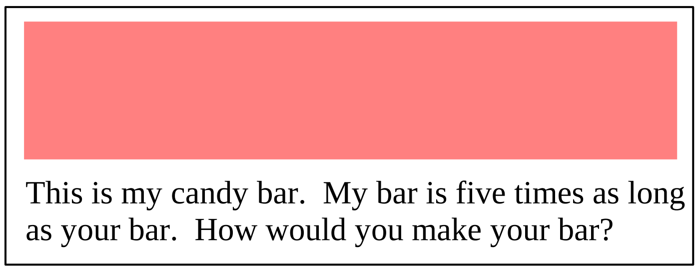

Figure 2.3. A splitting task.

In such a task, individuals who have constructed the splitting operation, who are informally referred to as splitters, *can anticipate* the size of their bar prior to engaging in any sensorimotor activity. A splitter would mentally posit a hypothetical bar, which could be iterated five times to constitute the given bar; because a splitter has unified iteration and partitioning, he would know that if five copies of the hypothetical bar exhaust the given bar, then the five copies constitute an equipartitioning of the given bar. Therefore, a splitter might solve the task by equipartitioning the given bar into five parts and producing a copy of one of those parts. Individuals who have not constructed the splitting operation often *need* to engage in sensorimotor activity (e.g., making an estimate for their bar) to solve splitting tasks, or they might interpret the task as a call to iterate the given bar five times.

Recursive partitioning. Splitting is the fundamental operation serving in the construction of the recursive partitioning operation (Steffe, 2003). *Recursive partitioning* involves “partitioning a partition in service of a non-partitioning goal” (Hackenberg & Tillema, 2009, p. 2). An example of a recursive partitioning task is shown in Figure 2.4.

> Imagine a long strip of candy. You share the candy fairly among yourself and three friends (4 people in total), and you take your piece. But, before you can eat it, four more friends show up. You share your piece fairly among the five of you. What amount is your little piece out of all of the candy?

Figure 2.4. A recursive partitioning task.

An individual who has constructed the recursive partitioning operation might solve the task by first mentally producing their piece of candy; she might imagine a strip partitioned into four equal shares, and equipartition one of those shares into five pieces, and take one of these small pieces as her own (Figure 2.5). Then she might establish a goal of determining how many copies of her piece she would need to re-constitute all of the original candy. Having established this goal, she would then distribute the second instantiation of her partitioning operation across the first instantiation, partitioning each of the original four shares into five pieces (Figure 2.6). Having partitioned each of the original four partitions into five parts, an individual might realize that she has partitioned the whole into twenty equal sized parts and conclude that of those parts, including her own, is one-twentieth of the original strip of candy.

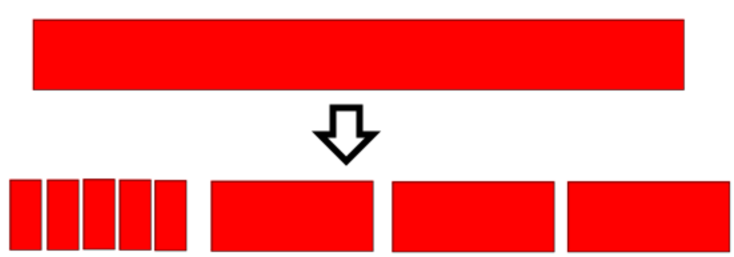

Figure 2.5. Enacting the initial sharing in a recursive partitioning task.

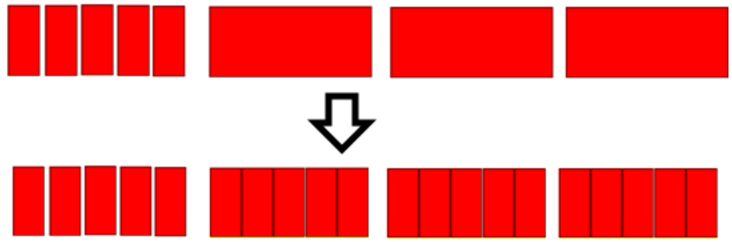

Figure 2.6. Partitioning a partition.

Levels of units coordination. In the previous section, I described operations that produce units from other units. Through repeated instantiations of partitioning and iterating, individuals can build up a quantitative structure of units. Units coordination is a construct characterizing the level of quantitative complexity an individual can hold in mind at a given moment (Ulrich, 2015; 2016a). If a student were to enact the solution to the recursive partitioning task as I described it above entirely in re-presentation “in one fell swoop,” that would be an indication that the particular student can assimilate situations with three levels of units. The first level of unit is produced by unitizing the entire strip of candy. The second level unit is produced in partitioning the strip into four parts, while maintaining an awareness that the four parts still constitute the entire strip. The third level unit is produced by partitioning the each of the four parts into five subparts all the while holding in mind that the five subparts constitute a part and four parts constitute the whole strip. A model of this three-level-of-units structure is presented in Figure 2.7.

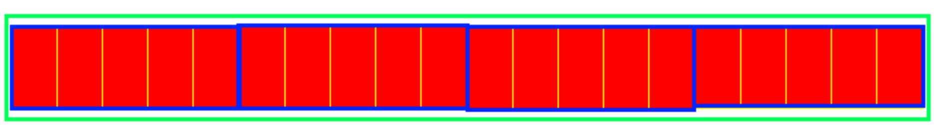

Figure 2.7. A unit-of-unit-of-units.

If an individual assimilates a situation with three levels of units, the quantitative structure is locally permanent within the immediate interaction. In other words, the units do not fade away from the individual's awareness, until she enacts some other operation. As such, a student who assimilates the recursive partitioning example with three levels of units would be able to simultaneously imagine the original strip as being composed of five units of four or twenty units of one.

In contrast, a student assimilating the recursive partitioning task with two levels of units (and working entirely in re-presentation) would lose track of some units as he produced new units. He might, for example, be able to view the original strip as a unit of four parts and then one of those four parts as a unit of five subparts. Such a student might respond, "one-fifth," since his piece is one part in a five-unit composite; such a response would indicate that the student lost track of largest unit containing the four parts. Alternatively, the same student might respond, "one-eighth," because he created five parts in the second partitioning and three parts were left untouched from the first partitioning.

Students who assimilate tasks with two levels of units can construct a third level of unit *in activity*. Through operating, a two-level-of-unit student can produce an additional level of units that was not available prior to operating. For example, suppose such a student physically enacted the sharing activity with a strip of paper producing four parts from the original strip and five parts from one of those four parts (see Figure 2.5). The student might then look at each of the four parts constituting the original strip counting, "five, ten, fifteen, twenty," as he looked at each part and then respond, "one-twentieth."

A student assimilating this recursive partitioning task with two levels of units would be supported by the strip within his visual field. Yet, students that can produce three levels of units in activity do not necessarily need perceptual material to produce a third level of unit through the enactment of sequential operations. In other words, the student might enact a similar solution in re-presentation and achieve the result, "one-twentieth." However, a student who produces three levels of units in activity might not be able to immediately describe how many instantiations of five he counted. In re-presentation, he would view the strip as being 20 ones or 4 fives, but not both simultaneously.

Initial Conceptual Analysis

Prior to conducting this dissertation research, I developed an elementary first-order conceptual analysis (Thompson, 2008; von, Glasersfeld, 1995) for how individuals might quantify angularity. In particular, I considered what early quantifications of angularity might entail and the modifications that might engender the development of a circular quantification of angularity advocated by Moore (2013) and Thompson (2008). This conceptual analysis served as the major hypothesis of my teaching experiment. The primary purpose of this section is to present this initial conceptual analysis, which will be reconsidered at the conclusion of this dissertation. First, I want to make a few remarks on angles as objects.

What is an Angle?

In a sense, angles are fundamental mathematical objects. However, mathematicians have long debated what to take as *the* definition of an angle (Keiser, 2004). According to Keiser, there are three long-standing, broad conceptions of angles that persist. First, an angle can be viewed as two rays sharing a common endpoint. Second, an angle can be taken as the intersection of two half-planes. Third, an angle can be considered a rotation. There are affordances and limitations in each of these conceptions of angle. The purpose of my study is to examine individuals' quantifications of angularity. As such, my use of "angle" throughout this dissertation should not be interpreted as a pointer to one specific formal mathematical definition. Instead, I am proceeding in the spirit described by Freudenthal (1973 as cited in Barabash 2017):

> As has been stressed several times, there is more than one angle concept. Some didacticians claim that there is only one which is correct. Love of order is fine unless it goes as far as to forbid important concepts because they do not fit into the system. Properly said such would be a bad mathematical attitude (p. 476).

An Awareness of Angularity

I turn now to an elementary first-order conceptual analysis of angularity that informed the design of this study. I claim any quantification of angularity relies on actions and operations enacted on the interior of an angle. Paralleling Steffe's (2013) analysis of the construction of length, I consider an awareness of angularity to involve motion of some kind through, what an observer would call, the interior of an angular object.

Prior to conducting the teaching experiment, I hypothesized an awareness of a radial sweep—the trace of a rotating ray constituting the interior of an angle—would be a critical development in students' construction of angularity (Figure 2.8). As I mentioned in the review of existing literature, Clements and colleagues (Clements et al., 1996; Clements & Burns, 2000) have also argued for the importance of such a motion. If individuals inserted such a rotational motion into angular objects, they might establish quantifications of angularity in four broad categories: gross, extensive, ratio, and rate.

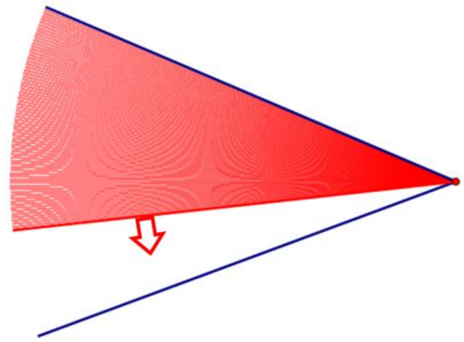

Figure 2.8. Radially sweeping the interior of an angle.

Gross Quantifications

According to Piaget, gross quantities "are nothing else than the relationships expressing 'more', 'equal to' or 'less' immediately perceived in the given qualities" (Piaget, 1965, p. 76). A gross quantification of angularity consists of the operations necessary for comparing two angles. If an individual were to insert the radial sweep into two angles and maintain an awareness of the duration of each radial sweep, then he might compare these durations to order the angles. Initially, the insertion of this sweep might need to be physically enacted (cf. Clements et al., 1996; Clements & Burns, 2000; Smith, King, & Hoyte, 2014). If the sweep were internalized (i.e., the individual could enact the sweep in visualized imagination), then the comparison could be made without physical actions.[15]

[15] Steffe (1991a) noted that Piaget's (1965) use of gross quantification was reserved for comparing items available within an individual's perceptual field.

Extensive quantifications

An extensive quantity arises when an individual introduces units into a gross quantity (Piaget, 1965; Steffe, 1991b). Therefore, extensive quantifications can be viewed as accommodations of gross quantifications. The operations introduced earlier in this chapter (e.g., unitizing, iterating, equisegmenting, partitioning, splitting) are extensive quantitative operations because these operations produce units. If an individual's quantification of angularity were to include iteration for example, then the individual might take a given angle as a unit for producing an angle four times as open as a given angle (Figure 2.9).

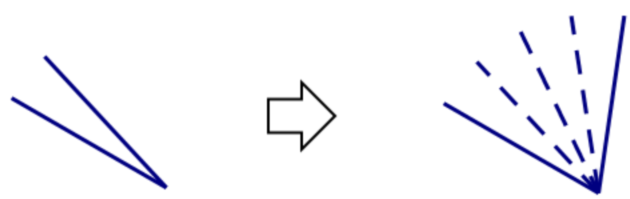

Figure 2.9. Producing an angle four times as open as a given angle through iteration.

As suggested by my discussion of non-circular quantifications of angularity in introductory chapter, I hypothesized that degrees as a standard unit of angular measure could emerge as a result of extensive quantifications of angularity. Specifically, I hypothesized that one-degree angles could emerge via the application of the splitting operation to a full angle. In other words, to produce a one-degree angle is to solve the following splitting task: A full angle is 360 times as open as another angle; how would you make this angle?

Ratio and Rate Quantifications

Thompson (1994) described a ratio as a quantity an individual constructs by making a multiplicative comparison of two other instantiated quantities.[16] If the individual conceives the ratio as applying outside the bounds in which it was originally constructed, then the individual has conceived a rate. In short, a rate is a generalized ratio in Thompson's view.

Ratio and rate quantifications of angularity arise when an individual considers angularity in terms of multiplicative comparisons of circular quantities (e.g., arc lengths), which entails proportional reasoning. If an angle is conceived in terms of a radial sweep, he might insert a circular arc into the angular context by imagining the trace of a single point along the rotating ray. If the individual could enact extensive quantitative operations on both an angle and a single circle centered at the angle's vertex, then he might quantify angularity as a ratio. For example, consider a circle with circumference 10 inches and central angle that subtends a minor arc that is 2 inches long (Figure 2.10). An individual might reason that five iterations of the central angle must produce a full angle since five iterations of the arc exhaust the circle. Such reasoning entailing a multiplicative comparison of an arc length and a circumference would constitute a ratio quantification of angularity.

[16] The multiplicative comparison of two quantities is an example of what Thompson (1994) calls a quantitative operation, which he contrasts with a numerical operation.

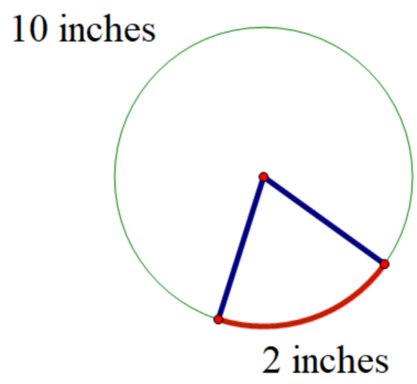

Figure 2.10. An arc of length 2 in a circle of circumference 10.

If the individual viewed such a multiplicative comparison of circular lengths as being invariant across all possible circles, then I would say the individual has established a rate quantification of angularity. In the context of the example provided in Figure 2.10, a rate quantification of angularity would be indicated if the individual recognized that *every* circle centered at the angle's vertex subtended a minor arc 1/5 the length of the corresponding circle's circumference.

Closing Remarks

The initial conceptual analysis presented is an organization of my own ways of thinking about angularity prior to conducting the teaching experiment described in this study. A purpose this study was to produce a second-order conceptual analysis based on the mathematical activities of students. To produce such a model, I conducted a teaching experiment with ninth-grade students. In Chapter 3, I discuss the teaching experiment methodology and specific characteristics of my teaching experiment.

CHAPTER 3

METHODOLOGY

I begin this chapter by describing the constructivist teaching experiment methodology. Subsequently, I describe specific characteristics of my teaching experiment.

Teaching Experiment Methodology

As described by Steffe & colleagues (Steffe & Thompson, 2000; Steffe & Ulrich, 2013), the constructivist teaching experiment is a methodology derived from Piaget's clinical interview and rooted in the principles of radical constructivism. The individual conducting a teaching experiment is known as a teacher-researcher because of the dual role he assumes throughout the study. The teacher-researcher engages in interactive communication with students as they solve mathematical tasks. As in the clinical interview, a goal of the teacher-researcher is to understand students' existing ways of reasoning. In contrast to the clinical interview, the teacher-researcher also intends to engender productive changes in students' ways of reasoning throughout the course of the teaching experiment. "Productive changes" are modifications that students make in their ways of reasoning that allow them to solve problems they could not previously solve. Thus, the teaching experiment is an exploratory method of inquiry wherein a researcher's goal is to explain students' ways of reasoning mathematically and changes in these ways of reasoning.

As I noted in the previous chapter, mathematics from the constructivist perspective exists in the minds of individuals. In a teaching experiment, the teacher-researcher is interested in developing an ontogenetic justification of mathematics—an elaboration of how individuals' mathematical operations and schemes develop overtime. Therefore, the teaching experiment is necessarily conducted with one or more students. A constructivist teacher-researcher positions the students as knowledgeable others in a mathematical sense. More specifically, the teacher-researcher attributes to each student a mathematics, which is necessarily different from the teacher-researcher's mathematics.

From the perspective of the student, a student's mathematics is viable simply because she developed specific mathematical ways of reasoning in the first place; in other words, the student's existing mathematical ways of reasoning have served in the student's organization of prior mathematical experiences. These ways of reasoning are inaccessible to the teacher-researcher as they exist within the mind of the student. To learn of students' ways of reasoning mathematically, the teacher researcher must interact with students over an extended period of time as the student engages in mathematical activity. The observable features (i.e., students' words and actions) accompanying student's mathematical activities are the data from which the teacher-researcher makes inferences about students' mathematics. Thus, the teacher-researcher engages in the creative analytic activity of model-building—developing a set of conceptual structures that account for the observable features of a student's mathematical activity when the teacher-researcher imputes these structures to the student. Although there is no prescribed number of interactions or duration for these interactions, a teaching experiment necessarily consists of a series of teaching sessions to enable the teacher researcher to test initial hypotheses and revisions of these hypotheses.

The teaching experiment is scientific because the teacher-researcher formulates major hypotheses to be tested prior to teaching the students. These major hypotheses may be formed through a combination of first-order conceptual analyses of the teacher-researcher's mathematical knowledge—imagining how another individual might come to understand a mathematical idea in a particular way—and experiential second-order models developed through exploratory teaching. The major research hypotheses formulated by the teacher-researcher prior to the teaching experiment inform the initial selection of the students' selected for participation in the experiment, as well as the overall scientific intentions of the teacher-researcher.

During the teaching experiment, the teacher-researcher holds major research hypotheses in mind while simultaneously holding them at bay. By holding them in mind, I mean the teacher researcher has an overall plan for tasks he might use throughout the teaching experiment to test portions of the major research hypotheses as well as particular follow-up questions he might ask. During teaching sessions however, interactions with students rarely unfold as the teacher-researcher might have anticipated as the students engage in surprising or otherwise unexpected activity from the perspective of the teacher-researcher. The students constrain the teacher-researcher because they think differently than the teacher-researcher; experiencing these constraints is the precisely the reason the teacher-researcher conducts the teaching experiment. As such, when interacting with students, the teacher-researcher foregrounds the present interaction by maintaining an adaptive disposition toward the students, modifying his pedagogical activities to be in harmony with the students' activities (Steffe & Wiegel, 1992). Major research hypotheses remain subordinate to students' contributions to the trajectory of teaching experiment throughout the series of teaching episodes. At the conclusion of all teaching sessions, the teacher-researcher tests the major research hypotheses by considering the collective activities of the students throughout the teaching experiment.

Throughout the course of a teaching experiment, the teacher-researcher may formulate and test additional or secondary hypotheses. These secondary hypotheses are formed between teaching sessions while reflecting on previous teaching sessions and are also formed during teaching sessions while interacting with students. When students say and do the unexpected in a teaching session, the teacher-researcher experiences a perturbation while remaining aware students are responding rationally for their individual perspectives. It is the responsibility of the teacher-researcher to "continually postulate possible meanings that lie behind students' language and actions" (Steffe & Thompson, 2000, p. 276).

Through generating and testing hypotheses, the teacher-researcher formulates the boundaries of students' mathematics. By observing what a student does *and doesn't do*, the teacher-researcher forms (sometimes fuzzy) delineations of what a student does and doesn't understand. Through observing students at a "higher or lower level" (Steffe & Ulrich, 2014, p. 4), the teacher-researcher can often form clearer boundaries around a particular student's mathematics.

At the conclusion of all teaching sessions, the teacher-researcher engages in retrospective analyses of the records of the teaching experiment. The purpose of these retrospective analyses is to refine the models of students' mathematics, which were developed throughout the teaching sessions. Essentially, the goal of the teacher-researcher is to develop a set of explanatory constructs (e.g., mental operations and schemes) that, when attributed to the student, account for the students' observable activities and changes in their activities throughout the teaching experiment. A major purpose of these models is to inform future teaching and research; the expectation is that the models will be useful to mathematics educators in teaching other mathematics students.

My Teaching Experiment

As described in Chapter 1, the purpose of my teaching experiment was to explore high-school students' quantifications of angularity and modifications in these quantifications. In the following sections, I describe the particular methodological characteristics of my teaching experiment including participant selection, data collection, and data analysis.

Major Research Hypotheses

In Chapter 2, I elaborated the major research hypotheses for this teaching experiment in the form of a first-order conceptual analysis. In brief, the major research hypotheses guiding this study are that (a) students can develop non-circular quantifications of angularity prior to developing circular quantifications and (b) these non-circular quantifications of angularity can emerge as a reorganization of the extensive operations constituting students' quantifications of length.

Site and Participant Selection

Participants for the teaching experiment were four ninth-grade students enrolled in a first-year algebra course at a small private school in the southeastern US. I selected the school as a site for the study due to the principal's willingness to allow for students' participation. After obtaining IRB approval for the study and a letter of cooperation from the principal, I communicated primarily with the assistant principal about recruiting students for participation in the study. I specifically recruited students enrolled in first-year algebra courses so that the students would not yet have taken a dedicated course on Euclidean geometry.

The assistant principal visited each first-year algebra class to solicit volunteers. I informed the assistant principal I was looking for a group of students who demonstrated a range of mathematical understandings. Additionally, I requested students with a history of attending school regularly and students who were comfortable verbalizing their thinking.

I hoped to interview eight students and, from these eight students, select four for participation in the study: two who could assimilate mathematical situations with three levels of units and two who could assimilate situations with only two levels of units. After approximately one month of soliciting volunteers, only five students expressed an interest in the study. Four of these students completed the required consent and assent forms for participation in the study—Camille, Kacie, Bertin, and Mac. These four students participated in the teaching experiment.

In this dissertation, I focus my analysis on the activities of two participants, Camille and Kacie. I selected these two participants for analysis in this dissertation for several reasons. First, Camille and Kacie participated in many paired teaching sessions together, which allowed for an analysis of the interaction between the two. Second, Camille and Kacie demonstrated the least sophisticated quantifications of angularity at the onset of the teaching experiment and demonstrated the most growth throughout the teaching experiment. Finally, the two students differed in terms of the levels of units they could assimilate in mathematical tasks. Camille could assimilate situations with only two levels of units as a given; Kacie could assimilate situations with three levels of units. In all subsequent sections, unless explicitly noted otherwise, all comments about the participants in the study refer only to Camille and Kacie.

Data and Tasks

Kacie and Camille participated in a total of 14 and 13 sessions, respectively, throughout the study. Each session was approximately 30 minutes in length. All sessions took place at the students' school during regular school hours and outside of their regular mathematics instruction. Sessions were scheduled approximately once per week throughout one school year.

I distinguish between two kinds of sessions, interview sessions and teaching sessions. In interview sessions, my intention as teacher-researcher was to investigate students' ways of reasoning without engendering changes in these ways of reasoning. In teaching sessions, I acted to both understand students' ways of reasoning *and* engender changes in their ways of reasoning. I conducted three interview sessions with each student during the study: two initial interview sessions per student at the onset of the study and one final interview session per student at the end of the study. All interview sessions were conducted with students individually. Between the initial and final interview sessions, Kacie and Camille participated in 11 and 10 teaching sessions, respectively, which were conducted individually or in pairs. In Table 5.1, each date a student attended a teaching session is indicated by "X." I served as teacher-researcher for all sessions throughout the teaching experiment.

Table 3.1. List of teaching sessions.

	11/12	11/17	12/3	1/11	1/19	1/25	2/1	2/8
Camille	X	X		X	X		X	
Kacie	X	X	X	X		X	X	X

	2/22	2/29	3/14	3/31	4/5	4/18	4/21
Camille	X	X	X	X		X	
Kacie	X	X			X		X

All sessions were videorecorded with at least two cameras: one camera with a wide focus on the participants and an additional camera with a narrow focus on each student's work. A witness-researcher, Hwa Young Lee, attended sessions as her schedule would permit and operated an additional camera during these teaching sessions. Recordings from all cameras during each session were mixed into a single synchronous video file to be used in retrospective and ongoing analyses. All written records produced by students during sessions were digitized.

Initial interview sessions. The initial interview protocol, which I developed with feedback from my advisory committee, can be found in Appendix A. The primary purposes of the initial interview sessions were to investigate (a) the extensive quantitative operations students had constructed prior to the teaching experiment including units coordination and (b) whether and how students applied these operations to angular material.[17] To investigate the former, I posed tasks to the students involving pieces of string and other linear material. To investigate the latter, I posed parallel tasks involving hinged wooded chopsticks. In addition to these tasks involving linear and angular material, I also posed tasks to investigate the students' proportional reasoning and

[17] In other words, were students able to assimilate linear and angular material to the same operations.

operations on curved lengths. An analysis of the students' activities during the initial interview sessions is presented in Chapter 4.

Teaching sessions. Between the initial and final interview sessions, each student participated in a series of teaching sessions either individually or in pairs. Through February, it was my intention for Camille and Kacie to participate in paired teaching sessions so that the students could interact with one another. It was my hope that this interaction would allow them to learn from one another; additionally, by observing how they interpreted each other's activities, I hoped to further refine my models of the students thinking. Occasionally, I taught the students individually prior to March when one student was unable to attend a teaching session due to other commitments. In March, I separated the pair in order to promote the greatest possible mathematical progress for each of the students.[18]

Tasks for the teaching sessions were developed throughout the teaching experiment based on students' activities in earlier teaching sessions. In some cases, tasks were developed collaboratively during weekly research group meetings throughout data collection.[19] Two angular contexts were used throughout the study—hinged wooden chopsticks and a rotating laser. Physical versions of these models were used initially during the teaching experiment; as the study progressed, these physical models were emulated using drawings or the Geometer's Sketchpad, a dynamic geometry software. In my analysis of students' activities presented in Chapter 5, I describe the tasks I used throughout the study in greater detail.

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[18] The rationale for separating the students is provided at the conclusion of Part 2 of Chapter 5.

[19] Members of the research group included Les Steffe, Hwa Young Lee, Jiyoon Chun, Eun Jung, and myself.

Final interview sessions. To conclude the study, each student participated in one final interview session. The primary purpose of the final interview session was to verify hypotheses that I had formed throughout the teaching experiment about each students' ways of reasoning. Some angular tasks posed during the initial interview sessions were repeated with students in the final interview session to offer a contrast between students' ways of reasoning at the onset and conclusion of the teaching experiment. The complete final interview protocol is available in Appendix B.

Data Analysis

As suggested by the earlier discussion on the teaching experiment methodology, I analyzed data throughout the teaching experiment and after the teaching experiment by conducting a second-order conceptual analysis (Thompson, 2008; von Glasersfeld, 1995). To conduct a second-order conceptual analysis is to devise a system of conceptual structures (i.e., operations and schemes) that account for students' mathematical activities throughout the teaching experiment, as well as changes in these activities. This process involved making and refining hypotheses about students' ways of reasoning through on going and retrospective analysis.

Ongoing analysis began while interacting with students. During interactions, I formed nascent hypotheses about students' ways of reasoning. After each session with students and prior to the subsequent session, I reviewed the video records of each session one or more times. In reviewing these video records, I focused my attention on students' words and actions as they engaged in mathematical activities to begin refining the nascent hypotheses. I discussed many developing hypotheses with Hwa Young Lee, who also reviewed the video records of most teaching sessions. Some of these developing hypotheses were recorded in a set of electronic research notes that I maintained during the teaching experiment; other hypotheses were recorded only mentally. Occasionally, these hypotheses were shared during research group meetings.[20] In these instances, I selected video excerpts to share with the research group to facilitate our discussion of students' activities and the developing hypotheses. Tasks for subsequent teaching sessions were designed with the intention of testing developing hypotheses. After designing tasks, I wrote plans for the subsequent teaching sessions, which included deliberate sequencing of tasks, particular phrasings to use when posing tasks to students, and potential follow-up questions based on hypothesized student responses. Throughout the teaching experiment, this process of ongoing analysis repeated with each teaching session.

At the conclusion of the final interviews, I began to retrospectively analyze the video records from all teaching sessions with the intention of refining models of students' quantifications of angularity. I began by analyzing each students' teaching sessions chronologically. In this first round of analysis, I watched every teaching session twice. I first watched the video its entirety without interruption to reactivate my records of the interactions contained in the teaching session. Afterwards, I restarted the video and attended carefully to students' observable activities, which included their verbal expressions, drawings, and gestures. I created detailed notes for each teaching session, which included descriptions of each student's activities as they solved each task as well as a list of tasks that the student attempted in each teaching session. I transcribed selectively on this second pass through the video, and I frequently paused and repeatedly

[20] Each member of the research group was conducting their own research; as such, each project could not always be discussed in detail each week.

watched particular excerpts in the session to try to understand how students might be thinking. During this process, I noted instances that supported or contradicted the developing hypotheses I made during ongoing analysis, and I also made new inferences to account for students' mathematical activities. For each student, I noted similarities in ways of reasoning across tasks and teaching sessions, as well significant instances where a student's activities seemed to indicate a shift in their reasoning. Additionally, I noted instances where a student's activities reminded me of the activities of another student in the study, either a difference or a similarity. When students participated in paired teaching sessions, I noted features related to social interaction (e.g., if one student seemed to be taking a passive role and imitating her partner).

When I finished this chronological pass through the data for each student, I began to further refine my models by looking across all sessions for each student and looking across students. I watched the videos repeatedly, refining my hypotheses through analytical notes along the way. Sometimes, I watched the teaching sessions in reverse order (i.e., watching session 4, then 3, then 2) to try to identify where students' thinking shifted during each teaching session. Other times, I organized my video analysis in terms of tasks that were similar from my perspective. Using the notes from the chronological analysis, I identified all instances where students solved similar tasks and watched students' activities on these tasks in sequence. Often, this method helped me to refine hypotheses about shifts in students' thinking as well as well as differences across students. Over time, I gradually refined my models of students' mathematics to account for much of students' observable mathematical activities. I present these models in Chapters 4 and 5.

CHAPTER 4

FINDINGS FROM INITIAL INTERVIEWS

In this chapter, I present an analysis of the students' individual initial interview sessions. My primary goals in the initial interview sessions were to investigate (a) the extensive quantitative operations students had constructed prior to the teaching experiment and (b) whether and how students applied these operations to angular material. To investigate the former, I posed tasks involving pieces of string and other linear material. To investigate the later, I posed parallel tasks involving hinged wooden chopsticks. In addition to these tasks involving linear and angular material, I also designed tasks to investigate the students' proportional reasoning and operations on curved lengths. None of the participants exhibited well-established schemes for proportional reasoning, and I did not find the analysis of the curved-length tasks to have much predictive utility in the context of the results presented in Chapters 5 and 6. As a result, I focus my analysis in this chapter on the extensive quantitative operations I inferred from the students' activities on linear material as well as the operations students enacted on angles.

Students' Extensive Operations Indicated by Linear Tasks

In this section, I describe inferences I made regarding ways of reasoning students constructed prior to the teaching experiment. In particular, these inferences involve the levels of units each student could coordinate prior to operating in quantitative situations and whether each student had constructed the splitting operation.

Units Coordination

Both students indicated they could produce continuous composite units from units of one in the initial interviews. For example, each student produced a string six times as long as a given string by repeating the given string into another piece of string six times and cutting off the result after six repetitions.

Both students also solved tasks that involved coordinating three levels of units during the initial interviews. In one such task, I asked each student to measure a green string using a pencil, measure the pencil using a wikki stick, and then determine the length of the green string in wikki sticks.[21] I prepared the materials so the green string was four pencils long and the pencil was six wikki sticks long. Both students immediately achieved a result of 24 and described the situation in terms of multiplication (i.e., four times six). I attribute the immediacy of this result to students' prior experiences assimilating tasks in terms of numerical operations. Although Camille and Kacie obtained the same result, I inferred differences in the ways the students had quantitatively structured the situations based on the discussions that followed.

Kacie indicated she immediately structured the situation so that the green string was a unit composed of four units of six and this structuring was permanent with respect to their current activities. After concluding the green string contained twenty-four wikki sticks, Kacie explained:

> So if there were four pencils in the green string then that means that for every pencil there was six wikki sticks. So six times four is twenty four.

Kacie indicated an awareness of simultaneously projecting six units of four into the green string. Furthermore, Kacie flexibly viewed the string as being composed of four

[21] A wikki stick is a piece of waxed string.

units of six or twenty-four units of one; additionally, Kacie remained aware of her original structuring of the situation (i.e., the green string as four units of six) after producing the result, 24. Because Kacie immediately structured the task using three levels of units and this structure did not atrophy as the students engaged in the task, I infer that Kacie assimilated the task with three levels of units.

The second student, Camille, also responded 24 immediately when I asked for the length of the green string in wikki sticks. Unlike Kacie, Camille indicated that she wanted to check her response by measuring the green string using the wikki stick. Before she could begin to measure, I encouraged Camille to explain her thinking. Camille indicated that she assimilated multiple two level of unit structures:

> I needed six of these [wikki sticks] to make [the pencil]. But I needed four [pencils] to make [the green string]. I just figured maybe I could get twenty four out of these [wikki sticks].

Camille described the pencil in units of wikki sticks, the green string in units of pencils, and the green string in units of wikki sticks. However, Camille never described the green string as being constituted by four units of pencils, each of which contained six units of wikki sticks. Camille lacked the simultaneity associated with the three-level-of-units assimilatory structure exhibited by Kacie. Additionally, Camille wanted to remeasure various lengths to reestablish the multiplicative relationships between the various units throughout our discussion, which indicated her composite structures atrophied as she created new ones. At my request, Camille demonstrated she could enumerate the units established in repeating a composite by counting the wikki sticks in the green string by using the pencil. As she moved the pencil in four motions along the green string, Camille counted: “six, twelve, eighteen, and then twenty-four.” From her activities, I inferred that Camille could produce three levels of units in activity, though only two levels of units were available to her in assimilation.

During the initial interview sessions, my inferences regarding the levels of units students could coordinate were informed by students' activities on multiple tasks. Looking across tasks at the end of the initial interview, my inferences were consistent with the analysis provided above. I inferred that Kacie could assimilate three levels of units prior to operating, while Camille could assimilate two levels of units prior to operating and produce three levels of units in activity.

Splitting

In addition to students' units coordination, I also investigated whether students had constructed the splitting operation at the onset of the teaching experiment. The splitting operation involves the unification of partitioning and iterating operations. I presented students with a piece of string and posed a splitting task:

> This is my piece of string. My piece of string is five times as long as your piece of string. How would you make your piece of string?

A student who has constructed the splitting operation can solve the task by partitioning the given string into five parts and copying one of these parts to create their string. Such a student would also be aware that five copies of their string would reproduce the original string provided by the teacher.

At the onset of the teaching experiment, I inferred that Kacie had constructed the splitting operation, while Camille had not.[22] I briefly discuss each student's activities on the splitting task in the sections that follow.

[22] In the ongoing analysis of subsequent teaching sessions and retrospective analyses, I later questioned whether Kacie had constructed the splitting operation. As such, I address Kacie's splitting operation in later chapters.

Kacie and the splitting task. Kacie began the splitting task by folding my string in halves. Kacie then suggested measuring my string and dividing the measurement by five. When I asked Kacie to solve the task without measuring, Kacie made an estimate at one end of my string and folded the estimate into my string, which indicated at least the equisegmenting operation (i.e., the sequential production of equal-sized parts intended to exhaust the whole). As she finished folding five equal parts into my string, Kacie remarked, “wait, that’s not five,” which I took as an indication that she was unsatisfied with a small portion of my string unaccounted for in any of the five parts.

Kacie then laid the string on the table and placed three fingers on the string creating (what I perceived to be) three approximately equal parts and a fourth noticeably longer part on the right end of the string. Kacie counted the parts, “one, two, three, four, five,” which indicated that she used the edge of the paper on which the string was resting to create five approximately equal parts in my string (Figure 4.1). Kacie emphasized that she was looking “to see if they’re all even” and remarked, “one section of your string equals my string.” When I asked which section she would take for her string, Kacie replied, “any of them.”

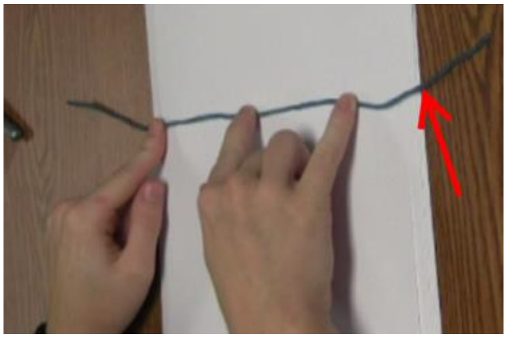

Figure 4.1. Kacie creates five sections in my string.

From Kacie's activities, I infer she formed a goal of producing five congruent parts within the given string. However, because Kacie clearly indicated the length of her string as one of these five parts and identified these parts as equivalent, I attributed the splitting operation to Kacie at the onset of the teaching experiment.

Camille and the splitting task. Camille engaged with the splitting task differently than Kacie. After hearing the task, Camille pointed to my string and asked, "what do you mean like make it longer?" Camille's question indicated she wanted to produce a string longer my string, which was an initial contraindication of the splitting operation. After I repeated the task two additional times, Camille remarked, "I think I understand…I'm going to make mine and then yours is going to be five times bigger."

Camille placed a long piece yellow string adjacent to the given blue string, which indicated she made a mental copy of the blue string. She indicated that she would "go back five" from the end of the blue string to produce her string. From her comment, I infer that Camille had established a goal of producing a string shorter than the blue string using her concept of five. Camille marked the yellow string in five places with her finger (as shown at left in Figure 4.2) counting, "one, two, three, four, five." Camille marked the string at unequal intervals, which indicated that she was not iterating a unit length into the string. Camille cut the yellow string at this fifth position and selected the piece she had not marked as her string (Figure 4.2 right). From my perspective, my string was approximately three times as long as her string.

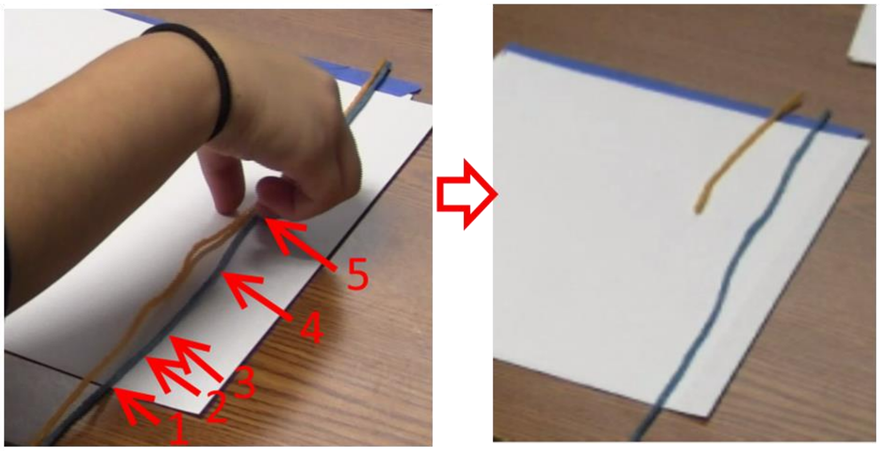

Figure 4.2. Camille creates a shorter string using her concept for five.

From Camille's actions, I infer she assimilated the task in terms of an additive structure; she made her string "five" shorter than my string. From my perspective, Camille neither partitioned nor iterated unit lengths as she engaged in the splitting task. As a result, I inferred that Camille had not constructed the splitting operation at the onset of the teaching experiment.

Summary of Inferences from Linear Tasks

From the initial interview tasks involving linear material, I made inferences about operations students had constructed at the onset of the teaching experiment. I inferred that Kacie could coordinate three levels of units in assimilation, while Camille could coordinate two levels of units in assimilation and three levels of units in activity. I attributed the splitting operation to Kacie, and I inferred that Camille had not constructed the splitting operation.

Students' Initial Angular Operations

In this section, I describe the initial inferences I made regarding the students' ways of reasoning with angular material. In this section, I use the phrase *students' angular operations* as an informal shorthand for the operations to which students assimilated angular material. I begin by arguing that both students demonstrated an awareness of angularity. Then, considering each student in turn, I characterize the students' ways of reasoning about establishing angular congruence, producing angular multiples, and splitting angular material.

Initial Findings Regarding Awareness of Angularity

Like the construction of length (Steffe, 2013), the construction of angularity involves motion of some kind. As I will show in this dissertation, the motions individuals enact in the angular case are more varied and complex than motion in the linear case. In my initial conceptual analysis, I posited a *radial sweep*—visualizing a single ray rotating about its endpoint—as an example of one such motion that would be propitious for the construction of angularity. In the initial interview sessions, I devised a task to examine if students could construct a figurative continuous unit out of such a sweep. In retrospectively analyzing students' initial interview sessions, I inferred students were accounting for the interior of angles using a second kind of motion, which I refer to as *re-presented opening*—imagining the sides of an angle model opening from a closed configuration. In the following sections, I argue that each student indicated an awareness of angularity through at least one of these motions.

Radial sweep. To investigate if students had constructed an awareness of angularity involving a radial sweep, I posed a rotating laser task to students.[23] This task involved a GSP sketch containing the image of a laser pointer. As I introduced the task to each student, I "turned on" the laser by clicking an action button, which showed a red ray emanating from the laser. I informed the students that I would "turn off" the laser and

[23] I purposefully posed this task after all other angular tasks to avoid leading the students to use the radial sweep on the other tasks.

then move the laser pointer. I asked the students to keep track of all the places on the screen the beam would have hit if the laser had been "on" as it moved. I designed the sketch so that the laser beam rotated 57° counterclockwise about the endpoint of the ray representing the beam. Initial and terminal positions of the laser beam are shown in Figure 4.3.

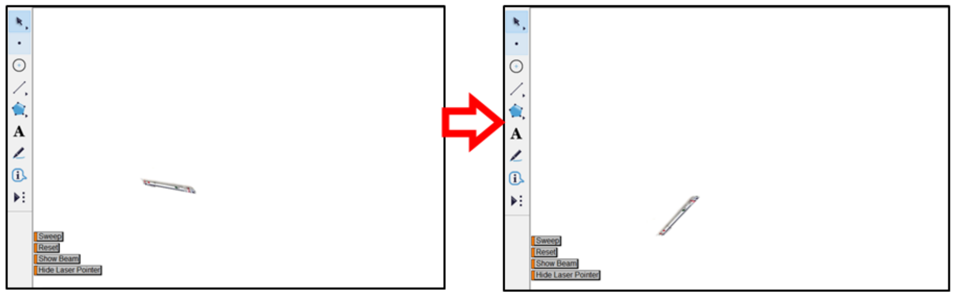

Figure 4.3. Initial and terminal positions of the laser pointer.

Both students shaded the screen accounting for three components: the initial location of the ray, the trace of the ray (i.e., the interior of the angle), and the terminal location of the ray. Camille shaded in the order that the motion occurred—shading the initial ray, then the interior, and finally the terminal ray (Figure 4.4 left). Kacie shaded in the opposite order (terminal ray, interior, and finally initial ray), as if re-presenting the experience in reverse (Figure 4.4 right).

As the students explained their shadings, each student indicated they intended to define the boundary as a ray. For example, Camille lamented, "I didn't draw it straight but that would be like where it would stop," which indicated her intent to create a linear boundary. Like Camille, Kacie also described the boundary lines in temporal terms, which indicated the boundedness of the re-presented motion. As Kacie shaded the interior, her remarks indicated she was reimagining the motion she imputed to the beam: "It would get like this whole area here." Camille's shading of the interior in a counterclockwise motion also indicated that she was reimagining the motion of the radial sweep.

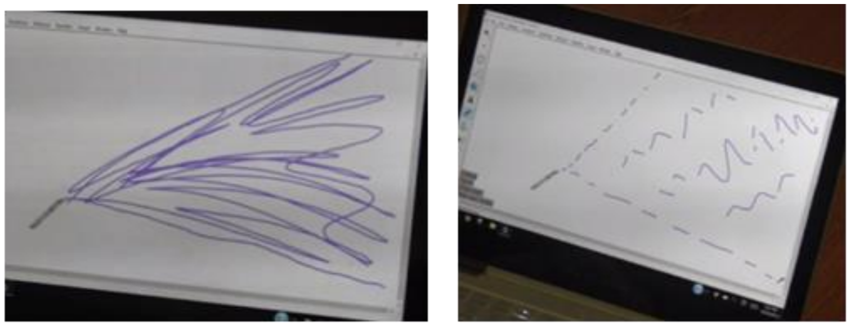

Figure 4.4. Camille's (left) and Kacie's (right) shadings

Each student re-presented the uninterrupted motion of the imagined beam, along with its trace, in visualized imagination. Therefore, I consider each student to have demonstrated an awareness of angularity via radial sweep. Camille's activities demonstrated at least a figurative awareness of angularity in that she re-presented the motion of the beam as she had observed it to occur; she internalized the rotational imagery (Les Steffe, personal communication, September 17, 2017). In contrast, Kacie demonstrated an operative awareness of angularity because she indicated she could imagine reversing the direction of the sweep in re-presentation, which indicated she had interiorized the rotational imagery of the rotating beam (Les Steffe, personal communication, September 17, 2017). Considering my initial conceptual analysis, I viewed the students' demonstrated awareness of angularity via radial sweep as a promising base for developing students' angular operations. In other words, I hypothesized the radial sweep would be beneficial for engendering future learning.

Re-presented opening. Although the students demonstrated an awareness of angularity via radial sweep, the students did not generate the sweeping motion spontaneously in that I designed the GSP sketch to encourage this motion. To understand students' existing ways of injecting motion into angular experiential items, I presented students with an angular re-presentation task.[24] In this task, I briefly displayed two pairs of hinged chopsticks (Figure 4.5) for each student and then covered them with a cloth. I asked each student to draw the chopsticks and describe the similarities or differences they noticed. A portion of Kacie's response is described in Excerpt 4.1.[25]

Figure 4.5. Obtuse and acute chopsticks for the re-presentation task.

Excerpt 4.1. Kacie describes differences in hinged chopsticks.

K: This [obtuse] one is like that [acute] one but it's farther out. [*Holds palms together and fingertips closed (Figure 4.6 left); then opens fingertips while keeping the bases of her palms pressed together (Figure 4.6 right)*]. So like they, [*gestures four times as if opening the hinged chopsticks from a closed position in the air*] uh, took it apart, I guess. But, it's still like together. And so, the same with like this [acute] one. It's just like somebody pushed it together [*holds hands open and then closes them some as if closing the obtuse chopsticks to the acute configuration*].

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[24] This was the first angular task I presented to each student to avoid leading students to descriptors (e.g., openness) in the phrasing of other angular prompts.

[25] In excerpts, K, C, and T are abbreviations for Kacie, Camille, and Teacher-Researcher, respectively. When a student's referent is unclear, I describe the referent in brackets. Descriptions of nonverbal actions and the lengths of pauses in seconds are italicized in addition to being included in brackets. An en-dash indicates a restart in speech. Omitted speech is indicated by ellipses.

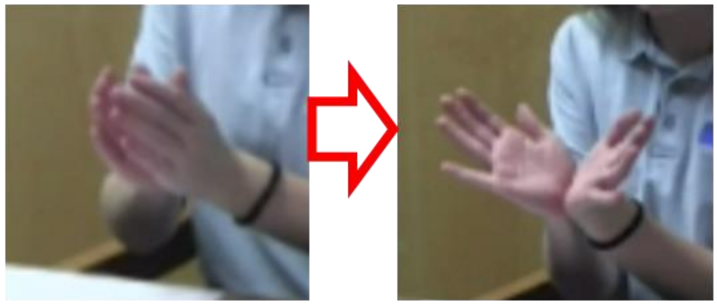

Figure 4.6. Kacie imagines opening chopsticks from a closed position.

In Excerpt 4.1, Kacie repeatedly opened and closed her hands. In the first instantiation of this gesture, Kacie began by holding her palms together with fingertips closed (Figure 4.6 left) and then opened her hands to a configuration like the obtuse chopsticks (Figure 4.6 right). From Kacie's gestures, I infer she imagined opening a pair of chopsticks while bounding this motion between two configurations: a closed configuration, which she represented, and the obtuse configuration she had drawn, which was perceptually available. [26] I interpret her verbal description, "so like they, uh, took it apart," as additional support for this inference. From her repeated gestures, I infer that Kacie re-instantiated this motion—opening a pair of chopsticks to the obtuse configuration from a closed configuration—in visualized imagination at least four additional times. Kacie's actions indicated she constituted the interior of the obtuse chopsticks by re-presenting the action of opening the chopsticks to the obtuse configuration from a closed configuration. Thus, Kacie demonstrated an awareness of angularity via re-presented opening.

As Kacie described the acute chopsticks, she did not use a closed configuration as a reference. Instead, she appeared to use the obtuse configuration as an initial state for

[26] I use "configurations" rather than "angles" to emphasize the transformation of a single angle model from one state to another.

mentally producing the acute configuration. After setting her hands to the obtuse configuration, Kacie moved her hands as if closing the chopsticks to produce the acute angle as she explained, “it’s like somebody pushed it together.” Therefore, I consider Kacie indicated an awareness of an experiential angular difference as she imagined the motion necessary to transform the obtuse configuration to the acute configuration. Kacie did not describe any other differences in the pairs of chopsticks. In fact, Kacie explained that the lengths of the chopsticks were approximately the same.

Although it is possible Kacie interpreted the re-presented opening of the chopsticks in terms of rotational motion, I interpret Kacie’s actions more conservatively. That Kacie re-presented the opening action does not necessitate that she also held in mind a fixed position for the vertex of an angle mode. Additionally, I distinguish an awareness of angularity via re-presented opening from radial sweep as the former involves motion on two distinct rays while the latter involves motion of a single ray through the interior of an angle.

In contrast to Kacie, Camille did not gesture in a way that indicated she engaged in mentally opening or closing the hinged chopsticks. She explained that the sides in the acute chopsticks were “close together,” while the sides in the obtuse chopsticks were “really wide.”[27] From her verbal description of the differences in the re-presentation task, I was not able to infer whether Camille was attending to the interior of the chopsticks or to the distance between the endpoints of the chopsticks. The features to which Camille attended will be addressed in a subsequent section in the context of other angular tasks.

[27] Camille named the chopsticks using the terms “acute” and “obtuse.” In contrast, Kacie gave no indication of assimilating the chopsticks as objects from her prior school experiences.

Summary of section. In this section, I presented two different motions through an angle's interior relevant in the construction of an awareness of angularity. A radial sweep involves an individual's image of the rotational motion of a single ray through the interior of an angle. Re-presented opening involves an individual's mental imagery of opening or closing an angle. Kacie and Camille each indicated an awareness of angularity via radial sweep. Kacie indicated an awareness of angularity via re-presented opening.

Camille's Initial Angular Operations

In this section, I analyze Camille's activities in the remaining angular tasks from her initial interview session. I designed these tasks to explore students' ways of reasoning about angular congruence & comparison, creating angular multiples, and splitting angles.

Angular congruence and comparison. To determine whether Camille had constructed a notion of angular congruence, I presented Camille with a fixed long pair of chopsticks and asked her to set a short pair of chopsticks to have the same openness as the long pair. Camille set the chopsticks as shown in Figure 4.7 below.

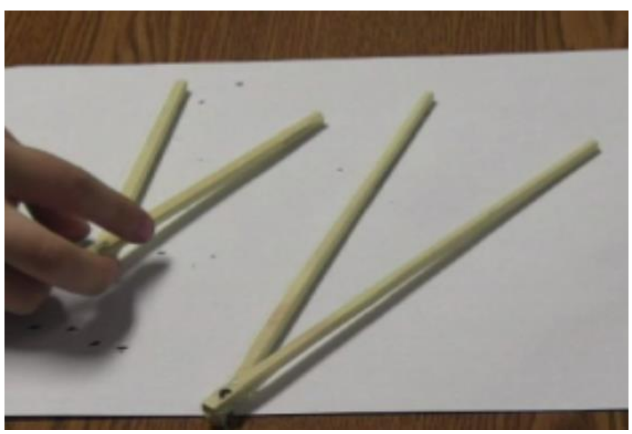

Figure 4.7. Camille sets the short pair to the same openness as the long pair.

When I asked Camille how she knew the chopsticks had the same openness, Camille explained she "tried to imagine this [shorter pair] as that one [longer pair] except smaller." As she explained, she picked up the short pair and gestured as if to place it atop the long pair. In excerpt 4.2 below, Camille elaborates on her reasoning in this angular congruence task.

Excerpt 4.2. Camille elaborates her reasoning on the congruence task.
T: It looked like you were moving that one [short pair] in some way.
C: Oh, yeah. I wanted to see if it would be the same amount. [*Moves the short chopsticks over the long chopsticks as shown in Figure 4.8*]
T: And so how did that help you see whether or not it would be the same amount?
C: Um, if it had the same like space. [*Repeats the superimposition of the short pair atop the long pair.*] Yeah, that's how I would.
T: So where are you looking when you're saying space there?
C: Right here [*slides her finger down one side of the long pair, through the vertex, and up the other side of the long pair*]. Yeah the inside [*points to the interior of the short pair*].

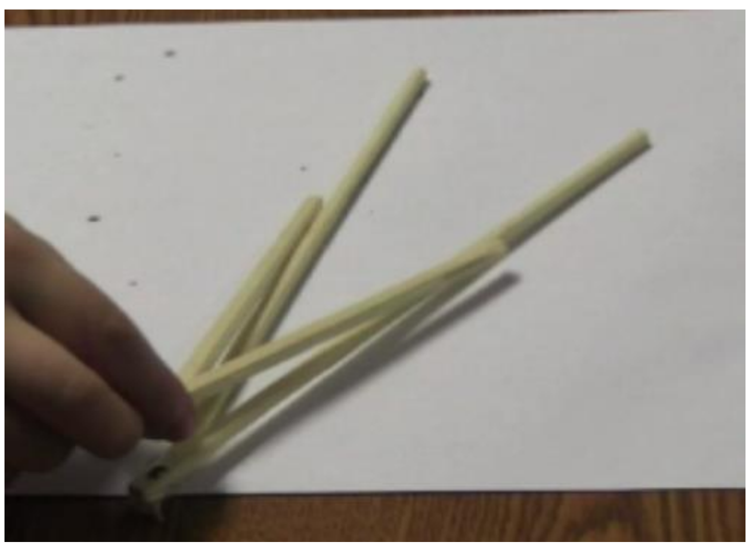

Figure 4.8. Camille superimposes the short pair atop the long pair.

Camille's activities in this angular congruence task clearly indicated that she was attending to the interior of the chopsticks, which is necessary for the construction of an angular unit. Camille identified the interior through her gestures and referred to this interior as "space" and "the inside." From the interaction in the initial interview, it is unclear whether the interior Camille attended to was finite, perhaps bounded by the ends of one of the pairs of chopsticks, or infinite, extending beyond the ends of the chopsticks (Figure 4.9).[28] In either case, Camille was attending to the interior of the chopsticks in such a way that the lengths of the chopsticks were inconsequential for her comparison. In other words, Camille set the chopsticks to have the same openness despite the differences in lengths.

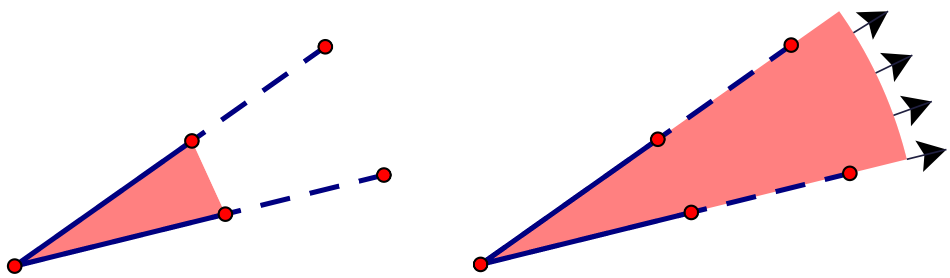

Figure 4.9. Finite (left) and infinite interiors (right) shown on superimposed angle models (solid and dashed).

Camille's activities on the task were an early indication she had constructed an angular unit. At the onset of the task, I infer that Camille established a goal of setting her angle model, the short chopsticks, to have an interior congruent to the given angle model, the long chopsticks. Camille set her angle model prior to physically overlaying one angle model atop the other. Therefore, I hypothesize Camille enacted this superimposition mentally as she set the chopsticks, which would require an abstract angular unit. Her subsequent physical superimposition, combined with her conclusion that both angle models "had the same like space," supports the inference that Camille unitized the angle, which included the interior. The result of Camille's activities was a copy of the given angle, which has been set to the same openness as the given angle.

[28] I will revisit these possibilities in Chapter 5.

The actions Camille initiated—superimposing one pair of chopsticks atop the other—were, in my view, the angular analog of placing two linear objects end to end to verify if they were the same size. Throughout the initial interview and in later teaching sessions, Camille enacted this superimposition activity to check for angular congruence. As such, I refer to this regularity in her reasoning as an *angular congruence via superimposition scheme*.

To investigate Camille's conception of angular comparison, I asked Camille to set the short pair of chopsticks to be more open than the fixed, long pair of chopsticks. With the chopsticks already set to the same openness from the previous task, Camille opened the sides of the short pair and explained, "I just made it a little bit bigger, " which indicated her progressive integration operation was activated.

Camille's actions and explanation indicate she viewed the short pair as more open than the long pair. However, I was unable to infer whether Camille viewed the final state of the short pair of chopsticks as being an additive composition of (a) the openness of the long pair of chopsticks and (b) the amount by which she had further opened the short pair. I pressed Camille for further justification to see if she would use superimposition in her justification as she had done in the previous angular congruence task, which would indicate whether she viewed the final state of the short pair of chopsticks as an additive composite. Camille elaborated, "it's wider from right here," and indicated the distance between the endpoints of the shorter pair using her thumb and middle finger (Figure 4.10). I refer to the linear segment between the endpoints of an angle model as the *span* of the angle model. In her continued explanation, Camille that she was comparing the span of the short pair to the span of the long pair. When I asked Camille to set the short pair to be less open than the long pair, she closed the short pair and noted the span of the long pair exceeded the span of the short pair.

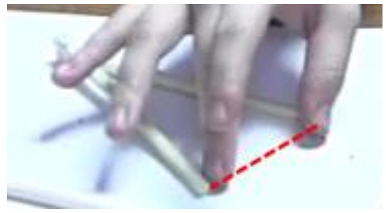

Figure 4.10. Camille indicates the span of the short chopsticks.

Camille's shift from focusing on the interior to focusing on the span may have been in part due to the novelty of the angular material. She may not have superimposed because she may have been unable to interpret the result of the superimposition in terms of an additive structure. Alternatively, social factors may have contributed to Camille's shift to focusing on the span of the chopsticks. In particular, Camille may have given an alternative explanation due to my requests for additional justification.

Angular multiple. When I asked Camille to set the short pair of chopsticks to be four times as open as the given longer pair, Camille began by setting the short pair to the same openness as the long pair. She then opened the short pair in several (3 or 4) short bursts of motion to the configuration shown in Figure 4.11 below. When I asked Camille to explain her thinking, she repeated her actions as she explained. First, she "went to get the same space," superimposing the short pair atop the long pair. Then she "just opened it—one, two, three four," rhythmically opening both sides of the short pair with each number word.

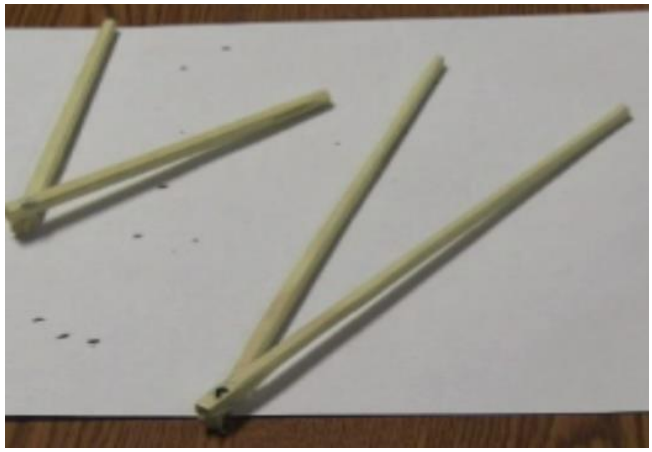

Figure 4.11. Camille sets the short chopsticks in the angular multiple task.

I interpret Camille's activities in this task as a novel application of her number sequence in the present situation. I infer that Camille established a goal of setting her chopsticks to be more open than the given chopsticks. Using her concept, four, with cadenced motions, Camille did set the short chopsticks to be more open than the long pair; however, Camille's motions were not accompanied by repeated projections of an angular unit (i.e., the given chopsticks) into the quantitative situation. Furthermore, Camille structured the situation additively—increasing the openness of the given chopsticks by appending "four"—rather than multiplicatively, which would have involved an insertion of four instantiations of the given chopsticks when setting the short chopsticks. Therefore, Camille had not established what I consider to be a scheme for angular repetition at the onset of the teaching experiment.[29]

Angular splitting. From the portions of the initial interview dealing with linear material, I inferred Camille had not yet established the splitting operation. In search of corroboration or contraindication for this inference, I posed the splitting task to Camille

[29] Angular repetition involves the physical action of producing an angle model *n* times as open as a given angle model, where *n* is a natural number.

in an angular context: This is my pair of chopsticks; my pair of chopsticks is five times as open as your pair of chopsticks; can you make your pair of chopsticks?

Camille solved the angular splitting task similarly to the angular multiple and linear splitting tasks she had previously solved. After superimposing her chopsticks atop my chopsticks for congruence, Camille bidirectionally decreased the openness of the chopsticks in five bursts of motion while counting, “one, two, three, four, five” (Figure 4.12). When I asked Camille if she could check to be sure my pair was five times as open as her pair, Camille opened her chopsticks in bursts while counting from one to five. As in the linear splitting and angular multiple tasks, Camille used cadenced motions to set the chopsticks and produced an additive rather than multiplicative structure. Camille’s activities contraindicated her construction of the splitting operation in the context of angles.

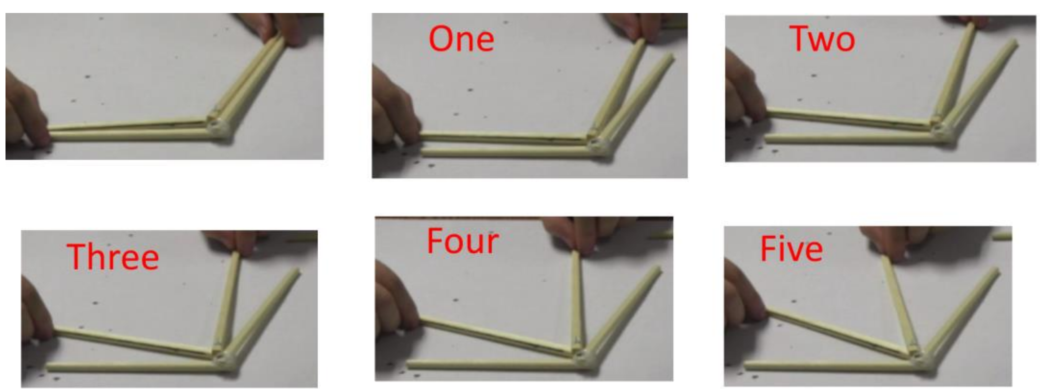

Figure 4.12. Camille sets her chopsticks using five cadenced motions.

Kacie’s Initial Angular Operations

In this section, I describe Kacie’s ways of operating on initial interview tasks involving angular congruence, comparison, multiples, and splitting. Additionally, I describe Kacie’s activities on an experiential dilation task.

Angular congruence and comparison. When I asked Kacie to set a short pair of chopsticks to the same openness as a long pair of fixed chopsticks (bottom right in Figure 4.13), Kacie opened her chopsticks as shown at top left in Figure 4.13. Kacie's explanation of how she set her chopsticks is presented in Excerpt 4.3 below.

Excerpt 4.3. Kacie explains her reasoning for the angular congruence task.
K: I looked at like the distance between the two chopsticks [*pointing to the endpoints of the long chopsticks*] and kind of looked at it and looked at mine and saw if they were similar or not. And just kind of went off that.
T: Oh okay. So can you show me what you mean by distance between the two chopsticks?
K: Like from this [*pointing to one end of the short chopsticks*] to here [*moves linearly and points to the other end of the short chopsticks*]

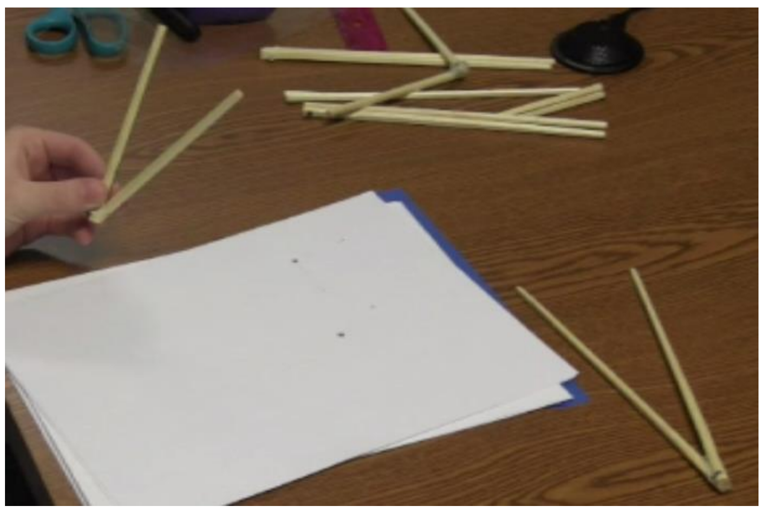

Figure 4.13. Kacie sets her chopsticks in the angular congruence task.

From Kacie's explanation of her activity in this angular congruence task, I infer that Kacie was attending to a linear distance when she set her short chopsticks to have the same openness as the long chopsticks. I refer to the linear segment between the endpoints of the sides of an angle model as the *span* of the angle model (see dashed lines in Figure 4.14 below). Kacie set the short chopsticks to have the same span as the long chopsticks (Figure 4.14). Although Kacie indicated an awareness of openness in the angular re-presentation task, her words and actions in angular congruence task did not indicate she was attending to the interior of the chopsticks in this context. The differences in Kacie's activities might be explained by two differences in the tasks. First, the chopsticks were hidden from view in the re-presentation task, whereas the chopsticks were perceptually available in the congruence task. Because the chopsticks were within Kacie's visual field in the congruence task, she may have focused on the static attributes of the chopsticks (i.e., the span) rather than the dynamic opening actions on the chopsticks, which she described in the re-presentation task. Second, Kacie assimilated the chopsticks as being equal in length in the re-presentation task and described opening and closing one pair of chopsticks as if to transform it into the other pair. Kacie may not have considered opening or closing the chopsticks precisely because she was aware of the differences in their lengths. In other words, opening or closing the chopsticks in the congruence task would not transform one pair of chopsticks into the other because the chopsticks were not the same length. As a result, Kacie may have focused on the span of the chopsticks as she could compare this attribute across both pairs of chopsticks despite the differences in length.

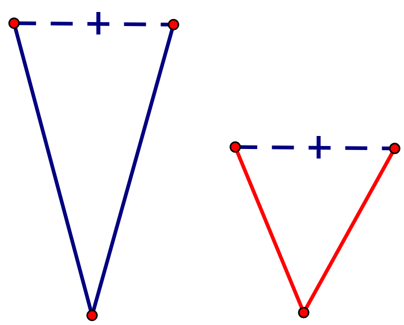

Figure 4.14. A model of Kacie's reasoning involving congruent spans on the congruence task.

In the angular comparison tasks, Kacie also operated on the span of the chopsticks. When I asked her to set the short pair to be more open or less open than the long pair, she set the short pair to have a longer or shorter span, respectively, than the span of the long pair.

Angular multiple. When I asked Kacie how she would set the short chopsticks to be exactly four times as open as the long chopsticks, Kacie again operated on the span of the chopsticks. She first suggested measuring the "distance of the space in between the two chopsticks [indicating the span of the long chopsticks] and then multiplying that by four." When I asked Kacie if it would be possible to accomplish the task without using numbers, Kacie set her short chopsticks as shown at left in Figure 4.15 to "get the same size." She then opened the chopsticks further (Figure 4.15 right) and explained that she "visualized that [span of the long chopsticks] going into [the span of the short chopsticks] four times."

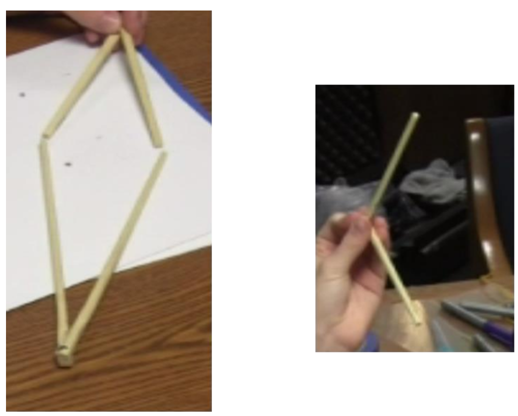

Figure 4.15. Kacie captures the span of the chopsticks and estimates.

Kacie's activities indicated that she assimilated the task as a situation for iteration; however, Kacie applied her iteration operation to linear material (i.e., the span of the chopsticks), rather than to angular material. Kacie set the short chopsticks to have the same span as the long chopsticks. Then, I infer that she mentally iterated this span a total of four times and set the short chopsticks to contain these four iterations of the span. A model of Kacie's span iteration is shown in Figure 4.16.

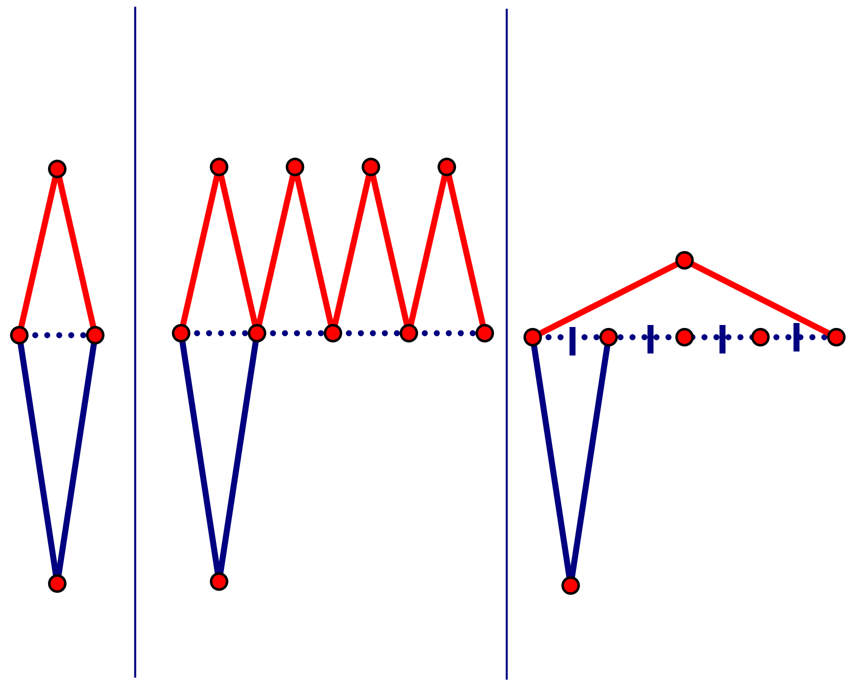

Figure 4.16. A model of Kacie's span iteration.

Angular splitting. In the angular splitting task, Kacie operated again on the span of the chopsticks. I presented Kacie with a fixed pair of chopsticks and asked her how she would set a pair of chopsticks so that the given pair was five times as open as her pair. Using a ruler, Kacie determined the span of the chopsticks was approximately 7 inches.[30] Using long division, she computed seven divided by five and set a new pair of chopsticks, which were longer than the given chopsticks, to have a span of 1.4 inches (Figure 4.17). After Kacie set her long pair of chopsticks, I asked how she would prove that the given chopsticks were five times as open as her pair. Kacie suggested two methods for verification. First, she described numerically verifying the result by

[30] I allowed Kacie to use a ruler when she requested one because the use of the ruler clearly indicated Kacie was operating on linear material.

multiplying the span of her chopsticks by five. Second, she demonstrated checking that five iterations of the span of her chopsticks would constitute the span of the given chopsticks. A model of Kacie's second method of verification is shown in Figure 4.18. In Figure 4.18, Kacie's chopsticks and the given chopsticks are shown in red and blue, respectively. The arrows indicate the successive iterations of the span of Kacie's chopsticks.

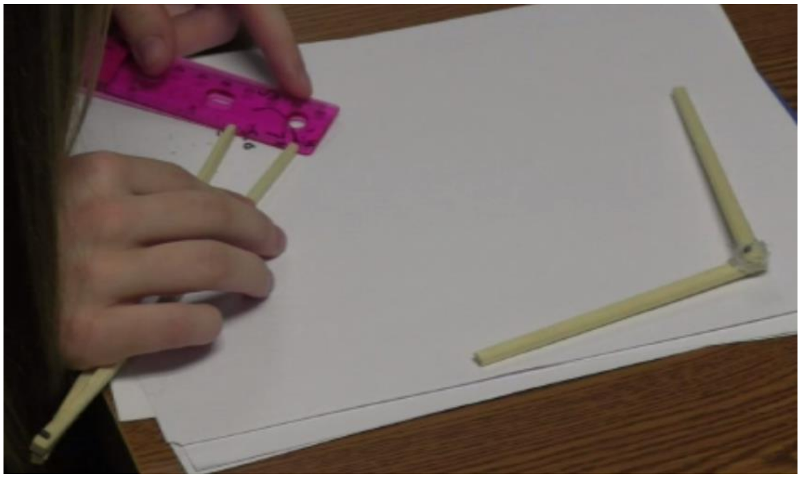

Figure 4.17. Kacie uses a ruler to set her chopsticks in the angular splitting task.

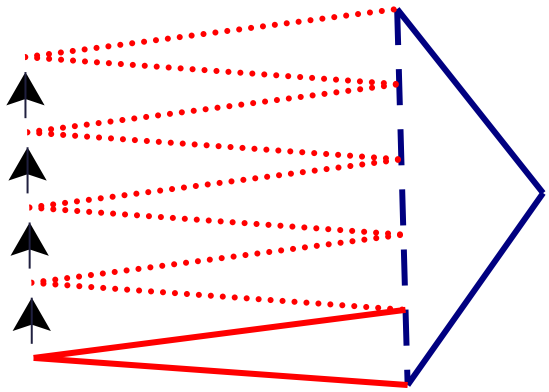

Figure 4.18. Kacie checks her estimate via span iteration in the splitting task.

Kacie used a ruler to measure and then performed numerical computations in this angular splitting task. Kacie's use of the ruler in this task provides compelling evidence that Kacie was operating on the span of the chopsticks. Because Kacie performed numerical computations, it is difficult to infer whether Kacie enacted the splitting operation in this task. If Kacie did implement the splitting operation, her second method of verification, which involved iterating the span of her chopsticks five times, indicates that she applied her splitting operation to linear material and not angular material.

Experiential dilation. Because Kacie demonstrated an awareness of openness in the re-presentation task but operated on the span of the chopsticks in all other angular tasks, I presented Kacie with an experiential dilation task. In this task, I showed Kacie a picture of the long chopsticks and asked her to set the long chopsticks to have the same openness as when I took the picture. The chopsticks in the image were perceptually shorter than the physical chopsticks available to Kacie.

Kacie set the chopsticks as shown in Figure 4.19 below and explained, "I just kind of went off the picture and looked to see it looked similar." Kacie continued, "I just kinda like took the distance in between those two [indicating the span of the photographed chopsticks] and just kind of put 'em in between the distance of these two [indicating the span of the physical chopsticks]."

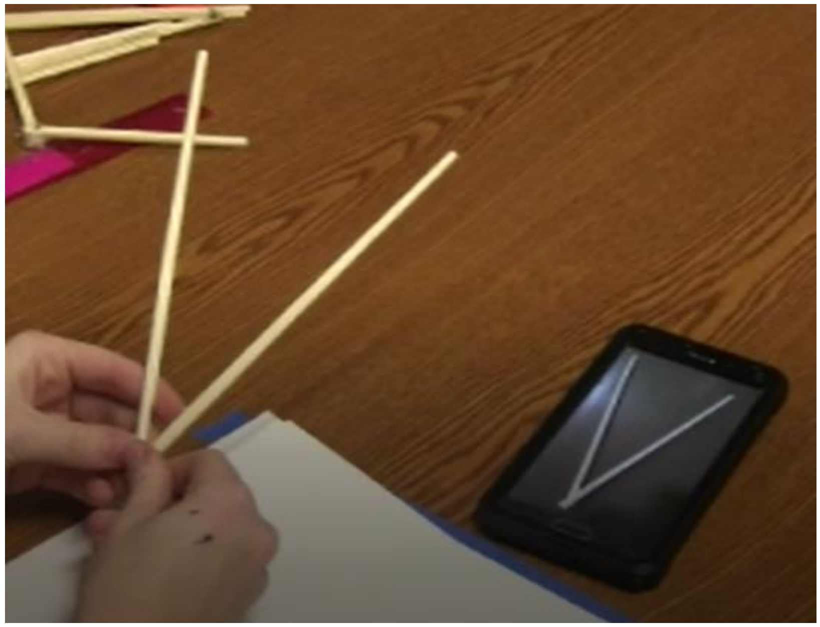

Figure 4.19. Kacie sets the long chopsticks in the experiential dilation task.

From my perspective, Kacie set the chopsticks to be approximately as open as they were in the photograph, which was different from her previous way of operating in the angular congruence task. I account for the difference in Kacie's way of operating by an experiential dilation operation. I hypothesize Kacie developed this experiential dilation operation through her experiences of the same object appearing to vary in size within her perceptual field. From my perspective, the application of the experiential dilation operation resulted in what I recognize as angular congruence. However, Kacie gave no indication of attending to the interiors of the chopsticks. Though I perceived a change in Kacie's activities on this task, Kacie asserted that she had set the physical chopsticks to have a span congruent to the photographed. For Kacie, I infer the distance was necessarily the same because the photograph contained an image of the physical chopsticks. Kacie was not perturbed in this task, though I perceived the span of the physical chopsticks to be noticeably longer than the span of the photographed chopsticks

Summary of Initial Interviews

In this chapter, I presented an analysis of students' solutions to the initial interview tasks; these analyses resulted in inferences about students' ways of reasoning at the onset of the teaching experiment. From the tasks involving linear material, I inferred Kacie could coordinate three levels of units in assimilation, while Camille could coordinate two levels of units in assimilation and three levels of units in activity. I attributed the splitting operation to Kacie, but not to Camille.

From the angular portions of the initial interview, I inferred that Camille had already constructed a scheme for angular congruence involving superimposition, which implied the construction of an angular unit. However, Camille did not use this angular unit in further operating for producing angular multiples or splitting a given angle. In angular multiple and splitting tasks, Camille produced additive structures through cadenced motions which did not indicate the projection of angular units into the angle models.

Although Kacie indicated an awareness of angularity via both re-presented opening and radial sweep, Kacie did not operate on the angular units throughout the remaining angular tasks. Kacie consistently applied her existing operations to the span of angle models throughout the other initial interview tasks. Therefore, I say she established a *span operating scheme*. I consider the span operating scheme to be a pseudo-angular scheme as the activity of the scheme involves operations on linear material, while the situation and results of the scheme involve angular material. Kacie implemented her span operating scheme in angular congruence, comparison, multiple, and splitting tasks.

CHAPTER 5

FINDINGS FROM TEACHING SESSIONS AND FINAL INTERVIEWS

In this chapter, I present second-order models of Camille's and Kacie's quantifications of angularity. Camille and Kacie were partnered for many of their teaching sessions, and I present my analysis of episodes from their teaching sessions chronologically. Through the chronological presentation of these results, I intend to foreground both stable and dynamic elements of each student's ways of reasoning during the study. In this chapter, I do not discuss every task or every teaching session in detail. Instead, I discuss significant episodes—portions of teaching sessions that I use as compelling evidence for my inferences about the students' quantifications of angularity, affordances and limitations of these quantifications, or changes in their quantifications.

The results in this chapter are presented in four parts. Kacie and Camille were partners throughout the portions of the study discussed in Parts 1 and 2. The results from Part 1 are from three teaching sessions prior to the school's winter break. The results from Part 2 correspond to seven teaching sessions after winter break. After the sessions described in Part 2, I decided to separate Camille and Kacie in subsequent teaching sessions to promote the greatest possible mathematical progress for both students. In Part 3, I analyze Camille's last four sessions in the teaching experiment. In Part 4, I analyze Kacie's last three sessions in the teaching experiment.

Part 1: Attributes and Operations Involved in the Construction and Comparison of Angular Units

In Part 1 of this chapter, I present data from three teaching sessions (Table 5.1). I taught the students as a pair for the first two sessions. In the third session, I taught only Kacie because Camille was absent. Throughout these three sessions, I posed angular congruence, comparison, and multiple tasks to the students, primarily using hinged chopsticks as angle models. In the course of teaching the students, I made modifications to these tasks in efforts to harmonize my teaching with the students' mathematical activities. In these sessions, the students attended to a variety of attributes they imputed to the angle models. One of my major goals in these sessions was to engender students' differentiation between angularity and other attributes. In Part 1, I discuss operations I attributed to the students as a result of their activities on angular congruence, comparison, and multiple tasks.

Table 5.1. Sessions and attendance for Part 1.

	11/12	11/17	12/3
Camille	X	X	
Kacie	X	X	X

Camille's and Kacie's November 12th Session

Three and four weeks after their respective individual interview sessions—Kacie and Camille participated in their first paired teaching session. In this session, the students worked on angular congruence and angular comparison tasks, which I designed for multiple purposes. First, I wanted to compare each student's responses to her responses from the congruence tasks in the initial interview, in particular to see if their ways on reasoning on these tasks were relatively stable. Second, Camille and Kacie reasoned differently on angular congruence tasks during their interviews, so I was eager to see how each would assimilate the actions of the other. Camille had operated on the length-independent interior of the angle models (what she called the "space"), and her actions suggested she had previously established a scheme for congruence that relied on superimposition; in contrast, Kacie had operated exclusively on the span of the chopsticks throughout congruence, comparison, multiple, and splitting tasks. Finally, Camille shifted from operating on the interior of the chopsticks to operating on the span of the chopsticks during the angular congruence tasks. As such, I was interested to see how she reasoned on comparison tasks in this session.

Congruence from photographs and the experiential dilation operation. In this teaching session, I presented the pair with an angular congruence task involving photographs, which was similar to the final task Kacie solved in her initial interview. I provided each student with a photograph of a pair of chopsticks and asked each student to set the chopsticks to be as open as when the photograph was taken. The chopsticks each student manipulated were the same pair of chopsticks in the photograph; however, the images of the chopsticks were perceptually different in length from the physical chopsticks available to the students. A primary purpose of the task was to engender a perturbation for Kacie if she activated her span operating scheme. Additionally, I hoped the situation would trigger Camille's superimposition for congruence scheme so that I could observe how Kacie assimilated Camille's activities.

Kacie and Camille set their chopsticks as shown in Figure 5.1 below, and both students expressed their reliance on visual perception. Kacie explained she was "just like looking at this picture, and then visually kind of basing it off of that picture, and then finding where it looked right." Camille offered a similar explanation, "well, I mean it looks the same so I'm just guessing it's the same." Kacie did not reference the span of the chopsticks, and Camille did not indicate using superimposition.

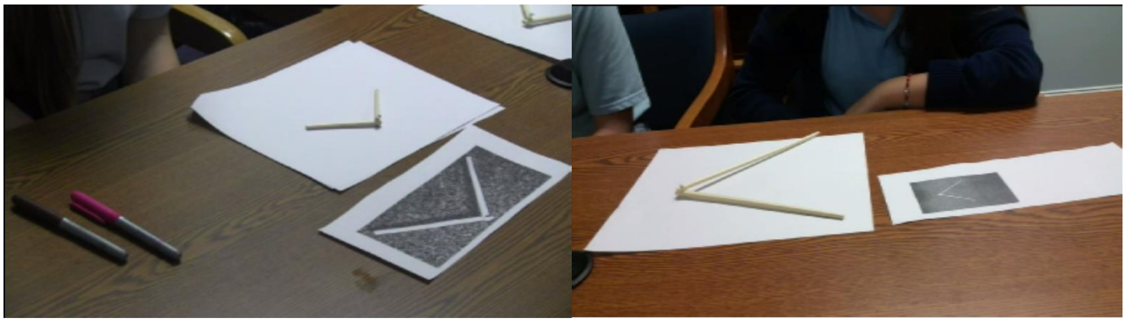

Figure 5.1. Kacie (left) and Camille (right) set their chopsticks to be as open as photographed chopsticks.

From my perspective, both students had set the chopsticks to approximately the same openness as the photographed chopsticks. However, I do not claim that either student explicitly attended to the angularity of the angle models in this instance as neither student gave any indication of having considered motion through the interior of the angle models. Instead, I attribute both students' activities in this task to the experiential dilation operation, which develops naturally from an individual observing the same object varying in perceptual size as her distance from the object changes. After assimilating the photographed and physical chopsticks as identical objects, the students needed to imagine varying their distance from the photographed chopsticks as they set the physical chopsticks.

Angular Comparison. After the experiential dilation task, the two pairs of wooden chopsticks on the table were set to different opennesses from my perspective (see Figure 5.1). To see whether Kacie would return to operating on the span of the chopsticks, I asked the students which pair of wooden chopsticks were more open, Kacie's short chopsticks (left in Figure 5.1 above) or Camille's long chopsticks (right in Figure 5.1 above). Because the span of the long chopsticks exceeded the span of the short chopsticks, I anticipated that Kacie would assert that the long chopsticks were more open if she activated her span operating scheme.

Camille truncates the long chopsticks. Camille responded before Kacie and asserted Kacie's short chopsticks were more open. After a three second pause, Kacie hesitantly agreed the short chopsticks were more open. To understand how Camille was comparing the opennesses of the chopsticks, I asked her to explain her reasoning. Portions of Camille's response are described in Excerpt 5.1 below.

Excerpt 5.1. The truncation operation.

C: I mean, I just kind of make this [long] one like if it was like that [short] one. So – kind of imagining it like that [*places marker across the top of her long chopsticks as if to render the long pair the same length as the short pair* (Figure 5.2)]. So like this right here [*indicating the long chopsticks from vertex to marker*] is just like that [short] one, except that [short] one's more wider.

…

T: How do you know which one's more open?

C: I mean that [short] one looks more [opens thumb and index from a closed position over the short pair, repeats a similar action over the long pair] wider than this [long] pair.

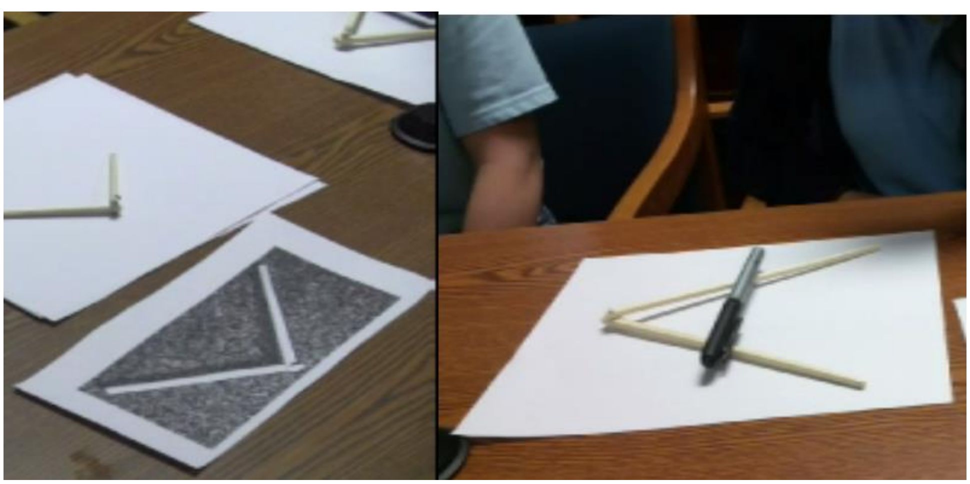

Figure 5.2. Camille truncates her chopsticks (right) to the length of Kacie's chopsticks (left)

From the interaction surrounding Excerpt 5.1, I inferred Camille transformed the long chopsticks into the short chopsticks via the *truncation operation*, mentally shortening the long chopsticks to the same length as the short chopsticks. That Camille instantiated the operation is supported both by her placement of the marker (Figure 5.2 right) and her explanation, “I just kind of make this one like it was like that one.” In truncating, Camille rendered the different lengths of the angle models inconsequential as she considered which angle model was more open. However, the truncation operation alone was insufficient for Camille to compare the openness of the two angle models. As Camille explained that her short chopsticks were “more wider,” she opened her thumb and index finger over the short chopsticks and repeated a similar gesture over the long chopsticks. From these gestures, I infer Camille re-presented opening each pair of chopsticks from a closed position and compared the duration of this motion to determine which chopsticks were more open.

***Kacie and the hyper-truncation operation*.** In this angular comparison task, Camille was the first to claim that the short chopsticks were more open than the long chopsticks. After a three second pause, Kacie hesitantly asserted the same claim. Had Camille not responded first, I would have expected Kacie to remark that her long pair of chopsticks was more open than Camille’s short pair since the span of the long chopsticks exceeded the span of the short chopsticks. I interpret Kacie’s hesitation as an indication that she was considering how Camille might have been thinking to have reached her conclusion. Following Camille’s explanation, Kacie explained her reasoning, which is described in Excerpt 5.2.

Excerpt 5.2. The hyper-truncation operation.

K: I was like looking at how open these were [*repeatedly tracing out the span of the long chopsticks*]. But then I realized that that probably wouldn't help me because it [the long pair] looks like it's more open than mine because it's bigger, like the chopsticks are longer than mine so it makes it look like it was more open. So then, I just kind of looked at like where they [*points to vertex of long pair*] cross or come together and saw that these [short pair] were like [*opens index fingers over the vertex*] not so like – I don't know how to explain it. [6 sec] That mine weren't so close together and these [long pair] were more – like there was littler space right here [*pointing to the interior near the vertex of the long pair as shown in Figure 5.3*] than there was in mine [*points similarly near the vertex of the shorter pair*].

Figure 5.3. Kacie points to the interior near the vertex of the long chopsticks.

From the interaction described in Excerpt 5.2, I infer Kacie was in the process of modifying her conception of openness. At the beginning of the excerpt, Kacie indicated she initially assimilated the angular comparison task to her span operating scheme. Though Kacie considered comparing the spans of the chopsticks, she discarded this possibility because she assimilated differences in the lengths of the chopsticks—an attribute that did not previously appear to be relevant to her based on the angular tasks in her initial interview session. I suspect Kacie's attention to the differences in lengths was occasioned, at least in part, by Camille's assertion that the shorter chopsticks were more open than the longer chopsticks. Kacie's attention to length may also have been occasioned in part by the photographs from the previous task, which were still within her visual field.

Like Camille, Kacie indicated she had established a goal of rendering the lengths of the chopsticks inconsequential for comparing openness. Rather than truncate the longer chopsticks to match the length of the shorter chopsticks, Kacie mentally shortened both pairs of chopsticks as if trying to free them of length entirely, which I refer to as *hyper-truncation*. Although it is possible Kacie might have been imagining some previous experience involving angle markings (i.e., arcs), this seems unlikely for two reasons. First, Kacie had not mentioned any such markings. Second, Kacie had not yet referred to the chopsticks as angles at this point in the teaching experiment. From her explanation and gestures, I was unable to infer whether "the space" referred to an area or a length.[31] However, Kacie was adamant that she limited her focus very near to the vertex: "I looked at like really close to where the chopsticks meet, but not like where they meet, like the space right after they meet."

Camille demonstrates an awareness of angularity via segment sweep. Following Kacie's demonstration of hyper-truncation, Camille indicated yet another means of comparing the opennesses of the two pairs of chopsticks. Camille held her thumb and index finger together over the vertex of the long chopsticks (Figure 5.4 left) and then dragged her thumb along one side of the chopstick and her index finger along the other (Figure 5.4 center and right). As she moved her fingers over the sides of the angle model, her thumb and index finger grew further apart. As she moved her hand, Camille explained, "Mine's starting off really small and getting bigger." Camille made a similar

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[31] From my perspective, Kacie essentially wanted to truncate the chopsticks to infinitesimal length which would make it more challenging to distinguish between length and area.

gesture as she referenced the short pair of chopsticks, “and that one’s just like – it’s just open really big,” moving her hands as if tracing out the short pair of chopsticks in mid-air.

Figure 5.4. Camille drags her thumb and index fingers across the angle model.

Camille’s gestures over the long chopsticks suggested a new motion through the interior of the angle. She moved as if sweeping a growing segment, whose endpoints were determined by her thumb and index finger, through the interior of the model (Figure 5.5). Camille’s explanation indicated she implicitly considered the growth of this segment as she moved her hand away from the vertex, as if constructing an experiential rate. I infer Camille compared the opennesses of the two pairs of chopsticks by considering which chopsticks would cause her thumb and index finger to separate more quickly as she moved her hands away from each vertex; she was reasoning about openness as an intensive quantity. She was not considering the duration of the motion she enacted as she moved her hands over the chopsticks; she was comparing the intensity of this motion.

Because the chopsticks were perceptually available and Camille gestured over the chopsticks, Camille demonstrated at least an awareness of experiential angularity via segment sweep. However, Camille compared openness of the two angles by comparing the growths of the sweeping segments without any noticeable simultaneous sensorimotor activity. This comparison indicated she re-presented the segment sweeping motion for at least one angle model and compared this growth to that of the corresponding sweeping segment for the other angle model. Because Camille held at least one of these rates of change in mind, Camille demonstrated an awareness of figurative angularity via segment sweep.

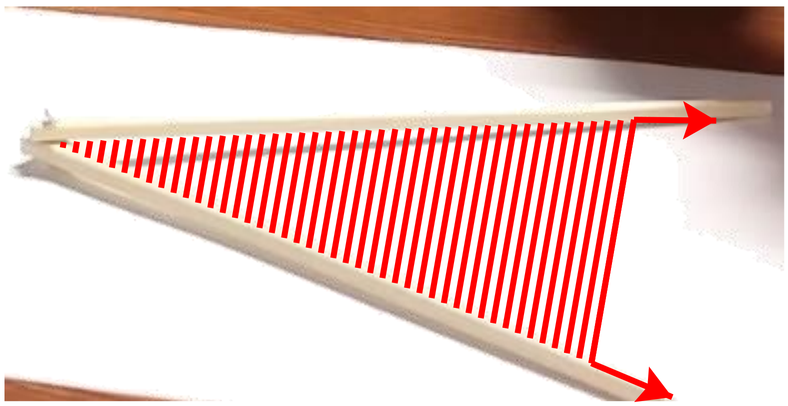

Figure 5.5. Segment sweep through the interior of an angle.

Camille compares angles via superimposition. In the final minutes of Kacie's and Camille's November 12th teaching session, I asked the pair to set the long chopsticks so that they would be more open than the short chopsticks. Camille opened the long chopsticks so that they were approximately congruent to the current configuration of the short chopsticks from my perspective.

When I asked how they could be "absolutely sure" that the long chopsticks were more open than the short chopsticks, Camille superimposed the short chopsticks atop the long chopsticks (Figure 5.6) and commented, "oh wait, never mind," which indicated her surprise at the perceptual result of the superimposition. In Excerpt 5.3, Camille explains the reason for her surprise, and Kacie responds to Camille's reasoning.

Figure 5.6. Camille compares the short and long chopsticks via superimposition.

Excerpt 5.3. Indications of comparing angles via superimposition in Camille and Kacie.

T: Wait. What just happened?

C: I just – well I put this [short] one on top of that [long] one. And this [short] one's sticking out more, so it's not like completely on it [*motions as if trying to entirely superimpose the short pair atop the long pair*]. So I think it – this [short] one's still more.

T: What do you [Kacie] think about what she's saying?

K: That makes sense because if one [side of the short pair] is like farther out than this [long pair] one, then that means it's more open [*opens hand over the chopsticks*].

Camille's explanation suggested she was constructing a scheme for comparing the openness of angles through modifying her angular congruence via superimposition scheme. Camille's comment that the short chopsticks were not "completely on" the long chopsticks indicated she was aware the short chopsticks were not open to the same amount as the long chopsticks because they could not be perfectly superimposed. Because Camille described the short chopsticks as "sticking out more," Camille assimilated some attribute of the short chopsticks as exceeding the same attribute in the long chopsticks. Camille did not give an explicit indication of opening the long chopsticks to match the openness of the short chopsticks within the present task. However, from her activities on the previous comparison task involving the truncation operation, I infer Camille was aware she would need to open the long chopsticks to match the openness of the short chopsticks.

As in the previous tasks, Kacie assumed a passive role in that she allowed Camille to explain her reasoning before she explained her own reasoning. In Excerpt 5.3, Kacie imitated Camille by providing a similar explanation "if one side is like farther out…that means it's more open." At this point, it was not clear if Kacie's assimilation of Camille's actions entailed an awareness that the models would have had the same openness if the sides of the angle models had been entirely coincident. Though subtle, Kacie's gesture, opening her hand over the chopsticks, at the end of the excerpt offered some indication she may have imagined opening the long chopsticks to be coincident with the short chopsticks, particularly since Camille did not demonstrate a similar action within the present interaction.

Kacie's and Camille's November 17th Session

In their second paired teaching session, Kacie and Camille continued to solve angular congruence and comparison tasks. At the beginning of the session, the students took turns setting short and long pairs of chopsticks to have the same openness. One student set her pair of chopsticks to a particular configuration, and then the second student set the other pair of chopsticks to have the same openness as the configuration her partner had established. From my perspective, each student reasonably set her chopsticks to match the openness of her partners configuration in these congruence tasks. Additionally, after producing an estimate, each student carried out the activity of superimposing one angle model atop the other while preserving the current configuration to check the accuracy of her estimate. As such, each student indicated she was constructing an angular congruence via superimposition scheme.

Evidence of an elongation operation. In the previous teaching session, Camille indicated she had enacted the truncation operation whereby she transformed a longer pair of chopsticks into a shorter pair of chopsticks by mentally shortening the longer pair. Following the turn-taking congruence tasks and at my request, Camille re-demonstrated her use of a marker to shorten the long chopsticks to match the lengths of the short pair.

During the teaching experiment, I hoped to foster in students a conception of angle that included an awareness of indefinite length. I hypothesized the elongation operation—the cognitive inverse of the truncation operation whereby an individual extends the sides of one angle model to match the lengths of another angle model—might play a critical role in such a conception of angle. To investigate whether the students had constructed such an operation and after Camille's demonstration, I asked the pair if it would instead be possible to think of a way to "turn the shorter pair into the longer pair."

Camille suggested cutting material from another chopstick and adding it to the shorter pair. Providing them with markers and paper, I asked the students to draw how they would add to material to the short chopsticks. Camille positioned the shorter chopsticks within the longer so that the vertices and sides were adjacent.[32] Using the marker, Kacie linearly extended one side of the short chopsticks to match the length of the corresponding side of the long chopsticks.

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[32] Though Camille did not place one angle model atop the other here, her actions indicate the same operations constituting the angular congruence via superimposition scheme. I attribute the change in Camille's physical actions on the chopsticks to her newly established goal of drawing an extension of the sides, which would be more difficult to physically enact if the short pair was not resting on the paper.

Camille's verbal description and Kacie's physical activity indicated that each student had constructed an elongation operation whereby she could imagine a continuation of a perceptually available segment along the line containing the segment. Prior to this point in the teaching experiment, neither student had spontaneously indicated such an operation in tasks involving the chopsticks; however, the students' activities on the linear multiple task from the initial interview, (making a string six times as long as a given piece of string), indicated each had constructed an elongation operation. I hypothesize it was not happenstance that the students spontaneously enacted the truncation operation but not the elongation operation on the angular tasks to this point in the teaching experiment. Truncation requires mentally discarding readily available perceptual material whereas the elongation operation requires the mental insertion of material that is not perceptually available.

Kacie and Camille experiment on angular comparison tasks. After inferring each student had constructed the elongation operation and could apply this operation to the angle models, I returned to investigating the students' ways of comparing opennesses. I gave short and long chopsticks to Camille and Kacie, respectively, and asked each to set her chopsticks to some openness. Once they had set the chopsticks as shown in Figure 5.7, I asked the students which pair of chopsticks was more open.

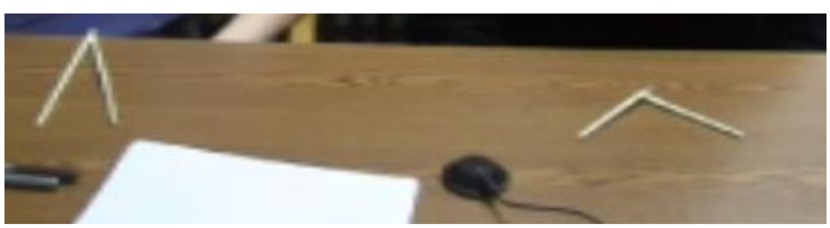

Figure 5.7. Kacie (left) and Camille (right) set chopsticks for an angular comparison task.

Both students immediately pointed to Camille's short pair of chopsticks, indicating each viewed the short pair as more open than the long pair. In their initial justifications for this conclusion, each student indicated through gestures that she had compared the lengths of the spans of the angle model. When I asked the students to show me how they would compare the angle models, Camille superimposed the chopsticks (Figure 5.8) and indicated comparing hyper-truncated angle models as she pointed near the superimposed vertices: "This [long pair] is more closed from the middle, from like where it begins, and this [short pair] is more open." Kacie assimilated Camille's way of reasoning involving hyper-truncation as she pointed near the superimposed vertices: "like there's more space right there [short pair] than there is on that one [long pair]."

Figure 5.8. Camille superimposes chopsticks

That Kacie compared the opennesses of the angle models using the span and hyper-truncation was not surprising as she had reasoned similarly on comparison tasks in previous sessions. Camille, however, had previously reasoned about comparison tasks by comparing interiors of the angle models (e.g., comparing the imagined actions of opening the chopsticks). Camille's consideration of the spans and hyper-truncation in this teaching session indicated she had internalized these operations from assimilating Kacie's activities.

The kite configuration. The configuration of the angle models in the previous comparison task was limiting from my perspective as teacher-researcher. The angularity of the short chopsticks exceeded that of the long chopsticks and the span of the short chopsticks was longer than the span of the long chopsticks. As such, students could reach the conclusion that the short chopsticks were more open than the long chopsticks whether they considered the spans or the angularity of the angle models.

To engender a perturbation that might cause the students to differentiate span length and openness, I positioned the angle models to form a kite (Figure 5.9).[33] In positioning the pairs of chopsticks end to end, I hoped that the students might assimilate differences in the opennesses of the chopsticks as well as congruent spans. After positioning the chopsticks as shown in Figure 5.9, I asked the students which chopsticks were more open.

Figure 5.9. The short and long chopsticks in the Kite configuration.

Kacie responded first and asserted that the short pair was more open. This assertion was a dramatic contrast from Kacie's reasoning in the initial interview session

[33] I am grateful to Hwa Young Lee for suggesting this kite configuration as a situation for angular comparison that might occasion students' constructions of span length and openness as distinct attributes.

wherein she positioned the pairs of chopsticks in a similar kite configuration to set pairs of chopsticks to the same openness. Camille disagreed with Kacie and said both pairs appeared to have the same openness. Each student indicated some uncertainty about her initial claim, so Kacie checked which pair was more open via superimposition. Afterward, both students agreed the short pair was more open.

After Kacie superimposed, Camille began to explain the short pair was more open because it had a longer span and explicitly attributed this strategy to Kacie. After I demonstrated that the spans were congruent, Camille abandoned that line of justification, and I asked the pair if comparing the spans was sufficient for comparing opennesses. Kacie explained, in the case of the kite configuration, comparing the spans was insufficient for comparing opennesses; however, she also explained that if a task involved pairs of chopsticks that were all the same length, then the pair with the longer span would be more open.

Kacie's explanation indicated she was constructing span length and openness as distinct attributes of the angle models. Instead of disregarding the span, Kacie was in the process of forming a comparative quantitative structure involving side length, span length, and openness. Camille also indicated she recognized a distinction between span length and openness; however, Camille did not appear to be establishing a similar comparative quantitative structure. Instead, Camille seemed to discard the span from her reasoning as she expressed her preference for superimposing to determine which chopsticks were more open.

Introducing body and laser rotations. As I indicated in my initial conceptual analysis, I conjectured that the mental insertion of rotational motion—the image of a ray sweeping from one side of an angle to another—was critical for constructing quantifications of angle measure. To this point in the teaching experiment, neither student indicated having imagined any such rotational motion in tasks involving chopsticks. To introduce rotational motion, I asked students to rotate themselves and to rotate a laser level. The students produced drawings to represent their body rotations; each student drew a circle. After the body rotations, I asked students to take turns rotating a laser level, which projected a ray of light across the tabletop. I intended to transition from using physical rotations of the laser level to rotations of a laser in Geometer's Sketchpad (GSP); however, time constraints prevented me from introducing the students to GSP during this session. Instead, I elected to present the students with an angular multiple task described in the section below.

Solving an angular multiple task. In the final minutes of Kacie and Camille's second paired teaching session, I presented each student with a pair of chopsticks, which I had fixed to the same openness prior to the session. I asked the students to verify the chopsticks did have the same openness; Kacie immediately obliged by superimposing one pair atop the other, which again supported that she had constructed an angular congruence via superimposition scheme. After the students agreed the two pairs of chopsticks were fixed to approximately the same openness, I asked each student to make a pair of chopsticks two times as open as her given pair of chopsticks. I provided students with additional free chopsticks, which the students could set as they deemed appropriate in the task. The students solved the angular multiple task simultaneously and independently. I describe and analyze each students' activities on this task in turn within the subsections below.

Reemergence of Camille's cadenced motions on angular multiple tasks. Camille proceeded tentatively in this angular multiple task. She began by superimposing the free chopsticks atop the fixed chopsticks (Figure 5.10 left) and remarked, "I'm not going to know how to answer the questions you're going to ask me when I do this." Camille then opened her free chopsticks in two short bursts; the results of the first and second bursts are shown at center and right in Figure 5.10, respectively. Following the bursts of motion, Camille carefully repositioned the free chopsticks next to the fixed chopsticks.

Figure 5.10. Camille superimposes (left) and opens in two bursts (center, and right).

Camille operated much the same way as she did on the angular multiple task in her initial interview session. Using superimposition, she established angular congruence between the fixed chopsticks and the free chopsticks. Because Camille opened the free chopsticks beyond the congruent configuration, I infer she established a goal of setting the free chopsticks to be more open than the fixed chopsticks. Additionally, Camille's two bursts of motion indicated she was using her concept of two to open the free chopsticks beyond the fixed chopsticks.

Camille's adjustment to the fixed chopsticks was additive in that she inserted the given chopsticks *and* two opening bursts into the free chopsticks. Camille had constructed an angular unit as indicated by her angular congruence via superimposition scheme. However, Camille did not take this angular unit as input for operations that would produce two levels of multiplicative composite units (e.g., iteration). Such a multiplicative structuring of the situation would entail the insertion of two copies of the given chopsticks into the free chopsticks, which was contraindicated by Camille's actions.

To check her result, Camille carefully superimposed her free chopsticks atop the fixed chopsticks. She then counted quietly, "one, two," as she decreased the openness of the free chopsticks in two bursts of motion. As she coordinated her rhythmic counting with the movements of the free chopsticks, Camille reversed her additive actions on the free chopsticks to return them to the same openness as the fixed chopsticks. With the free chopsticks superimposed atop the fixed chopsticks at the end of the second burst, Camille looked up with a smile, presumably indicating her satisfaction. Camille's actions in checking her result indicated an element of reversibility in her way of reasoning about this task; these actions underscored the additive structure to which Camille had assimilated the task in that Camille reduced the openness of the free chopsticks by two bursts to return to the configuration of the fixed chopsticks. From my perspective, Camille did not establish a multiplicative structure involving composite units in this angular multiple task.

Kacie checks an angular multiple task via repetition. Within 15 seconds of my request to set a free pair of chopsticks to be twice as open as the fixed pair, Kacie set her free chopsticks next to the fixed chopsticks and opened them as shown in Figure 5.11. Unlike Camille, Kacie gave no discernable physical cues indicating how she might have used her concept of two in setting the free chopsticks. From annotating screenshots from the video records of the session using Euclidean constructions in GSP, I inferred Kacie might have set the free chopsticks by either duplicating the span (Figure 5.12 left) or duplicating the angle (Figure 5.12 right).

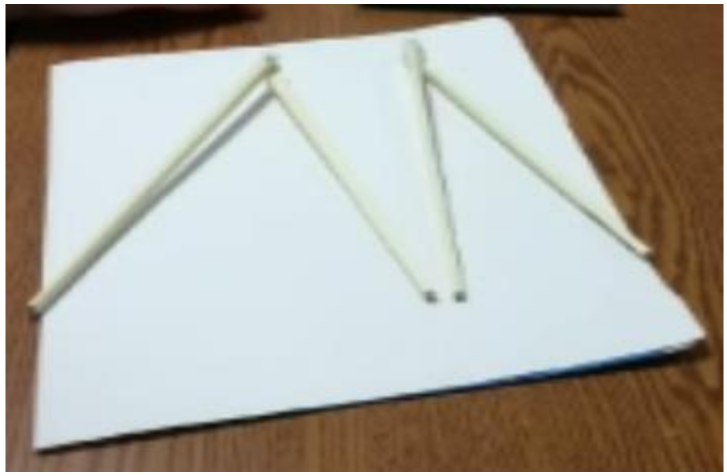

Figure 5.11. Kacie sets the free chopsticks (left) to twice as open as the fixed chopsticks (right)

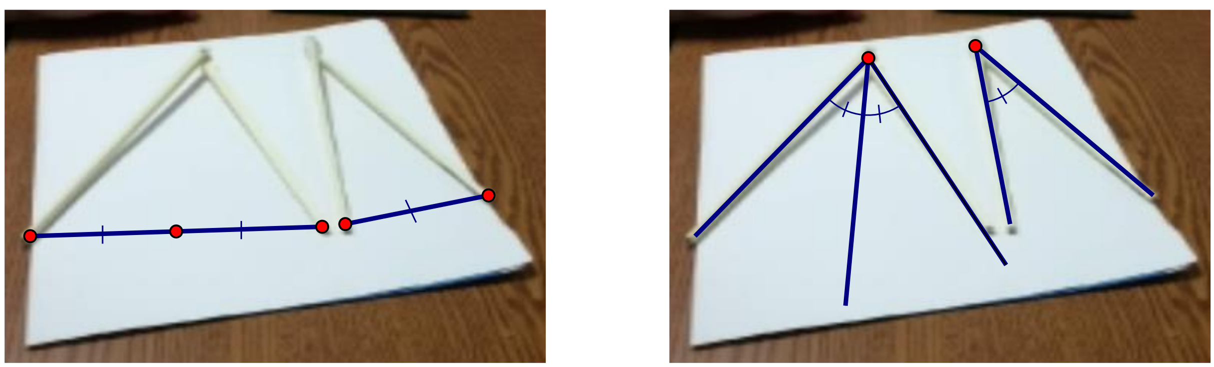

Figure 5.12. Illustration of the plausibility of Kacie's span (left) or angular duplication (right).

Although Kacie's rationale for setting the free chopsticks was unclear, Kacie's subsequent actions to verify her estimate indicated at least a temporary modification in her way of reasoning with angular multiple tasks. Kacie checked her result by repeating the fixed chopsticks into the free chopsticks. She began by first superimposing the fixed chopsticks atop with free chopsticks so that the vertex and one side were coincident

(Figure 5.13 left). I will refer to the sides of the fixed and free chopsticks that Kacie chose to initially align as the *primary sides* and the other sides of the chopsticks as the *secondary sides*. Kacie spontaneously used a pen to mark the end of the secondary side of the fixed chopsticks, which laid in the interior of the free chopsticks (Figure 5.13 center). She then repositioned the fixed chopsticks so that the endpoint of the primary side was coincident with this mark, while keeping the vertices of both pairs of chopsticks aligned. With the secondary side of the fixed chopsticks just beyond the secondary side of the free chopsticks (Figure 5.13 right), Kacie removed her hand and remarked “Oh! I was like really close.” Kacie’s activities indicated that she was checking to see if the free chopsticks were set to contain two copies of the fixed chopsticks.

Figure 5.13. Kacie checks if the free chopsticks contain two fixed chopsticks.

Although Kacie did not verbally describe her reasoning for determining how to set the free chopsticks at the onset of this task, her verification method and subsequent satisfaction indicated a different way of operating than did her activities in the initial interview. In the initial interview, Kacie operated exclusively on the span of the given chopsticks: she repeated the span to both set and check her estimate. In this task, I infer that Kacie established a new goal: to set the free chopsticks so that exactly two copies of the fixed chopsticks would exhaust the interior of the free chopsticks. Because Kacie carefully attended to the vertex and marked the end of the fixed chopstick’s secondary side before repositioning them, I infer that Kacie's progressive integration operation was activated as she checked her result; Kacie viewed the free chopsticks as containing two united copies of the fixed chopsticks.

Kacie's activities in this angular multiple task offered the first indication that she was on the verge of modifying her assimilatory structure for iteration to include angular material. However, as I will support with my discussion of Kacie's next teaching session, Kacie was most likely operating on the span of the free chopsticks when she first produced her estimate in this angular multiple task.

Kacie's December 3rd Session

On December 3rd, Kacie participated in a solo teaching session because Camille was absent. At the onset of the session, we briefly discussed the kite task from the previous session. In our discussion, Kacie reflected on when comparing the chopsticks' spans was a viable method for comparing the chopsticks' opennesses:

> I think I said that if the chopsticks were the same size and like length and everything and, then you could look at the distance between the ends and you could tell [which one was more open] by that. But that's not the case for every one, like every chopstick, because they're going to vary in sizes.

Kacie's reflection indicated that if the two angle models had equal-length sides, then the angle model with the longer span would also be the more open angle model. Thus, Kacie had established a comparative quantitative structure for span and openness when all sides of the angle models were congruent.

After revisiting the kite task and for much of the teaching session, Kacie solved angular multiple tasks. In the course of solving these angular multiple tasks, Kacie indicated she was considering an attribute we had not previously discussed—spanned area. I discuss Kacie's activities from these portions of the teaching session in the sections below.

The emergence of oblique span iteration in Kacie. For the first angular multiple task of Kacie's December 3rd session, I asked Kacie to set a pair of free chopsticks to be twice as open as a given fixed pair of chopsticks; this was the same task she had solved at the end of her November 17th teaching session with Camille, and we used the same fixed pair of chopsticks. As she began the task, Kacie reflected on her activities from the previous teaching session:

> Mm. I think last time I kind of like measured it in my head, like I took the distance between these two [*indicating the span of the given chopsticks*] and then I just put it like another [*motioning along the path of the red arrow shown in Figure 5.14*] that distance over here [*in the direction of the red arrow shown in Figure 5.14*]. And just kind of open them and see if they were kind of correct.

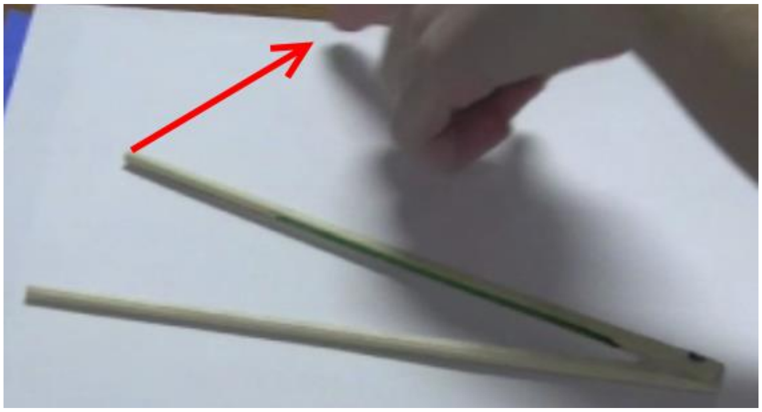

Figure 5.14. Kacie indicates where she intends to copy the span of the given chopsticks.

After her initial explanation, I provided Kacie with a free pair of chopsticks, which were the same length as the given pair of chopsticks, and Kacie set these chopsticks to be approximately twice as open as the given chopsticks, from my perspective. As she had done in the previous teaching session, Kacie checked her result by repeating the fixed chopsticks into the free chopsticks. She aligned the vertices and primary sides of the free and fixed chopsticks while remarking, “so that’s one”. She marked the end of the secondary side of the fixed chopsticks with her index finger [Figure 5.15 left] to “see where it stops.” Then Kacie placed the primary side of the fixed chopsticks so that the endpoint coincided with her index finger, while also keeping the vertices of the chopsticks aligned [Figure 5.15 center]. Afterward, Kacie noted the free chopsticks were “too big” and adjusted the terminal side of the free chopsticks to be coincident with the terminal side of the final repetition of the fixed chopsticks [Figure 5.15 right].

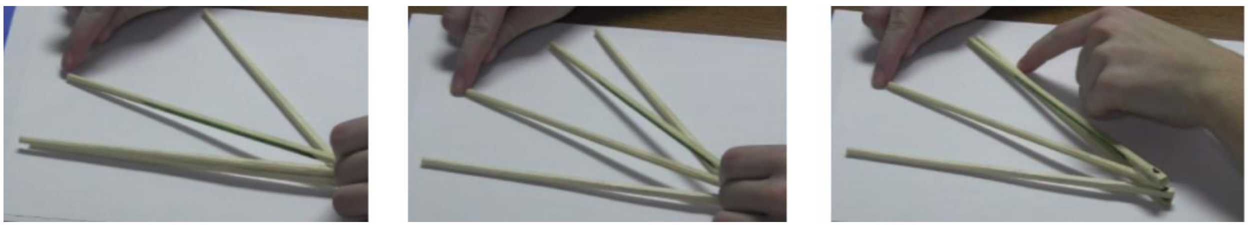

Figure 5.15. Kacie repeats the fixed chopsticks into the free chopsticks and adjusts.

Kacie set the free chopsticks by mentally uniting copies of the fixed chopstick’s span, but she checked her estimate by repeating one angle model into the other. As she reflected on her activities from the previous teaching session, Kacie expressed her intention to duplicate the span of the fixed chopsticks to set her free chopsticks; she gave no explicit indication of attending to the interior of the chopsticks. However, through her gestures (Figure 5.14 above), Kacie indicated she would *obliquely* join the copies of the span. On the angular multiple task in the initial interview, Kacie had adjoined copies of the span linearly. Thus, Kacie’s gesture in the present task indicated a subtle shift in her way of operating on angular multiple tasks. I attribute this modification in Kacie’s span operating scheme to her reflection on the same task 16 days earlier.

As Kacie regenerated her previous experience solving the task, she likely recalled both setting her estimate and checking the estimate. In particular, she may have remembered using her finger to mark the endpoint of the secondary side of the fixed chopsticks before repositioning them to form two united copies while keeping the vertices coincident, which would result in the same oblique span iteration. Thus, while the span was Kacie's primary focus in setting the chopsticks, her actions indicated she was attending to more than just the span as she set the chopsticks. From my perspective as an observer, Kacie's oblique span iteration *implicitly* suggested duplicating the angle, though I do not claim Kacie was aware of anything beyond duplicating the span when she set her estimate.

As she checked her estimate, Kacie carefully ensured the vertices of two angle models were coincident. Because she maintained coincident vertices, I infer that the final position of fixed chopstick's secondary side simultaneously represented the end of a second copy of the fixed chopsticks *and* the end of two adjacent copies of the fixed chopsticks. As in the November 17th teaching session, Kacie's actions, including her final adjustment of the free chopsticks to contain two repetitions of the fixed chopsticks, indicated her progressive integration operation was activated.

Kacie conflates spanned area and openness. To further investigate Kacie's reasoning on angular multiple tasks, I asked Kacie to set a shorter pair of chopsticks to be twice as open as the given pair of chopsticks. I intentionally provided a shorter pair of chopsticks to impede Kacie from operating exclusively on the span of the fixed chopsticks. Kacie paused in thought for 10 seconds before saying, "I guess I would just kind of do the – the same thing I did last time, but that's probably not right." When I pressed Kacie to show me how she would set the short chopsticks, Kacie gestured over the span of the fixed, long chopsticks before remarking, "but that kind of goes against everything I was just saying," presumably referring to her generalization from the kite configuration regarding openness, span, and length. Kacie's initial responses to this task indicated that span length was still prominent in her way of reasoning and the different-length chopsticks, in conjunction with her generalization from the kite task, had resulted in a perturbation.

I encouraged Kacie to proceed, and set the short chopsticks as shown in Figure 5.16 below. From my perspective, Kacie had set the span of the short chopsticks to be slightly less than twice the span of the fixed chopsticks, while the angularity of the short chopsticks exceeded twice the angularity of the fixed chopsticks.

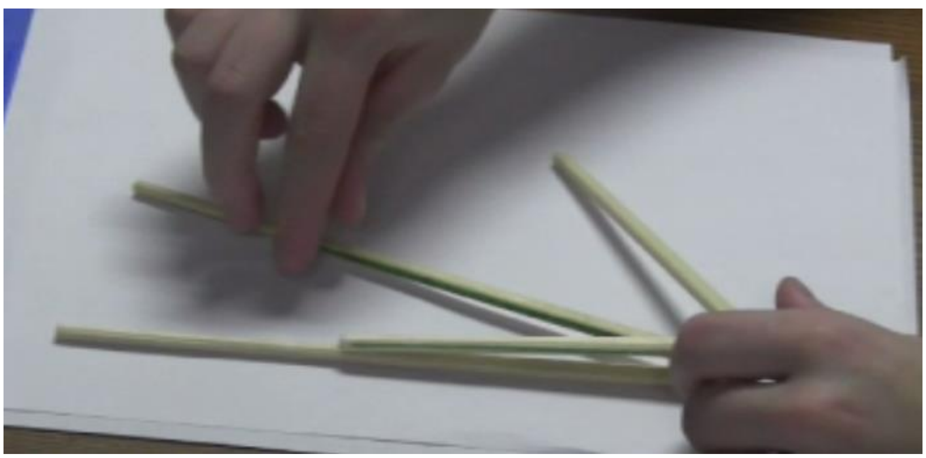

Figure 5.16. Kacie sets the short chopsticks to be twice as open as the long chopsticks.

Without prompting, Kacie began to check her result by examining whether two repetitions of the given chopsticks would be contained in the short chopsticks. As she had done previously, she aligned the vertices and initial sides of the chopsticks and used her finger to mark the endpoint of the terminal side of the given chopstick. After enacting the second repetition, Kacie evaluated her estimate remarking, “And then that’s too open again.”

Rather than adjust the short chopsticks, Kacie began reflecting on equivalence of two regions she delineated through checking her estimate. She commented, “But, that doesn’t seem right because this right here [region 1 shown in Figure 5.17] this ang – space right here isn’t the same size as this [region 2 shown in Figure 5.17], that space.” Kacie’s concern about the differences in the size of the “space” indicated to me that she was considering an additional attribute relevant to her in this situation: the areas bound by the sides of the angle model and the span, which I will refer to as *spanned area*. At my request, Kacie shaded the spaces to which she was referring as shown in Figure 5.17, and I acknowledged those areas were indeed different.

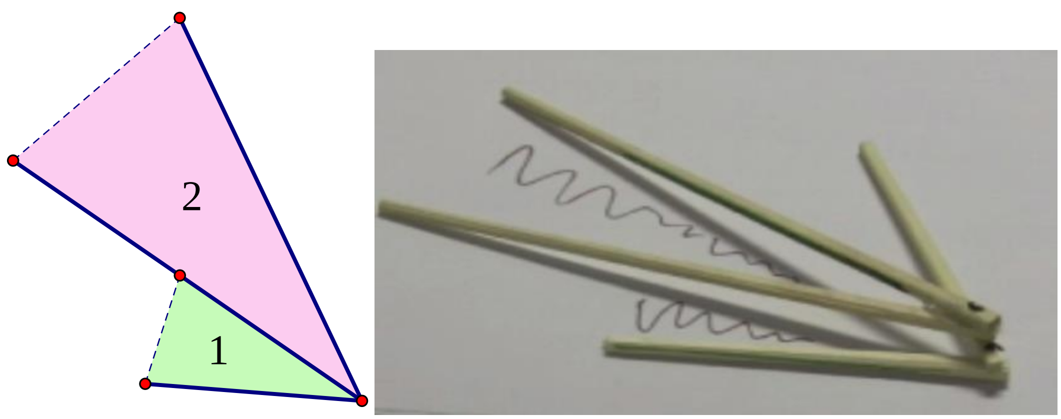

Figure 5.17. Kacie compares spanned areas.

Because Kacie seemed to be considering how spanned area and openness were related, I asked Kacie to set the short pair of chopsticks to have the same openness as the long pair of chopsticks. By posing this task, I hoped to give Kacie an occasion for observing a case where two chopsticks could have the same openness, but different spanned areas. She superimposed the short chopsticks atop the long chopsticks, and I asked her to show me the areas that she had considered in the previous task. Using a different color for each pair of chopsticks, Kacie indicated that the spanned areas for both chopsticks “would be the same thing” as shown in Figure 5.18.

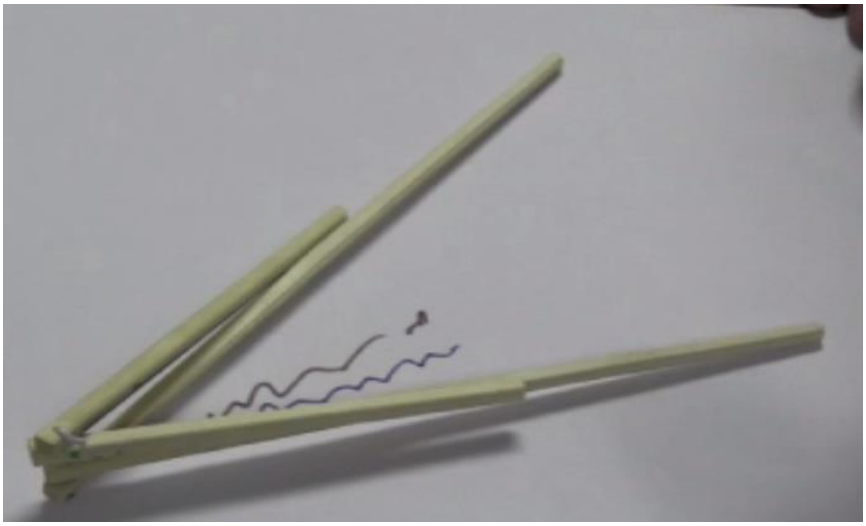

Figure 5.18. Kacie initial shading for the spanned areas of the short and long chopsticks.

Rather than consider the full spanned area of the long chopsticks, Kacie limited the area she associated with the long chopsticks to the spanned area of the short chopsticks, which suggested Kacie mentally truncated the long chopsticks when setting them to have the same openness as the short chopsticks. When I asked Kacie why the shading corresponding to the longer pair of chopsticks stopped at the end of the short chopsticks, Kacie said, “oh, okay I see what you’re saying,” and extended the shading to the end of the long chopsticks, as shown in Figure 5.19.

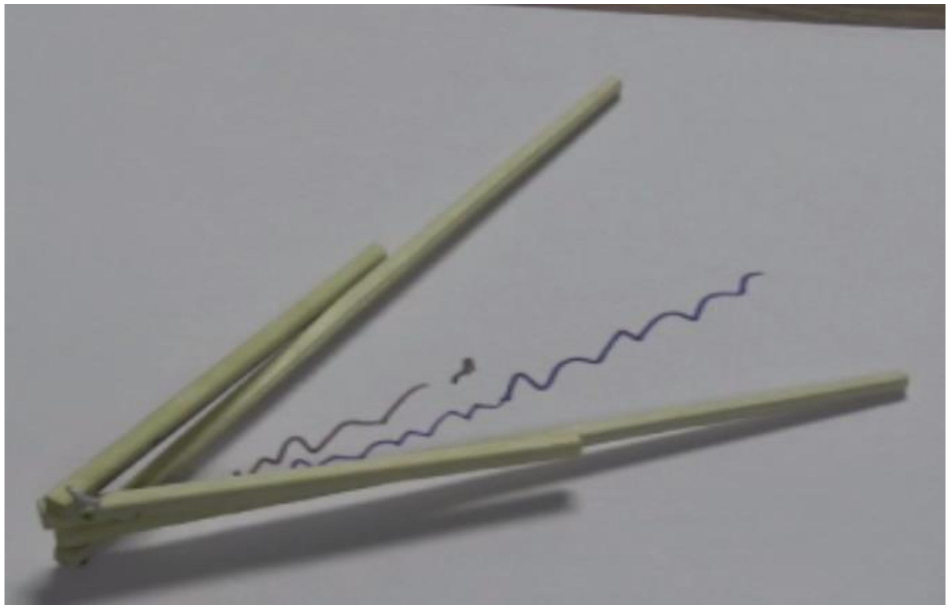

Figure 5.19. Kacie's revised shading for the spanned areas of the short and long chopsticks.

Kacie continued, "But for, um, the area of openness if you just like cut it off right there [*placing a marker at the end of the short chopsticks*], then it would be the same." Kacie also noted the areas would not be the same if the long chopsticks were not truncated, but that the openness would still be the same. From Kacie's explanations, I inferred she had at least temporarily distinguished between openness and spanned area as distinct attributes of the chopsticks.

Dominance of the perceptual over the operative. To further investigate whether Kacie had distinguished between these attributes, I returned to the previous angular multiple task wherein I asked Kacie to set the short chopsticks to be twice as open as the long pair. Kacie repeated the long chopsticks twice and adjusted the short chopsticks to contain the two repetitions as shown in Figure 5.20. She remarked, "it still just doesn't look the same to me." As she spoke, she pointed near the vertices of the two smallest perceptually available angles that constituted the short chopsticks as a two-unit composite while she referenced “the space at the beginning.” Kacie’s words and actions indicated she had mentally hyper-truncated the chopsticks and, despite having repeated the long pair twice, was still doubtful that the openness of these repetitions was the same.

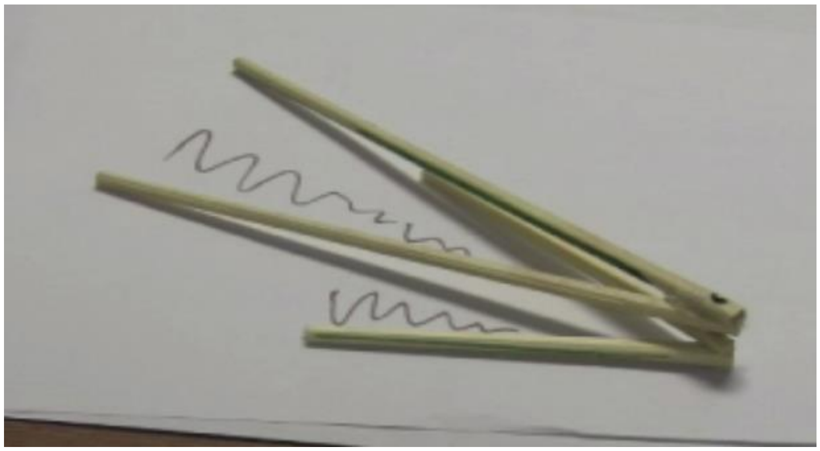

Figure 5.20. Kacie adjusts the short chopsticks to contain two repetitions of the long chopsticks.

In continuing her explanation, Kacie noted that one of the chopstick legs was “straight” (indicated by Kacie’s index finger in Figure 5.21) while the others were “at like a diagonal.” In this interaction, I infer Kacie used “straight” to indicate one leg of the chopsticks was contained in the vertical plane through her line of sight; by “diagonal,” Kacie meant the other legs were not contained in her vertical line-of-sight plane.

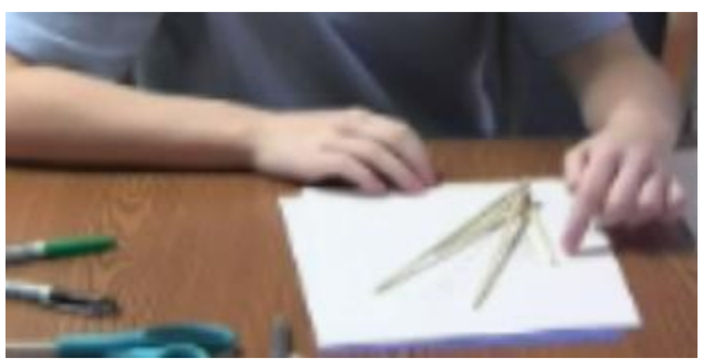

Figure 5.21. Kacie notes that one chopstick is “straight” while the others are “diagonal.”

Without any instructions from me, Kacie rotated the configuration so the vertex was pointing directly in her direction (Figure 5.22), at which point Kacie said, “then I guess they look the same because I like straightened it out.” When I asked Kacie to shade to show what was now the same, Kacie shaded the chopsticks as shown in Figure 5.23.

Figure 5.22. Kacie reorients the chopsticks and perceives the constituent units as the same.

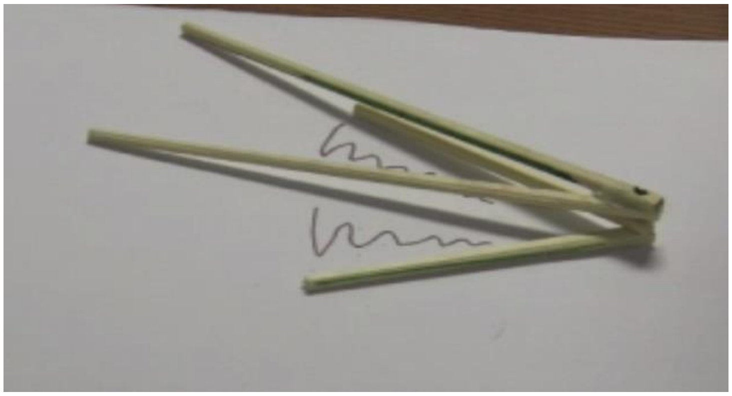

Figure 5.23. Kacie shades truncated spanned areas for constituent units.

Although Kacie had set the short chopsticks to contain two repetitions of the long chopsticks, she still perceived differences in the constituent angular units comprising the interior of the short chopsticks. However, when Kacie reoriented the entire configuration so the angle bisector of the short chopsticks (i.e., the primary side of the long chopsticks) was contained in her vertical line-of-sight plane, Kacie determined the constituent angular units were indeed the same size. Kacie’s shading indicates she mentally truncated the long chopsticks to the length of the short chopsticks. Although Kacie engaged in the physical action of repeating the long chopsticks, any mental records of this repetition were subordinate to the perceptual features of the angle models available in her visual field.

Kacie distinguishes openness from area: the hyper-elongation operation. To further examine how Kacie was thinking about openness and spanned area as attributes of the chopsticks, I presented Kacie with a pair of chopsticks, placed small pieces of pipe cleaner on the table, and asked her if each piece of pipe cleaner was in the openness and spanned area of the chopsticks.[34] In our conversation, we referred to the pieces of pipe cleaner as "dots" and the spanned area as "area." When I placed the first dot within the spanned area of the chopsticks (from my perspective), Kacie explained that the dot was in both the openness and the area of the chopsticks (see the pipe cleaner labeled "1" in Figure 5.24).

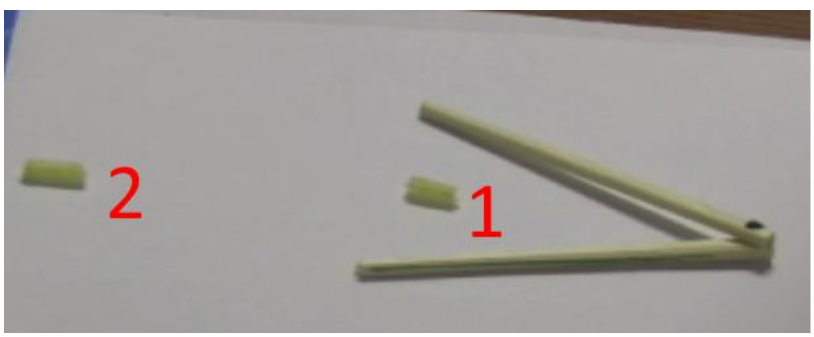

Figure 5.24. Two "dots" Kacie considered when distinguishing openness from spanned area.

When I placed a second dot beyond the endpoints of the chopsticks and approximately in line with the vertex and first dot (see pipe cleaner labeled "2" in Figure

[34] I am grateful to Les Steffe for suggesting the use of this task for examining Kacie's operations for extending the sides of the chopsticks. After the teaching experiment, I discovered Silfverbeg & Joutsenlathti (2014) investigated prospective teachers' conceptions of angles as objects using a similar task.

5.24), Kacie explained, “I wouldn’t consider it in the area, but I would consider it in the openness.” When I asked Kacie for additional elaboration, she continued, “The way I think of it it’s like these [sides] are still going on [*motions as if extending the sides of the chopsticks away from the vertex*].” At my request, Kacie used a marker to further illustrate her thinking and she drew in the segments shown in Figure 5.25 below as if to extend the sides of the angle model indefinitely.

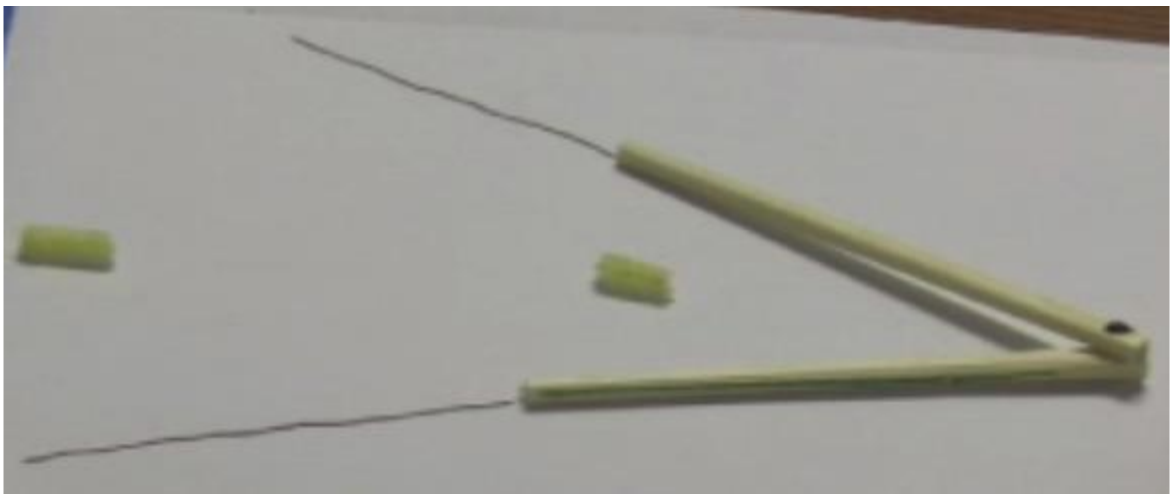

Figure 5.25. Kacie hyper-extends the chopsticks.

Kacie’s activities indicated she had made at least a temporary distinction between two different attributes: the openness and the spanned area. Previously in this session Kacie had used truncation to compare congruent spanned areas; by truncating the long chopsticks, Kacie had rendered the lengths of the angle models’ sides *inconsequential*. In this task, Kacie mentally produced unidirectional *indefinite elongations* of both sides of the angle model, which I refer to as the hyper-elongation operation.[35] In this interaction, Kacie indicated her conception of openness involved the points in convex interior bounded by the rays containing angle model’s sides.

[35] In the case of the elongation operation, the sides of an angle model are mentally extended to some particular length (e.g., the length of another pair of chopsticks). For the hyper-elongation operation, the sides of an angle model are extended indefinitely.

Further contraindications of angular iteration in angular multiple tasks. Toward the end of Kacie's December 3rd session, I transitioned from posing tasks involving physical chopsticks to posing tasks involving drawings of chopsticks (i.e., two line segments sharing a common endpoint). Although the physical chopsticks provided a relatable context for triggering the students' mathematical activities, the chopsticks were at times difficult for the students to manipulate (e.g., precise superimposition was difficult for students to enact). In contrast, the drawings could be more readily manipulated to whatever configuration the students desired.

I presented Kacie with a drawing of a pair of chopsticks and asked her to draw a pair of chopsticks that would be three times as open as the given drawing. To engender making physical copies of the given drawing, I provided Kacie with blank transparencies and explained that she could use them if she liked. The given drawing and Kacie's estimate for the three times as open task are shown at left and right, respectively, in Figure 5.26.

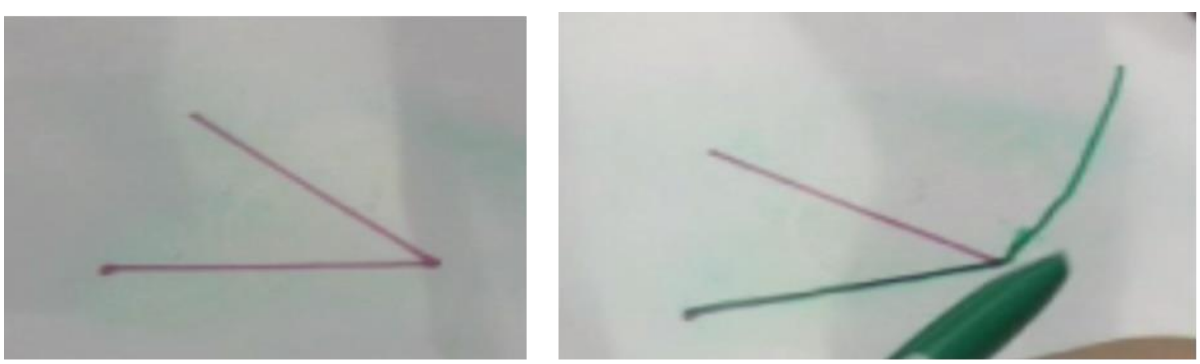

Figure 5.26. Given drawing (left) and Kacie's estimate (right in green) for three times as open task.

At the start of the task, Kacie placed the transparency over the given angle and began by tracing one side of the given angle, which she then used as one side for her estimate. As she drew the second side of her estimate, she remarked, "I'm going to take a guess here." After producing her estimate, I asked Kacie how she would check her result; she proceeded as she had done in previous tasks by checking to see how many times the given angle could be repeated into the estimate she had drawn.

In the three times as open task, Kacie did not physically repeat the given drawing three times to produce her desired drawing, though she did engage in repetition to check her estimate. Thus, Kacie judged the appropriateness of her estimate by whether it contained three repetitions of the given drawing. Yet, because she offered no explanation for how she drew her estimate, I was unable to infer if Kacie was operating on the span or the angularity of the given angle model when she initially drew her estimate.

During the teaching session, I conjectured she might have mentally iterated the given angle model due to the accuracy of the estimate she produced. To test this conjecture, I provided Kacie with a new given angle (Figure 5.27) and asked Kacie "to draw a pair of chopsticks that would be five times as open…and to do it in a way so that [she] wouldn't even need to check." With this additional restriction, I hoped Kacie might engage in angular repetition to produce the desired drawing.

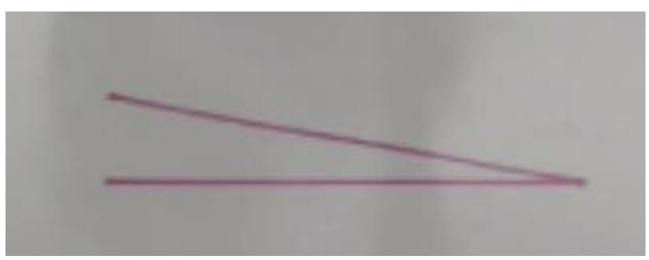

Figure 5.27. The given drawing for the five times as open task.

Kacie immediately indicated her desire operate on the span of the given drawing remarking, "you could try measuring the distance at the end of each and then multiplying that by five." This initial response contraindicated angular iteration and indicated Kacie was operating on the span of the given drawn model. To impede Kacie from operating on the span, I asked Kacie if it would be possible to complete the task without having to use a ruler. After an eight second pause, Kacie responded, “nothing comes to mind.” Due to the momentary standstill in Kacie’s progress, I relaxed the requirements and asked her to draw an estimate, stipulating that she could check afterwards. As in the previous angular multiple task, Kacie began by tracing over one side of the given drawn model using the transparency. She then drew the second side of her estimate, producing the green drawn angle model shown in Figure 5.28. As shown via the annotations in Figure 5.29, Kacie’s green estimate had a span almost exactly five times as long as the span of the given pink drawing, which further indicated Kacie was producing estimates using her span operating scheme.

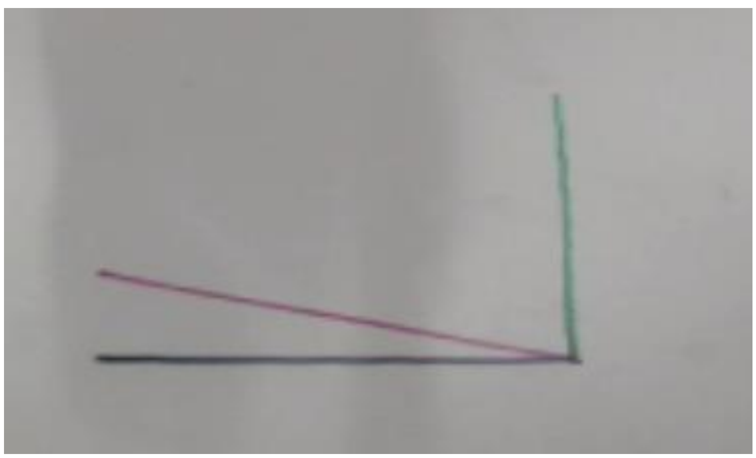

Figure 5.28. Kacie’s estimate for the five times as open task.

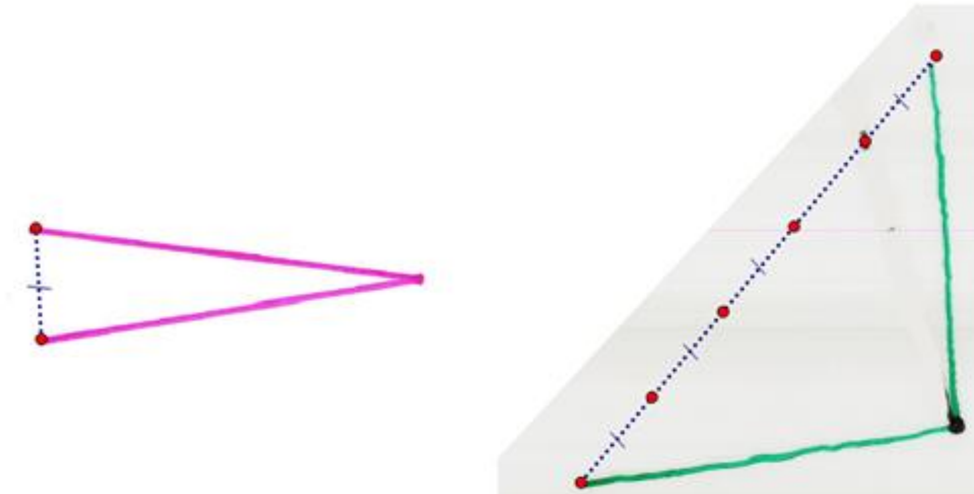

Figure 5.29. An annotated comparison of the spans of two angle models.

After producing her drawing, Kacie reflected on her estimate as described in Excerpt 5.4:

Excerpt 5.4. Kacie reflects on the angular multiple n=5 estimate.
K: I don't think that's going to be right, but –
T: Why not?
K: I don't know. I just have this feeling. I feel like it's either going to be – I feel like it's going to be too small.
T: Okay.
K: Okay.
T: Do you have a – why do you think it's too small?
K: Because I don't think five of these [*pointing in the interior of the given drawing*] is going to fit in here [*pointing in the interior of her estimate*].
…
T: How would you check?
[*Kacie repeats the given pink drawing into the green drawing, marking the endpoint of the given drawing's secondary side for each repetition while keeping the vertices of both drawings coincident. She adjusts her estimate, which was too open, by marking the terminal side of her revised estimate using a black marker (Figure 5.30)*]

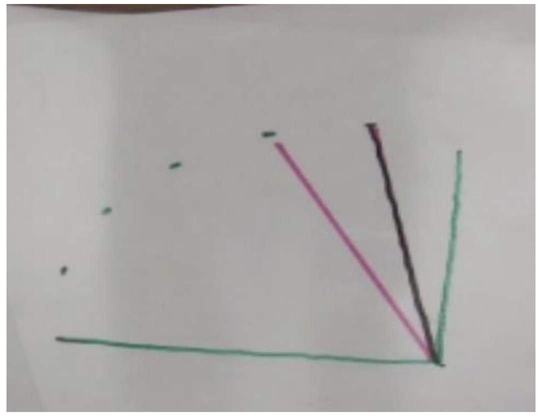

Figure 5.30. Kacie checks and adjusts her estimate for the five times as open task.

In Excerpt 5.4, Kacie's verbal reflection on her estimate—"I don't think five of these is going to fit in here"—and gestures (i.e., pointing to the angle models' interiors) suggest that her goal was to produce an angle that would contain five copies of the given angle. Kacie's repetition of the given angle model to check her estimate further supports this inference. However, Kacie's process for making her estimate involved iterating the span of the drawn chopsticks rather than the interior.

From an observer's perspective, Kacie's methods for producing and checking estimates in angular multiple task appeared contradictory. Kacie produced estimates by iterating the given angle model's span, but she checked these estimates by repeating given angle model's interior. Yet, Kacie did not exhibit any observable signs of cognitive conflict. Thus, Kacie's methods for producing and checking estimates were unproblematic from her perspective. Because she was unperturbed, I infer Kacie expected that if she produced an angle model with five times the span of the given drawing's span, the angle model would necessarily contain five of the given angular units. Figure 5.31 below shows Kacie's initial estimate for this task, which I have overlain with annotations. The dashed span of Kacie's estimate is composed of five congruent segments, which represent Kacie's production of the estimate through iterating the span of the given drawing. The pink segments, which emanate from the vertex of Kacie's estimate and pass through the endpoints of the five congruent segments, delineate the five angular units I hypothesize Kacie intended to form. I use green arcs marked with tildes, rather than the usual linear marks, to convey that while these angles are not congruent from my perspective, I hypothesize that Kacie thought these angles were congruent.

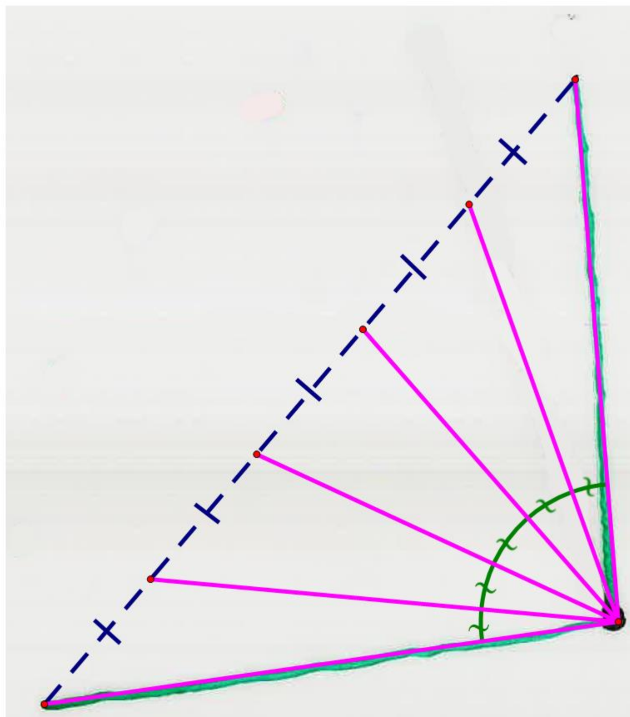

Figure 5.31. A model of Kacie's reasoning for producing an estimate in the n=5 angular multiple task

Based on her words and actions in this teaching session, I infer that Kacie viewed the results of her activities on the span and angularity as equivalent; for example, if angle *A* was five times as open as angle *B*, then the span of angle *A* would be five times as long as the span of angle *B*. I further support this inference by returning to Kacie's generalization about openness from the kite task:

> I think I said that if the chopsticks were the same size and like length and everything and, then you could look at the distance between the ends and you could tell [which one was more open] by that. But that's not the case for every one, like every chopstick, because they're going to vary in sizes.

At the beginning of the angular multiple task, there was no variation side length because there was but a single given angle model. Additionally, her method for checking and revising her estimate, which is illustrated in Figure 5.32 below, involved repeatedly manipulating one angle model. Thus, the spans of the five constituent angular parts were congruent, as were the five constituent angles. Without variation in length, Kacie may have viewed operating on angle model's span and operating on the angle model's interior as equivalent because that proved to be a viable on angular comparison tasks without different-length sides. Although Kacie had disentangled openness and span length when comparing two distinct angular units, Kacie had not yet considered how the openness of a "composite" angle produced through iterating a given angle's span would compare to the openness of the given angle.

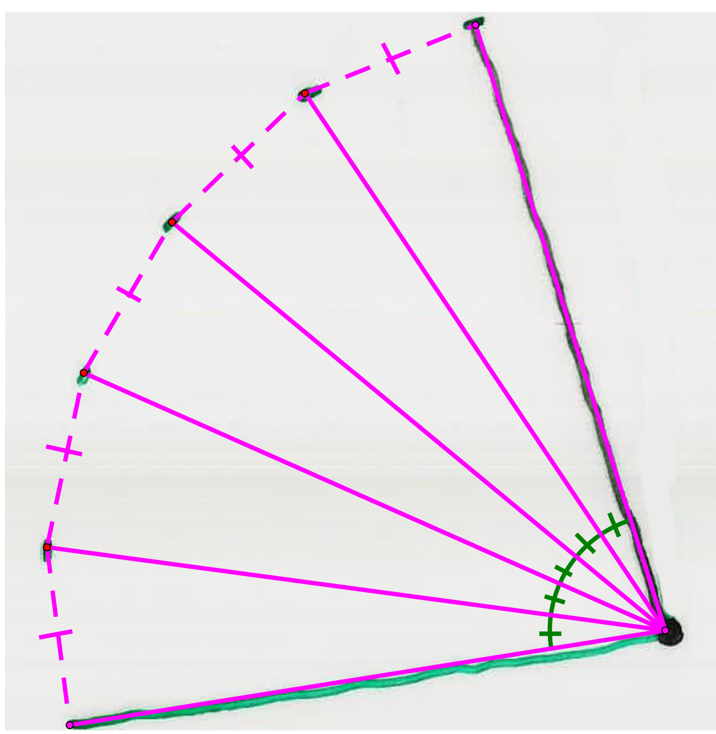

Figure 5.32. A model of Kacie's reasoning for checking and revising her estimate in the *n*=5 angular multiple task.

Summary of Part 1

In Part 1, Kacie and Camille both demonstrated an angular congruence via superimposition scheme. From angular comparison tasks, I inferred the students used four different operations when reconciling difference in length between angle models: truncation, elongation, hyper-truncation, and hyper-elongation. When comparing opennesses of two angle models, both students attended to non-angular attributes including span length and spanned area. Camille indicated an additional motion, sweeping a growing segment through the interior of an angle, that might serve in students' construction of angularity. Neither student developed a scheme for angular iteration during Part 1 of the teaching experiment. Camille used cadenced motions to solve angular multiple tasks; Kacie operated on the span of angle models when solving angular multiple tasks, but repeated the angle model when checking her solution.

Part 2: The Construction of Composite Angular Units and Degrees as a Unit of Angular Measure

In the second part of this chapter, I discuss Camille's and Kacie's activities in the seven teaching sessions immediately following winter break (Table 5.2). In these sessions, I focused on engendering students' angular operations. In the early sessions, we focused on iterating angles a specified number of times, establishing planar coverings, and working on plane splitting tasks. In the later sessions, we transitioned to discussing tasks I designed to investigate students' conceptions of degrees as an angular unit of measure.

Table 5.2. Sessions and attendance for Part 2.

	1/11	1/19	1/25	2/1	2/8	2/22	2/29
Camille	X	X		X		X	X
Kacie	X		X	X	X	X	X

Camille's & Kacie's January 11th Session

On January 11, Camille and Kacie joined me for their first teaching session after the winter break. Prior to the winter break, both students last worked on angular multiple tasks; however, I chose to begin with an angular measurement task, rather than an angular multiple task for at least two reasons. First, from my perspective, Kacie had experienced a lacuna in her reasoning on angular multiple tasks. She produced estimates by iterating the span of a given angle model, but checked her estimates by repeating the given angular unit. By asking Kacie to engage only in the activity of measuring (i.e., repeating one perceptually available angle model into another), I hoped Kacie might reflect on her activities and make an accommodation in her way of reasoning with angular multiple tasks. Second, Camille had not yet engaged in angular repetition; since I observed Kacie to repeat angles for measurement prior to repeating angles for generating angular multiples, I hypothesized Camille's reasoning might follow a similar trajectory.

The angular measurement task. For the reasons described above, we began the session with an angular measurement task wherein I presented each student with a fixed pair of wooden chopsticks and a pair of drawn chopsticks (Figure 5.33); I asked each student to measure her drawn chopsticks using the wooden chopsticks. Prior to the session I had prepared Kacie's drawn chopsticks to be three times as open as her wooden chopsticks and Camille's chopsticks to be four times as open as her wooden chopsticks.

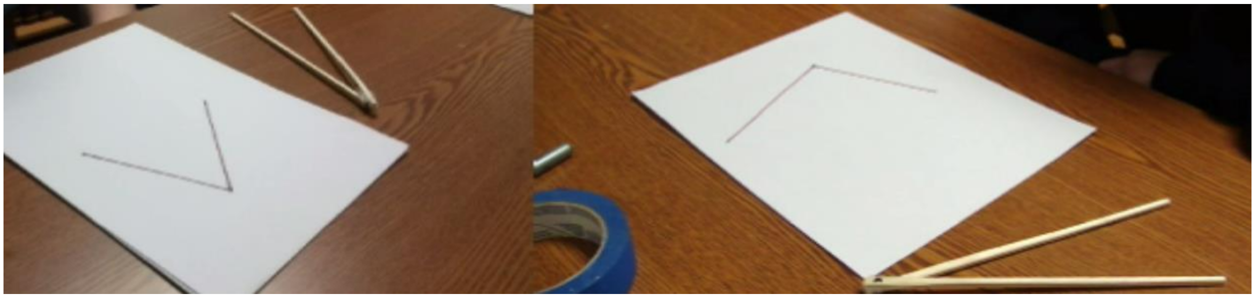

Figure 5.33: Kacie’s (left) and Camille’s (right) angle models for the angular measurement task

Kacie immediately repeated the wooden chopsticks into the drawn chopsticks, marking where one of the chopsticks ended each time using her finger. In contrast, Camille was unable to proceed in the task and remarked, “these [wooden chopsticks] don’t open…if they were able to [*gestures as if opening the wooden chopsticks*] you probably could.” I asked Camille to elaborate on what she wanted to do with the chopsticks. Camille explained, “Oh, you know, if I were just able to open them, I would just make it [the wooden chopsticks] the same as this one [the drawn chopsticks].” At my request, Kacie repeated her strategy so Camille and I could observe it. Kacie explained at the conclusion of this reenactment of her measurement activity that the drawn chopsticks were “three of these [wooden chopsticks] open.”

I infer Camille was unable to proceed in the task because she did not view the drawn chopsticks as being composed of units of the physical chopsticks. In contrast, Kacie viewed the drawn chopsticks as being composed of an indefinite number of the wooden chopsticks. Through her repeating activity and final comment that the drawn chopsticks were “three of these [wooden chopsticks] open”, I infer that Kacie viewed drawn chopsticks as a composite unit containing three units of the wooden chopsticks.

I asked Camille to try Kacie's strategy to see how she had assimilated Kacie's measurement activities. Camille began by placing the one side of the wooden chopsticks to be coincident with one side of the drawn chopsticks. She then repositioned the wooden chopsticks two more times: once to middle of the interior of the drawn chopsticks and again so that the secondary side of the wooden chopsticks was coincident with the secondary side of drawn chopsticks. A model of Camille's repeated placements of the wooden chopsticks within the drawn chopsticks is provided in Figure 5.34. The wooden chopsticks are represented with dashed segments and blue interiors, and the drawn chopsticks are represented with solid orange segments.[36] The numbers within each blue interior indicate the order in which Camille placed the wooden chopsticks.

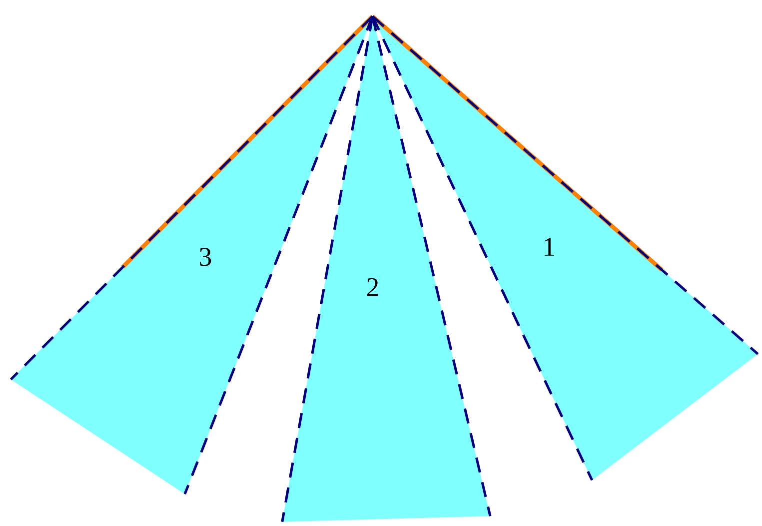

Figure 5.34. A model of Camille's emulation of Kacie's measurement strategy.

Camille concluded that "you would need three" of the wooden chopsticks to measure the drawn chopsticks. When I asked Camille how she knew where to place the wooden chopsticks on her second and third counts, Camille explained that the second

[36] The blue interiors are used to clearly indicate the placement of the chopsticks to the reader, but I do not claim that Camille was necessarily aware of these interiors as she engaged in this task.

placement "looked like the middle" and the third placement needed to "stop on that line," referring to the secondary side of the drawn chopsticks.

As illustrated in Figure 5.34, Camille attempted to imitate Kacie's measurement strategy. Although Camille counted from one to three as she placed the wooden chopsticks within the drawn chopsticks, she did not exhaust the interior of the drawn chopsticks through these three placements. Camille did not have an observable method for tracking the end of each placement of the wooden chopsticks, unlike Kacie who used her index finger for this purpose. As such, Camille did not view the drawn chopsticks as a composite unit constituted by an unspecified number of units of the wooden chopsticks.

After Kacie demonstrated her measurement activity again, Camille noted that she didn't use her finger to mark the end of the chopsticks as Kacie had done; Camille referred to Kacie's marking strategy as "the finger thing." The students' discussion about whether "the finger thing" impacted the result of the measurement is described in Excerpt 5.5 below

Excerpt 5.5. The finger thing. (Onset)
C: I didn't do the finger thing that she [Kacie] did.
T: Oh. So does that matter?
C: I don't think so. I mean – yeah, I don't think so. [*Turns to Kacie*] Does it matter?
K: [*Shakes head from side to side as if she isn't sure and laughs*]
C: I don't think so. [*Begins to remeasure her drawn chopsticks with the wooden chopsticks. She uses her finger to mark the end of the first repetition, and then repositions the wooden chopstick with the primary side at her finger. She marks the end of the second repetition with a finger, and repositions the wooden chopstick with the primary side at her finger. She doesn't mark the end of the third repetition, but moves the wooden chopstick beyond the third repetition toward the terminal side*]. Yeah, I think it does matter. Um.

Initially, Camille did not think marking the end of the chopstick would impact the result of measuring, which underscored that she did not view the drawn angle model as being exhaustively composed of units of the chopsticks. Camille's uncertainty about her measurement processes is indicated by her request for Kacie's evaluation of the "the finger thing" at the onset of Excerpt 5.5. After Camille spontaneously attempted to implement Kacie's marking technique, Camille concluded that Kacie's technique would change the result of measuring. Following the onset of Excerpt 5.5, I asked the students to comment on the purpose of "the finger thing." The ensuing interaction is described in the continuation of Excerpt 5.5 below.

Excerpt 5.5. The finger thing. (Continuation)

…

T: What does using your finger to mark – what does that help you to do?

C: Like where to place this [wooden chopstick] next.

T: Okay.

K: To mark where one – like that [wooden] chopstick ended and then start a new one.

C: Like if I say it starts right here. [*Camille begins to measure the drawn chopstick with the wooden chopstick again. She begins with primary sides coincident and marks the secondary side of the wooden chopstick with her finger. She moves the primary side of the wooden chopstick to coincide with her finger and keeps the vertices aligned.*] Start a new one right here. [*Hesitates and turns to Kacie*] I forgot what [inaudible].

K: And then – And then, you put your other finger here. [*Kacie continues the measurement activity. She marks the secondary side of the second repetition of the wooden chopsticks and repositions the wooden chopsticks so the primary side is coincident with her finger.*] And then, like so

C: Oh yeah.

K: Yeah. And then you just [*Kacie marks the secondary side of the third repetition of the wooden chopstick with her finger*] keep going. [*She repositions the wooden chopsticks so that the primary side is coincident with her finger and the secondary sides of both chopsticks are coincident.*]

T: And tell me one more time about why – I just want to hear it once more. What was the reasoning behind using the finger? What are you marking there?

K: Um, where the chopstick ended.

C: Like, yeah, like where we should put it.

K: Yeah like

C: Like to know where to put it.

As indicated by the continuation of Excerpt 5.5, Camille appeared to assimilate Kacie's usage of a finger to mark the onset of the next repetition of the chopstick (e.g., "like where to place this next" and "like to know where to put it"). However, at no point during the measurement task did Camille characterize the finger as marking *the end* of the previous repetition, which indicates her progressive integration operation was not activated. Furthermore, Camille "forgot" how to move her finger twice during angular measurement tasks in this teaching session, once during the protocol and again after the protocol. Although Camille attempted to imitate Kacie's physical actions, she did not mentally project exhaustive repetitions of the wooden chopsticks into the drawn chopsticks.

In contrast, Kacie noted her finger was being used to simultaneously track the end of one chopstick and the beginning of the next chopstick (e.g., "To mark where one – like that chopstick ended and then start a new one"). After the excerpt, Kacie instructed Camille to always put her finger on "the outside one" when marking where to place the chopstick next. Because Kacie referred to one side of the chopsticks as "the outside one," I infer the other side was in some way interior to the measurement activity from Kacie's perspective; I interpret Kacie's instructions to focus on the "outside one" to indicate that, as she was repeating the wooden chopsticks, she was constituting the drawn chopsticks by progressively integrating copies of the wooden chopsticks. Thus, the "outside one" was not contained in the interior of the portion of the drawn angle she had already exhausted through earlier repetitions.

For clarity, a model of Kacie's reasoning about this angular measurement task after she instantiated three repetitions of the wooden chopsticks is shown in Figure 5.35. In the figure, the solid orange and blue segments represent the actual positions of the drawn and wooden chopsticks, respectively, after three repetitions. The dashed blue segments represent the previous positions of the wooden chopsticks during the first and second repetitions. The solid, closed arrow represents the position of Kacie's finger pointing to "the outside one;" the dashed, open arrow indicates the interior side, as this side is contained in the union of three repetitions she has previously executed.

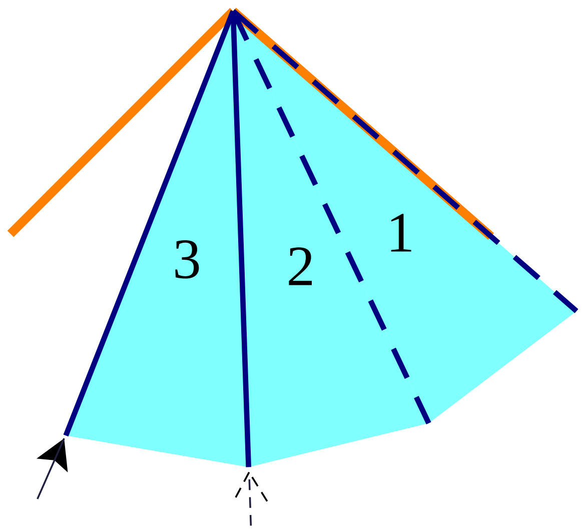

Figure 5.35: A model of Kacie's angular measurement after three repetitions.

Progress for Kacie and Camille in angular multiple tasks. As I mentioned in the discussion of the previous task, I hypothesized both students might make modifications in their ways of operating in angular multiple tasks in part due to their activities in the angular measurement task. After the angular measurement task, I presented the students with an angular multiple task. I distributed a new drawn pair of chopsticks to Kacie and another to Camille (Figure 5.36); I also provided each student with a transparency and asked each to draw a pair of chopsticks that was exactly two times as open as the given drawing. The students solved the tasks simultaneously and independently of one another. In the following two subsections, I present an analysis of each student's activities on this angular multiple task.

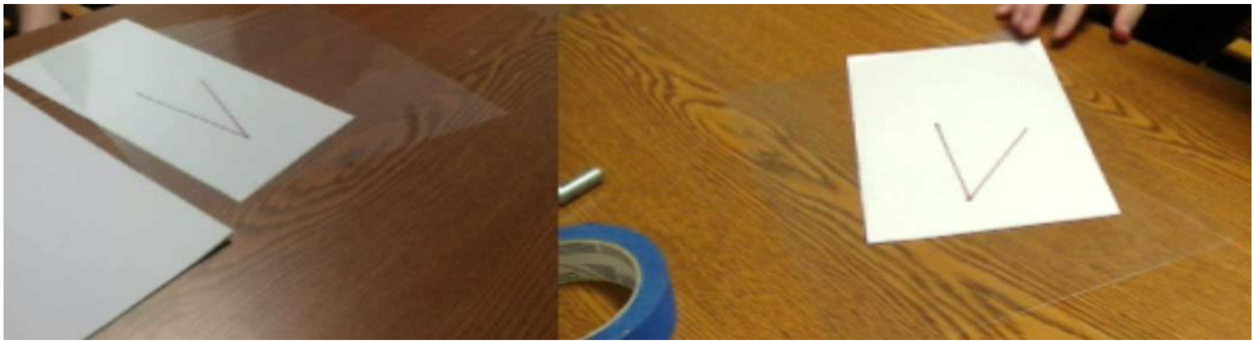

Figure 5.36. Kacie's (left) and Camille's (right) given drawings for the angular multiple task.

Camille duplicates an angle. In prior angular multiple tasks, Camille enacted rhythmic opening actions on physical chopsticks and these actions were verbally coordinated with her counting sequence. By posing the tasks using drawn angle models, I (unintentionally) introduced a constraint for Camille as I had not provided a physical pair of chopsticks that she could open. Camille's uncertainty and frustration is indicated by Excerpt 5.6 below.

Excerpt 5.6. Camille duplicates an angle (Onset)

T: Could you draw a pair of chopsticks that's exactly two times as open as the one that you've got?

C: [3 sec] Hm. [*Hits marker on the table and looks toward T in innocent frustration*] [5 sec] Okay. [5 sec] Um. [4 sec.] Alright, I have to think about this. [10 sec]. Hmm. [4 sec] Okay. I'm thinking of something, I just don't know how to –

T: Tell me about it if that would – you can talk through it if you want.

Thirty-five seconds elapsed between the statement of the task and Camille's first utterance indicating potential progress: "I'm thinking of something." During this time, Camille focused intently on the drawn chopsticks and occasionally looked at me in innocent frustration; she did not attend to the actions of her partner, Kacie, who was working on the same task.

I infer that Camille, having encountered a constraint in the drawn chopsticks, was searching for relevant operations among those she had previously constructed. I interpreted her utterance, "I'm thinking of something," as an indication Camille may have assimilated the situation to some of her existing conceptual operations. I encouraged Camille to verbalize her thinking so that I could infer which operations had been activated and also so Camille might reflect on these operations by verbalizing her thinking. Camille's response is presented in the first continuation of Excerpt 5.6.

Excerpt 5.6. Camille duplicates an angle. (First Continuation)

C: Like if I make, wait. Hmm. If I make two of the exact ones, but like make it [*motions over the drawn chopsticks as if opening them*], because I'm supposed to make it – okay, never mind. Um. [5 sec] Hmmm. I can't do this. Wait.

T: So you – you mentioned something about making two of the exact ones and then you stopped. Can you tell me what you were thinking?

C: Yeah. Like, it's cause I don't know how to put this into words. Um. So if I make two of the exact ones [*motions down one side of the given drawing (see 1 in Figure 5.37) and then up the other side (2)*] and then just, um, [*motions down one side of the given drawing (1), up the other side (2), back down the second side (3), and then motions a line segment (4) exterior to the given angle as shown in Figure 5.37*] yeah. Let's say I add another one right here [*motioning over the terminal dashed segment (4) in Figure 5.37 repeatedly; then she uses her pinky to motion down part of the initial segment (1) and over the terminal segment (4) with her hand*], to me that would just be like two times as big.

T: I see.

C: Do you understand?

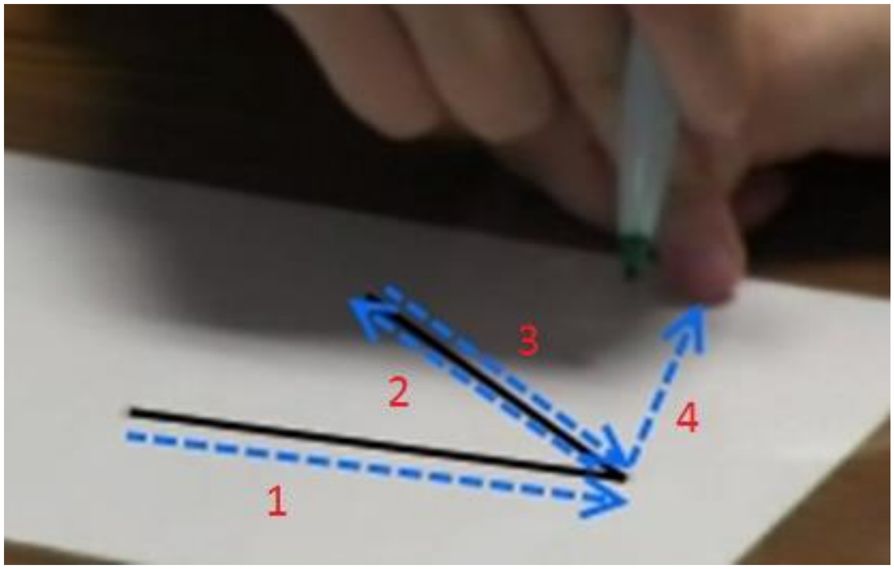

Figure 5.37. Camille unites two copies of an angular unit.

In the first continuation of Excerpt 5.6, Camille activated her concept for two, which contained the unitizing operation. Because of Camille's utterance, "If I make two of the exact ones," I infer Camille abstracted the given drawn angle model as a unitary item and imagined an unconnected pair of angular units. Camille's pause and comment, "I can't do this," indicated that, while Camille found her concept for two to be relevant, she was initially unable to anticipate how the unconnected pair of angular units might be leveraged to produce a single angular unit; I infer her uniting operation was not yet activated. Camille's hesitancy indicated the novelty of mentally producing a pair of angular units in this context. To encourage reflection on the operations she had activated, I repeated a portion of Camille's utterance, "you mentioned something about making two of the exact ones," and asked Camille to elaborate.

Camille's subsequent gestures (indicated by arrows 1–4 in Figure 5.37) indicated she applied the uniting operation to the two abstract angular units as she projected the adjacent units onto the transparency. The gestures indicated by arrows 1 and 2 in Figure

5.37 designate the first angular unit, and the gestures indicated by arrows 3 and 4 designate the second angular unit. I infer Camille mentally projected adjacent angular units because she gestured up then down the same segment as shown by arrows 3 and 4, which indicated the second angular unit began where the first angular unit ended. The final gesture described in the first continuation of excerpt 5.6—motioning down a portion of arrow 1 with her pinky before using her hand to gesture along arrow 4—and her comment, "to me that would just be like two times as big," suggested Camille had applied the unitizing operation the two adjacent angular units and therefore created a (potentially segmented) composite angular unit. I describe the combination of operations outlined above as *angular duplication*, which is a special case of angular iteration.[37]

From my perspective, Camille's words and actions indicated that she instantiated the requisite mental operations for solving the duplication task. However, Camille had not yet produced a drawing of the desired chopsticks. As such, I suspected that Camille had encountered a new constraint in the physical materials available to her. In the first continuation of Excerpt 5.6, Camille exhibited the operations constituting angular duplication sequentially, without anticipation, and with occasional uncertainty. Therefore, Camille was unable to anticipate how to use a single transparency to produce two adjacent copies; a single transparency would not permit making two copies first and then joining these copies later.

I hypothesized that if I provided her with a second transparency, Camille might be able to sequentially engage in physical actions that mimicked the mental operations I

[37] Individuals often construct "two" as a composite unit before being able to construct composite units more generally (Steffe & Olive, 2010). As such, I distinguish angular duplication from angular iteration in general.

inferred from her activities. Our interaction continued as described in the second continuation of Excerpt 5.6 below.

Excerpt 5.6. Camille duplicates an angle. (Second Continuation)
T: Would a second piece of – would this help if I gave you two of these [transparencies]? Or no?
C: Mm [*expressing uncertainty*]. Oh yeah.
T: If you think it might be helpful [*gives C another transparency*].
C: Well, maybe. Okay, cause like, okay. So let's say I make one of these [*traces one copy of the given drawing on the first transparency*]. And then, can I do this on this one [*points to second transparency*] too?
T: Yeah, do whatever you need to do.
C: I think I'm doing this right. [*Removes first transparency and places the second transparency over the given drawing, then traces a copy of the given drawing on the second transparency as shown in Figure 5.38 left*]. Kind of like that. [*overlays the two transparencies so that one side from each copy is coincident as shown in Figure 5.38 right*] It's going to be the – yeah.
T: That's really nice.
C: Yeah. [*Smiles in satisfaction and adjusts the transparencies ensuring that the vertices are coincident.*]

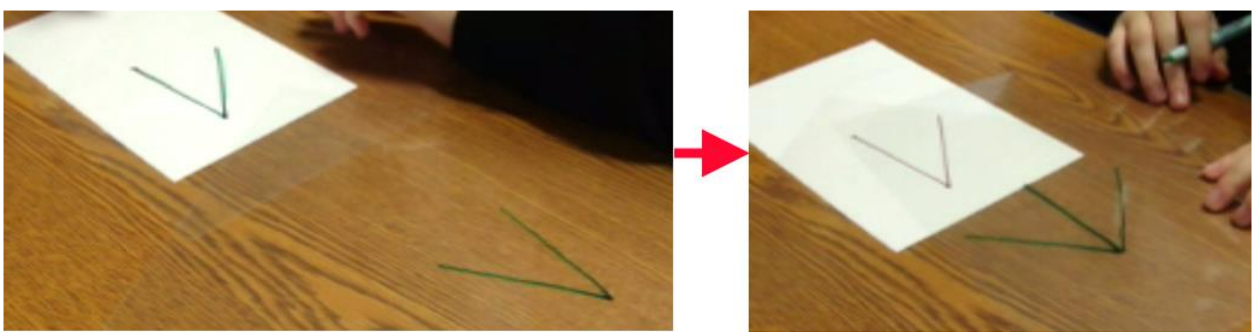

Figure 5.38: Camille creates (left) and unites (right) two copies of the given chopsticks

Although Camille initially appeared unaware whether a second transparency might be useful, copying the given angle on the first transparency seemed to reactivate the sequence of operations I inferred from her description moments before. Prior to making the second copy, Camille's expression, "I think I'm doing this right," was uttered with a modicum of cautious optimism as if she viewed creating the copies as productive even though she had yet to physically produce a final result. The availability of two transparencies supported Camille in sequentially executing the physical actions of making two copies of the given angle and joining them together; these actions mimicked the mental production of two angular units and subsequent uniting of these units.

After Camille joined the two copies of the given angle by overlaying the transparencies so one side from each copy was coincident, Camille did not give an explicit behavioral indication of taking the joined copies as a single composite unit. Although I interpret her careful attention to ensuring the vertices of the unit angles as weak indication she formed a segmented composite unit, Camille gave a stronger indication of unitizing the result two minutes after the conclusion of Excerpt 5.6 when I asked her to explain her solution. During her explanation, Camille spontaneously erased the coincident segments in the interior of the segmented composite (Figure 5.39) while remarking, “that’s not really supposed to be there…it’s just supposed to be like that.” This erasing was a physical indication of Camille’s application of the unitizing operation to create an unsegmented composite angular unit.

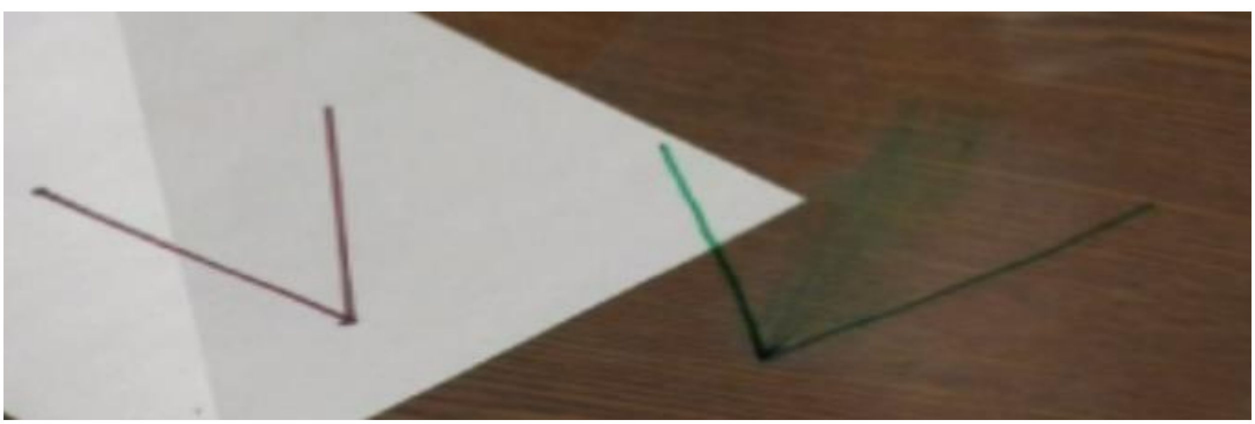

Figure 5.39. Camille creates an unsegmented composite angular unit.

Kacie repeats an angle. Kacie began the same task by starting to trace an initial side of the given drawing. Before completing the tracing of the primary side, Kacie paused for four seconds in thought, remarked, “I think I did this wrong,” and then finished tracing the primary side. Without moving the transparency, she marked the endpoint of the secondary side. Figure 5.40 illustrates Kacie’s progress to this point. The primary and secondary sides of the given drawing, which is underneath the transparency, are labeled in red as *p* and *s*, respectively. Kacie’s inscriptions on the transparency are labeled in black uppercase. *A* refers to the marked endpoint of *s*, and *P* refers to Kacie’s initial tracing of *p*.

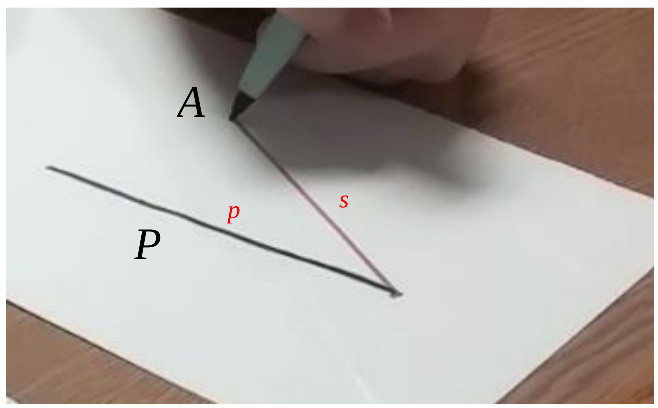

Figure 5.40. Kacie begins tracing her drawing.

Kacie’s marking of *A* was a dramatic contrast to her activities on angular multiple tasks in the previous teaching sessions; she had previously produced visual estimates, which I inferred were a result of her span operating scheme. While Kacie’s primary focus was likely still on the span as indicated by her marking a single point rather than tracing the entire angle, Kacie’s marking indicated a reflected abstraction in that she was now aware of the length she wished to copy. Kacie’s physical actions in marking of *A* mimicked the use of her finger in the previous angular measurement task.

Kacie had barely begun to trace when she remarked, “I think I did this wrong.” Because she had made so little progress, I interpret Kacie’s comment as an indication

Kacie was reflecting on her activities in prior teaching sessions. She was comparing her current plan of action to how she solved similar tasks in a previous teaching session and now found those previous methods problematic. In short, Kacie's comment foreshadowed an accommodation in her way of operating

Next, Kacie repositioned her transparency so that the endpoint of *p* was coincident with A, and she marked a point, *B*, at the endpoint of side *s* (Figure 5.41 below). Kacie's actions indicated she had unitized at least the span of the given drawing, and she had obliquely united two copies of the span. However, Kacie did not appear to explicitly attend to the vertex of the given drawing when she moved the transparency; when she marked *B*, this vertex was not contained in *P*, the segment she initially traced on the transparency. Thus, Kacie was not engaged in angular repetition from my perspective; however, her oblique span repetition was implicitly suggestive of angular iteration.

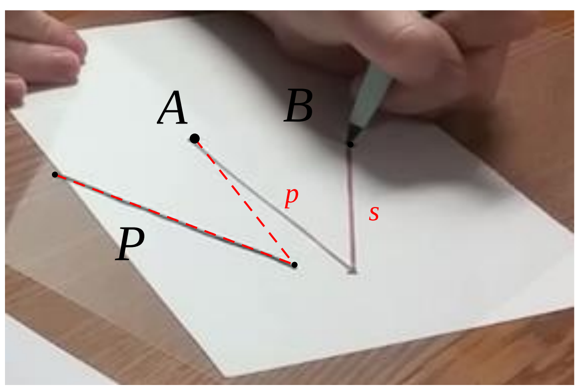

Figure 5.41. Kacie fails to attend to the vertex.

After marking B, Kacie began to reposition the transparency as if to enact another oblique repetition of the span, but she stopped suddenly (Figure 5.42) whispering, "oh, gosh darn it." Although I asked Kacie to produce a drawing twice as open as the given drawing, her attempt to repeat the span a third time indicated she had established a goal of producing a drawing three times as open as the given drawn model. The sudden halt of Kacie's activities coupled with her remark of dismay, "oh, gosh darn it," signaled Kacie had experienced a perturbation. I infer Kacie was moving the given drawing to a position like the one indicated by the dotted blue segments in Figure 5.42. While repositioning the drawings, I conjecture Kacie imagined completing the repositioning and tracing over the side *s* to complete her drawing of a model three times as open as the given drawing. I infer the source of Kacie's perturbation was that the side she anticipated tracing did not intersect *P* to create a single angle. In other words, I infer Kacie anticipated her activity would produce two disjoint segments rather than a single angle model.

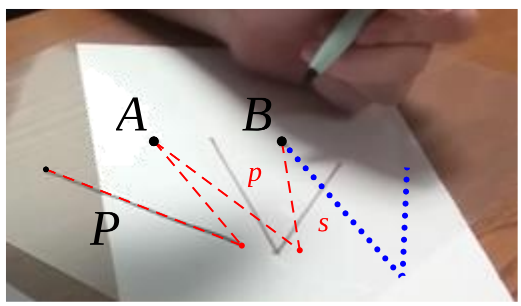

Figure 5.42. Kacie begins a third oblique span repetition.

Kacie spontaneously restarted the task without erasing *B*.[38] She moved the transparency so the given chopsticks were back in the original position shown in Figure 5.40. She placed her right hand on the transparency as if to hold the vertex in place and then began carefully adjusting the transparency and paper. With her right hand "holding down the vertex," Kacie deliberately adjusted the transparency and paper for thirty seconds until arriving at the position shown in Figure 5.43; she had positioned the

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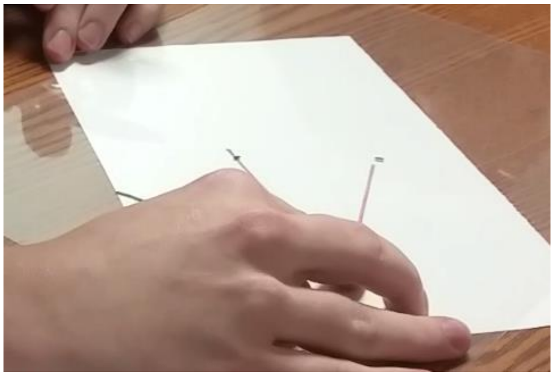
[38] Throughout Kacie's activities on this task, my attention is focused primarily on Camille's activity.

transparency so that the line containing *p* passed through *A* and the line containing *s* nearly passed through *B*.

As on her first attempt, Kacie was engaged in oblique span repetition. On her second attempt, Kacie made a functional accommodation to her oblique span repetition in that she maintained a fixed vertex while repeating the span. Because she maintained a fixed vertex, I infer Kacie was applying her progressive integration operations to both the span and the interior of the angle models.



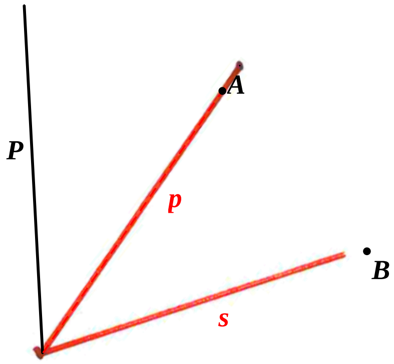


Figure 5.43. The results of Kacie's deliberate adjustments.

Kacie did not stop after her duplication; instead she enacted another repetition. Maintaining a fixed vertex at the endpoint of *P*, she moved *p* to nearly pass through *B* and marked the endpoint of s, creating point *C* (Figure 5.44). She then traced *s* on the transparency creating a segment, *S*, which intersected the endpoint of *P* (Figure 5.45). *P* and *S* formed the primary and secondary sides, respectively, of an angle model three times as open as the given angle model.

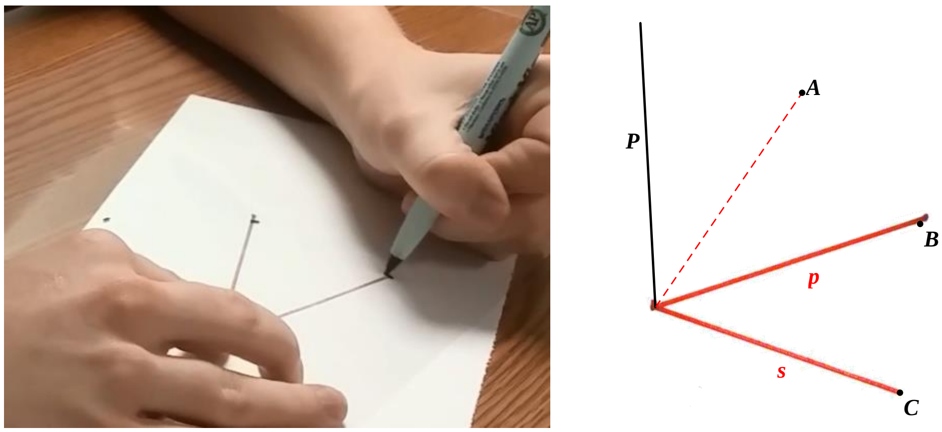


Figure 5.44. Kacie's enacts a third repetition.

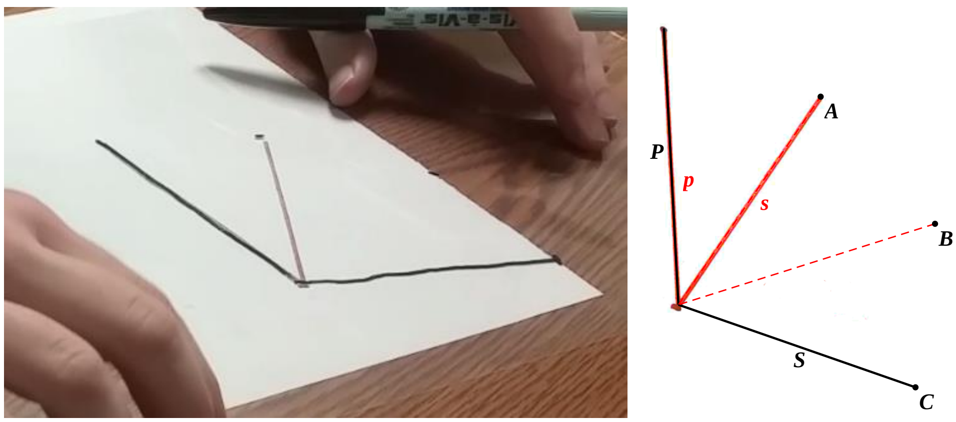


Figure 5.45. Kacie makes an angle three times as open as a given angle.

In her subsequent description of her activity, Kacie indicated she found her previous activities on the angular measurement task relevant when solving the angular multiple task, remarking, "I did what I kind of did with the chopsticks." Using her drawing, Kacie explained using point *A* to mark "where the first chopstick ended."

I consider Kacie's functional accommodation in way of operating to be significant. Kacie did not experiment or falter after experiencing a perturbation in this task. Different from Camille, Kacie did not need to make separate copies and then unite them. Instead, Kacie's actions indicated she knew immediately how to position successive instantiations of the given angle model to produce a composite unit, although manipulating the paper and transparency was cumbersome for Kacie. Since Kacie was marking the endpoint of the angle model and not tracing the entire side, Kacie had to have imagined copying and joining the three angles in visualized imagination prior to acting.

Encouraging angular repetition and reflection. After the duplication task, Camille and Kacie worked on some additional tasks I thought might encourage angular repetition, as well as reflection on their activities. Immediately following the duplicating task, they worked on another angular multiple task wherein each student was to make a drawing three times as open as their given drawing. The witness and I provided a different additional constraint for each student to encourage each student's reflection on her previous activities. For Kacie, the witness provided a drawing of chopsticks with incongruent sides. The intention of the witness was to test the hypothesis that Kacie was applying her operations to the interior of the angle model and not just to the span. I challenged Camille to produce the desired drawing by using only one transparency because it would encourage concatenation of her operations, which I hoped would in turn foster the sense of simultaneity in operating requisite for angular iteration.

Camille modified her physical actions to meet the new constraint by engaging in angular repetition. Within one minute, she produced an unsegmented angular composite unit containing three copies of the given drawing using a single transparency (Figure 5.46). A minute later, I asked Camille how the openness of the two drawings compared and she remained aware that her drawn angle was three times as open as the given angle. Camille was establishing a scheme for angular repetition and was now making composite angular units, at least at the perceptual level.

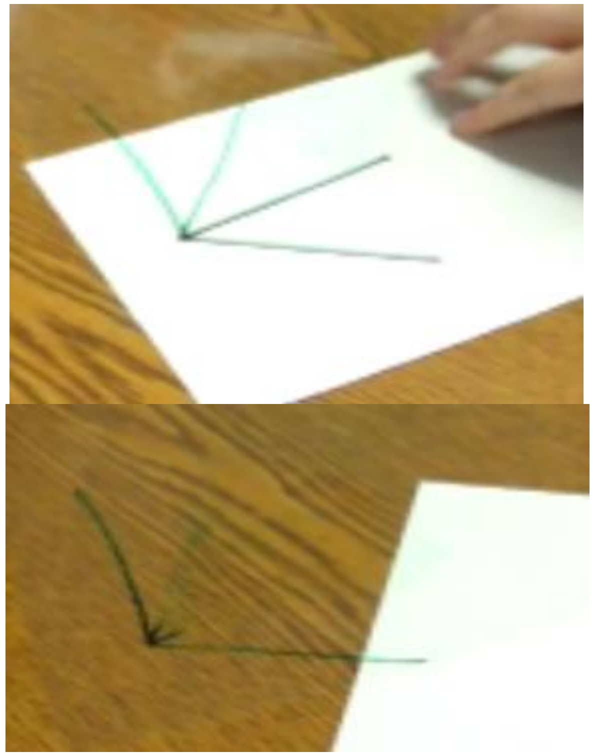


Figure 5.46. Camille repeats the angle three times (left) and then erases the interior to produce an unsegmented composite (right).

Kacie worked for approximately three minutes on the angular multiple task involving an angle model with incongruent sides.[39] During this time, Kacie spontaneously experimented with marking the end of the repetitions at different distances. As shown in Figure 5.47, Kacie sometimes marked the end of a repetition using the length of the given model's shorter side and at other times she used the length of the longer side. Because Kacie took the longer side of the given model as the primary side, marking the end of a repetition further away indicated the implementation of her elongation operation. Kacie described "imagining that [the short side] was longer."

[39] Repositioning the physical materials was cumbersome for Kacie.

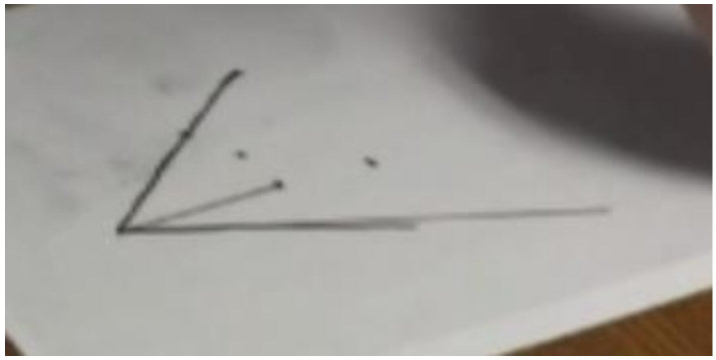


Figure 5.47. Kacie experiments with marking at different distances.

By applying the elongation operation, Kacie rendered the incongruent sides of the angle model inconsequential in the production of an angular composite. In contrast to her activities in the December 3rd session, Kacie never considered the differences in the spanned area of the constituent perceptual units she produced on the transparency. When I asked Kacie if it mattered where she chose to put the marks, explained, “no…they still line up the same.” Kacie’s activities clearly indicated she was engaged in angular repetition; she was operating on the angularity and not on the span. Kacie’s words and actions indicated she had constructed openness in such a way that the length of the angle model was inconsequential for her way of operating.

Following the angular multiple tasks and to continue to encourage students’ angular repetition, I provided the students with a drawn angle model and asked the students to take turns repeating the angle on a transparency. After three repetitions, I asked the students what would happen if they kept making copies to see whether the students would be able to anticipate the results of future repetitions. The students’ discussion is described in excerpt 5.7 below.

Excerpt 5.7. Repeating an angle to establish a planar covering.
T: And if you keep going, what do you think’s going to happen?
K: They’re eventually going to come back together.

C: Oh yeah [*signaling agreement with Kacie*]. [*Draws a fourth repetition (Figure 5.48)*]
T: Do you see that? So my question is, how many copies do you need -
K: After that point? [*Pointing to the four repetitions*]
C: Oh.
T: How many copies in all –
K: Oh, eight.
T: How many copies of this one in all do you think you would need?
K: Eight.
T: What do you think, Camille?
C: Probably eight, yeah.
K: Seven or eight.
C: Yeah. It would probably be seven.
T: [*To Kacie*] Why did you say eight so quickly?
K: Because there's four here and that's a half of the circle.
C: I feel like eight might not fit, but I'm not sure so I say seven.

Figure 5.48. Four repetitions of a drawn angle model.

Although I did not ask Kacie to elaborate on what she meant when she said, "they're eventually going to come back together," what she meant was clear to me during that interaction. As the subsequent text in the excerpt indicates, Kacie anticipated further repetitions of the angle model would ultimately exhaust the plane. Kacie's initial choice of words, "they're eventually going to come back together," is interesting. "They" does not have a clear referent in the surrounding text. To say that "they" would eventually "come back together" implies that "they" were together at some earlier instance. To this point in the teaching experiment, the students had worked nearly exclusively in contexts involving physical or drawn chopsticks. Now, I infer "they" referred to the sides of such an angle model. When Kacie said, "they're eventually going to come back together," I infer she was imagining a pair of chopsticks that started in a closed configuration and opened more with each repetition they drew on the transparency[40]. As she thought about continuing the repetitions of the drawn angle model, she imaged the sides of her re-presented angle model "coming back together."

When I asked Kacie how many copies would be needed in total, Kacie's response of 8 was nearly immediate, and, when I asked for clarification, Kacie indicated that she had produced a three-levels-of-units structure. She recognized the four perceptually available repetitions nearly constituted a straight angle. She took the four united copies as a composite unit and duplicated this composite in order approximately exhaust the plane.[41]

Camille adopted a passive role throughout the interaction surrounding excerpt 5.7 above. When Kacie anticipated continued repetitions of the angle would ultimately exhaust the plane, Camille agreed with her. Camille also agreed when Kacie suggested that 8 repetitions of the given angle would be needed to exhaust the plane. When Kacie explained her reasoning, "because there's four here and that's a half of the circle," Camille did not appear to assimilate the three-level-of-unit structure Kacie described. Camille remained uncertain as to whether eight copies "would fit," which indicated that Camille had not assimilated the plane as two half planes in this context. I infer Camille did not take the four repetitions as a composite unit for duplication. Instead, she may have been imagining continuing the repetition of the given angle model to exhaust the remainder of the plane.

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[40] Based on her physical repetition, I suspect she imagined one side fixed in place while the other side moved to open the chopsticks.

[41] Kacie's use of "circle" may indicate that she was imagining a circular path containing the endpoints of the sides. A second (and I think more) viable alternative is that Kacie may have used "circle" in reference to a complete revolution.

What's is it that you're measuring when you measure an angle? In the final moments of the January 11th teaching session, I asked Kacie and Camille to reflect on their prior experiences with angle measure in mathematics classes. Specifically, I asked the students what they were measuring when they measured an angle. Kacie immediately gestured across the span of a drawn angle model on the table, and later drew a small curved mark near the vertex of another angle model (Figure 5.49). Camille said, "the lines," as she motioned over the sides of another drawn angle model and demonstrated how she would measure with a ruler (Figure 5.50).[42] In their immediate responses, both students associated angle measure with linear attributes of angle models. We had never discussed "angle measure" to this point in the teaching experiment. Neither student mentioned openness, which indicated a disconnect between their prior school experiences and their experiences to this point in the teaching experiment.

Figure 5.49. Kacie draws a curved mark near the vertex of an angle model.

[42] Following her demonstration with the ruler, Camille also recalled using an L-shaped protractor that opened to measure angles.

Figure 5.50. Camille demonstrates using a ruler to measure an angle.

Summary of January 11th session. In the January 11th session, Kacie demonstrated that she could use a given angle model to measure the openness of another model by repeating the latter using the former. Though Camille tried to mimic Kacie's actions, Camille never produced an exhaustive segmentation in the angular measurement tasks. Both students modified their way of reasoning on angular multiple tasks, demonstrated angular repetition, and formed composite angular units. When asked what was being measured when they measured an angle, both students initially indicated a length.

Camille's January 19th Session

Since Kacie was absent on January 19th, Camille participated in a solo teaching session. My primary goal in this teaching session was to foster and explore Camille's angular repetition. At the beginning of the teaching experiment, I returned to the discussion we started at the end of the preceding session and asked Camille about her prior experiences with angle measure.

Previous experiences with angle measure and degrees as a unit of measure. In the first five minutes of Camille's January 19th session, I asked Camille to talk a bit more about her past experiences with angle measure. In the ensuing discussion, Camille mentioned three separate attributes that were salient to her from her previous experiences: length, openness, and orientation.

To start the conversation, I asked Camille what was being measured when we talked about measuring an angle. Camille first indicated that to measure an angle was to measure the length of the sides of an angle. Camille drew an example and drew a line parallel to one side emphasizing the length she would measure (Figure 5.51). When I asked her to estimate the measure of her example, she estimated the angle would measure about four inches.

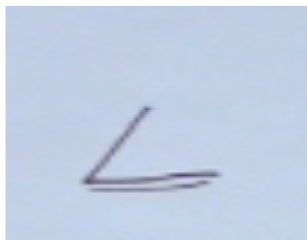


Figure 5.51. Camille indicates measuring side length.

To investigate what else Camille remembered from her past experiences with angles, I asked Camille if she had ever heard of measuring angles in degrees. Camille confirmed that she had, and so I asked her what it would mean for an angle to have a measure of one degree. Camille recounted measuring angles in eighth grade by opening an L-shaped protractor and reading measurements. When I asked how far she would need to open the protractor for a one-degree angle, Camille explained she would open the protractor “probably just one time like this” as she demonstrated by opening two markers as shown in Figure 5.52 (left). Camille also drew an example of what she thought a one-degree angle would look like (Figure 5.52 right), but was unsure of how she might check that the angle had a measure of one degree without using a protractor.

Figure 5.52. Camille's illustrations of one-degree angles.

In addition to length and openness as "measured" by a protractor, Camille also considered orientation a salient feature when thinking about the term "angle measure". Camille drew a second angle next to the "one-degree" angle she had previously drawn (Figure 5.53) and wondered if angle measure described the way an angle was "faced." Camille said the two drawn angle models would have different measures because "one's going this way and the other's going that way." Orientation was salient attribute of angle models for Camille due to her experiences outside of the classroom. Specifically, Camille discussed being directed to kick a ball or stand at different angles while on the soccer field.

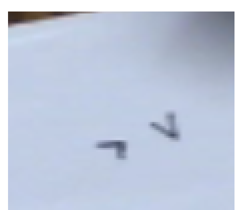


Figure 5.53. Two angles facing different ways.

In contexts involving chopsticks, Camille never indicated orientation or side lengths as relevant attributes when discussing the openness of the angle models. She had regularly repositioned and reoriented angle models when comparing opennesses and indicated side lengths of the angle models were inconsequential for comparing openness.

Therefore, I infer orientation and side length were relevant to Camille in this teaching session due to the use of the terms "angle" and "angle measure." The numerical values Camille associated with angle measure were not uniquely associated with angularity from my perspective. Instead, Camille indicated angle measurements might describe length, orientation, or, when an L-shaped protractor was used, openness.

Confirmation of angular repetition. Following the initial discussion about Camille's prior experiences with angle and angle measure, I returned to posing angular multiple tasks. In these tasks, I referred to the drawn angle models as angles rather than drawn chopsticks. To examine whether Camille had constructed a scheme for angular repetition, I asked Camille to make an angle six times as open as a given angle, which I drew to have incongruent sides (Figure 5.54).

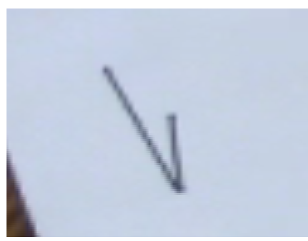


Figure 5.54. An angle model with incongruent sides.

Camille was initially unsure how to proceed in the task, presumably due to the incongruent sides. Camille elected to use the transparency paper, traced a copy of the given angle model and then erased it. Approximately 45 seconds after I initially posed the question, Camille remarked, "I can't do it," before asking suddenly, "can I try it one more time?" In the subsequent 90 seconds, Camille repeated the angle six times (Figure 5.55 left) and erased the interior segments (Figure 5.55 right).

Figure 5.55. Camille produces an angle model six times as open as a given model.

Camille's activities indicated that she had constructed at least a scheme for angular repetition in that she could produce angular multiples through her physical activities. However, her repetitions were not automatic. Camille did not appear to anticipate where she needed to move the transparency to create the next copy, and sometimes she traced over lines she had already drawn. The lack of anticipatory physical coordinations contraindicated Camille was engaging in mentally iterating the angle. At this point in the teaching experiment, I infer Camille needed to enact the physical repetition in order to produce composite angular units.

Contraindications of operating on composite angular units. Camille could produce composite angular units from angular units of one through her physical actions. To investigate whether Camille could take composite units as input for further operation and action, I asked Camille to solve a series of composite angular multiple tasks—tasks that involved creating multiples of composite angular units, from my perspective. For clarity, I use the abbreviation CAM(m, n) to refer to a composite angular multiple task wherein I requested an angle n times as open as previously established m-unit composite. In this section, I briefly describe Camille's activities on three of these tasks and then provide an interpretation of her activities looking across these three tasks.

CAM(6, 2). In the first composite angular multiple task, I asked Camille to produce an angle that was two times as open as the resultant angle from the previous task (Figure 5.55 right above). Camille did not take the composite angle as input for her angular repetition scheme. Instead, she acted on the original given angle (Figure 5.56), joining two additional repetitions to her previous result (Figure 5.56 left); she then erased the interior segments as shown in Figure 5.56 right. From my perspective, Camille's actions resulted in an unsegmented angular composite containing 8 units of the original angle model; however, Camille explained she had produced an angle two times as open as the last angle she produced.

Figure 5.56. Camille combines units additively.

CAM(3, 2): version 1. In the second composite angular multiple task, I used smaller numbers in the problem statement. I asked Camille to produce an angle three times as open as a given angle model and instructed her not to erase anything. After she repeated the given model three times (Figure 5.57 left), I asked her to produce an angle that was twice as open as the result. Camille appended two additional copies of the given angle to her intermediate result (Figure 5.57 right), which created an angle five times as open as the given angle from my perspective. Camille described her activities in this task saying, "this one [intermediate result] was three times as big and then I added two more."

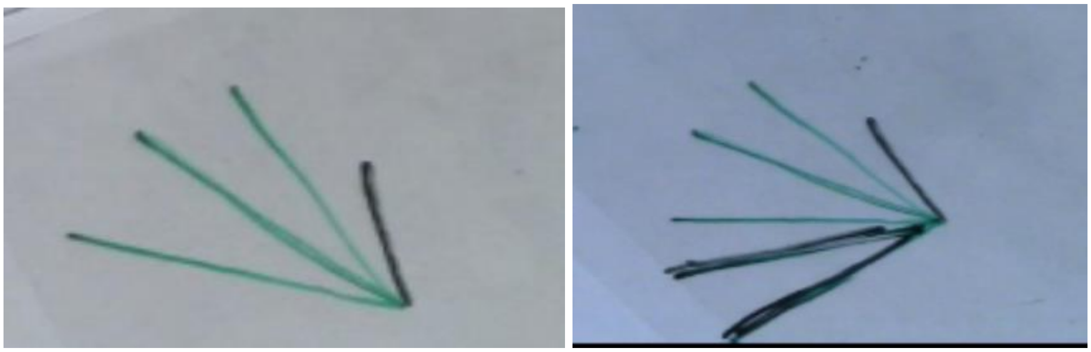


Figure 5.57. Camille combines units additively again.

CAM(3, 2): version 2. For the third composite angular multiple task, I asked Camille to make an angle three times as open as a new pink angle model (Figure 5.58 left). Camille repeated the pink angle three times and erased the interior segments to create a new green angle (Figure 5.58 center). After I removed the pink angle model from the table to prevent Camille from physically using it, I asked Camille to produce an angle twice as open as the green angle. Rather than repeat the green angle, Camille appended two smaller angles to the intermediate result (5.58 right). The two angles Camille appended to the green angle were closer in magnitude to the original pink angle than to the intermediate green angle.

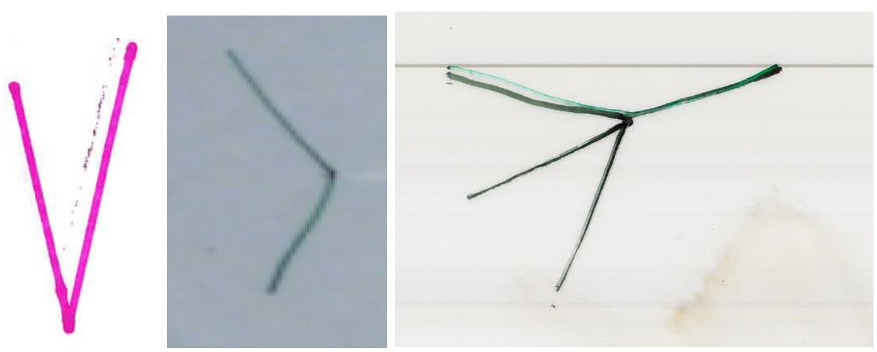


Figure 5.58. Camille combines units additively a third time.

Analysis of Camille's activities on CAM(m, n). For each CAM(*m*, *n*) task, Camille used her angular repetition scheme to produce an *m*-unit composite. Following the production of this intermediate result for each task, Camille continued the activity of her angular repetition scheme by instantiating *m* additional repetitions of the original unit and produced an (*m*+*n)*-unit angular composite. Even when the original given angle was not perceptually available (as in CAM(3, 2): version 2), Camille held the original given angle model in mind and appended copies of the original angle model to her intermediate result.

In creating the intermediate *m*-unit composite, Camille produced two levels of units in activity and did not assimilate this *m*-unit composite as input for her angular repetition scheme. Camille seemed unable to mentally "let go" of the initial unit she established in each CAM task. After she produced the *m*-unit composite in each task, I infer Camille established a new goal of using her concept for *n* to produce an angle more open than the *m*-unit angular composite in her visual field. Unable to establish an additional multiplicative level of unit, Camille implemented her concept of *n* additively to achieve her goal in each task. In CAM(3,2): version 1, Camille's remark, "this one [intermediate result] was three times as big and then I added two more," highlights the additivity of Camille's process. As such, Camille's activities were a novel attempt to solve a problem involving openness that she had not previously encountered.

Camille's activities on the CAM tasks in this session paralleled her activities on angular multiple tasks in the early sessions of the teaching experiment. Using the notation of composite angular multiple tasks, angular multiple tasks can be represented as CAM(*m*, 0). In these tasks, I requested Camille produce a composite angular unit containing *m* constituent units of a specified magnitude. Early in the teaching experiment, Camille established an angular unit, *A*, of the specified magnitude through superimposition and then increased the magnitude by inserting *m* bursts of motion to establish a new angular unit, *B*. At that time, Camille did not insert *A* into her template for *m*. Instead, she inserted the *motion* that served in the constitution of *A* (e.g., re-presented opening) into *m*. In doing so, she created a unit, *B*, which was at the same multiplicative level of unit as *A*, (i.e., one level of unit).

In this session when I requested Camille produce a unit containing *m* composite units of *n*, Camille established an *n*-unit angular composite, *A*, and then increased the magnitude of *A* by inserting *m* units to establish a new angular unit, *B*. In this session, Camille did not insert *A* into her template for *m*. Instead, she inserted the activity that served in the constitution of *A* (i.e., repeating the original unit) into *m*. In doing so, she created a unit *B*, which was at the same multiplicative level of unit as *A*, (i.e., two levels of units).

In the initial interview session, Camille could coordinate 3 levels of units in activity when dealing with linear material; however, Camille did not produce a three-level-of-units structure in this session. Throughout her activities in the composite angular multiple tasks, Camille reasoned with two levels of multiplicative units.

Summary of Camille's January 19th session. In her January 19th session, Camille found orientation, openness, and side length to be relevant attributes when I asked about her prior experiences with angle and angle measure. Camille had not previously conflated side length and orientation with angularity when working in chopsticks contexts framed in terms of openness.

Camille demonstrated she had established an angular repetition scheme for producing composite angular units. However, Camille was limited to establishing two levels of angular units in activity in this session. She did not take composite angular units as input for her angular repetition scheme on composite angular multiple tasks.

Kacie's January 25th session

On January 25th, Kacie participated in a solo teaching session because Camille was absent. During this session, I asked Kacie to discuss her previous experiences with angle measure and degrees as a unit of measure. Kacie solved composite angular multiple tasks and planar covering tasks. The latter tasks involved enumerating the number of times a given angle would need to be repeated in order to cover the plane.

Previous experiences with angle measure and degrees as a unit of measure. As I had done in Camille's January 19th solo teaching session, I began Kacie's teaching session by asking about her prior experiences with angle measure. I re-posed the question, "what is it that you're measuring when you're measuring an angle." In her response, Kacie mentioned the space in between two sides of an angle and drew a curved mark near the vertex of the angle in red (Figure 5.59). Kacie explained that the curved mark indicated "the space that you would measure." I asked Kacie if it mattered if she drew the curved mark further from the vertex, and she replied, "I don't think so." At my request, Kacie drew another curved mark in green (Figure 5.59) further from the vertex.

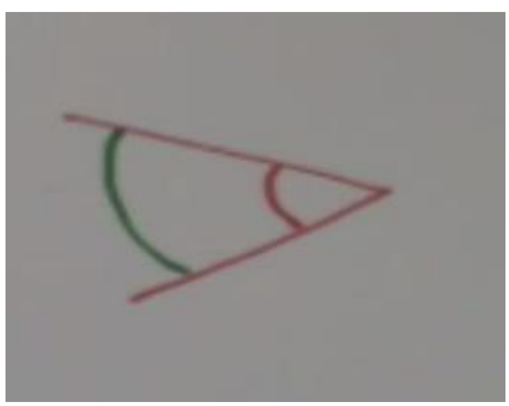


Figure 5.59. Kacie draws two curved marks on an angle model.

After Kacie drew the second mark, I asked what would happen if she measured the angle using the red curved mark and then measured the angle using the green curved mark. Kacie explained that the angle would have different measures because the red mark was smaller than the green mark. Before the curved marks were drawn, however, Kacie explained that the angle would have a single measure.

Until the introduction of the terms “angle” and “angle measure,” Kacie had not drawn curved marks near the vertex of an angle; therefore, I infer these markings were an artifact of Kacie’s prior school experiences. While it was not entirely clear whether measures Kacie associated with the marks were linked to length or area, Kacie’s meaning for “angle measure” involved a conflation of some other attribute dependent upon these marks.

Kacie also explained that she knew angles were measured in degrees. When I asked Kacie what it would mean for an angle to have a measure of one degree, she replied, “I would say that it’s like open to one degree,” and mentioned opening chopsticks to that amount. I asked Kacie how she might make an angle whose measure was one degree also what such an angle might look like. Kacie explained that it would be a “a very small angle” and drew an example as shown in Figure 5.60. Kacie explained that she could check if her estimate was correct if she had another angle that she knew to be a one-degree angle to compare it to. When asked if there would be a way to do it without using another such angle, Kacie says, “if you had like a formula or something.”

Figure 5.60. Kacie's estimate for a one-degree angle.

Kacie's response she had connected this standard unit of angle measure, at least in name, to the attribute openness and her prior activities in the teaching experiment. Kacie's characterization of a one-degree angle as "a very small angle" was a preliminary indication her conception of degree measure included at least a loose coordination between her number sequence and relative extents of an attribute.[43] She understood a one-degree angle comparatively; however, Kacie's conception of a one-degree angle was not operative in that she could neither produce (or verify the measure of) such an angle without using another known one-degree angle.

Kacie operates on composite angular units. Following our brief discussion of Kacie's knowledge of angle measure rooted in her prior experiences, I returned to posing tasks involving composite angular units. In Kacie's previous paired session (January 11th), I inferred that she could duplicate an angular composite. To further examine how Kacie operated with composite angular units, I asked Kacie to engage in tasks could be solved through operating on composite angular units from my perspective.

CAM(3,2). First, I presented Kacie with a composite angular multiple task. I provided Kacie with a brown angle and asked her to produce a black angle three times as open as the brown angle. After Kacie repeated the brown angle to produce the black angle, I asked Kacie to produce a green angle twice as open as the black angle. Kacie

[43] I elaborate upon what I mean by a "loose" coordination in my discussion of Camille's and Kacie's February 22nd teaching session.

repeated the brown angle six times in order to produce the green angle (Figure 5.61 below). Kacie explained her production of the green angle as follows:

> Well I knew that this one [black angle] was three of the brown ones. And you said it needed to be twice as open as this [black one]. So I just did two of those [black angles] by using this [brown angle], since I knew that three plus three is six and we used three of these [given angles] to get that [black angle]. So I just did the same thing I did there [black angle] but six times.

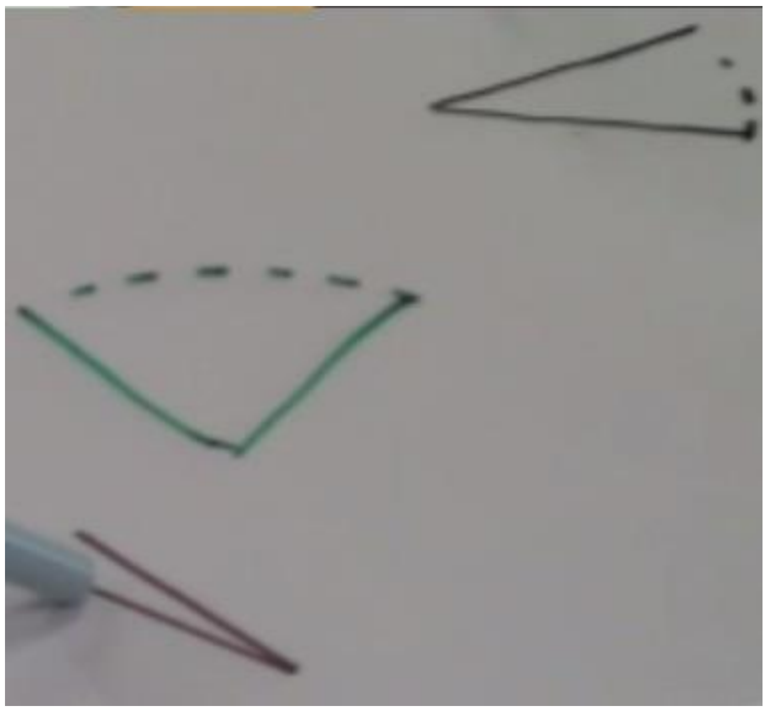


Figure 5.61. Given angle (bottom left), triple angle (top right), double the triple angle (middle)

Kacie chose to repeat the brown given angle rather than the black three-unit composite to produce the green angle. Kacie's explanation indicates she viewed six repetitions of the brown given angle as being equivalent to two repetitions of the black angle. Because Kacie did not physically produce two copies of the black angle, I infer that she unitized the first three repetitions and last three repetitions of the given brown angle, which ultimately constituted the green angle as a six-unit composite. Kacie's explanation offered additional corroboration that she could produce three levels of units within angular contexts as she viewed the green angle as being simultaneously composed of two black angles and six green angles.

Establishing a planar covering. At this point in the teaching experiment, I wanted to investigate whether Kacie could mentally iterate angular composites while remaining aware of the constituent units. To encourage Kacie's operations on composite angular units, I asked Kacie to anticipate how many repetitions of the original brown angle would be needed to "go all the way around," which from my perspective entailed covering the plane using the brown angle. Specifically, I was curious whether Kacie might use one of the composite angles she had already produced (black or green) to reason about this planar covering task. Kacie's response is outlined in Excerpt 5.8 below.

Excerpt 5.8. Kacie creates a planar covering using an angular composite

T: How many copies of the brown one do you think you would need to like go all the way around?

K: [4 sec] I'm going to say four. Oh wait, no. To go all the way around?

T: Yeah.

K: Of the brown one?

T: Mm hmm.

K: Oh. [6 sec] Probably like twenty-something.

T: Tell me how you know that.

K: Well cause I – I took this [green angle] and I was – that's why I said four first cause I was like, okay well if I did that over here [motions a segment from the vertex as if repeating the green angle] that'd be two. And then again over here [*motions another segment as if repeating the green angle again*], that'd be three and four (Figure 5.62).

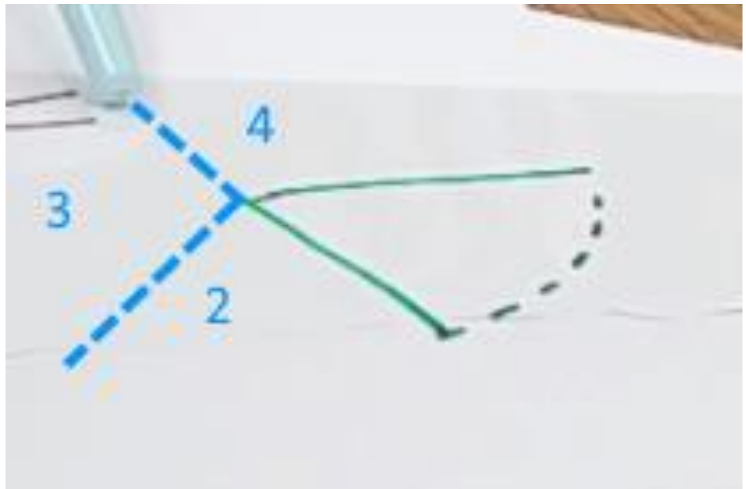


Figure 5.62. Kacie covers the (truncated) plane using an angular composite.

Kacie initially solved this task by covering the (truncated) plane with a composite angular unit. Kacie's initial response, "four," indicated that she expected four united copies of the green angle would exhaust the plane. After unitizing the green angle, I infer she projected at least two of these units adjacent to the perceptually available unit on the transparency as indicated by the two segments she motioned over the transparency. I infer Kacie formed the fourth and final unit by taking the remainder of the plane not accounted for by the three adjacent units as perceptually equivalent to one green angle.[44]

Kacie revised her initial response, "four," moments later, presumably having realized that she had enumerated the number of green, rather than brown, angles required for the covering. Kacie's revised estimate "twenty-something" offers some indication, albeit weak, that Kacie projected six units of the brown angle into each of the four united green angles. To better understand Kacie's estimate, "twenty-something," I asked Kacie to illustrate her strategy on the transparency. Using a second transparency, Kacie traced

[44] While these four units were adjacent, Kacie may not have she united these adjacent units into an object that she viewed as a single angle. While Kacie was aware that the four adjacent copies covered the truncated plane, she did not indicate that she viewed the (truncated) plane as the interior of a single angle. As such, I do not characterize Kacie's mental operating as angular iteration.

four adjacent copies of the green angle and, after doing so, perceived a fifth angle, which was the remainder of the plane left uncovered by the four adjacent copies (Figure 5.63). Because Kacie treated this fifth angle as if it were equivalent to the traced copies without verifying the equivalence (e.g., via superimposition), I refer to the fifth angle as a perceptual copy. In the continuation of excerpt 5.8, Kacie describes her solution.

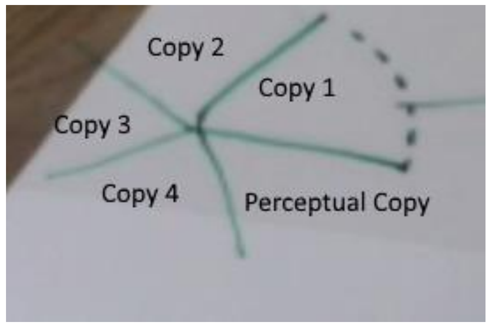


Figure 5.63. Kacie covers the plane with four traced copies and one perceptual copy.

Excerpt 5.8. Kacie covers the (truncated) plane using an angular composite (Continuation)

…

T: So about how many of the brown one do you think it would take to go all the way around, based on what you just did?

K: [3 sec] Thirty.

T: About thirty? And how'd you get that thirty?

K: Cause I knew that one of these [green angles] took six of the brown ones. And there's five, [motioning over the copies of the green angle shown in figure 5.63] so I just did thirty.

After producing figure 5.63 above, Kacie adjusted her estimate of the number of brown angles from "twenty-something" to thirty. In obtaining this estimate, Kacie projected six angular units into each of the five copies of the green angle—four traced copies and one perceptual copy. Kacie produced a three-level-of-units structure in that she deduced the total number of brown angles (30) from the five green angles, each of which contained six brown angles.[45]

Contraindications of plane splitting. In the preceding task, Kacie indicated she could enumerate angles needed to cover the plane by operating on angular composites. To examine whether Kacie might take the plane as input for her operations, I posed plane splitting tasks to Kacie. The goal of a plane splitting task is to produce an angle so that a specified number of adjoined copies of the angle would cover the plane.

Attempting a 3-split of the plane. For example, in the first plane splitting task I asked Kacie to make an angle so that it would take exactly three copies to "go all the way around." Kacie spent about 90 seconds in thought before drawing an approximately right angle (Figure 5.64). After looking intently for 20 seconds at the angle, Kacie motioned two linear segments from the vertex of the angle (see blue and red segments in figure 5.64). Kacie's gaze remained fixed on the angle for another ten seconds before remarking, "I really don't know." Kacie erased her estimate, and I asked her to elaborate on her thinking; Kacie's elaboration is presented in Excerpt 5.9 below.

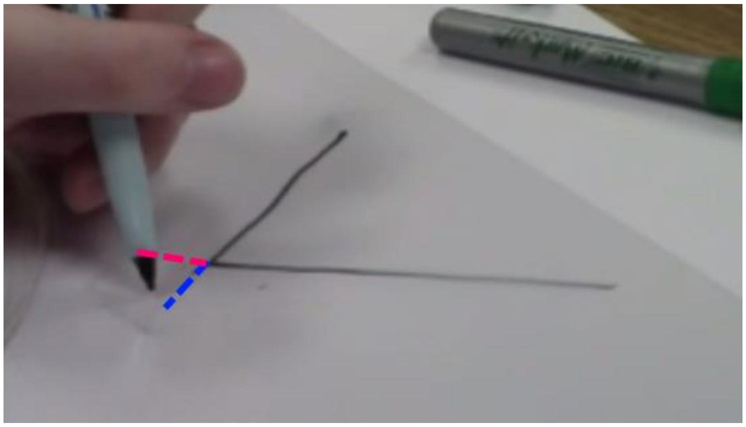


Figure 5.64. Kacie's estimate for an angle that would take three copies to cover the plane.

[45] However, Kacie may not have produced three levels of *angular* units as she gave no clear indication that the five units of six constituted a single angle.

Excerpt 5.9. Kacie discusses making an angle so that three copies would cover the plane

K: Well, at first I thought about like a pie graph since it's circular and it's split into a bunch of different parts. But then when you split, um, like a pie graph into three different parts some, most of the time they're not always equal. So that wouldn't be right for this because the angles have to be equal. So then – then I just drew that [*motions as if to recreate the angle shown in figure 5.64*] and, but then if I did another one over here [*motions downward like the dashed blue segment in Figure 5.64*] then it would be – then this side over here [*indicating the half plane not accounted for by two adjoined copies*] would be much bigger than the other two angles.

T: I see. So do you think that the one you originally drew was too open or not open enough.

K: [*Hesitantly*] I think not open enough.

T: Okay.

K: But I don't know how to make one open enough that I can do it three times around.

T: I see. So it may not be possible.

K: Well I think it is. I just don't know how to do it.

Kacie's intense focus and pause before acting indicated the novelty of this task for Kacie. In making her initial estimate for the desired angle, Kacie did not appear to take the plane as input for radial partitioning, which contraindicated plane splitting. I use *plane splitting* to refer to the application of the splitting operation to a full angle (i.e., an angle whose interior is the entire plane), which entails the simultaneous implementation of partitioning and iterating. Were Kacie to split the plane, she would need to imagine the simultaneous insertion (i.e., partitioning) of three equiangular parts exhausting the plane prior to operating. Kacie did not enact such a simultaneous insertion.

Instead, Kacie produced an angle and then mentally produced adjacent copies, which I inferred from her gestures shown by the dashed segments in Figure 5.64. Because Kacie erased her initial estimate, I infer that she found it unsatisfactory for achieving the goal she had established, which suggests she was counting the number of copies required to cover the (truncated) plane. Thus, Kacie relied on mentally adjoining copies of her estimate to cover the plane without any evidence of partitioning the plane to achieve her initial estimate. Kacie was engaged in segmenting activity.

In her elaboration (Excerpt 5.9 above), Kacie indicated she found her previous experiences with pie graphs relevant because these graphs are "circular" and "split into a bunch of different parts." Kacie's attention to the circularity of pie graphs suggests she was focused on a finite portion of the plane during this task. Kacie verbalized a need to create three distinct angular parts that were equal in size, but she did not have an established way of operating allowing her to accomplish this goal. Through her words and actions, Kacie re-illustrated the angle she had previously drawn could be used to segment the plane into three angular parts but that one of those parts was "much bigger" than the other two. Thus, Kacie could cover the plane with three angular parts, but she recognized that only two of those parts were equivalent.

When asked to evaluate the openness of her angle, Kacie indicated, albeit hesitantly, that her angle was not open enough, which suggested Kacie might soon establish a coordination between the openness of an angle and the number of copies required to cover the plane. At this point, Kacie was unable to use such a coordination involving openness and number of copies to anticipate the magnitude of the angle she needed to draw. Kacie seemed aware that she needed an angle more open than a right angle, but she never produced an initial angle more open than a right angle when working on this task.[46]

Evidence of a template for a 4-paritioned plane. Following the excerpt, I asked Kacie to make another estimate for a 3-split of the plane, and she produced another

[46] I present a possible explanation for this reluctance in the discussion of Kacie's February 8th teaching session.

approximately right angle. After erasing her new estimate, Kacie explained, “um, if I did that angle …you would have to do it four times and not three.” Kacie spontaneously illustrated that four such angles would cover the plane by sequentially illustrating them on the transparency as shown in Figure 5.65.

Figure 5.65. Kacie demonstrates a template for a 4-partitioned plane

Kacie’s revised estimate appeared to be essentially identical to her first estimate from my perspective. Her rationale for why the estimate was inappropriate was slightly different. Rather than stopping at three repetitions, Kacie continued to produce copies to exhaust the plane. In doing this, Kacie indicated she had established a template for a plane partitioned into four congruent angular parts, which I refer to as a 4-partitioned plane.[47] This time, Kacie was satisfied with congruence of the parts, but she was aware she produced the wrong number of parts.

Attempting a 5-split of the plane. Because Kacie was struggling to produce a 3-partitioned plane, I asked Kacie to try to make an angle so that five of them would be needed to “go all the way around.” As in the previous plane splitting task, Kacie gave no discernable indication of partitioning, which contraindicated the splitting operation. Instead, she produced an estimate (Figure 5.66 right) and then repeated the estimate to

[47] A planar covering formed from *n* congruent, disjoint, adjacent angles is an *n*-partitioned plane.

establish a planar covering (Figure 5.66 left). Kacie was again engaged in segmenting activity.

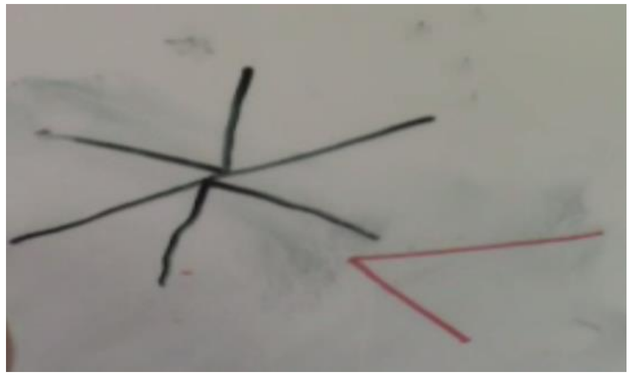


Figure 5.66. Kacie's estimate (right in red) and planar covering (left in black) for a 5-split of the plane.

The planar covering Kacie produced was composed of six angular parts rather than five, and Kacie did not spontaneously evaluate the appropriateness of her estimate as she had done in the previous plane splitting tasks. When I asked Kacie about the appropriateness of her initial estimate, Kacie counted the disjoint angles in her planar covering and expressed her frustration with these tasks: "Gosh darn it. I feel like if I did a smaller one it would take more to get it around. But then when I try with a bigger one, I can't get it to do three."

Reflecting on Kacie's challenges in plane splitting tasks. During this teaching session, Kacie had difficulty coordinating the number and size of the angular parts in her planar coverings; she demonstrated accounting for the number of parts, or size of the parts, but not both at the same time. Kacie's challenges in these plane splitting tasks were surprising to me because I had inferred she constructed the splitting operation based on her initial interview. Kacie's difficulties in these tasks indicated either (a) she hadn't constructed the splitting operation or (b) she was not able to take a full angle as input for her splitting operation. As I will show in subsequent sections, both explanations are viable.

Summary of Kacie's January 25th session. In her January 25th solo teaching session, Kacie indicated that, to some extent, she associated the term "angle measure" and degrees as relating to "openness" and her experiences with chopsticks in the teaching experiment. However, Kacie indicated she associated the term "angle measure" with a curved mark near the vertex of an angle model. In Kacie's view, the measure of the angle was dependent upon where this mark was placed. Kacie's conception of a one-degree angle was comparative in that she knew a one-degree angle was "small." However, Kacie did not indicate an operative conception of a one-degree angle as she did not describe how to produce or check the measure of such an angle using mental operations.

In this session, Kacie indicated she could mentally copy and adjoin composite units to produce planar coverings and three-level-of-unit structures. However, Kacie struggled with plane splitting task and did not indicate the splitting operation. Kacie engaged in segmenting activity on these tasks; she produced an estimate and then checked the appropriateness of this estimate based on sequentially producing copies to exhaust the plane. Kacie indicated she had constructed a template of a 4-partitioned plane.

Camille's & Kacie's February 1st Session

To this point in the teaching experiment, the students had worked almost exclusively in physical or drawn angular contexts. A major goal in Camille's & Kacie's February 1st teaching session was to provide students with some experience working within a dynamic geometry environment, Geometer's Sketchpad (GSP). In this session, I asked student to explore two sketches: (a) Five Points on an Angle (FPA) and (b) Rotating Laser.

Five Points on an Angle. I designed FPA to resemble a dynamic pair of chopsticks. The sketch contained a single angle model, on which there were five draggable points (Figure 5.67). From my perspective, each draggable point controlled one attribute of the angle model: green (first side length), blue (second side length), pink (orientation), red (openness), and yellow (position). To introduce students to GSP, I asked Kacie and Camille to experiment with dragging each of the points and to note what each point changed or didn't change.

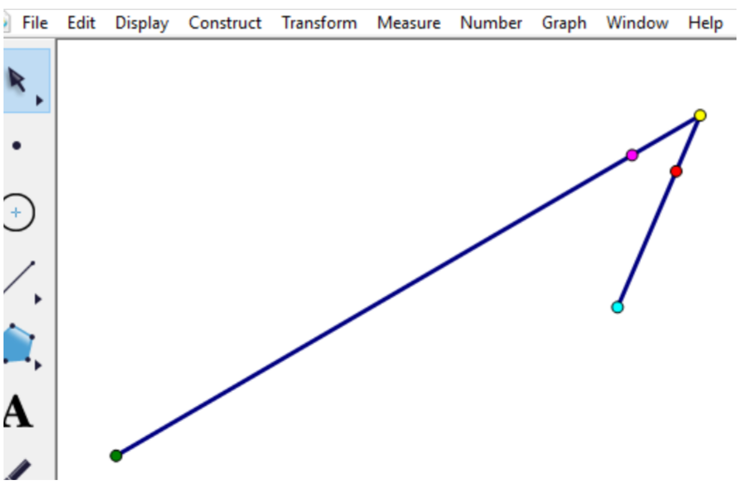


Figure 5.67. Five Points on an Angle (FPA)

The students spent six minutes independently experimenting within the sketch. After their independent examination of the sketch, we discussed what students had noticed for each point. As we discussed the effect of dragging each point, I explicitly asked students whether the point in question changed the openness of the angle if students did not mention openness spontaneously. Both students agreed that the red point changed the openness of the angle and all other points did not change the openness.[48]

Rotating Laser. The second sketch contained an image of the laser level students had rotated on a table in a previous teaching session (Figure 5.68). There were two action buttons in the sketch. Clicking the “Rotate Laser” button stopped or started a counterclockwise rotation of the laser. The “Show/Hide Ray” button displayed or hid the “laser beam,” which was a ray emanating from a green point at the front of the laser. Students could reposition the laser by dragging a blue point at the center of the laser. Additionally, students could rotate the laser without using the action button by dragging the green point.

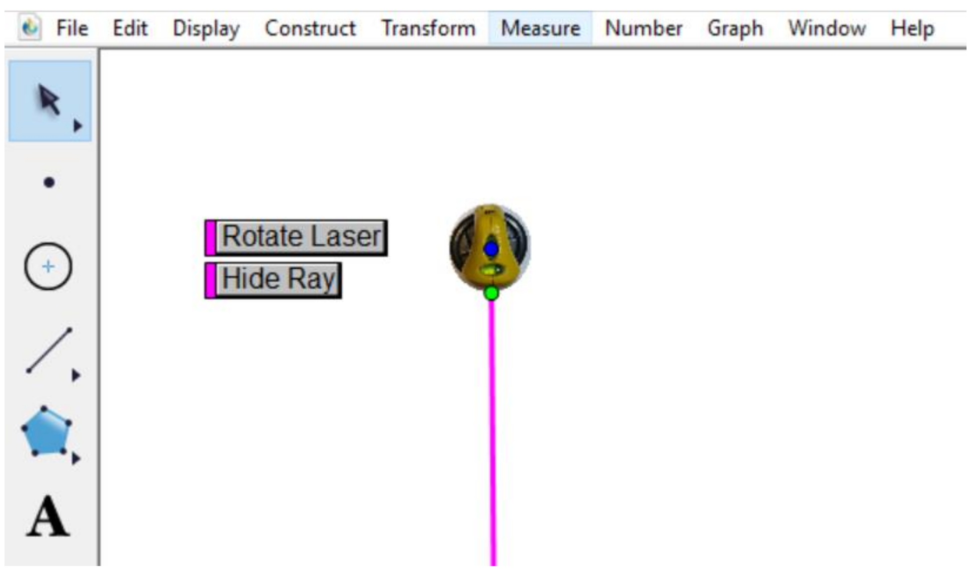


Figure 5.68. The Rotating Laser Sketch.

After the students had some time to explore rotating the laser, I asked if the laser reminded them of anything we had previously done. Both students mentioned previous

[48] Just before discussing the effects of dragging the red point, Camille mentioned the openness changed when discussing the blue point; she quickly changed her mind after dragging the blue point again during the conversation. My interpretation is that Camille was anticipating talking about the red point, which did change the openness.

activities with the physical laser level; neither student connected the rotational motion of the laser to the chopsticks contexts. Because the students did not associate the laser with the chopstick context, I inferred neither student had previously mentally enacted a radial sweep in the chopstick contexts.

Kacie's February 8th Session

On February 8th, Kacie participated in a solo teaching session because Camille was absent. In this session, I investigated whether Kacie could make connections across the laser and chopstick contexts. Throughout this discussion, I use *rotational* to refer to angle contexts that explicitly involve experiential rotational motion from my perspective as an observer (e.g., the rotating laser pointer); I use *non-rotational* to refer to angle contexts do not explicitly involve experiential rotational motion from my perspective as an observer (e.g., a pair of hinged chopsticks or drawn angle model). In addition to investigating Kacie's connections between these contexts, I also examined the domain of angularity Kacie associated with each context. Additionally, I presented Kacie with an angular splitting task and asked her to discuss a one-degree angle.

Kacie connects laser and chopstick contexts. In her previous paired teaching session, Kacie had examined dynamic sketches I designed to imitate chopstick and laser contexts. When I asked Kacie if the rotating laser in one of the GSP sketches was related to anything else we had done in the teaching experiment, Kacie connected the virtual laser to her previous activity with a physical laser in the teaching experiment; Kacie did not assimilate the laser and chopsticks contexts as being related to one another, which contraindicated the insertion of a radial sweep into angle models. At the beginning of her

February 8th teaching session, I wanted to investigate whether Kacie could make connections between these two contexts.

Connecting from the laser to the chopsticks. At the beginning of Kacie's February 8th session, I asked Kacie to demonstrate two consecutive quarter rotations using the rotating laser in GSP. For each quarter rotation, Kacie moved the laser from corner to corner within the rectangular GSP window. Kacie asserted that the two rotations she enacted were the same size, although this was not the case from my perspective. Following these rotations, I asked Kacie, "Does this remind you of any other stuff we've done throughout our time working together?" Kacie explained that it was "kind of like the chopsticks…like the openness of the angle and then like an angle two times as big as that one."

This was the first time Kacie likened the rotating laser to the activities with the chopsticks. To investigate the similarities she saw in these situations, I handed her a pair of wooden chopsticks and asked how she would relate the chopsticks to the laser situation; Kacie related the chopsticks to the laser as described in Excerpt 5.10.

Excerpt 5.10. Kacie relates the wooden chopsticks to the rotating laser

K: Like the laser started here [*holding the chopsticks in a closed position as shown in Figure 5.69 left*]. And then like moved it to a fourth [*moves one side of the chopsticks counterclockwise while holding the other fixed until she reaches an approximately right angle; see Figure 5.69 middle*]. And then I moved it to another fourth, which would be like that [*keeping one side fixed, moves the other side counterclockwise until she reaches a straight angle; see Figure 5.69 right*].

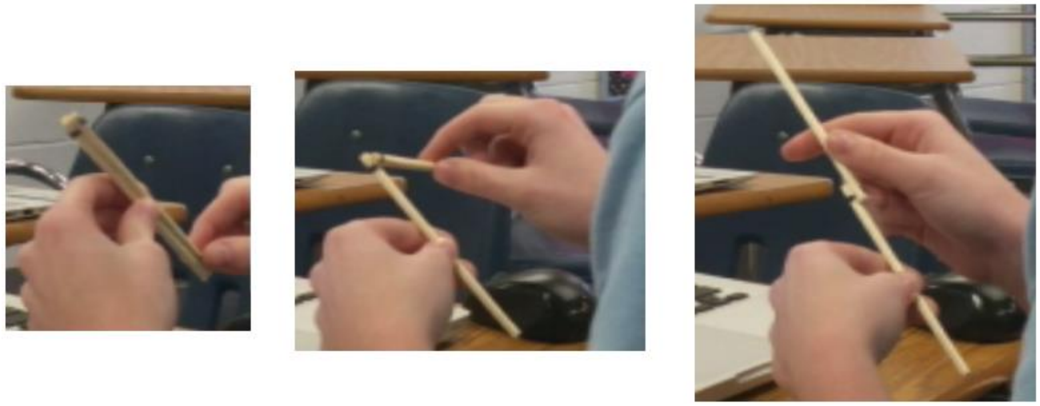


Figure 5.69. Kacie demonstrates consecutive quarter rotations using the chopsticks.

In the relating the wooden chopsticks to the rotating laser, two elements of Kacie’s reasoning were critical as she related the chopsticks to the laser. First, she marked the initial location of the beam using one side of the angle model. Second, she re-presented the motion of the beam, which she physically represented using the other side of the angle model. Thus, Kacie treated the two sides of the chopsticks differently. She used one side of the chopsticks to represent the position of the laser prior to the enactment of any rotation; she used the other side to demonstrate the laser’s rotation, which she paused at three instances: prior to rotation, after the first quarter rotation, and after the second quarter rotation (Figure 5.69 above).

From my perspective, Kacie’s sudden connection between the laser and chopstick contexts was occasioned by the enactment of multiple rotations in the laser context. Her repeated instantiation of quarter-rotations activated the same operations Kacie had used when adjoining copies of the wooden chopsticks (e.g., copying and uniting). Kacie’s comment, “an angle two times as big as that one,” supports this hypothesis. Furthermore, she held the initial side of the chopstick fixed in a single location as she instantiated two quarter rotations with the other side, as if setting the chopsticks to contain two quarter rotations.

That the rotations I asked Kacie to enact were quarter rotations may have also played a supporting role in Kacie's connection between the two angle contexts. In Kacie's January 28th session, she established a template for a 4-partitioned plane when attempting plane splitting tasks. In the present session, the enactment of successive quarter rotations may have activated her template for a 4-partitioned plane and, in turn, engendered her sudden awareness of a similarity between the two contexts.

To further investigate Kacie's connections from rotational to non-rotational angle contexts, I presented Kacie with a GSP sketch involving the rotating laser (which I had set to rotate counterclockwise through 144°) and asked Kacie to draw a picture represent the rotation. After viewing the rotating beam, Kacie began by drawing a point representing the laser followed by a segment representing the initial position of the beam. From this initial segment, she drew a curved path in a counterclockwise direction remarking, "and then it went all the way to about, let's say there;" she then drew another segment which met the end of the curved path and emanated from the point representing the laser (see Figure 5.70).

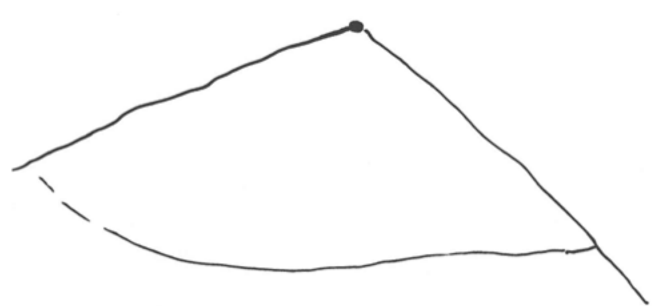


Figure 5.70. Kacie draws a picture to represent a laser rotation.

In creating a drawing to represent the rotation, Kacie indicated the same mental operations she had instantiated when relating quarter rotations of the beam to the wooden chopsticks. In particular, she held in mind the initial location of the beam, which she represented with a segment; she re-presented the motion of the rotating beam, which she illustrated using the curved path; and she held in mind the terminal location of the beam, which she represented with another segment. In this task, Kacie explicitly noted the vertex represented the location of the laser.

Connecting from chopsticks to the laser. Kacie had indicated she could represent angles in rotational contexts using non-rotational angle models. To investigate whether she could interpret angles in non-rotational contexts using the rotating laser, I presented Kacie with an angle model within the rotating laser GSP sketch and asked where she could place the laser so the beam could rotate from one side to the other. Kacie immediately moved the laser to the vertex of the angle explaining she would place the laser "like where the angle meets." Kacie then rotated the laser to align the beam with one side of the angle model and rotated the beam through the convex interior until the beam was coincident with the other side. Kacie's activities indicated she could insert the laser into a non-rotational angle context and rotate a ray emanating from the vertex to sweep out the interior of an angle model.

Comparing openness and rotation. Kacie had demonstrated she could relate non-rotational and rotational angle contexts to one another. However, until this point in the teaching experiment, we had used different descriptors when discussing the angular attribute in these contexts. In chopstick contexts, Kacie and I had talked about openness; with the rotating laser, Kacie and I talked had about the amount of rotation. As indicated in the preceding sections, Kacie could relate both contexts to angle drawings. To investigate if and how these two attributes—openness and amount of rotation—were different for Kacie, I drew two angles on a whiteboard (Figure 5.71) and asked Kacie to compare the angles in terms of each attribute. Kacie's response is described in excerpt 5.11 below.

Figure 5.71. Two angles for discussing openness vs. rotation.

Excerpt 5.11. Kacie rotates a beam through reflex angles

T: Let me ask you this. I've got two angles drawn up there on the board (Figure 5.71). Which of those would you say is more open, the black one or the blue one?

K: The blue one.

T: The blue one… If you imagine doing that laser rotation on those two, on which one would you have to rotate the laser more?

K: The black one.

T: So tell me about that. Can you show me with this [*hands K the laser level*].

… [*K walks to the board. W turns off the lights*]

K: So if it started there [*places laser at vertex of the black angle and aligns the beam with one side*]. And you rotated it all the way around until it met the black one's other side [*rotates beam counterclockwise through the reflex angle ending at the other side of the black angle; see Figure 5.72*], it would take longer than starting here and then going like this [*rotates beam counterclockwise through the reflex angle implied by the blue angle; see Figure 5.73*].

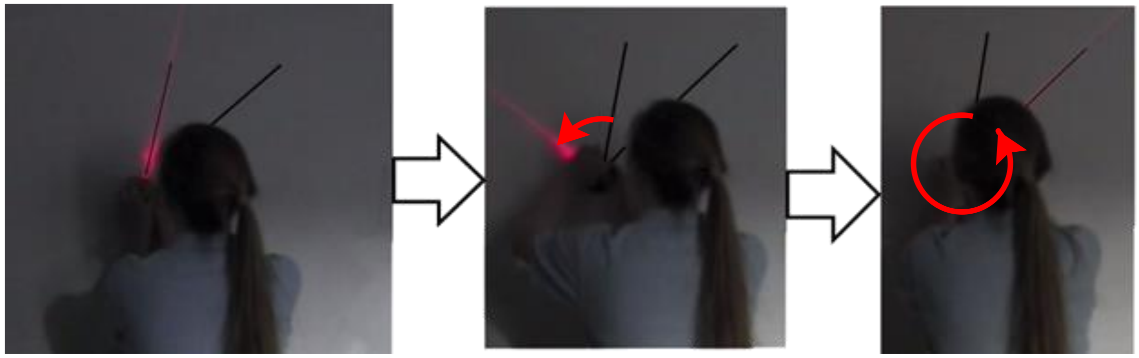


Figure 5.72. Kacie rotates the beam through the black reflex angle.

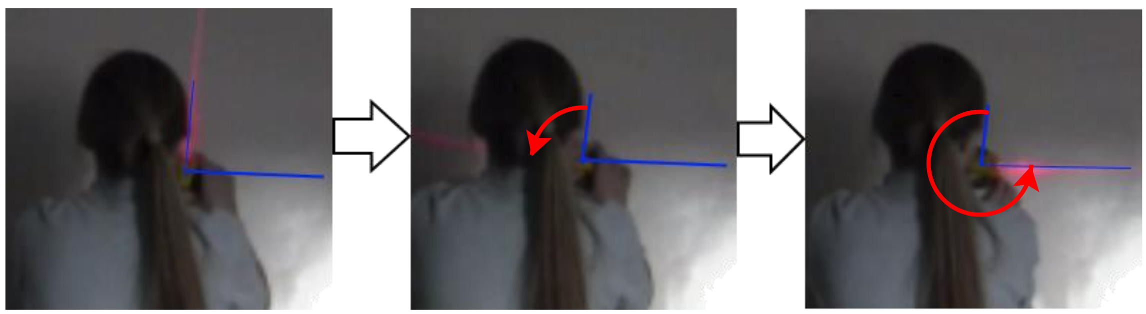


Figure 5.73. Kacie rotates the beam through the blue reflex angle.

Openness and amount of rotation were not synonymous attributes for Kacie: the blue angle was more open, but the black angle required more rotation. As I expected based on her previous activities, Kacie identified the blue angle as being more open than the black angle. However, I was initially surprised at Kacie's assertion that the black angle would require more rotation than the blue angle. As indicated in Excerpt 5.11, Kacie's decision was based on mentally inserting a ray rotating counterclockwise through the reflex angles associated with the black and blue drawn angle models. For each drawn model, she mentally inserted imagery of the rotating ray where one side marked the start of the rotation and the other side marked the end. By holding in mind the duration of the two rotational motions, Kacie compared the amounts of rotation and concluded the blue angle required more rotation.[49]

Following Kacie's comparison of reflex angles, I asked Kacie to consider sweeping out the conjugate angles for the blue and black drawings.[50] We marked these convex angles using solid curved path (Figure 5.74) similar to the ones Kacie had used to mark her angle drawings earlier in a previous teaching session. When comparing the rotations for the convex angles, Kacie asserted that the blue angle would require more rotation than the black angle. We also marked the reflex angles using dashed curved paths as shown in Figure 5.74. To further investigate differences in Kacie's conceptions of rotation and openness, I asked Kacie if she could think about the reflex angles in terms of openness (see Excerpt 5.12).

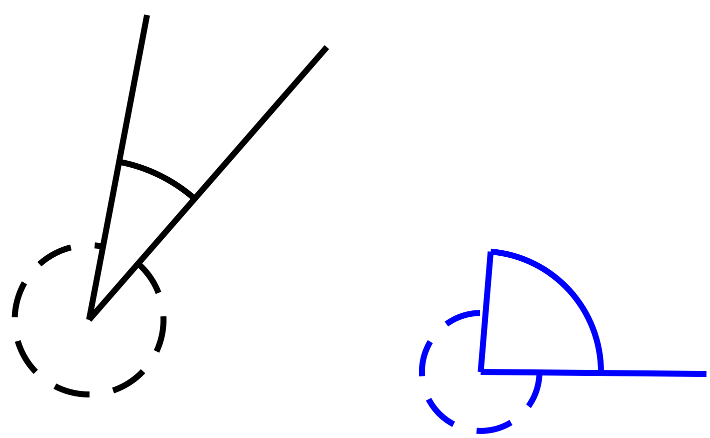


Figure 5.74. Reflex and non-reflex angles indicated by dashed and solid arcs, respectively.

Excerpt 5.12. Openness does not apply to reflex angles for Kacie.

T: Would it make sense to think about openness like in this [*pointing to the reflex cases*].

[49] Kacie's reflex interpretation was surprising to me in the moment and a fortuitous anomaly in the teaching experiment at large; in fact, this was the only time in the entire teaching experiment that any student spontaneously interpreted a drawn angle model as a reflex angle.

[50] During our conversation, I used both "rotation" and "sweep" interchangeably.

K: No [*slightly laughing*]. Not to me it wouldn't.
T: Okay. Not to you. Can you talk about that?
K: Um, because when I see like an angle I think of the space in between it and not the space outside of it. So it would be kind of weird to change what I've been doing .

As excerpt 5.12 indicates, Kacie did not find it appropriate to discuss openness for the reflex angle. For Kacie, openness entailed the "space in between" the sides of the angle; the reflex case, however, required consideration of "the space outside of" the black angle drawing. Thus, when considering the given angle drawings in terms of openness, Kacie assimilated the drawn angle models as convex angles only. For Kacie, I infer openness was an attribute she conceived as involving the least motion required to obtain a current angular configuration from a closed configuration. In fact, up to this point in the teaching experiment Kacie had worked with many drawings of angles, and never had she taken a given angle drawing to indicate a reflex angle. Instead, she interpreted all given angle drawings as non-reflex angles (i.e., convex angles).

From my perspective, the activities with the rotating laser beam engendered Kacie's construction of a new class of angles—reflex angles—which she had not assimilated without rotational imagery. Having abstracted the rotating beam, Kacie was able to insert the beam spontaneously (Excerpt 5.11 above), construct two reflex angles, and compare the rotation required to sweep through the interior of each.

Reflections on Kacie's activities in plane splitting tasks from prior teaching sessions. At this point in the teaching experiment, I inferred, for Kacie, openness was a characteristic of angles between a closed angle and a straight angle. The domain of Kacie's openness explained, in part, her challenges with the plane splitting tasks in her January 25th teaching session. Kacie had demonstrated she could establish a planar covering by repeating a convex angle in these tasks; however, Kacie did not take a full angle (i.e., an angle whose interior is the entire plane) as input for partitioning in these tasks. As indicated by her activities in the present teaching session, openness was nonsensical for Kacie in the context of a full angle. As such, Kacie had no magnitude to partition when she previously engaged with plane splitting tasks.

In prior sessions, Kacie solved plane splitting tasks through segmenting activity: she produced an estimate (without any observable indication of partitioning) and repeated the estimate to enumerate the number of copies required to cover the plane. In her attempts to produce a 3-split of the plane, the estimates Kacie produced were never more open than a right angle. That Kacie did not repeat an obtuse angle is sensible based on the domain of openness she indicated in this teaching session. If Kacie viewed the initial repetition as creating an angle twice open as a given angle, then repeating an obtuse angle would have necessitated that Kacie interpret openness for a reflex angle, which, as she explained, was not sensible from her perspective.

Additional activities in Kacie's February 8th session. After Kacie demonstrated that she could make connections between non-rotational and rotational contexts, my goal was to investigate Kacie's operations involving rotational imagery and engender these operations in non-rotational angle contexts.

Five points on a lasered angle. After our discussion about openness and rotations on the whiteboard, Kacie and I examined Five Points on a Lasered Angle (FPLA), a modification of the FPA sketch she and Camille had investigated in an earlier session. As in FPA, I designed FPLA to contain a single model defined by five draggable points, where each of these points (red, blue, green, pink, yellow) changed one attribute (openness, one side length, the other side length, orientation, and position, respectively). Different from FPA, FPLA had a round laser centered at the vertex of the angle model. To re-familiarize Kacie with the sketch, I asked her which points, when dragged, would change the openness. After dragging each of the five dots, Kacie remarked, “the only one that changes the openness is the red one.” After this assertion, I introduced Kacie to the new feature in this version of the sketch: the “Sweep Beam” button. Clicking this button showed a beam emanating from the laser at the vertex of the angle. Starting from one side of the angle model, the beam rotated counterclockwise through the convex angle while leaving traces of the beam (see figure 5.75).

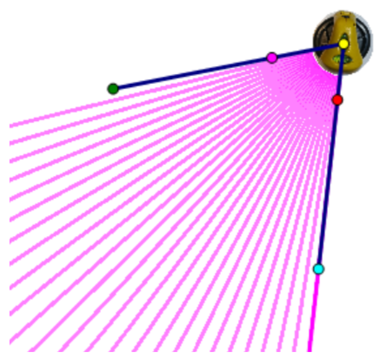


Figure 5.75. Five points on a lasered angle.

After she clicked the “Sweep Beam” button, I asked Kacie which of the five points would change the sweep of the laser. Kacie quickly asserted “the red one,” and I followed up asking if any of the other points would change the sweep. After a brief pause, Kacie noted that changing the length of the sides would not change the sweep, but moving the pink dot would reorient the sweep to another part of the screen. Kacie acknowledged that while altering the orientation produced a different sweep, the amount of sweep remained unchanged. This acknowledgement indicated Kacie had constructed amount of sweep as an attribute independent from side length and orientation. Kacie’s demonstrated way of operating in this context indicated her assimilation of sweep was compatible with my conception of angularity.

Splitting tasks revisited. Following the introduction to FPLA, I revisited plane splitting tasks with Kacie in the context of this sketch. This time, I phrased tasks in terms of “amount of sweep” rather than “openness”. In particular, I asked Kacie to adjust the angle so that 3, 5, and 8 sweeps would be required to sweep out the entire page. For the 3-split of the plane, Kacie reasoned using previously established re-presentable templates for partitioned planes. Specifically, she adjusted the red point to form a straight angle, which she knew would require two sweeps, and a right angle, which she knew would require four sweeps because it was “like a cross.” From these templates for partitioned planes, Kacie reasoned the angle needed to be set between the angles occurring in the 2- and 4-partitioned planes; however, Kacie gave no indication of having partitioned the plane into three parts when setting her estimate (Figure 5.76 left), which was a contraindication of applying the splitting operation.[51]

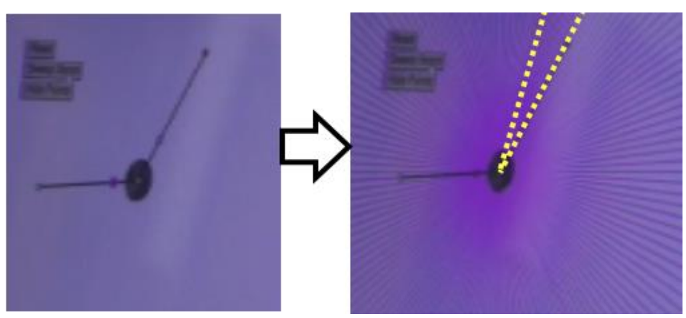


Figure 5.76. Kacie’s estimates a 3-split of plane; overage shown in yellow.

That Kacie set an obtuse estimate for a 3-split of the plane is significant as Kacie had not previously produced an estimate exceeding a right angle in these tasks. I attribute

[51] Splitting the plane would require the simultaneous insertion of three angular parts, which might be indicated, for example, by gesturing three approximately equal radial segments emanating from the vertex prior to making an estimate.

this difference in Kacie's activities to differences in Kacie's quantifications of sweep and openness. Concave sweeps were sensible for Kacie, whereas openness had no sensible interpretation for a concave case from her perspective.

Kacie checked her estimate for a 3-split of the plane by sweeping the angle out three times, which slightly exceeded the plane (see Figure 5.76 right where the overage after three sweeps is delineated by dotted yellow segments). Afterwards, Kacie adjusted her estimate appropriately, meaning she reduced the openness of the angle since three sweeps of her previous estimate more than exhausted the plane. In her adjustment, however, Kacie did not attempt to distribute this overage equally across all three angular parts. Instead, she reduced her initial estimate by the overage, which resulted in a portion of the plane being unaccounted for after three sweeps of her second estimate. Kacie again adjusted appropriately without distributing the difference.

Kacie made reasonable estimates for the 5- and 8-splits as well, but again gave little indication of partitioning, which contraindicated the splitting operation; in other words, Kacie did not indicate simultaneously projecting 5 or 8 angular parts to subdivide the plane prior to producing an estimate. After checking her estimates for each case, she adjusted appropriately, but not distributively. In the n=8 case, Kacie explained she wanted to half a right angle, but gave no indication she viewed the plane as being composed of eight parts as she made this remark. As she checked her estimate the n=8 case, Kacie indicated she had structured the plane in terms of two half-planes. For example, after the sixth sweep of an estimate exceeded a straight angle, Kacie lamented, "oh, this is way off," and explained two additional sweeps would not exhaust the plane. After opening the angle more for her second estimate, Kacie stopped after three sweeps exceeded the half-plane, noting that "three is like half already." When enacting a fourth sweep for a third estimate, Kacie remarked, "hopefully it'll stop in the middle," and indicated she wanted the fourth sweep to cover half the plane.

As indicated in the preceding paragraph, Kacie's decisions about the appropriateness of her estimates were often based on the half-plane. This indicated that Kacie had structured the plane as a unit consisting of two half planes. Thus, when checking if an estimate was appropriate for an 8-split of the plane for example, Kacie did not need to carry out all eight sweeps. Instead, she could evaluate her estimate once she had swept enough angles to cover the half-plane. Kacie's structuring of the plane in this manner appeared to be unique for two half-planes. For example, when I asked Kacie if she could check the reasonableness of her estimate for an 8-split of the plane using only two sweeps, Kacie did not consider how two sweeps compared to a right angle.

In addition to the plane splitting tasks, I also posed an angle splitting task to Kacie asking her to set a blue angle so that three sweeps of the blue angle would sweep out a given red angle. As with the plane splitting tasks, Kacie engaged in segmenting activity: she made a reasonable estimate and checked her result by enacting three sweeps of the blue angle. However, she gave no indication of splitting the given red angle into three parts when making her initial estimate. Looking across the splitting tasks in this session, Kacie's activities contraindicated she could assimilate angles to the splitting operation.

One degree. In the closing minutes of the session, I asked Kacie again what it would mean for an angle to have a measure of one degree. Kacie explained, "I think last time I said it would be open that much," and I asked Kacie if it would be possible to describe a one-degree angle in terms of the sweeping actions we had used within GSP during the session. Focusing her attention on a previously enacted sweep, Kacie suggested that a one-degree angle would be "one of the lines" comprising the sweep. It is unclear whether Kacie was referring to a single ray or to the space between two of the rays. In either case, it was clear that Kacie was not leveraging extensive operations (e.g., iteration) in characterizing a one-degree angle. Instead, she continued to leverage comparative properties (e.g., a one-degree angle is "small").

Summary of Kacie's February 8th session. In her February 8th session, Kacie indicated (a) she could represent rotational motion using a non-rotational angle model and (b) she could insert rotational motion into a non-rotational angle model. Openness for Kacie was not sensible for reflex angles, but Kacie spontaneously applied rotational imagery to reflex angles. Kacie presented further contraindications of having constructed the splitting operation and indicated a non-operative conception of a one-degree angle.

Camille's & Kacie's February 22nd Session

Camille and Kacie participated in a paired teaching session on February 22nd. Until this point in the teaching experiment, I had primarily investigated Kacie's and Camille's quantifications of angularity in terms of extensive quantitative operations involving arbitrary unit angles. Other than asking Camille and Kacie about one-degree angles, I had not explicitly asked them to reason using degrees as a unit of angular measure. A major goal of this teaching session was to examine if and how students' conceptions of degrees as a unit of angular measure were related to their extensive quantitative operations, including units coordination. As such, I asked students to reason about angles using two new GSP sketches: one involving an equipartitioned plane and the other involving an equipartitioned right angle. In addition to tasks within these GSP sketches, I examined both students' domain for openness at the beginning of the teaching session.

Domain of openness. Due to the differences in domain for openness and sweep that Kacie had discussed in the previous session, I began the paired session by examining Kacie's and Camille's domain for openness. Specifically, I handed Camille and Kacie each a pair of hinged chopsticks and asked each to set the chopsticks to be as open as possible. Kacie immediately opened her chopsticks to a straight angle and confidently rested them on the desk. As Camille opened her chopsticks, she hesitated as she reached a straight angle and started to open them further; she then looked to Kacie (and the desk on which Kacie's chopsticks were resting), set the chopsticks at a straight angle, and rested them on her desk.

After setting their respective chopsticks, each student offered an explanation for her actions. Kacie explained that a straight angle was "as big as you can get, because if you keep going [*opening the chopsticks beyond a straight angle*], they [the sides] kind of come back like that," indicating that the sides started to near each other again. Immediately following Kacie's explanation, Camille discussed her thinking:

> [I think] the same. Cause, I kept going like that [*opening the chopsticks beyond a straight angle*] and I realized it would just end up like this [*moving chopsticks to a closed position*] again. And then I looked over at Kacie [*laughs*], and then – yeah.

As in her previous teaching session, Kacie indicated her conception of openness for angle models included the convex case alone. Kacie's conception of openness in this task did not include an accumulation of the totality of motions she physically enacted when opening the chopsticks (i.e., opening to a straight angle and then opening more), else she might have considered cases beyond a straight angle to be more open than a straight angle. However, Kacie's activated conception of openness still involved motion. As Kacie opened the chopsticks beyond a straight angle in her explanation, she assimilated the chopsticks as being less open because they were approaching a closed state. Kacie reasoned about openness as if considering an opposing attribute, "closedness." Kacie took her template for a closed configuration as the referent state for ordering the openness of the chopsticks, rather than a re-presentation of the totality of the motions she enacted when opening the chopsticks. For Kacie, once she moved the chopsticks beyond a straight angle, the chopsticks were getting closer to closed and were therefore also becoming less open. Put succinctly, the openness of a particular angular configuration, for Kacie, entailed the least amount of motion required to close the configuration.

Like Kacie, Camille also decided that a straight angle was the most open setting for the chopsticks. Unlike Kacie, Camille physically extended the chopsticks beyond a straight angle prior to settling on a straight angle. As in some of the previous teaching sessions, Camille's reasoning was likely influenced by her observation of Kacie's activities; she appeared to be imitating Kacie in both actions and words. Camille finalized setting her chopsticks immediately after looking at the Kacie's chopsticks, which were set at a straight angle and resting on Kacie's desk. Camille's subsequent explanation was remarkably similar to Kacie's, which further indicated her imitation of Kacie. In particular, Camille mentioned considering cases beyond a straight angle and noted that such continuations would ultimately close the chopsticks. Furthermore, Camille good-naturedly acknowledged Kacie's influence (e.g., "I looked over at Kacie").

If Camille thought differently about the maximum for openness, she was perhaps unwilling to take the intellectual risk of disagreeing with Kacie in this instance. However, Camille's interpretations of openness throughout the teaching experiment to this point supports the hypothesis that the straight angle exhibited the maximum openness for a non-rotational angle model. Just like Kacie, Camille never interpreted a non-rotational angle model as indicating a concave angle throughout the teaching experiment. In all her activities with such angle models, Camille always took the convex angle as the object of her reasoning. As such, even though Camille was imitating Kacie throughout this task, there is no evidence throughout the teaching experiment that her maximum for openness exceeded a straight angle.

After discussing the most openness for the chopsticks, I also asked each student to set the chopsticks to the least possible openness. Both students closed the chopsticks completely, though Camille, as in the previous task, was slower to act and eyed Kacie. Trying to impede any additional imitation for the moment, I asked Camille to explain her thinking first. Holding up the closed chopsticks, Camille explained, "well, I can't make them any smaller, so yeah." After Camille, Kacie explained, "there isn't any like space in between the two chopsticks...kind of what she said [pointing to Camille], they can't go any more closed."

Both students decided that the closed configuration was the least open configuration for the chopsticks. From Camille's perspective, the closed chopsticks could not be made any less open, though she used the term "smaller." In addition to referencing the empty interior of the chopsticks, Kacie also continued to reason about openness in terms of closedness. For Kacie, achieving the least openness was the same as achieving the most closedness.

The partitioned plane sketch. After discussing openness, I presented Kacie and Camille with *The Partitioned Plane* (TPP), a GSP sketch they had not used in previous sessions. In TPP, the students could set a parameter, *n*, to any whole-number value to instantly partition the screen into *n* equiangular parts (examples of TPP for *n* = 10, 4, 1, and 100 are shown in Figure 5.77).

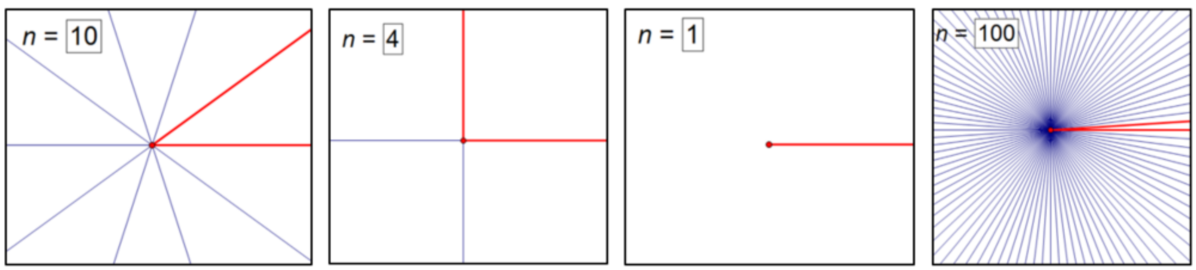


Figure 5.77. The Partitioned Plane with *n* = 10, 4, 1, & 100.

One-degree angle. After introducing the sketch and giving the students a few moments to experiment with the value of *n*, I asked Camille and Kacie how they would set *n* to produce one-degree angles. Camille began hesitantly and set *n* = 1, 2, 4, 3 before ultimately remarking, "I'm not sure." In a previous teaching session (January 19th), Camille had asserted that length, openness, and orientation were relevant attributes when she thought about measuring an angle using degrees. Due to the multitude of attributes she associated with degrees, Camille's uncertainty as to how to proceed in this task using TPP was reasonable. With Camille at a standstill, Kacie, who had immediately set *n* = 100 (see right in Figure 5.77 above), explained her thinking at my request:

> Um well at first, I just thought of like – Since you said one-degree angle, I was like, oh maybe I should put N equals to one. But then I was like, wait. The smaller the number like the bigger the angle. So then I was like, okay well maybe I'll try a hundred. And then I clicked it and I was like, oh this

looks about right.

In her explanation, Kacie indicated she coordinated the number she input for the parameter with the size of one of the angles in the sketch. With this coordination established, I infer Kacie's goal was to produce a "very small angle," a description for a one-degree angle she had given in her January 25th teaching session. I suspect that Kacie's choice of 100 for a parameter value was a result of her desire to use a large number combined with her previous experiences with powers of 10. Kacie evaluated her progress via the visual feedback provided by the sketch: she was satisfied with her result because the small angle looked "about right" to her.

Although Kacie coordinated the numerical parameter with the size of one of the angles in the sketch, she did not, however, give an indication that she had coordinated this parameter with the specific number of angles into which the screen was partitioned. In fact, when setting the parameter for the one-degree task, the exhaustive partitioning of the plane did not seem relevant to Kacie. Instead, as was the case in her previous sessions when we discussed one-degree angles, Kacie was fixated on the magnitude of angularity without implementing extensive quantitative operations that would produce units from other units.

90-degree angles and right angles. Following the one-degree angle task, I asked Kacie and Camille to set n to produce a 90-degree angle. Within 6 seconds, Kacie set $n = 4$. In previous sessions, Kacie had indicated that she had established a re-presentable template for a 4-partitioned plane. Her immediacy in setting $n = 4$ in this task was further confirmation of this template. Additionally, Kacie indicated at least a numerical association between the number 90 and this template. In contrast to Kacie, Camille proceeded hesitantly and tried n = 90, 50, and 20. Camille settled on n = 20, which from my perspective produced 20 eighteen-degree angles. Camille describes her thinking about this task in Excerpt 5.12 below.

Excerpt 5.12. Discussing 90-degree angles and right angles.
T: So tell me about that one.
C: Uh, well I just remember it being kind of like that [points *to the computer screen and laughs*]
T: Kind of like that?
C: But I'm not sure though.
T: So if I asked you to draw a ninety-degree angle, what would you draw?
C: [*Points to the computer screen with n = 20*] Same thing as here.
T: Something like that. What about a right angle? Would that be any different?
C: Wouldn't that be more open? [*points upwards*]
T: Yeah. What would a – So if I asked you to draw a right angle, what would that look like?
C: You want me to draw?
T: Yeah. Go ahead and draw one.
C: I'm not sure, but [*draws a right angle as shown in Figure 5.77*]
T: Ah, okay. Yeah. So, Kacie, what would you draw for a right angle?
K: The same thing she drew.

Figure 5.78. Camille draws a right angle.

As indicated by Excerpt 5.12, a 90-degree angle was not synonymous with a right angle for Camille. Camille viewed one the 20 equiangular parts on the computer screen as a 90-degree angle, and she knew a right angle was more open than one of the angles on her screen. By subsequently drawing a right angle, Camille indicated she had established a re-presentable template for a right angle. However, Camille did not assimilate this right-angle template as having a measure of 90 degrees. Though Camille had established a right-angle template, she had not permanently assigned a measure of 90 degrees to this template.

Excerpt 5.12. Discussing 90-degree angles and right angles. (Continuation)
T: So another name that people often use for a right angle is like ninety-degree angle.
C: Oh.
T: So would there be a way – so does that change how you would set N, Camille?
C: Yeah.
T: So how would you set it then?
C: Um. I'm just going to try different numbers until I get that.
…
[*Camille inputs n = 15, 10, 5, and then n = 4*]
C: Probably that, or [*inputs n = 3*]. Nope. Yeah, four. [*C inputs n = 4.*]

As described in the continuation of excerpt 5.12, Camille used a trial and error strategy as she worked through the task. As she tried various parameter values, I infer Camille compared the perceptually available angles on the computer screen to her re-presentable template for a right angle. Unlike Kacie, she had not established a template for a 4-partitioned plane. Her adjustments were systematic in that she gradually increased the openness of the angle by decreasing the parameter values until she reached $n = 4$ for the first time, which suggested she assimilated Kacie's remark from the one-degree task (i.e., "the smaller the number the bigger the angle"). After reaching $n = 4$, Camille remarked, "probably that," expressing hesitant approval. However, Camille went on to try $n = 3$, which underscored both her dependence upon the visual feedback from the GSP sketch and the lack of a re-presentable 4-partitioned plane template. Camille was searching for the number that made one of the angles in the sketch look most like her right-angle template.

Following Camille's activities, Kacie explained how she knew to set $n = 4$ to produce a 90-degree angle:

> Well, that's just kind of like one of the, the main angles that you learn, I guess. So it was just kind of like, I knew the shape of it and that – that two, like N equals two it'd just be like a straight line across…and I don't really know why I chose four. I just knew, I guess.

Through her explanation, Kacie confirmed she had established re-presentable templates for 2- and 4-partitioned planes. These re-presentable templates for 2- and 4-partitioned planes were static images Kacie could bring forth in visualized imagination at will. Although I refer to these as templates of partitioned planes, I do not intend to convey Kacie was engaged in operative partitioning activity, which would require taking the plane as the interior of a single angle.[52,53] Notably absent from Kacie's explanation were any references to a three-level-of-unit structure involving 360°. For example, Kacie did not mention that 360 degrees divided into four equiangular parts would necessitate that each of the four parts contained 90 degrees. Instead, Kacie "knew the shape of it," which indicated a dominance of figurative over operative thought. Nevertheless, Kacie associated this 4-partitioned plane template with a measure of ninety degrees.

Determining degree measure when n = 5. To this point, I had asked Kacie and Camille to provide a parameter value so that TPP displayed angles of specified degree measure. Neither student indicated she viewed the plane as an angle composed of 360 degrees. Instead, both had relied on particular templates and angular comparison schemes. To investigate whether and how students might use 360—the composite unit through which degrees are often introduced during classroom instruction—in their reasoning, I asked the pair to determine the measure of the angles displayed when the parameter in TPP was set to $n = 5$. I purposefully selected $n = 5$ because uniting the perceptually available angles in a 5-partitioned plane produces neither right nor straight

[52] I hypothesize that these particular partitioned plane templates may emerge earlier than other templates due to (a) repeated halving of the plane and (b) experiences with everyday objects; for example, in a different interaction Kacie described a 4-partitioned plane as "like a cross."
[53] In contrast to the more passive adjective form (i.e., partitioned plane), I use partition as a verb when describing the application of partitioning operations to an angle. For example, if an individual described simultaneously breaking the interior of a full angle into 4 equiangular parts, I would say that the individual partitioned the plane.

angles. Thus, I hoped to encourage reasoning that relied on a 360-unit composite by impeding students from operating on straight or right angles.

At the onset of the TPP $n = 5$ task, Kacie suggested that the angle had a measure of 80 degrees. Camille agreed and explained one side of the angle "went down" from the $n = 4$ case. "If it was ninety minus like ten it would be eighty," Camille continued. Thus, Camille assimilated Kacie's estimate to her own ways of reasoning. Camille held in mind a right-angle from the previous TPP task with $n = 4$. She imagined decreasing the openness of the right angle by moving one of the sides to be coincident with the perceptually available angle. Thus, Camille additively decomposed the right-angle template into two parts: the desired angle, which was perceptually available, and a difference angle, which was the amount by which the right angle needed to be decreased to produce the perceptual angle. Camille assigned a measure of 10 to the difference angle spontaneously, which offers additional evidence that she incorporated Kacie's estimate for the desired angle into an additive structure. Thus, Camille's numerical computation (i.e., $90 - 10 = 80$), paralleled her quantitative operations (i.e., decreasing the openness).

Additionally, Camille's computation was consistent with the scheme for angular comparison she had established in that the measures she assigned to the three additively structured angles were ordered appropriately based on the relative opennesses of the corresponding angles. In other words, the right angle measuring 90 degrees was more open than the 80-degree angle, which was more open than the 10-degree difference angle. Although Camille produced an additive structure with these three angles, she did not produce a multiplicative structure, which would have required for example constituting the right angle and perceptually available angle in terms of units of the difference angle.

Following Camille's explanation, I asked Kacie to explain her thinking, which essentially paralleled Camille's argument:

> I just kind of was like, well if four was ninety degrees. And then I typed in five and looked at this one and it was smaller than the four. And it didn't look like drastically smaller than four so I just chose ten. Or eighty, yeah.

As was the case with Camille, Kacie's explanation indicated she had assimilated an additive structure using a right angle, the perceptually available angle, and the difference angle. Like Camille, Kacie had not multiplicatively structured the angles within the additive structure. Despite my attempts to provoke reasoning involving a 360-unit composite, both students justified their estimates via a right-angle referent. Neither student appeared to leverage the five perceptually available congruent angles in their reasoning. Instead, both students focused on a single perceptually available angle in relation to a right-angle template, which each student held in mind from the previous task.

Figurative degree schemes. A *Figurative Degree Scheme (FDS)* emerges when an individual coordinates her number sequence, in an ordinal sense, with relative extents of angularity. Across three teaching sessions, Kacie had characterized a one-degree angle in comparative terms; Kacie knew that a one-degree angle was "a very small angle," which indicated she had coordinated ordinal properties of her whole number sequence and relative extents of angularity. This coordination constituted a modification of her schemes for congruence and comparison and was likely occasioned by her prior classroom experiences measuring angles in degrees. As she reasoned with degree measure in TPP, Kacie's activities further indicated she had established such a coordination. In TPP with *n*

= 5, Kacie indicated the FDS because she determined the measure of a given angle was 80° because it was "smaller" than a previously perceptually available 90° angle.

Different from Kacie, Camille had not previously indicated the same coordination between her number sequence and angularity when discussing one-degree angles. Camille had not previously associated degrees with angularity alone and instead also associated this unit of measure with orientation and side lengths of an angle model. However, in TPP with $n = 5$, Camille indicated she was in the process of establishing a figurative degree scheme precisely because she assimilated a decrease in angularity (from $n = 4$ to $n = 5$) along with a lesser degree measure (90° versus 80°).

A critical difference in Kacie's and Camille's way of reasoning with degree measures at this point in the teaching experiment involved each student's assimilation of a right angle. Both students had established a right-angle template.[54] For Kacie, the phrases "right angle" and "ninety-degree angle" were synonymous; her capacity to interchange these terms indicated Kacie had *permanently assigned* her right-angle template to have a measure of 90° and could use this as a referent for comparison when assigning degree measures to other angles. As such, I consider Kacie to have established a *Templated Figurative Degree Scheme* (TFDS). Having established a TFDS, angles less open than a right angle necessarily measured less than 90 degrees for Kacie; similarly, angles more open than a right angle necessarily measured more than 90 degrees for Kacie.

In this teaching session, Camille did not view "right angle" and "ninety-degree angle" as phrases referring to the same extent of angularity; therefore, Camille had not yet permanently assigned her right-angle template to have a measure of 90°. As such,

[54] I will show in later sections that Camille's right-angle template was orientation dependent.

Camille's activities indicated she was in the process of establishing at the FDS and had not yet constructed the TFDS.

Kacie induces a property of partitioned planes. In her explanation for TPP with *n* = 5, Kacie indicated an element of choice in her method: "I just *chose* ten." To encourage the students to construct a multiplicative structure involving a 360-unit composite, I asked, "Would there be any way to convince somebody that it was *exactly* eighty degrees just based on what you know from this picture?" In pressing the students to justify the exact measure of the angle in degrees, I occasioned a different way of reasoning in Kacie, which is indicated by Excerpt 5.13.

Excerpt 5.13. Kacie's numerical association of 360 and a 4-partitioned plane

K: Oh. When – Okay, so. I remember learning about this like last year or something, but I don't know if it's right, but. When you add a ninety-degree angle [*motions as if considering a 4-partitioned plane*], something about like the half of it's one-eighty and then the other half is one-eighty and you add those together and you get like three-hundred and sixty or something

…

T: Would it help to draw a picture?

K: I – I guess. But I think it might just be for circles or something. But, I remember [*picks up a marker to begin drawing*].

…

K: I remember somebody saying something about like if you had a ninety-degree angle [*draws a 4-partitioned plane as shown in figure 5.79 below*]. Let's just say that's ninety degrees. Like this part of it [*shades top half of the 4-partitioned plane from right to left*] is a hundred and eighty degrees. And then this part of it is also a hundred and eighty degrees [*shades bottom half of the 4-partitioned plane from right to left as shown in Figure 5.80 below*]. And those two together are three hundred and sixty.

T: And when you're saying those two together, what's the those two together?

K: The top and the bottom like added.

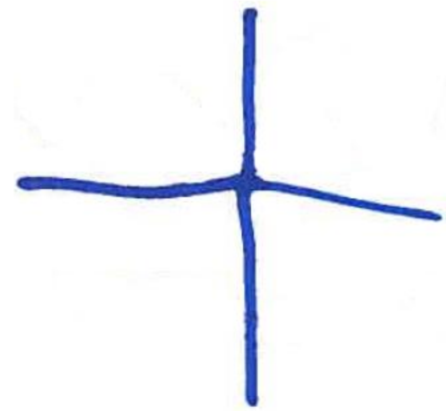


Figure 5.79. Kacie draws a 4-partitioned plane, which she calls a 90-degree angle.

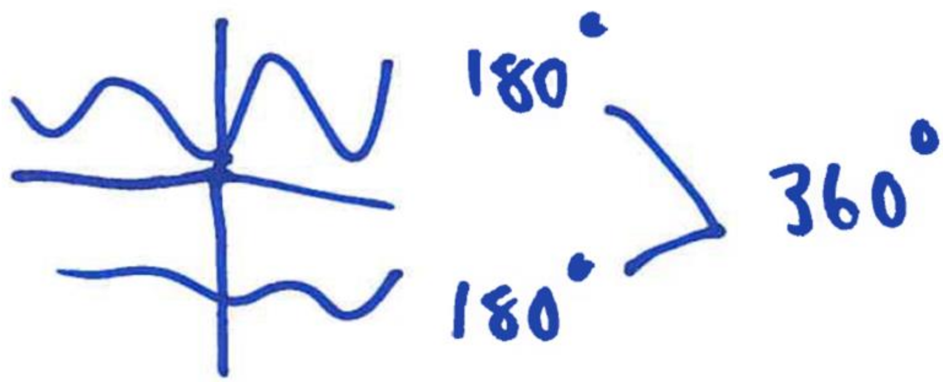


Figure 5.80. Kacie indicates two half planes and computes 180° + 180° = 360°.

After being pressed to justify an exact measurement, Kacie activated records of (what I presume to be) a prior school experience involving a 4-partitioned plane. Though Kacie named the entire 4-partitioned plane as a "ninety-degree angle," I infer Kacie used the term unconventionally to refer to two perpendicular lines. Kacie indicated at least a numerical association between 180 and half the plane (e.g., "half of it's one-eighty").[55] Using this numerical association and computations that paralleled her spatial operations, Kacie "deduced" a numerical association between 360 and the plane: she viewed the

[55] By numerical association here, I mean Kacie used 180 to refer to the half-plane, and I was unable to infer whether she quantitatively structured the plane using composite angular units. In other words, it was unclear whether Kacie viewed the half plane as being composed two 90-degree angles, for example.

plane as being the result of uniting two half planes, and computationally she represented this uniting via the addition 180° + 180° = 360°.

Kacie re-presented a 4-partitioned plane—a planar covering consisting of four congruent angles; however, as in her January 25th session, she did not indicate an awareness of the plane as the interior of a single angle. In the context of two adjacent half-planes, such an awareness would require a single, temporally bounded motion resulting from the concatenation of motions through each half-plane (e.g., a full rotation composed of two successive half-rotations). Kacie's two shadings, each from right to left, contraindicated such a concatenation as one shading did not begin where the other ended.

In excerpt 5.13 above, Kacie's comment, "but I think it might just be for circles or something," provides additional evidence that Kacie did not view the right angles as constituting the plane as the interior of a single angle. Kacie had previously used "circle" to refer to a complete rotation; in this teaching session, Kacie established a verbal contrast between her meaning for "circle" and the 4-partitioned plane she had drawn. Therefore, I infer that 360°—the result of Kacie's addition—referred to a property of the 4-partitioned plane and did not refer to the measure of a single angle.

Reasoning by analogy from this 4-partitioned plane and the corresponding numerical calculations, Kacie spontaneously extended her reasoning about planar coverings to TPP with $n = 5$ as described in the first continuation of Excerpt 5.13 below.

Excerpt 5.13. Kacie's numerical association of 360 and a 4-partitioned plane (First Continuation)

K: So you could do the same kind of thing for this one [*pointing to the 5-partitioned plane on the computer screen*]. If this one's [*pointing to one angle on the screen*] – if we think that it's eighty degrees, then you could do like eighty plus eighty plus eighty plus eighty plus eighty [*pointing to one of the five angles on the computer screen with each utterance of eighty*]. And see if it added – if it added up to be three hundred and sixty.

Because Kacie applied the additive numerical property she imputed to a 4-partitioned plane to a 5-partitioned plane, I infer that Kacie induced that the measures of angles in an arbitrary partitioned plane sum to 360°. This induction resulted in at least a temporary modification in her scheme for measuring angles in degrees. In her initial assimilation of the 5-partitioned plane task, Kacie did not indicate that the 5 congruent angles were relevant to her way of operating. Additionally, she had initially assimilated the 5-partitioned plane task in terms of a single right-angle referent. As she established a new way of operating for the task, Kacie assimilated five equiangular partitions and established a goal of assigning a measure to each partition such that the measures of all five equiangular partitions summed to 360°.

Camille's assimilation of TPP and imitation of Kacie. At this point in the session, I invited Camille to comment on Kacie's strategy. Camille noted one of the angles on the screen "looked kind of like it [was] as big as" another one of the angles, and she assigned a measure of 180° to each of these angles. Camille's remarks were enlightening to me in several ways. First, Camille apparently assimilated two of the angles as congruent after, not before, Kacie described assigning equal measures to each angle in the sketch. Thus, Camille's indicated she did not assimilate the multiple congruent angles at the onset of the 5-partitoned plane task and likely throughout prior TPP tasks in the session. Second, Camille assigned a measure of 180° to two of the angles in the 5-partitioned plane, just as Kacie had twice assigned 180° to the half-planes in a 4-partitioned plane. In her imitation, Camille indicated she had not assimilated the additive partitioned plane property that Kacie had introduced. Finally, Camille's designation of 180° to an acute angle underscored that Camille's right-angle template was not permanently assigned a measure of 90°, which in turn supported the hypothesis she had not yet constructed the TFDS.

Further contraindications of a full angle for Kacie followed by resolution. As I talked with Camille, Kacie calculated the sum of 5 eighties using pencil and paper. Kacie's use of numerical addition paralleled the equisegmenting operation she used when solving splitting tasks; she posited an estimate, 80, and repeated the estimate to see if the desired number of repetitions, 5, exhausted the whole, 360. After she found this sum to be different from 360, Kacie remarked, "nope, that's wrong," and I asked her to explain her reasoning to Camille.

Excerpt 5.13. Kacie's numerical association of 360 and a 4-partitioned plane. (Second Continuation)

…

K: Since we think that, um, all the angles equal eighty degrees, we could add all of them up and see if they make three hundred and sixty degrees.

T: And can you say one more time why they would need to equal three hundred and sixty degrees all together?

C: Isn't that as big as it goes? I mean like – it's like when they say do a [*spinning finger in the air*] um three hundred and sixty degree turn, I guess what you say. You know you do a full turn so you go back to where you started.

T: I see.

K: Yeah. Like I said, I don't know if this is with angles. It might be with circles or something [*looking at Camille*], but I don't know.

As Kacie explained her reasoning, I pressed her for why the sum of the angles should be 360°. Camille, and not Kacie, offered a response equating a three-hundred-sixty degree turn with a full rotation, though neither student explicitly made a connection to the 5-partitioned plane on the computer screen. As she had done at the onset of excerpt 5.13, Kacie expressed doubt about her method and questioned whether 360° was related to circles or angles. Kacie's distinction between circles and angles was in response to Camille's mention of rotational imagery. Thus, in Kacie's view, Camille's remarks about a full turn referred to circles and not angles. Kacie's comments here underscored the additive property she induced was indeed a property of the partitioned plane and not of a single angle. Therefore, I interpret this exchange as further confirmation Kacie did not assimilate the plane as the interior of a full angle in this context.

Following Excerpt 5.13, Kacie proceeded via computational trial and error to find a number that would satisfy the additive partitioned-plane property. Rather than use addition, Kacie began to use multiplication to search for a number so that five times that number was 360. Again, this trial-and-error calculational approach paralleled her previous equisegmenting activity. During this time, Camille was uncertain how to proceed and made little discernable progress on the task. After computing products using 75, 70, and 72 (see figure 5.81), Kacie remarked, "it's seventy two," when she found the product of 72 and 5 was 360.

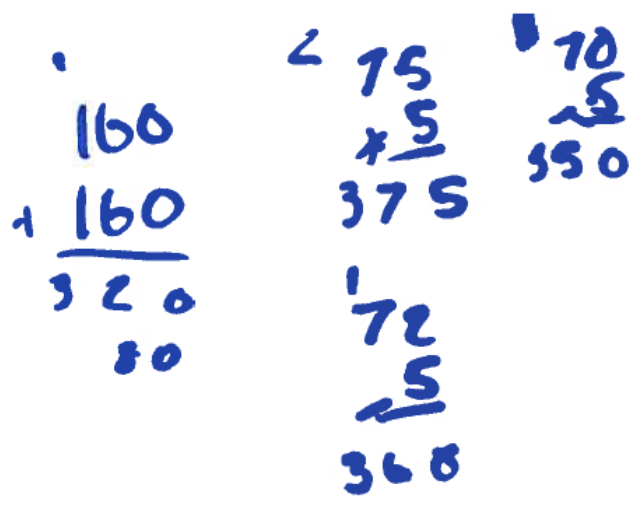


Figure 5.81. Kacie's computational search for a number so that five times the number is 360.

Determining degree measure when n = 8. I selected the 5-partitioned plane task to encourage students to develop a way of reasoning to leverage a 360-unit composite.

Kacie had managed to develop such a way of reasoning through assimilating a 5-partitoned plane and applying an induced property about the sum of the measures of this partitioned plane. In contrast, Camille did not assimilate a 5-partitioned plane and did not operate with a 360-unit composite.

Camille had previously relied on a right-angle referent, and to accommodate this previous way of operating, I asked the students to determine the measure of a single angle for an 8-partitioned plane. I hypothesized that Camille would assimilate a right angle by unitizing two adjacent angles in the 8-partitioned plane and then perhaps use the right angle in her operating. In particular, I wanted to investigate if Camille would assimilate a right angle partitioned into two congruent angular parts and conclude that each of these angular parts must have the same measure.

Kacie immediately noted the measure of one angle would be smaller than 72° since the angles were smaller than those in the previous task; I interpret Kacie's immediate recognition as confirmation of her (Templated) Figurative Degree Scheme. She began calculating on her paper starting with 50×8, which indicated that she had established a goal of determining whether 8 fifties would total 360 to satisfy the planar sum property she had previously induced. Camille agreed with Kacie that the angle was smaller than the previous case, but expressed frustration: "I don't know how to figure this out." Camille looked at Kacie's desk and, presumably after seeing Kacie's initial calculation, then suggested one of the angles would have a measure of 50°. Camille explained her reasoning as follows:

> Um, well, I was kind of looking at this cause this is a right angle [*traces finger the sides of the solid red right angle shown in Figure 5.82 below*]… And, um, well I thought about it, if it went like in the middle [*moves*

cursor over the angle bisector of the right angle] it would be I guess fifty… But ninety divided by two is not fifty.

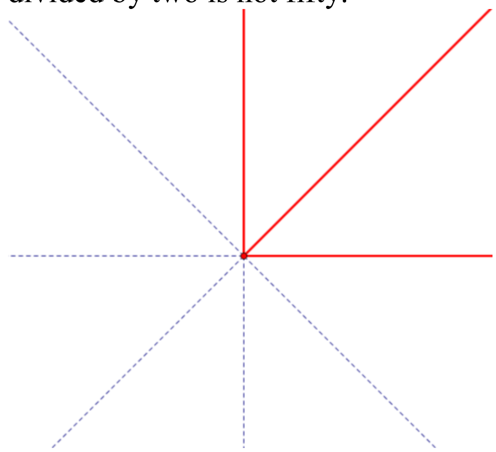


Figure 5.82. A model of Camille's assimilation of the 8-partitioned plane task.

Although Camille may have borrowed her initial estimate, 50°, from Kacie, she interpreted this estimate in terms of her assimilation of the task, which involved a 2-partitioned right angle. As in the previous task, Camille did not use the entire 8-partitioned plane in her reasoning. The dashed lines in Figure 5.81 indicate the segments in the 8-partitioned plane that were not salient in Camille's assimilation of the task. Instead, Camille assimilated a right angle and assigned the right angle a measure of 90°. Upon the recognition of a right angle segmented into two parts (see red segments in Figure 5.81 above), Camille recognized the situation as an occasion for division, and she used long division to compute 90÷2 = 45.[56] I infer that she divided the measure of the right angle by two because the interior of the angle was segmented into two congruent parts.

Kacie solved the task independently from Camille. When Kacie explained her solution, she noted she "did what [she] did last time," which indicated that she was

[56] Camille's numerical division was a critical moment in the teaching experiment *for Kacie*. Having observed Camille's computational strategy, Kacie later used numerical division to more efficiently determine the degree measure of angles in partitive division contexts like TPP.

establishing at least a procedural scheme for partitioned plane tasks.[57] As on the previous task, Kacie used multiplicative trial and error to determine the appropriate measure. Specifically, she multiplied 50, 48, and 45 by eight, checking each time to see if the product was 360.[58] Having determined that 45×8 = 360, Kacie achieved resolution in the task.

Figure 5.82. Kacie's trial and error computations.

The partitioned right angle sketch. Following TPP, I presented Camille and Kacie with a second GSP sketch, *The Partitioned Right Angle* (TPRA), which included a similar parameter, *n*, that the students could set to partition a given right angle into *n* equiangular parts (examples of TPRA for *n* = 1, 5, and 18 are shown in Figure 5.83). The right angle within the sketch was in a non-standard orientation (i.e., sides neither horizontal nor vertical) so I could investigate whether students' right-angle templates were orientation dependent.

[57] In a procedural scheme, a portion of the scheme's activity is enacted outside of the particular quantitative situation that activated the scheme (Les Steffe, personal communication, November 4th, 2017).
[58] Kacie did not carry out the entirety of the multiplication for 48×8 as she realized that the product of the unit's digits would not render a zero in the units digit of the product. Thus, her computational choices were informed by both her TFDS, which guided her initial estimate of the degree measure, and her extant schemes for whole number computation, which allowed her to strategically refine her estimate.

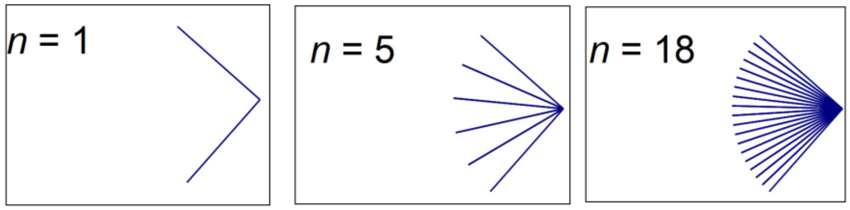


Figure 5.83. The Partitioned Right Angle with *n* = 1, 5, & 18.

When I presented TPRA with *n* = 1 to Kacie and Camille, I asked them how they would describe the angle on their screens. Simultaneously, both students acknowledged that the angle was a 90-degree angle, though Kacie verbalized her thoughts more confidently than did Camille. To familiarize them with the sketch, I asked the pair to try out different values of *n*. Kacie input *n* = 8 and reacted with surprise: "Whoa! It puts little angles in the big angle." Kacie's exclamation indicated she viewed TPRA as different from TPP, which further supported the conjecture she did not view the plane as the interior of a single angle in TPP. In TPRA, however, Kacie immediately assimilated the smaller angles as being contained the right angle. Although Kacie did not initially mention congruence or number of the smaller angles, she soon asked, "Is it like an eighth of the angle?" and counted the number of smaller angles while pointing to each interior. Because Kacie used fractional language, I infer she recognized the eight smaller angles were congruent, which I attribute to her existing scheme for angular congruence. Additionally, Kacie's spontaneous use of fraction language signaled she viewed the smaller angles as parts of an angular whole, which was fundamentally different from her assimilation of TPP wherein she did not view the smaller angular parts as constituting the plane as a single angular whole. Kacie's initial remarks regarding TPRA suggested she assimilated the 8-partitioned right angle as a two-level-of-units angular structure.

In contrast to Kacie, Camille said she didn't "know how to count" the smaller angles in the sketch. With *n* set to 8, Camille initially wanted to count the "lines that were added," by which she meant the six lines not constituting the sides of the right angle. However, she quickly and spontaneously stopped, noting these lines were not angles. Next, Camille counted interiors of the smaller angles while excluding the two outermost smaller angles and thus counted six smaller angles that were interior to the right angle (Figure 5.84 below). Camille was counting the angles that had been "added" within the right angle. I infer that Camille had already "used" the sides of the right angle, shown in red in figure 5.84, to constitute the right angle, and she seemed reluctant to re-use those sides to constitute smaller angles as well. Camille acknowledged that if she did "have to add those two" outermost smaller angles, then she would count eight in total. Although Camille assimilated smaller angles and a right angle, she did not indicate a two-level-of-unit assimilatory structure wherein the smaller angles immediately constituted the right angle without deliberate activity.

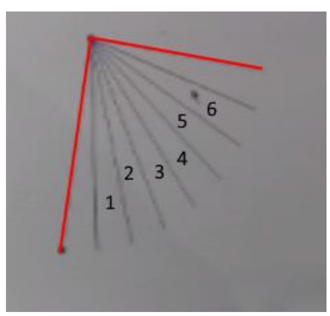


Figure 5.84. Camille counts small angles interior to the right angle.

Making one-degree angles in TPRA. Within TPRA, I asked Kacie and Camille to each set *n* so that one of the small angles would have a measure of one degree. Camille acted more quickly than Kacie and set *n* = 100, 50, 20, and 10. Camille appeared reliant on the visual feedback provided by the sketch as she set the parameter each time, as if trying to find the number that would produce an angle matching some mental image of a one-degree angle from a prior experience. Kacie on the other hand was slow to act in setting n =100 and seemed uncertain how to proceed.[59] She remarked, “I don’t really know what a one-degree angle looks like,” as if a re-presentable template for such an angle was essential for her to complete the task. From my perspective, both students were focused on primarily on the figurative aspects of a one-degree angle—what a one-degree angle would look like alone—rather than the operative—transforming the right angle into 90 one-degree angles through partitioning.

Because both students had named the right angle as a 90-degree angle, I called the students’ attention to the degree measure of right angle to see if that might engender different ways of reasoning. I asked the students, “how many one-degree angles do you think you could fit into a 90-degree angle?” Just before I asked the question, Kacie adjusted her parameter to n = 90. Following the question, Kacie immediately responded, “ninety.” Then, Camille adjusted her parameter to n = 9. I repeated the question again. Kacie again responded, “ninety.” Camille agreed with Kacie, but left her parameter set to n = 9. I validated Kacie’s verbal responses remarking, “you should be able to make 90 one-degree angles fit into one 90-degree angle.” Neither student made additional adjustments to her value of n, which was an indication each was satisfied with her parameter: Camille had set n = 9; Kacie had set n = 90.

Camille described her reasoning first and explained that n = 10 “was too small” (i.e., the angles produced were not open enough to be one-degree angles from her perspective). So, she had adjusted her parameter to n = 9, which she said, “looked kind of

[59] Both students initially set n = 100, which I conjecture was due to Kacie setting n = 100 in TPP to produce “one-degree” angles.

like a one-degree angle from the first time we did it." In her explanation, Camille confirmed that she was varying the parameter with the goal of producing a perceptual angle that matched visual records of some prior experience, though it was unclear to me the particular experience to which she was referring. Camille then referenced the number of one-degree angles contained in a 90-degree angle:

Excerpt 5.14. Nine 1-degree angles in one 90-degree angle.
C: And then what you just said the – How many does it take to fit and you said – how many did you say? Nine right? [*Looks to Kacie*]
K: Ninety
C: Yeah, Ninety. Well, I'm counting these as one, two, three, four [*points to the interior of each of the nine smallest angles*]. Yeah, there's nine here. And that would be
T: There's nine there. So if there's nine that fit into that, how many degrees is that in all? Like if you – how many degrees is like the biggest angle? Didn't we say that was a right angle?
C: Mm hmm. Ninety.
T: That's a ninety-degree angle. And if each one of those is the same size and there are nine of them, how many degrees would one of those be?
C: [3 sec] One [*hesitantly*]. I think. I don't know.

With $n = 9$, Camille considered there to be nine constituent angles within the sketch, which indicated at least a temporary modification in her reasoning as moments before when she was uncertain of how to count the angles for TPP with $n = 8$. Although she recognized the nine smallest angles, it is doubtful these nine angles also comprised the right angle for Camille, particularly considering her difficulty counting the smallest angles in the previous task. Camille's claim that each of these nine angles had a measure of one-degree contraindicated she multiplicatively structured the situation with three levels of units, which would require an individual's recognition of a right angle composed of nine angular parts each containing ten one-degree subparts. From my perspective, Camille's explanation indicated precisely nine 1-degree angles were contained in one 90-degree angle; however, Camille remained unperturbed, which indicated she did not view the nine constituent angles and 90 one-degree angles as simultaneously constituting the right angle.

The measures Camille associated with the right angle and the constituent angles were *independently* established from perception. Camille stopped adjusting the parameter when she set n = 9 because she perceived the small angles as one-degree angles. Camille also recognized the 90-degree angle from perception. Thus, I consider Camille's FDS to have been activated as she assigned measures to angles. These measures were reasonable in that the right angle was more open than one of the constituent angles and Camille assigned a greater degree measure to the former than the latter, which is consistent with the FDS.

Following Camille's explanation, Kacie confirmed she too was initially focused on figurative aspects of the sketch:

> I sent mine to a hundred at first, because that's what we did last time. And then I was like, wait, that looks super small. So I was like, uh, let's go ten down and try ninety. And I tried ninety and that's when I said, I don't know what a one-degree angle looks like. And you said, that there should be ninety – there should be ninety one-degree angles in a ninety-degree angle. So then I was like, oh okay, this must be right.

In her response, Kacie indicated that her initial parameter value of 100 was an imitation of her previous actions in TPP. Like Camille, Kacie decreased this initial value based on the visual feedback provided by the sketch; the angles *looked* too small to be one-degree angles with n =100, so Kacie decreased n to 90 to increase the size of the angles. Kacie wished for a re-presentable template for a one-degree angle before I called attention to the number of one-degree angles a ninety-degree might contain. However, Kacie assimilated the conversation regarding the number of one-degree angles contained in a 90-degree angle differently from Camille. Specifically, Kacie indicated she now assimilated TPRA with $n = 90$ as a two-level-of-units structure: the 90-angle contained ninety 1-degree angles.

Summary of Camille's and Kacie's February 22nd session. In this session, Camille and Kacie each made at least temporary modifications in their respective way of thinking. Camille indicated she was in the process of constructing a *Figurative Degree Scheme* (FDS), which entailed a coordination of her scheme for angular comparison and her existing number sequence. A critical component of the activity of the FDS is assigning numerical measures to particular angles such that the numerical measures and the opennesses of the corresponding angles are consistently ordered. The construction of the FDS implies an individual has imputed a basic additive structure to angles. That is, if an individual assigns an angle A to have a measure of m degrees and a less open angle B to have a measure of n degrees with $m < n$, implicitly the individual has conceived of at least an experiential difference in the angularity of angles A and B. Both Kacie and Camille indicated such an additive structure in this session when they argued an angle measured 80 degrees because it appeared to be about ten degrees less open than a right angle. Although angles and measures are ordered in the FDS, neither is multiplicatively structured and, in this sense, the angles are independent, as are the numerical measures. For example, Camille could take a right angle to have a measure of 90 degrees and produce a less open angle, which she would assign a smaller numerical value (e.g., 1-degree). In previous teaching sessions and in the present session, Camille indicated she had established a right-angle template; however, this template was not permanently assigned a measure of 90°.

In contrast to Camille, Kacie indicated she had constructed a Templated Figurative Degree Scheme (TFDS) wherein a right-angle was permanent referent for 90°. For Kacie, angles more open than her right-angle template necessarily had a degree measure greater than 90°; similarly, angles less open than her right-angle template necessarily had a degree measure less than 90°. Working in the context of TPP and TPRA, Kacie indicated she was in the process of establishing a new way of reasoning which involved structuring angles measured in degrees using composite units. Kacie induced the measures of the angles in a partitioned plane summed to 360. Thus, when subsequently solving TPP for particular values of *n*, Kacie searched for a number *d* such that 360 was the sum of *n d*'s. Kacie's use of repeated addition to check her estimates for the measures of a single angle in paralleled her implementation of the equisegmenting operation to solve spitting tasks in earlier sessions.

Camille's & Kacie's February 29th Session

At the end of the February 22nd teaching session, Camille had informed me that she would not be present during the February 29th teaching session due to another appointment. As a result, I had anticipated working exclusively with Kacie on February 29th and planned tasks that I thought were appropriate for her. In particular, I planned to revisit the notion of a laser sweeping through the interior of an angle, which I hypothesize might engender the construction of a plane as the interior of a single angle for Kacie. Additionally, I wanted to examine Kacie's units coordination in tasks that involved planar recursive partitioning from my perspective. Finally, I also wanted to examine whether Kacie's domain for angularity in rotational contexts included angles exceeding a full angle.

Camille's appointment was unexpectedly canceled, and she arrived at the teaching session with Kacie. Based on my current model of Camille's thinking, I hypothesized the tasks I had planned for Kacie would be outside of Camille's zone of potential construction. I had not observed Camille assimilate tasks with three levels of units as given, which many of these tasks required from my perspective. I proceeded with the tasks largely as planned but with a few minor variations to limit Camille from becoming entirely frustrated during the session. The pair shared a single computer during this session as I had anticipated working a single student.

Conjugate sweeps. To explore the students' conceptions of conjugate angles and angles that exceed a full angle, I designed a GSP sketch consisting of a laser and three buttons: Rotate1, Rotate2, and Reset. When a student clicked Rotate1, the laser rotated counterclockwise through a 144° angle. The reset button returned the laser to its original position. When a student clicked Rotate2, the laser rotated clockwise through a 216° angle. As both rotations had the same initial and terminal rays, the two rotations swept out the interiors of two conjugate angles.[60] In this task, I first asked students to enact the first rotation, draw a picture to represent this rotation, enact the second rotation, and draw a picture to represent the second rotation on the same sheet of paper.

When I asked the students to click Rotate1 and draw the rotation, the students enacted the rotation using GSP several times and each produced a drawing (Figure 5.85). Kacie used segments to represent the initial and terminal sides of the rotation. Camille used a single segment to represent the initial side and a curved path to represent the rotation.

[60] When I say the interiors of two conjugate angles here, I am referring to the two sets of points (one concave and the other convex) into which the plane is divided by the initial and terminal position of the ray.

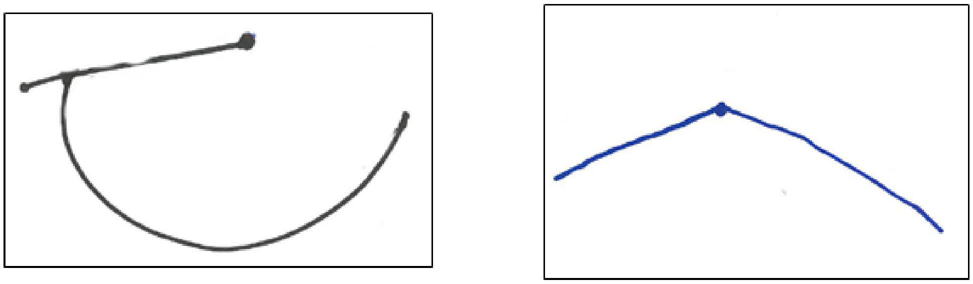


Figure 5.85. Camille's (Left) and Kacie's (Right) drawings for Rotate1.

After drawing Rotate1, I asked students to draw a picture to represent the rotation shown when they clicked Rotate2 and, if possible, to use the picture they already had drawn. After enacting the second rotation in GSP, Camille recognized the initial position of the laser was the same for each rotation and drew a second curved path clockwise from her initial segment (Figure 5.86). Camille also annotated her drawing to indicate the beginning and end of each rotation. Camille denoted two distinct ending points for the rotations, which suggested she did not recognize the terminal rays for both rotations were coincident.

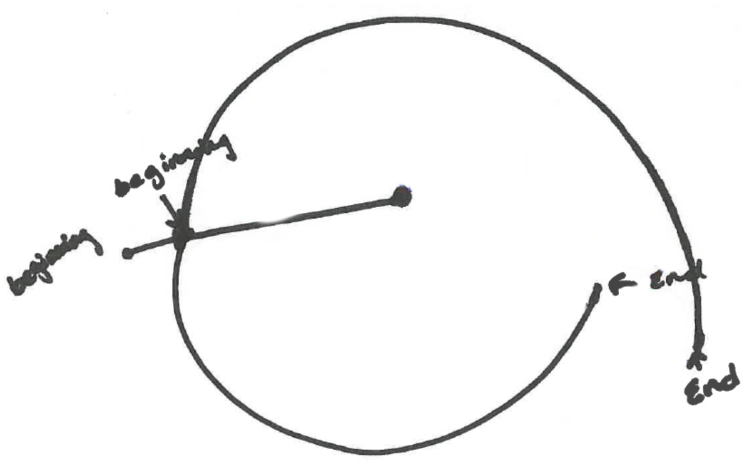


Figure 5.86. Camille's drawing of both two rotations.

While Camille produced a different drawing for the second rotation, Kacie sat quietly without changing her drawing. When I asked her to explain her thinking, Kacie responded:

> They both start at the same place, so the starting point would be the same…they look like they stop in the same place too. So, I didn't really change anything to my drawing.

Kacie's explanation indicated she viewed the ray's terminal positions (as well as the initial positions) as the same for both rotations. After hearing Kacie's explanation, Camille acknowledged she perceived the rotations ending in different locations, and after re-enacting the rotations, she acknowledged the rotations ended in the same place. I asked the students to imagine other ways to rotate the laser that would still involve the same terminal and initial positions shown in their drawings. My intent was to examine if either student would consider angles in excess of a full angle; for example, rotating the ray through a full rotation, continuing the rotation further, and stopping final at the terminal position.

Kacie explained, "You could just really pick any angle…and rotate it one way and have an ending point and rotate the other way and make sure it just ends in the same spot." Kacie was adjusting the terminal and initial positions of the ray and then re-enacting the two rotations, which indicated she had constructed a general notion of conjugate angles. In particular, she effectively asserted that every angle has a conjugate. Neither Kacie nor Camille voiced any other possibilities for rotations containing the same starting and ending point. Therefore, neither student indicated considering rotations exceeding a full rotation.

Two lasered angles. To examine compositions of the students' extensive angular operations, I presented students with a new GSP sketch: Two Lasered Angles (TLA). TLA contained two angles, each with a laser positioned at the vertex. For each angle a "Sweep Beam" button, when clicked, rotated the laser counterclockwise from one side of the angle to the other while leaving behind a trace of the beam (see Figure 5.87). If a student clicked the "Sweep Beam" button *n* times, *n* sweeps of the same magnitude would be enacted in succession, leaving behind a trace *n* times as large as the original trace. To distinguish between the angles, one sweep appeared in red and the other sweep appeared in blue. A "Reset" button was available to reset the rotation and clear the traces. As in a previous sketch, five draggable points (yellow, red, pink, blue, and green) could be dragged to vary particular attributes of the angle (position, amount of sweep, orientation, length of one side, and length of the other side).

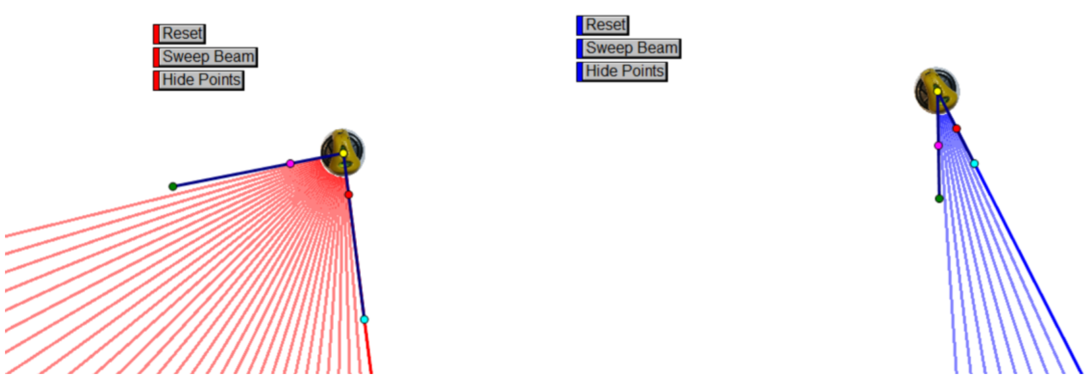


Figure 5.87. The Two Lasered Angles GSP sketch.

Planar recursive partitioning tasks. A planar recursive partitioning task involves two angles, *A* and *B*, such that *n* sweeps of A exhausts the entire plane and *m* sweeps of *B* exhausts *A*. I denote such a task by PRP(*n*, *m*). The goal of the task is to determine the number of sweeps of *B* required to exhaust the entire plane. An additional goal that can be established is to determine the measure of angles A and B in degrees (or any other unit of angular measure). I designed these tasks to examine students' units coordinating operations in angular contexts. Immediate recognition that $n \times m$ sweeps of angle B will be required to sweep out the plane is an indication that the individual has assimilated the task with three levels of units. An example of the first such task, PRP(4, 3), that I posed to the pair follows.[61]

> Set the red angle so that four sweeps of the red angle sweep out the entire page. Set the blue angle so that three sweeps of the blue angle sweep out one red angle. How many blue angles would it take to sweep out the entire page? What are the degree measures of the red and blue angles?

Producing the red and blue angles. Because Camille had not previously assimilated tasks with three levels of units, I modified the implementation of the PRP(4, 3) task by first asking Camille to produce the red angle, and then asking Kacie to produce the blue angle. Thus, Camille effectively was tasked with an m = 4 plane splitting task, and Kacie was tasked with an n = 3 angular splitting task.

Camille initially opened the red angle so that it was four times as open as its original configuration. Camille's actions indicated she had not established a goal of covering the plane with four adjacent copies of the red angle. Instead, she assimilated the red angle to her angular iteration scheme. To help Camille form the goal I had intended, I asked Kacie to demonstrate setting the blue angle so that six sweeps of the blue angle would sweep out the entire page. Kacie made an estimate using her existing partitioned plane templates. She first adjusted the blue angle to a straight angle and then to a right angle, which she referred to as "like half" and "a fourth," respectively. Kacie continued,

[61] As I describe in the subsequent sections, I presented this task in sequential parts to the students; this action was intentional on my part to support Camille.

“and then I know that a sixth like is smaller than a fourth so I knew that it had to be less open than the four, the fourth angle,” as she made the angle slightly less open.

Kacie’s use of fraction language as she set the blue angle to straight and right angles confirmed that she had established 2- and 4-partitioned plane templates. These were interiorized templates which she knew produced planes partitioned into specified numerosities. Kacie did not have to take the plane as input for partitioning operations when she re-presented these templates. In setting the blue angle to “a sixth,” Kacie leveraged properties of order but did not appear to simultaneously project six equiangular parts into the situation. As in previous tasks involving angular partitioning, I consider Kacie’s sequential activities (i.e., making and checking an estimate without evidence of projecting units) to be consistent with the equisegmenting operation.[62]

Kacie checked her result by sweeping out six copies of the blue angle and, noticing that a portion of the plane remained uncovered after four sweeps, deemed her estimate not open enough. Kacie adjusted her estimate by making it more open, but again gave no indication of partitioning. Six sweeps of Kacie’s second estimate more than exhausted the plane, and she concluded her second estimate was too open.

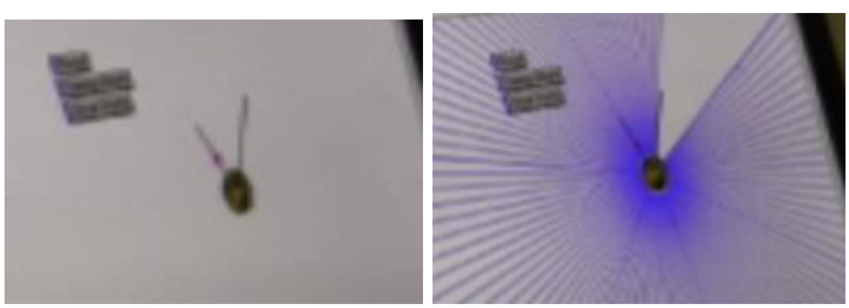


Figure 5.88. Kacie sets (left) and checks (right) her first estimate for the 6-split plane.

[62] The sequential element of the enacted rotation may have engendered the equisegmenting operation; still, an individual could solve this task via partitioning if she had interiorized a full rotation.

Following Kacie's demonstration, I asked Camille to attempt the 4-split of the plane again. Camille's first estimate (Figure 5.89 left) exceeded a right angle, and, after sweeping out four copies of the angle, she recognized her original estimate was too open. Camille adjusted her estimate by making the red angle less open (Figure 5.89 middle). After sweeping out her second estimate four times, Camille recognized this estimate was too small. Camille's third estimate (Figure 5.89 right) was nearly a right angle, and four sweeps of the angle nearly exhausted the plane.

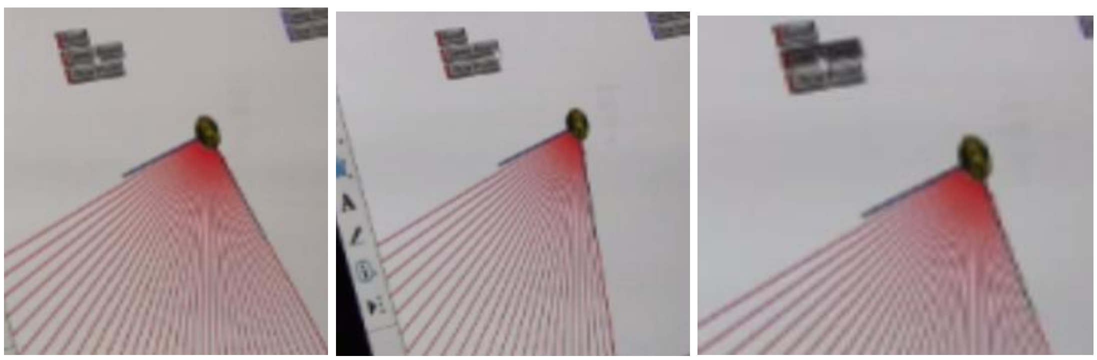


Figure 5.89. Camille's first (left), second (center), and third (right) estimates for a 4-split of the plane.

Camille's actions indicated she still had not established a re-presentable template for a 4-partitioned plane. Like Kacie, Camille was engaged in segmenting activity and judged the appropriateness of her estimate based on the visual feedback provided after instantiating all four sweeps of each estimate.

Following Camille's actions, I asked Kacie to set the blue angle so that three sweeps of the blue angle would sweep out one red angle. After setting the blue angle, Kacie spontaneously superimposed the blue angle atop the red angle to check the accuracy of her result. With the angles superimposed and before sweeping out the blue angle, Kacie noticed that only two sweeps of the blue angle would be needed to exhaust the red angle (Figure 5.90).

Figure 5.90. Kacie superimposes the blue angle atop the red angle to check her first estimate.

Kacie adjusted the blue angle to be less open (Figure 5.91 left) and enacted three sweeps of the blue angle (Figure 5.91 right). Kacie revised her estimate again and three sweeps of this final estimate nearly exhausted the red angle perfectly.

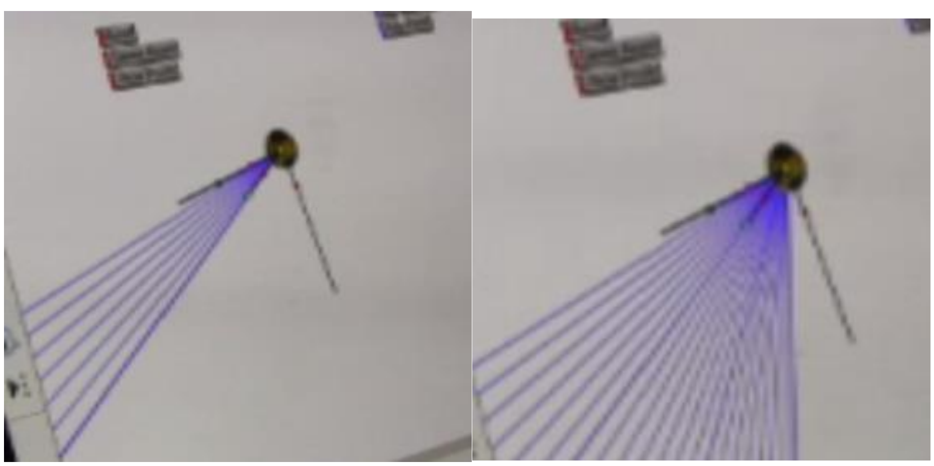


Figure 5.91. Kacie’s second estimate for $n = 3$ angular splitting task.

In this angular splitting task and as in other splitting tasks, Kacie did not give an indication she was simultaneously projecting equiangular parts into the given angle prior to producing an estimate for the desired angle. The inaccuracy of her first two estimates also supports the hypothesis Kacie did not project three equiangular units into the red angle prior to setting her estimate. Kacie's actions in this task were again consistent with the equisegmenting operation.

Determining the total number of blue angles in PRP(4, 3). With three sweeps of the blue angle perceptually embedded within the red angle, I asked Kacie and Camille how many blue angles would be needed to sweep out the entire page. Camille sat quietly in thought for ten seconds before remarking "this is hard to figure out." After another twenty seconds, Camille remarked, "I want to say nine." Kacie quickly answered twelve after Camille responded. In this interaction, I inferred Kacie was able to solve the task quite quickly because she had previously exhibited no difficulty coordinating the number of units within units within a unit. I interpreted Kacie's patience as courtesy to her partner—she anticipated the problem was more challenging for Camille than it was for herself. Camille explained her reasoning as described in Excerpt 5.15 below.

Excerpt 5.15. Camille produces three levels of angular units in activity.

C: Well, I was going to double it and do six, but that would only go half way. And if it's half way then it would just need [2 sec] – oh it might be twelve.

T: Tell me what you were thinking then.

C: Cause, I thought, if it was six it would probably end like right here [*motions as shown in red in Figure 5.92*]. And for some reason I was thinking once I get to three more it would go the whole way but – but if you do three more – no if I do it – if I go to nine I'll probably be like right here [*motions with cursor as shown in red in Figure 5.93.*]

…

C: And then I thought about what she said and it'll go – if I put three more, which would be twelve, it'll do the whole thing.

T: So how do you know that that's going to do the whole thing then?

C: I – I just figured. I mean I was thinking cause – cause I said three plus three would be six and that would be halfway so – or three times four which is what it took cause mine was four and that takes three to do, I guess, as much as mine and that's twelve.

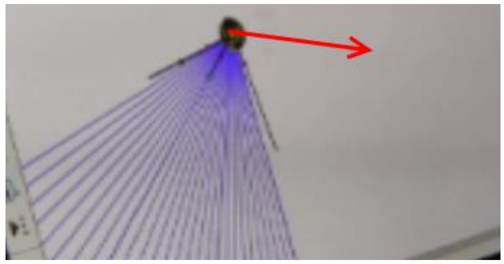


Figure 5.92. Camille motions where six sweeps of the blue angle would end.

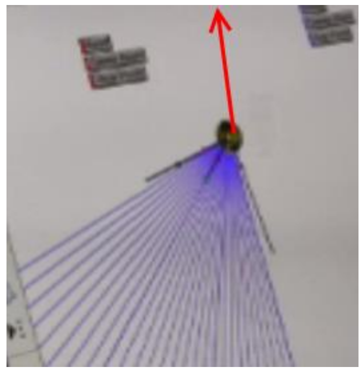


Figure 5.93. Camille motions where nine sweeps of the blue angle would end.

Camille's initial response, "nine," contraindicated she had assimilated the task with three levels of units. As she described her initial way of reasoning, Camille indicated she had taken the red angle as a composite unit of three and was iterating the red angle while tracking the number of blue angles contained in each iteration. After producing two iterations of the red angle, Camille lost track of the units she was coordinating. She exhausted the half plane with two sweeps of the red angle and knew she had one half-plane remaining uncovered. Camille then imagined sweeping out the half plane, but counted this as a single red angle (and three blue angles), which explains her initial response of nine.

As she verbalized her thinking, Camille produced three levels of angular units in activity. Using the cursor to mark each adjacent copy of the red angle, she counted by threes to track the number of blue angles. In fact, this task marked the first time Camille took a composite unit as an iterating unit in the teaching experiment. Though Camille did not explicitly double count as she imagined each copy of the red angle, her cursor motions combined wither her assertion, “mine was four and that takes three to do…as much as mine,” indicate that Camille was counting four instantiations of three. When she had finished counting, Camille described accounting for four groups of three: “mine was four and that takes three to do I guess as much as mine and that’s twelve.” Additionally, she verbally described this as multiplication (i.e., three times four). However, these four groups of three were only available to her after she sequentially inserted the red angles while tracking the blue angles.

Following Camille’s explanation, Kacie succinctly described her reasoning: “since there’s three in her angle and it took four of her angle to get around, I just multiplied three by four and got twelve.” Kacie’s explanation supported the hypothesis she assimilated this task using three levels of units. She did not appear to require any sensorimotor activity (e.g., gesturing like Camille) to coordinate these units. Kacie had already constructed the red angle as a composite unit containing three blue angles through her equisegmenting operation. In determining the number of blue angles required to sweep out the plane, Kacie iterated the red angle as a composite unit to fill each of the four right angles in her 4-partitioned plane template. Kacie’s language, “it took four of her angle *to get around,*” suggested that she viewed resultant iteration as single bounded motion, which indicated she unitized the full rotation (i.e., she created a full sweep of the plane containing four red sweeps each containing three blue sweeps). Kacie’s response, twelve, indicates she viewed these twelve sweeps as four sweeps of three smaller sweeps.

Determining degree measures for PRP(4,3). After the pair had determined twelve blue angles would be needed to sweep out the entire plane, I asked Kacie and Camille to determine the degree measures of the blue and red angles, respectively. I asked Camille to sweep out the red angle and then Kacie suggested separating the angles (See Figure 5.94).[63] The subsequent discussion of the degree measures of these angles is presented in Excerpt 5.16 below.

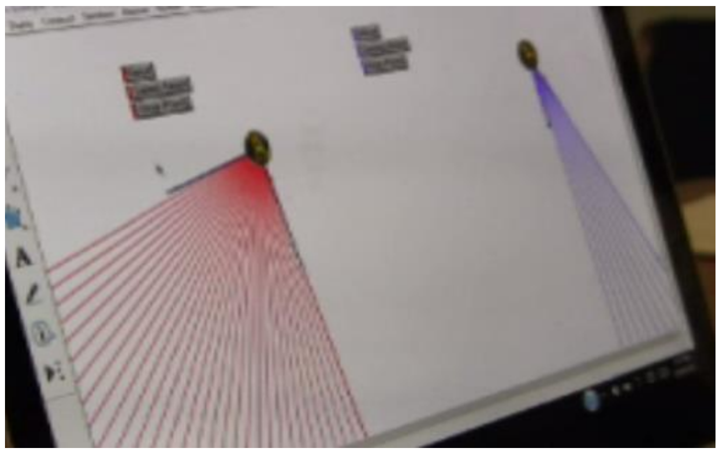


Figure 5.94. Kacie separates the angles

Excerpt 5.16. Kacie and Camille determine degree measures in PRP(4,3). (Onset)
K: So we're finding the degree of each angle?
T: Yeah. I'm wondering if each of you could find what the measure would be in degrees.
K: Mine's thirty.
C: I was going to say the same for mine, but that's wrong.
T: Why do you think that's wrong?
C: Well, I was going to make each one of my like [pointing to the blue angle] – I was going to grab hers – well, I wasn't going to grab it, but I was imagining [laughs]
K: [Laughs]
T: No that's good. Tell me what you were imagining.
C: I was imagining hers on mine and saying it would take three of hers.
T: Yeah. That's right.
C: Then I will say thirty.
T: I see. Interesting. So how big were you imagining hers to be?
C: What do you mean? Ten.
T: You were imagining hers to be ten?

[63] Due to technology issues, it took the pair 75 seconds to separate and sweep out both angles.

C: Yeah.

Each student arrived at a measure of 30 degrees for her angle. For Kacie, 30 degrees referred to the measure of the blue angle. For Camille, the 30° referred to the measure of the red angle. Camille assimilated the task to her figurative degree scheme in that she *assigned* a measure of ten degrees to the blue angle. For Camille, the blue angle was a ten-degree angle because it looked like one. In solving the task, Camille made a functional accommodation in her figurative degree scheme. Rather than assigning a measure to both the blue and red angles, Camille used the measure of the blue angle to determine the measure of the red angle, which indicated Camille was iterating the blue angle. Camille held in mind that three copies of the blue angle were contained in the red angle and reasoned the measure of the blue angle must be three times as large as the red. In producing this result, Camille produced three levels of units in activity in that she viewed thirty as composed of three units of ten. However, I do not claim that Camille was aware of 10 one-degree angles inside the blue angle or 30 one-degree angles inside the red angle. Instead, these numerical values, 10 and 30, were linked to the perceptually available blue and red angles.

In the continuation of Excerpt 5.16 below, Kacie explains how she determined the measure of the angles in PRP(4,3).

Excerpt 5.16. Kacie and Camille determine degree measures in PRP(4,3). (Continuation)

T: So, Kacie, tell me how you were imagining yours.

K: Well, since hers went four times around, um, I know that a right angle is ninety degrees and it takes four right angles to go all the way around.

T: How do you know that?

K: Um, it's just kind of like common knowledge I guess.

T: Yeah? Really?

K: I guess. And so I knew that mine went three times into her and thirty plus thirty is sixty, and then sixty plus thirty is ninety. So I know that mine is thirty degrees.

…

T: So tell me again how you knew Camille's angle would have a measure of like ninety degrees.

K: Cause it went around four times and ninety – or four times ninety is – well, okay – yeah. Ninety times four or forty times ninety, whatever, is three hundred and sixty. And that's what it takes to go all the way around [gestures a circle with index finger in the air].

…

T: [To Camille] Do you think that yours is actually a right angle?

C: [tilts head sideways looking at the computer screen] Hmm. [3 sec] I mean maybe if it went a little bit more up [gesturing with the cursor as if to pull one side to the horizontal]. Like this one (inaudible). I mean I think it is [tilting head sideways again] actually. Yeah. I think it, yeah.

The continuation of Excerpt 5.16 above evidences a significant modification of Kacie's Templated Figurative Degree Scheme (TFDS). Kacie immediately recognized the red angle as a right angle due to her template for a 4-partitioned plane. Kacie named a right angle as a 90-degree angle, which again indicated Kacie's right angle template was permanently assigned a measure of 90 degrees. Had Kacie only activated her TFDS, she would have simply assigned some measure less than 90 degrees to the blue angle; Kacie did more than this by leveraging her equisegmenting operation. Since Kacie knew three copies of the blue angle were contained in the red angle, she searched for a number *d* so that three *d*'s added together would be ninety. Kacie's additive numerical reasoning here—thirty plus thirty plus thirty—parallels the structure of the equisegmenting activity she demonstrated in prior angular splitting tasks. The shift in Kacie's reasoning here is significant in that her equisegmenting of the red right angle allowed her to operatively determine the measure of the blue angle. Therefore, Kacie's combination of the equisegmenting operation and the TFDS constituted a new scheme for measuring angularity in degrees—the Equisegmenting Degree Scheme (EDS).

In the continuation of Excerpt 5.16, Kacie explicitly and quantitively justified why a full angle has a measure of 360°, which differed from her previously induced property of a partitioned plane. Kacie viewed an entire sweep of the plane as being composed of four right angle sweeps. In this rotational context, Kacie had established quantitatively structured templates for full and right angles, which she had permanently assigned measure of 360° and 90°, respectively. Because she permanently assigned a right angle a measure of 90 degrees, Kacie was able to quantitatively justify why there were 360° in a full angle: "cause it went around four times and ninety … is three hundred and sixty."

The continuation of Excerpt 5.16 illustrates Camille's template for a right angle was orientation dependent. In previous teaching sessions, Camille indicated she had established a right-angle template; however, Camille did not immediately recognize the red angle as a right angle, which indicated the perceptually available red angle had not activated Camille's right-angle template. Ultimately, Camille's recognition of the red angle as a right angle was occasioned by Camille turning and tilting her head, which altered the orientation of the red angle within her visual field. As such, I infer the orientation of Camille's right angle template was fixed in a standard position (i.e., with one side horizontal extending right and the other vertical extending upward).

TPRA with n = 10. As a closing task in their February 29th session, I asked Kacie and Camille to determine the degree measure of each angle in a 10-partitioned plane. The pair sat in silence for seventeen seconds before Camille remarked, "I don't know; I'm so bad at angles." That Camille's intellectual frustration peaked at this point in the teaching experiment was understandable. Throughout the teaching experiment, Camille had not assimilated tasks with three levels of units.

For Kacie, this task activated her Equisegmenting Degree Scheme. She posited each angle had a measure of 10 degrees, which was likely due to the value of the parameter (n = 10). After computing that the sum of the measures in the partitioned plane would be 100 rather than 360, Kacie then mentally computed 360 divided by 10 arriving at 36. Kacie's use of division in this context was a numerical shortcut of sorts in that she did not provide a quantitative interpretation for her division. Additionally, Kacie initially doubted the measure was appropriate and explained she thought the angle was not open enough to measure 36°. Kacie then checked her result via multiplication and addition, adding five 36s to get 180 and then doubling 180. Kacie's reliance on addition as a surefire method of verification evidences her reliance on the equisegmenting operation.

Summary of Part 2

In the teaching sessions described in Part 2, both Kacie and Camille developed schemes for angular iteration and formed composite angular units. Neither student initially associated rotational angle contexts (e.g., rotating laser) with non-rotational angle contexts (e.g., chopsticks). Neither student initially indicated operative conceptions of angles measured in degrees. When discussing the attribute described by the measure of an angle, both students initially mentioned lengths.

Kacie indicated she could assimilate non-rotational angle models as rotational angle models and vice versa. In non-rotational contexts, Kacie's domain for angularity included convex angles, along with closed and straight angles; in these contexts, Kacie interpreted openness in terms of the least amount of motion required to close an angular configuration. In other words, Kacie took the nearest closed configuration as a referent for the openness of a given angle model and therefore did not consider concave angles in non-rotational contexts. In rotational contexts, Kacie's domain for angularity extended to include concave angles (i.e., reflex angles), and she indicated an awareness of conjugate angles.

Throughout Part 2, Kacie did not indicate the splitting operation and instead relied on equisegmenting, which is the sequential production of parts intended to exhaust a whole. By equisegmenting the plane using a composite unit, Kacie produced planar coverings as three-level-of-unit structures. Kacie indicated she had constructed re-presentable templates for a 2- and 4-partitioned planes as well as a right angle. Kacie indicated she had permanently assigned her right-angle template a measure of 90°. In part 2, Kacie indicated she had constructed a templated figurative degree scheme (FDS) whereby she coordinated her number sequence in an ordinal sense with relative extents of angularity. Because Kacie had permanently assigned her right-angle template a measure of 90°, Kacie constructed a templated figurative degree scheme (TFDS). Due to the construction of the TFDS, angles less open than a right angle always measured less than 90°, and angles more open than a right angle always measured more than 90°. Not long after constructing the TFDS, Kacie applied her equisegmenting operation to her degree-designated angles, which allowed her to determine the measures of angles in an n-partitioned plane or m-partitioned right angle as $360 \div n$ or $90 \div m$, respectively. Kacie's incorporation of the equisegmenting operation into her TFDS constituted a new scheme for measuring angles in degrees, the equisegmenting degree scheme (EDS).

Like Kacie, Camille also viewed non-rotational angle contexts stated in terms of openness as being restricted to convex angles. For Camille, degrees described several different attributes including orientation, openness, and side length. Camille indicated the construction of a figurative degree scheme, though degrees were associated with both angularity and orientation for Camille from my perspective. To be clear, it was only upon the introduction of degrees that orientation became salient for Camille in the teaching experiment; when discussing angle contexts in terms of openness, Camille regularly reoriented angle models without changing the openness. Camille indicated she had formed a re-presentable template for a right angle, but this template was not permanently assigned a measure of 90°. Camille did not assimilate tasks with three levels of units, but did produce three levels of angular units in activity in her February 29^{th} paired session with Kacie.

By the end of the pair's February 29^{th} session, Camille was noticeably frustrated. From my perspective, the differences in Camille's and Kacie's ways of operating had grown to a point where it was no longer appropriate to keep the pair together. As such, for the remainder of the teaching experiment, Camille and Kacie participated in separate sessions so that I could work to engender the greatest possible progress for each student.

Part 3: Camille's Last Four Sessions

Camille's final four sessions in the teaching experiment consisted of three solo teaching sessions and one individual final interview session. Camille experienced two persistent challenges throughout these sessions. First, Camille could assimilate tasks with two, but not three, levels of units. Second, Camille's template for a right angle and her scheme for understanding degrees were orientation dependent. My primary goals in the remaining teaching sessions were to (a) engender Camille's disentangling of orientation and angularity measured in degrees (b) foster Camille's establishment of a template for a 4-partitioned plane and (c) for Camille to construct a right-angle as a 90-unit angular composite.

Camille's March 14th Session

Camille's first solo teaching session after her separation from Kacie took place two weeks after her previous teaching session. In this session, Camille worked on angular units coordinating tasks involving lasered angles in GSP. Additionally, I investigated Camille's conception of a right angle and posed tasks for Camille in The Partitioned Right Angle (TPRA) GSP sketch.

Contraindications of iterating composite units. In Camille's last teaching session with Kacie two weeks prior to the present session, I inferred Camille engaged in taking a composite angular unit as input for iterating for the first time in the teaching experiment. To further investigate whether Camille could produce composite angular units in activity, I asked Camille to first set a blue angle so that two sweeps of the blue angle would sweep out a red angle. [64] After several attempts and some issues with the technology, Camille set the blue angle so that two sweeps of the blue angle exhausted the red angle (Figure 5.95).

[64] The red angle was set to a right angle, though Camille gave no indication of recognizing it as such.

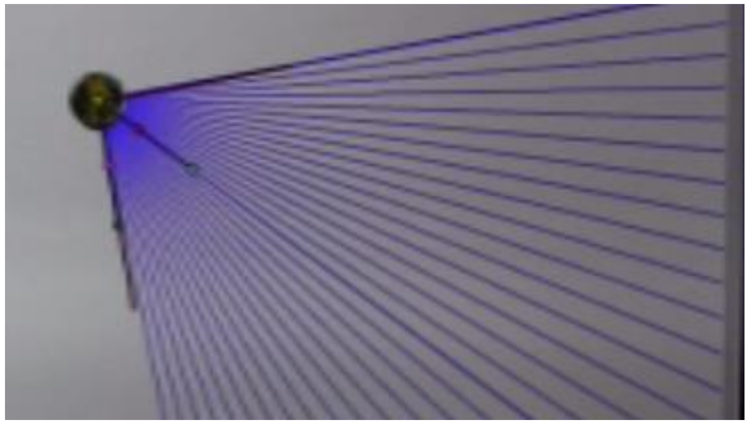


Figure 5.95. Two sweeps of the blue angle contained in the red angle.

To investigate whether Camille could operate with composite units, I instructed Camille to imagine sweeping out the red angle three times and asked her how many blue sweeps that would be the same as. Camille responded, “probably six.” In her explanation of this result, Camille motioned segments from the vertex as shown in Figure 5.96 as she counted from one to six, which indicated she was mentally uniting copies of the blue angle, though some of these instantiations were noticeably larger in magnitude than others from my perspective. When I asked Camille how she knew to stop at six, Camille indicated she was stopping where the first sweep of blue angle started. Thus, Camille had established a goal of covering the plane with copies of the blue angle and was not instantiating three sweeps of the red angle.

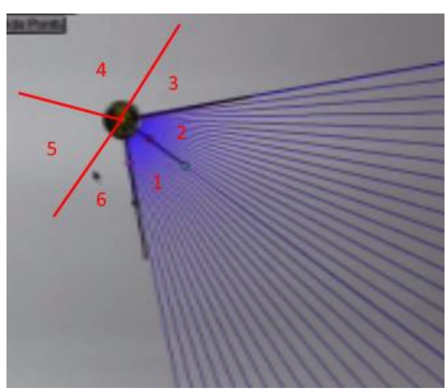


Figure 5.96. Camille mentally unites copies of the blue angle.

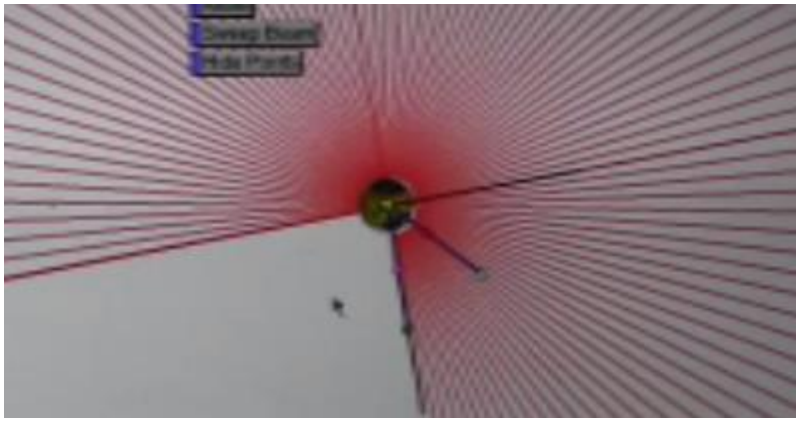


Figure 5.97. Three sweeps of a red (right) angle.

Because Camille established a goal enumerating the number of blue angles needed to cover the plane, I asked Camille to enact three sweeps of the red angle in GSP (Figure 5.97). With three sweeps of the red angle displayed, I asked Camille how many blue angles she would need to sweep out the same amount shown in red. Rotating the blue angle into each copy of the red angle, Camille counted from one to six in pairs (i.e., "one, two; three, four; five, six). However, Camille wasn't finished, and she moved the blue angle into the white quadrant which had not been covered by three sweeps of the red angle as she finished counting, "probably eight." Despite the perceptually available red trace, Camille maintained a goal of enumerating the number of blue angles required to cover the plane. I pressed Camille again, "how many would you need to sweep out just the red amount?" Camille replied, "five or four."

Camille's activities on this task were a solid indication she had not assimilated the task with three levels of units. After setting the blue angle so that two sweeps of the blue angle swept the red angle, Camille did not immediately recognize three sweeps of the red angle as six sweeps of the blue angle. Because Camille actively inserted two blue angles into each of the three red angles (and the white quadrant), I infer the repetitions of the red angle were not immediately composite units of two for Camille. Camille's response of eight indicated she had produced a three-level-of-unit structure in activity involving the plane as a unit of four red angles each containing two blue angles. That Camille replied, "five or four," when I asked her again to enumerate the number of blue angles required to exhaust three sweeps of the red angle indicated the previous insertion of the blue angles had atrophied for Camille.

Contraindication of a 90-degree designated right angle. In the previous task, Camille never identified the red angle as a right angle or a 90-degree angle. In a follow-up task, I asked Camille to set the blue angle so that three sweeps of the blue would be the same as one sweep of the red, which was still set to a right angle from my perspective. After a reasonable first estimate for the blue angle and two rounds of appropriate adjustments, Camille set the blue angle so that three sweeps nearly perfectly exhausted the red angle (Figure 5.98).

Figure 5.98. Camille sets a blue angle so three blues is congruent to one red.

I asked Camille to consider the measures of the blue and red angles. Camille was unsure about the measures of both angles. Camille's verbalization of her thinking about the measures of these angles is described in Excerpt 5.17 below.

Excerpt 5.17. Camille considers the measures of the blue and red angles.

T: How many degrees do you think that blue angle is if you measured it in degrees?

C: Mm. I don't know. Mm mm. I have trouble with that.

T: Oh okay. What about the red angle? How many degrees do you think that red angle is?

C: Probably ninety – no. I want to say ninety but I'm probably wrong

T: Tell me why you want to say ninety?

C: Cause I forgot what a ninety-degree angle looks like, but I know it's something like that. Yeah, cause a one eighty would probably be half, I think, since the whole thing is three sixty [Motions a circle with the cursor around the screen.]

… [Camille computes 180 + 180 = 360 using a vertical addition on paper] …

C: Yeah. I'm guessing half of the three sixty would be one eighty. So I think the red one is a one eighty.

T: You think the red one is a one eighty?

C: Yeah. If it were to be half. Well, yeah – if this were to be a little [motions as if to close the red angle until it is twice as big as the blue angle]

T: Okay. So how much would the blue be then?

C: Mm. Ninety I think. Well if it were a little bit more in the middle. So, probably eighty degrees.

As described in Excerpt 5.17 above, Camille was debating between a measure of 90° or 180° to the red angle and ultimately decided on 180°. As in previous teaching sessions, Camille conceived a full rotation as 360° angle. Camille reasoned that half of the full angle would measure 180°. In trying to relate the 180° to the image available on her screen, I infer Camille became fixated on the blue angle, which divided the red angle into two incongruent angular parts. I interpret Camille's motion to close the red angle as an indication Camille was shifted from imagining half of a full angle to half of the red angle. Camille's decision the red angle measured 180° was a clear contraindication of a 90-degree designated right angle template. Furthermore, Camille's assignment of 180° to the full angle contraindicated Camille had remained aware four instantiations of the red angle exhausted the plane, which is consistent with the hypothesis she had yet to form a template for a 4-partioned plane.

After designating the measure of the red angle as 180°, Camille implemented her figurative degree scheme to assign the blue angle a lesser measure, namely 80°. With my prompting, Camille later recalled three sweeps of the blue angle would be required to sweep out one red angle. She then decided to compute 180 ÷ 3 using long division and asserted the angle would measure 60°. When I asked Camille if her new angle measure made sense, Camille explained, "yeah, cause sixty times three would be one eighty." Camille's numerical computations and her subsequent explanation suggested she viewed the result as appropriate because three blue angles exhausted one red angle just like three sixties exhausted 180. However, I do not claim that Camille was aware of 60 unit angles inside the blue angle nor 180 unit angles inside the red angle.

Return to TPRA. Following Camille's activities on the previous task, I asked Camille to draw a right angle and asked her what the measure of the angle would be in degrees. Camille produced a right angle in standard orientation and again seemed to express uncertainty over whether the angle measured 180° or 90°. I reminded Camille in her previous teaching session with Kacie we discussed that a right angle had a measure of 90°. I instructed Camille to open the TPRA GSP sketch and asked Camille whether the angle on the screen looked like a right angle to her (see Figure 5.99).

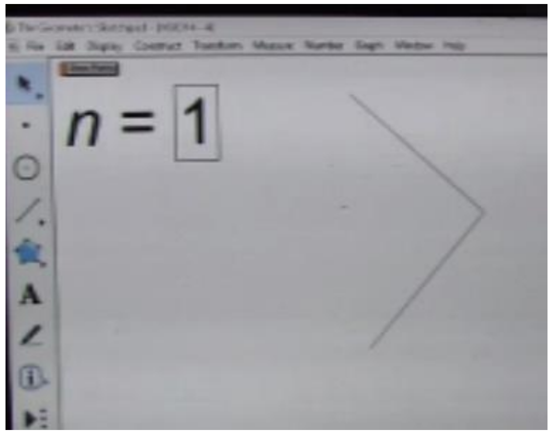


Figure 5.99. TPRA (Non-standard orientation)

Camille immediately tilted her head to the side and indicated she was troubled by the angle on the computer screen. Camille explained the angle was “kind of like an acute angle” and drew an acute angle with vertex pointing to her right on the paper in front of her. Camille explained that acute angles “were small” and that the angle on the screen was “not as open as a right angle.” I challenged Camille hold the paper on which she had previously drawn a right angle to the screen and prove the angle shown was not as open as a right angle. Camille tilted her head to the side again and explained the angle on the computer screen could be a right angle if you changed the orientation. Camille then rotated her drawing of a right angle to match the orientation of the angle on the screen. Ultimately, Camille agreed the angle on the computer screen was indeed a right angle because it had the same openness as a right angle.

Camille’s initial assimilation of the angle in TPRA indicated that her right-angle template was orientation dependent. At least temporarily, Camille allowed for a right angle to be independent of orientation. I attribute this temporary modification to Camille focusing on the openness of the angles in question; specifically, establishing a goal of comparing of the openness of these angles occasioned a perturbation for Camille, which she momentarily resolved by allowing angles as open as a right angle in standard position to be included in her class of right angles.

Following this discussion of right angles and openness, I asked Camille to adjust the parameter in TPRA to $n = 3$ and describe what she noticed. Camille explained, "it made three" angles that were "all the same and they made it so they could fit in between these [gesturing to the sides of the right angle]." I interpret Camille's remarks as an indication she assimilated the task as a two-level-of-units structure. Different from when she first examined TPRA with Kacie, Camille viewed three congruent angles as being inserted within the right angle. Camille adjusted the parameter to several other values and explained that for larger numbers the angles got "smaller…so they could fit," which was a weak indication Camille coordinated the number of angles inserted into the right angle with the magnitude of the angles inserted.

In the closing minutes of the session, I asked Camille to determine the measure of the smallest angles for $n = 3$ and $n = 2$ in TPRA. In each case, Camille computed the quotient $90 \div n$. In explaining why she elected to use division for the $n = 3$ case, Camille explained it would give her "the answer of one of the angles." Though Camille did not provide an explicit quantitative interpretation involving partitive division (e.g., establishing 3 equal groups of one-degree angles from 90-degree angles), Camille's activities indicated she was developing at least a procedural scheme for solving tasks in TPRA. Camille interpreted the angular situation as an occasion for numerical division, which was productive from both her perspective and mine.

Summary of Camille's March 14th teaching session. In this teaching session, Camille did not take composite units as input for iterating. Camille's activities contraindicated (a) she could assimilate tasks as three-level-of unit structures and (b) she had constructed a re-presentable template for a 4-partitioned plane. Camille's activities throughout the session indicated her right-angle template was not permanently designated as a 90-unit composite and this template was orientation dependent. By comparing the opennesses of differently oriented angles, which were right angles from my perspective, Camille made a temporary modification to allow her right-angle template to include right angles in non-standard orientations. In the context of TPRA where a right angle had been partitioned into *n* equiangular parts, Camille indicated that she was establishing at least a procedural scheme for determining that each of the smallest angular parts had a measure of $(90 \div n)°$.

Camille's March 31st Session

My primary goals in this teaching session were (a) explore whether Camille maintained an orientation-independent right-angle template, (b) to engender Camille's construction of a right angle composed of 90 one-degree unit angles, and (c) to engender Camille's construction of a template for a 4-partitioned plane.

Orientation-dependent right angles. To investigate Camille's template for a right angle, I asked Camille to produce right angles in two different GSP sketches. Each of these sketches contained two lines: a fixed blue line and an adjustable redline. In each version, I asked Camille to adjust the red line to make a right angle.[65] The orientation of the fixed blue line was the differentiating factor between the two versions of the sketch. In the first version, the fixed blue line was oriented horizontally (Figure 5.100); in the second version, the fixed blue line was oriented obliquely (Figure 5.101).

[65] Initially, I asked Camille to make a "square corner;" Camille indicated this terminology was not immediately sensible for her in the context of the present task, though she did ultimately relate a right angle to a square in the subsequent discussion.

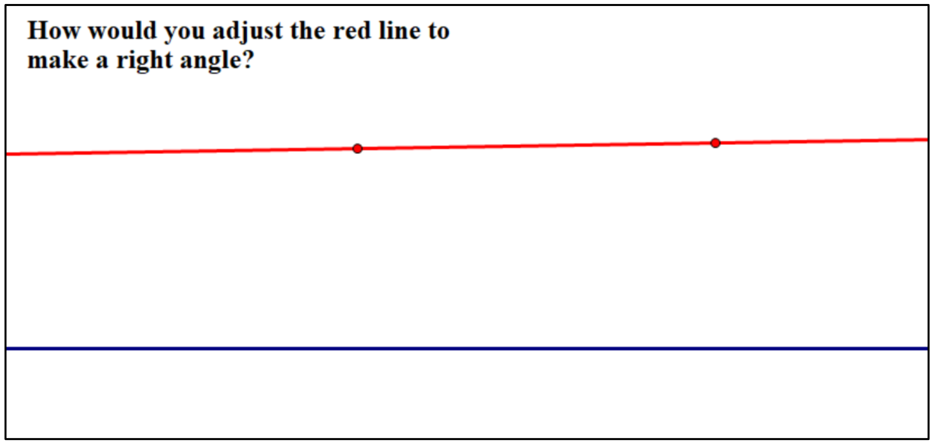


Figure 5.100. Horizontal referent right angle task.

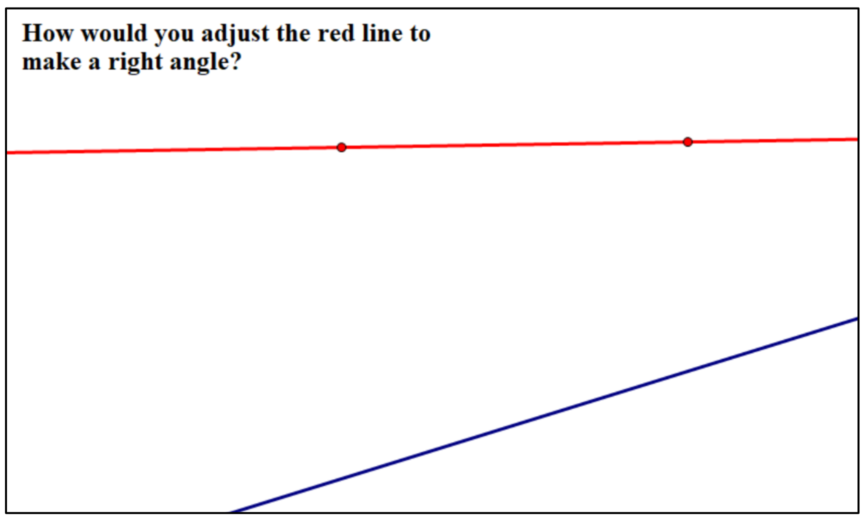


Figure 5.101. Oblique referent right angle task.

In the first version of the sketch, Camille immediately oriented the red line vertically (Figure 5.102). Afterwards, I asked Camille how many right angles she saw, and Camille replied, “one.” Camille identified the angle delineating the top right quadrant as the only right angle she perceived (Figure 5.102). When I asked Camille if it would be possible to draw any other right angles on the screen, she responded in the negative. That Camille did not identify any other right angles confirmed she had not made a lasting modification to her right-angle template, which remained orientation dependent in this session.

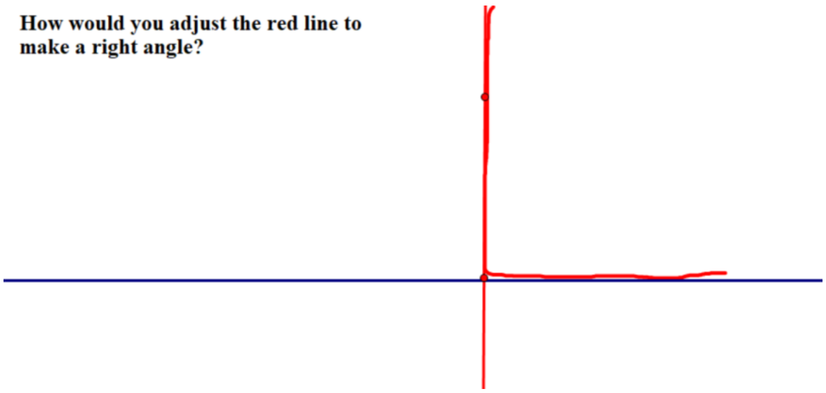


Figure 5.102. Camille sets the red line and identifies one right angle.

When asked to adjust the red line to make a right angle in the second version of the sketch, Camille immediately responded, “I don’t think you can.” After 28 seconds of adjustments accompanied by head tilting, Camille indicated it might be possible. After making additional adjustments for 22 more seconds, she settled on the configuration shown in Figure 5.103 below, which left the lines nearly perpendicular from my perspective. Camille’s tilting head and deliberate efforts as she solved this task again foregrounded the orientation dependent nature of her right-angle template.

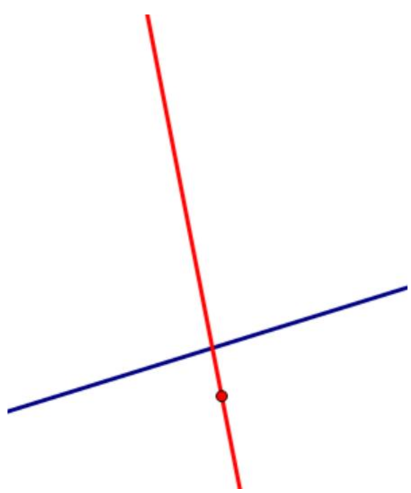
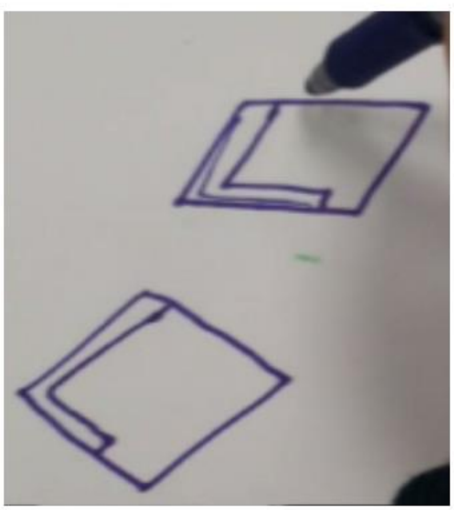


Figure 5.103. Camille sets the red line to make a right angle with the oblique blue line

During the 50 seconds of adjustments, Camille noted she was "trying to make it straight" and the blue line was "making it harder to do one." When I asked her to elaborate on what she meant by "straight," Camille related the situation to a square. Drawing on a piece of paper, Camille indicated a right angle formed an L-shape in the lower left corner of a square. After repeating the activity with a second square (Figure 5.104 left), I asked Camille if she could see any other L-shapes in the squares. Camille explained there were more L-shapes and drew two additional L's within the squares as shown at right in figure 5.104.[66] However, these two additional L-shapes were upside-down from Camille's perspective, and Camille stipulated they needed to be "facing so that [she] could tell it's an L" in order to be right angles. As I rotated the paper in front of Camille on the desk, which L-shapes were and were not right angles changed depending on the orientation of the L-shapes from Camille's perspective. Again, Camille's activities indicated her right-angle template was orientation dependent.



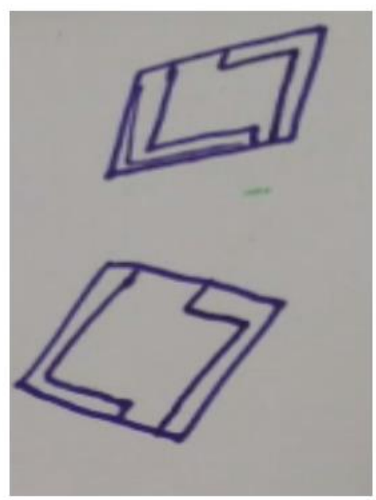


Figure 5.104. Camille indicates two right angles (left) and two additional L-shapes (right)

[66] That Camille drew exactly two L shapes in each square indicates in some sense that all four sides of each square were exhausted from Camille's perspective.

Inserting 90 one-degree angles into a right angle. Having obtained multiple confirmations of Camille's orientation-dependent right-angle template, I moved to a new GSP sketch depicting a right angle in standard orientation. The sketch contained a single draggable point, which altered the orientation of the right angle, and one action button, which showed 90 one-degree angles within the right angle when clicked (Figure 5.105). In Camille's previous March 14th teaching session, Camille temporarily modified her template for a right-angle to be orientation independent after she focused on the openness of the angle. My intention with the present GSP sketch was to again draw Camille's attention to the invariant openness of a right angle in various orientations while also encouraging Camille to structure the right angle as a unit composed of 90 one-degree angles.

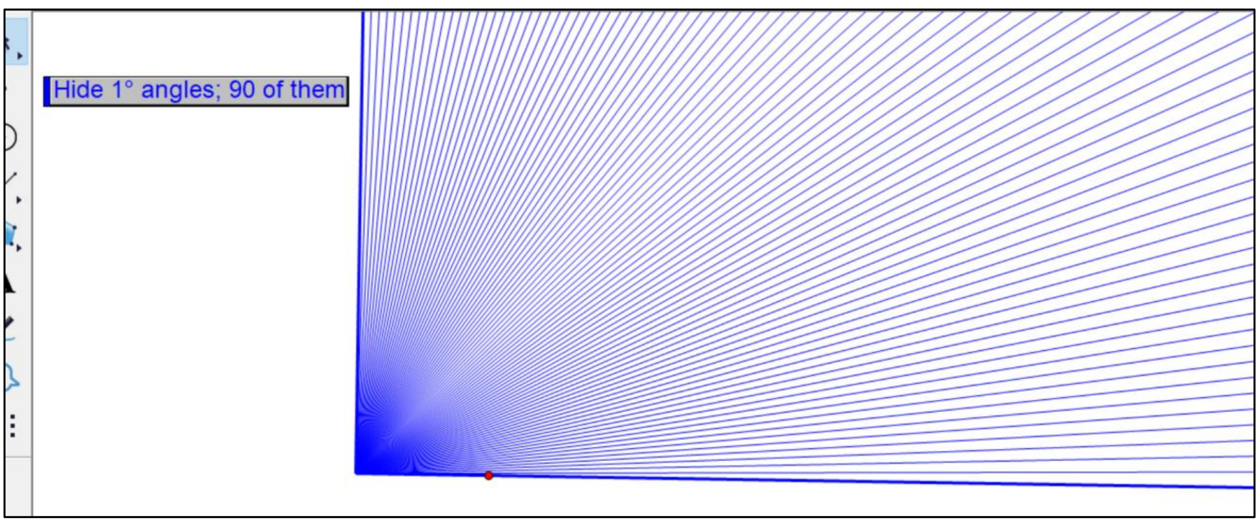


Figure 5.105. 90 one-degree angles inside one 90-degree angle.

At the onset of the task with the 1° angles hidden, Camille indicated that the angle was a right angle in standard orientation but not a right angle in other orientations. After reminding Camille that another name for a right angle is a 90-degree angle, I asked Camille to show the 1° angles and explained it would take 90 one-degree angles together to make one 90-degree angle. Following this explanation, I asked Camille to move the draggable point and, with the angle in a non-standard orientation, asked if the angle was still a 90-degree angle. Camille reluctantly replied, “yeah,” and continued, “I was probably wrong back then with all the things I was saying, but because there’s ninety one-degree angles in there it makes a ninety-degree angle.” For the moment, Camille had again temporarily modified her conception of a right angle to be orientation independent, which I attribute to Camille attending to perceptually available unit angles comprising the interior of the angle. As Camille continued to reorient the angle on the screen, she confirmed she viewed the angle as containing 90 one-degree units, regardless of the orientation.

Camille structures the plane with three levels of units in activity. Following the insertion of 90 one-degree angles into one right angle, I asked Camille to return to the GSP sketch wherein she counted one right angle after orienting a red line vertically to intersect a horizontal blue line. Camille immediately counted four right angles in the configuration, and I asked Camille how many one-degree angles there would be in all of the right angles combined. Camille explained there would be ninety one-degree angles in each of the four right angles, and I asked again how many that would be all together. Camille calculated four times ninety and, arriving at 360, remarked, “Three hundred and sixty. Oh yeah. Three sixty.” I asked Camille what she meant by “oh yeah,” and Camille explained, “cause a whole turn is three sixty,” as she gestured her hand in circular motions around the computer screen. Through her activity of inserting ninety one-degree angles into each of the four adjacent right angles, Camille had structured the plane with three levels of units through her activity. The plane was accounted for by four adjacent right angles, each of which contained 90 one-degree angles.

To encourage Camille's reflection on the units coordination she had just made, I asked Camille to count the one-degree angles again. This second time, I asked Camille to count the one-degree angles as she showed the 90 one-degree angles in each quadrant using four hide/show buttons in an additional GSP sketch (Figure 5.106). After showing the 90 one-degree angles in the first quadrant, Camille showed the angles in the other quadrants working clockwise counting, "a hundred eighty…two seventy…three sixty."

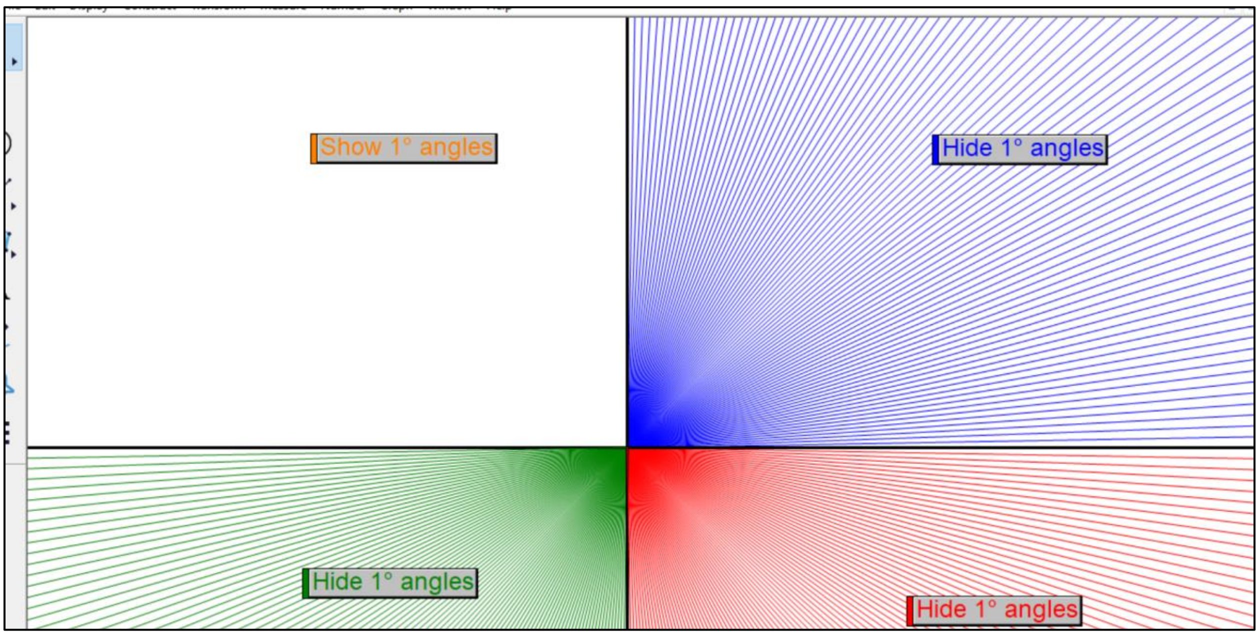


Figure 5.106. Inserting 90 one-degree angles into four right angles.

Summary of Camille's March 31st teaching session. In this teaching session, Camille again indicated an orientation-dependent right-angle template. By the end of the session, Camille indicated this template was temporarily modified to be independent from orientation, which was facilitated by Camille's attention to the invariance of a right angle's 90 one-degree constituent angular units despite changes in the right angle's orientation. Through inserting 90 one-degree subparts in re-presentation and reenacting this insertion perceptually using GSP, Camille structured the plane with three levels of angular units: a full angle composed of four right angles, each of which contained 90 one-degree angles.

Camille's April 18th Session

In her final teaching session, I again examined whether Camille had made any lasting changes to her right-angle template. I also investigated whether Camille had permanently designated her right-angle template as a 90-unit composite angle and whether she could structure the plane using this 90-unit composite. Additionally, I presented angular units coordinating tasks that involving partitioning a right angle from my perspective. To start this teaching session, I presented Camille with an angle identification task to determine whether Camille assimilated reflex angles in non-rotational angle contexts.

Angle identification task. I provided Camille with figure showing four distinct segments sharing a common endpoint (Figure 5.107).[67] I asked Camille how many angles were in the picture and instructed her to identify as many as she could. Camille initially responded, "four," and identified the three smallest convex angles and then largest convex angle. When I asked Camille if she could count any more angles, Camille identified the convex angles bounded by pairs of sides {a, c} and {b, d}. Following Camille's response, I asked again if it would be possible to count any more angles; Camille indicated she did not see any additional angles in the figure. Camille's response to this task offered a contraindication that Camille assimilated reflex angles in non-

[67] Segments were not labeled in the version presented to Camille.

rotational contexts, which was consistent with Camille's activities throughout the teaching experiment.

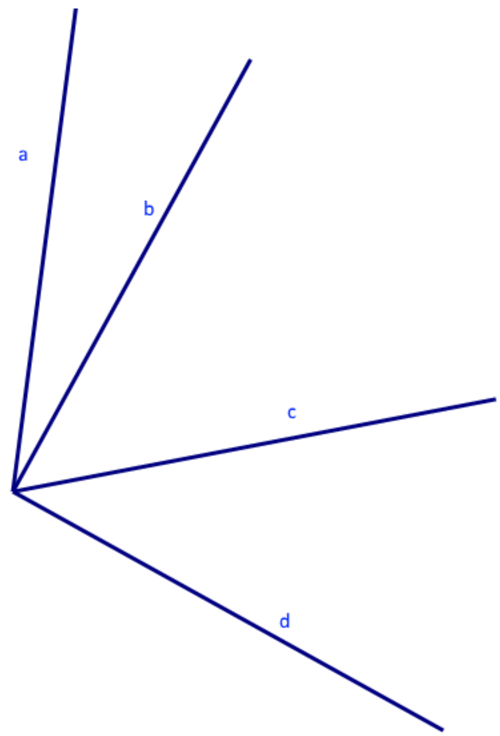


Figure 5.107. The angle identification task.

Reemergence of the orientation-dependent right-angle template. When Camille had finished identifying all possible angles from her perspective in the angle identification task, I asked Camille if she noticed any right angles in the figure. Camille immediately indicated her uncertainty. In the subsequent three minutes, Camille and I began a discussion very similar to the one we had in the previous teaching session. Camille mentioned a right angle as being L-shaped and drew an example. When I rotated the L-shaped angle, Camille asserted the angle was no longer right. When I drew a square and asked about the number of right angles, Camille remembered "that tricky question" from our previous teaching session 18 days ago. As she demonstrated rotating the square and the L-shaped angle, Camille explained she remembered saying previously the angles would still be right angles. Camille indicated, even though she wasn't sure about right angles in the present moment, she wanted to change her answer based on her recollection of the prior experience. Ultimately, Camille identified 4 right angles in the square and concluded that changing the orientation did not impact whether an angle was right.

Camille's right-angle template was still orientation dependent, which continued to be evident on some of her later tasks within this teaching session. However, I viewed Camille's initial activities in this session as an indicator of progress. When I rotated the angles, Camille re-presented her previous activities along with the decision she had made regarding the effects of changing the orientation of the angles. Moreover, Camille recalled this experience after a relatively long duration (2.5 weeks) and within a relatively short amount of time (three minutes).

A modest indication of a 90-degree designated right-angle template. To encourage Camille's re-presentation of her experiences from the previous teaching session, I positioned a right angle in standard orientation in front of Camille and asked her to consider the measure of the angle in degrees. Camille tentatively suggested ninety degrees. I asked Camille to explain, and she responded as described in Excerpt 5.18.

Excerpt 5.18. Camille relates 90° to 360°

C: How do I explain this? Okay, so a full one is three sixty [*circular motion over the page*]. And I know half of that would be one eighty. And half of that would be ninety. And I'm guessing this [right angle] would be ninety.

T: And when you're saying this [right angle] is half of something, what's this half of?

C: Uh. That's where I'm going to be confused…I was thinking of like making it a straight line, but I don't know if that would be an angle or not. [Camille extends a segment downward from the vertex of the angle as shown in Figure 5.108.

T: Hmm. Yeah. So where would the angle be then?

C: That's – I don't know. That's where I'm – I'm confused there too.

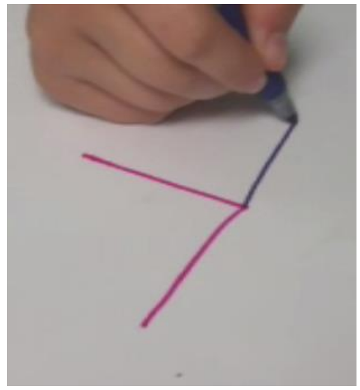


Figure 5.108. Camille considers whether a straight line is an angle.

Camille's numerical computations indicated she was leveraging her template for a full angle, which she associated with a measure of 360°, in order to deduce the measure of a right-angle. Camille's repeated halving of the 360° led her to a measure of 90° for the right angle. I hypothesize Camille's numerical division was occasioned by her re-presentation of the 4-partitioned plane in her previous teaching session wherein she used GSP to visually insert 90 one-degree angles into each of the four right angles. In drawing the purple segment shown in Figure 5.108, Camille attempted to parallel her numerical operations with angular operations; however, her uncertainty over whether the line was an angle suggested Camille did not imagine a concatenated motion through two right angles to constitute the half plane. In other words, the line did not have an interior composed of two right angles for Camille.

Coordinating angular units. In the six minutes following Camille's self-expressed confusion over whether a straight line was an angle, evidence of Camille's orientation-dependent right-angle template emerged again in the context of setting a line perpendicular to an oblique line. As such, I returned to tasks that involved the insertion angular units into a right angle, as these tasks had previously engendered temporary modifications in Camille's right-angle template. To start, I asked Camille how she could use a right angle in standard orientation to make a ten-degree angle. Camille's response is described in Excerpt 5.19 below.

Excerpt 5.19. Camille estimates a ten-degree angle.

C: I would probably have to make it, um – well I don't know how big a ten-degree angle, but I'm guessing it's really small. So, I mean – I guess I would just add another one … [*Camille draws the pink segment shown in Figure 5.109*]

C: I mean I don't know how big they are, but I'm guessing they're really small. And then, it's probably a little bit smaller. Something like that.

…

T: Let's say that you're right, and that's a ten-degree angle. What would be true about - What would have to be true about that?

C: What do you mean, like?

T: How many copies of that ten-degree angle could you fit into the right angle?

C: Oh. Nine probably.

T: How do you know that?

C: Cause ten times nine is ninety and I was guess – I mean, cause I said that the right angle is a ninety. I remember we did one a long time ago [*pointing to the computer*], I think it was with Kacie too. I remember we had to fill one up and it was with ten-degree angles or one-degree, I don't know how big they were. Yeah. So that's what I was thinking about.

Figure 5.109. Camille draws a 10-degree angle.

Camille's initial estimate of a ten-degree angle indicates the operations of a figurative degree scheme. Knowing a right angle had a measure of ninety-degrees, Camille knew a ten-degree angle would be much smaller than that. Camille was reasoning comparatively about the openness of a ten-degree angle in reference to the right angle, and gave no indication of inserting units into the right angle when making her initial estimate. When asked to consider the number of ten-degree angles that would fit into the right angle, Camille's response, "nine," and justification, "ten times nine is ninety," indicated a shift in her reasoning. At least at least at the numerical level, Camille was structuring the measures of the two angles in question. Camille's subsequent recollection about an earlier paired session with Kacie suggested she was recalling a task involving a partitioned right angle or partitioned plane in GSP. This recollection combined with her justification indicated at least a two-level-of-unit structure: Camille was imagining ten nine-degree angles inserted into a right-angle.

Following the ten-degree angle task, I asked Camille to imagine an angle so that it would take ten copies of that angle to make a right angle. Camille determined the measure must be 9° remarking, "if I need ten to make a right angle, and I'm guessing it's ninety degrees…they would each be nine degrees." Following that, I asked Camille to explain how she would make a one-degree angle using a right angle. Camille explained she would close one side of the right angle to be "really close" to the other.

Excerpt 5.20. Camille checks a one-degree angle via equisegmenting

T: And if you had to convince somebody that you had done it right, what would have to be true?

C: Um, [6 sec], probably like how many would – I would probably put in, uh like, eighty-nine more of those to make it fit.

T: Say just a little bit more about that.

C: Like since I've already – since I already have one in [pointing to the right angle], I'd probably do eighty-nine more like exactly the same to see if it was a ninety-degree angle.

T: And how would you know whether or not it was a ninety-degree angle at the end of that?

C: If they all fit.

T: If they all fit?
C: If they fit perfectly.

Excerpt 5.20 describes the first time in the teaching experiment Camille indicated an awareness of a 90-degree angle as a unit composed of 90 one-degree angles in without a recent experience of a perceptual composite unit (e.g., TPRA with n = 90) within her visual field. Camille's initial estimate of a one-degree angle was not based on partitioning the right angle. Instead, she wanted to make the sides of the angle "really close" together. Holding this angle in re-presentation, Camille described inserting a total of 90 copies to see the ninety-copies perfectly exhausted a right angle. The process Camille described for making/checking a one-degree angle indicated the equisegmenting operation. Camille's six second pause and multiple restarts in speech indicate this process was not immediate for Camille, (i.e., she had not already established a scheme for making or checking one-degree angles). Furthermore, Camille's comment, "since I already have one [one-degree angle] in," contraindicates the simultaneous insertion of 90 one-degree angles into the right angles and is consistent with the equisegmenting operation.

With Camille's extensive quantitative operations activated in this context, I asked Camille what would happen if she continued to make 90 more one-degree copies beyond the 90 she had already imagined. Camille added a new pink segment, which is shown in Figure 5.110. In Figure 5.110, the interiors of the initial and second right angle are labeled 1 and 2, respectively; segments are labeled a-c. Our conversation after the addition of the pink segment c is described in Excerpt 5.21.

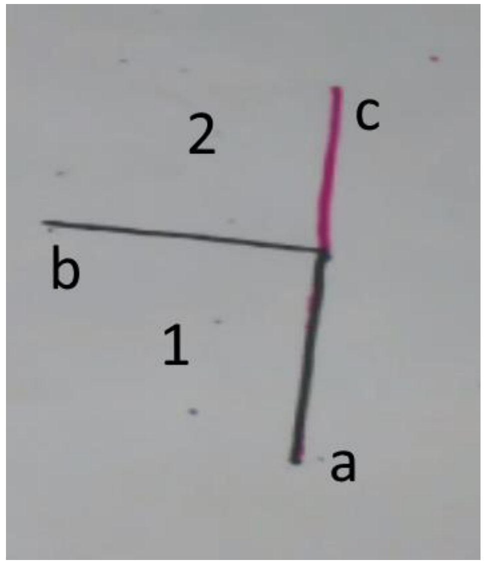


Figure 5.110. Camille adds 90 more one-degree angle to a right angle.

Excerpt 5.21. Camille makes adjacent copies of a right angle as a 90-unit composite.

C: Uh, I think it's be a straight line. Like it'd end right here [*pointing to the end of segment c in figure 5.110 after motioning through the interior of right angle 2.*]

T: I see. So how many one-degree angles would there be in all there?

C: A hundred and eighty.

…

T: What if you did ninety more copies? Where would you end up?

C: Uh probably – let me get the purple one [marker] – And so out of this line [*pointing to segment c*] it'd probably be another right angle, facing my way like this [*draws purple segment d in figure 5.111*]. And then it would start here [c] and then end here [d]. And it would probably be um …[Camille computes 180 + 90 using vertical addition] …I think it's two hundred seventy. Yeah, two hundred seventy all from these three.

T: From those three [pointing to the interiors of each right angle 1-3 in figure 5.111.]

C: Yeah.

T: And what if you put ninety more in?

C: Well it'll [pointing to the interior of unlabeled fourth right angle in figure 5.111], uh. [calculates 270 + 90 = 360]. Yeah, it would be three sixty.

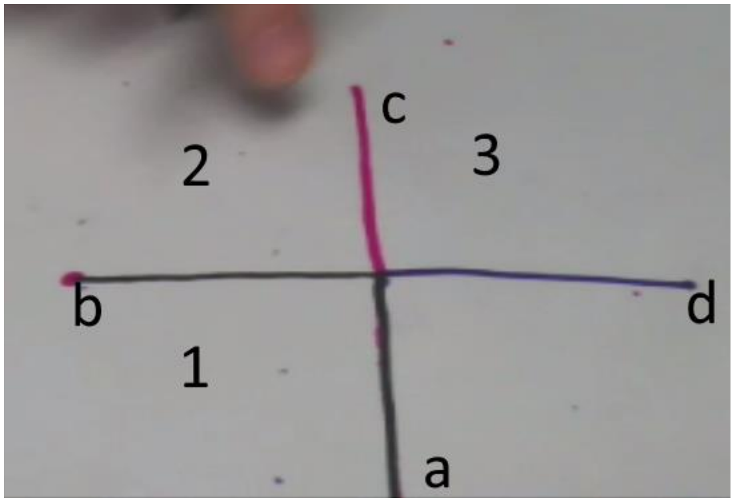


Figure 5.111. Camille unites 3 composite angular units, each containing 90 one-degree angles.

Continuing to make copies of 90 one-degree angles at a time, Camille took a right angle as a composite unit and repeated adjacent copies of right angles to exhaust the plane. In engaging in this activity, Camille structured the plane with three levels of units through her activity, as she had done in the previous teaching session. When considering the fourth repetition of 90 one-degree angles, Camille pointed to the interior of the remaining right angle and then computed the sum of 270 and 90. Because Camille resorted to this calculation, I infer she was not immediately aware that four adjacent right angles would exhaust the plane, which is consistent with the hypotheses Camille had not yet established a re-presentable template for a 4- partitioned plane and was not yet able to assimilate tasks with 3 levels of units.

Summary of Camille's April 18th teaching session. In Camille's final teaching session, her activities indicated her right-angle template was still orientation dependent, though she again made temporary modifications during the session. Camille described checking the measure of a one-degree angle via the equisegmenting operation and, in doing so, structured a right angle as a 90-unit composite. By repeating this composite unit in this instance, Camille structured plane as a three level of unit structure containing 4 right angles, each containing 90 one-degree angles.[68]

Camille's Final Interview

In her final interview session on April 25th, Camille's activities confirmed she had maintained schemes for angular congruence and comparison, which she had indicated at the beginning of the teaching experiment. Additionally, Camille confirmed she had established an angular repetition scheme for producing multiples of a given angular unit. At the onset of the teaching experiment, Camille had not constructed an angular repetition scheme. In the following sections, I discuss Camille's activities on angular splitting, angular units coordinating, estimation, and making a one-degree angle tasks. In discussing these tasks, I briefly compare Camille's activities to her engagement with similar tasks from earlier teaching sessions.

Angular splitting. In the angular splitting task, I provided Camille with a pair of wooden chopsticks and asked her how she would set a pair of chopsticks so that the given pair was five times as open as the desired pair. Camille sat in thought for ten seconds before remarking, "that's hard…I don't know." With Camille at a standstill, I offered her a free pair of long chopsticks. As she had done in her initial interview session, Camille set the free chopsticks to be as open as the given chopsticks and indicated she would reduce the openness of the free chopsticks in five bursts of motion. Camille indicated she wanted to produce a less open pair of chopsticks using her concept of five; however, she did not partition the given chopsticks into five equiangular parts. Thus, Camille's activities on this task contraindicated the splitting operation as I anticipated.

[68] Additional corroboration would be needed to confirm whether Camille could re-enact this structuring.

Angular units coordinating tasks. In her last three teaching sessions, Camille and I had focused a good deal of our attention on right angles. As such, in Camille's final interview session, I asked her to solve a units coordinating task involving a right angle. Specifically, I asked Camille to determine the measure of a purple angle if five sweeps of the purple angle swept out a right angle. After 20 seconds of silent thought, Camille commented, "probably sixteen each...a right angle is like ninety degrees so probably that divided by five." Camille had assigned a right angle a measure of ninety degrees and established a goal of at least producing the numerical quotient of ninety and five. Camille proceeded to check her result, 16, using multiplication written vertically on the paper in front of her. Dissatisfied with the result, Camille computed 90 ÷ 5 using long division and asserted the purple angle would measure 18°.

Camille's activities on this task were surprising to me. [69] Camille was in the process of establishing at least a procedural scheme for determining angle measure when asked to consider a partitioned right angle. To examine whether Camille's numerical operations had quantitative counterparts, I asked Camille to draw a picture of the situation. Camille attempted to draw the situation twice, each time drawing a right angle in standard orientation. Camille was dissatisfied with her first attempt because angles she inserted were not congruent. On her second attempt, Camille made four linear gestures emanating from the vertex of the right angle using the tip of her marker, which suggested was projecting five angular units into it. Following these gestures, Camille produced the drawing shown in Figure 5.112 and explained that she was "trying to put five angles in a

[69] In the moment, I was initially surprised when Camille solved this task after not solving the splitting task. This task involves (a) an element of sequentiality due to the sweeping language and (b) suggests an insertion of the purple angle into the right angle; therefore, this task is less cognitively demanding than the previous angular splitting task with n = 5.

right angle." Camille was not entirely satisfied with the sizes of the angles in this second drawing and indicated that she intended for the five smallest angles to be congruent. Because I infer Camille was imagining the simultaneous insertion of five equiangular parts into the right angle, I consider Camille to have partitioned the right angle in this task.

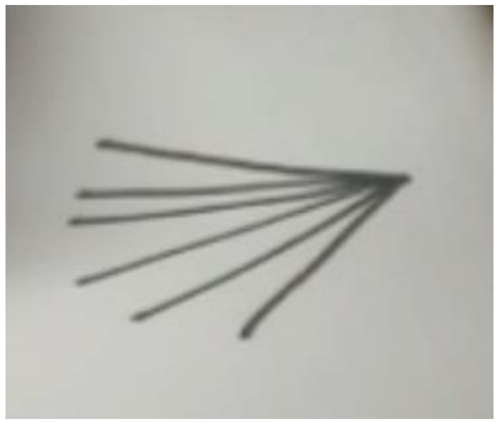


Figure 5.112. Camille draws a right angle partitioned into five angular parts.

Following the preceding units coordinating tasks, I presented Camille with a second units coordinating task. Specifically, I asked Camille to determine the measures of a yellow angle and a red angle if four sweeps of the yellow angle exhausted the plane and three red angles exhausted one yellow angle. To determine the measure of the yellow angle, Camille computed the quotient of 360 and 4 via long division, which contraindicated she had established a 4-partitoned plane template. After finding the yellow angle would measure 90°, Camille asserted the red angle would measure 30° since "thirty times three is ninety." Consistent with the previous task, Camille's activities indicated she could take a suggested partitioned right angle or full angle as an occasion for numerical division. Camille's assimilation of the task as an occasion for numerical division may have been due to her activities in the previous teaching sessions involving the insertion of one-degree angles into the interior of right angles; though Camille did not explicitly indicate considering the number of one-degree angles contained in the 90° angle.

When I asked Camille to draw this situation, Camille began remarking, “well, I don’t know what type of angle it is,” referring to the yellow angle. She paused briefly and commented, “Oh! It’s this one,” drawing a right angle in standard orientation. Camille’s surprise confirmed she had not immediately assimilated a 4-partitioned plane constituted by four adjacent right angles. Camille partitioned the right angle into three congruent angle parts and labeled the angles as shown in Figure 5.113 below.

Figure 5.113. Camille draws a yellow angle and three red angles.

When I pressed Camille to draw all four yellow angles mentioned in the prompt, Camille produced a new drawing by sequentially drawing four adjacent right angles and did not spontaneously insert the red angles. To examine whether Camille had produced a locally permanent three-level-of-unit structure, I removed Camille’s drawings and asked her how many red angles it would take to sweep out the entire plane. If Camille had maintained an awareness of all three levels of units—the plane composed of four yellow angles each containing three red angles—Camille might have reasoned that there must have been 3×4 = 12 red angles. Instead, Camille computed 360 ÷ 30 = 12, which contraindicated she had assimilated the task with three levels of units. From her drawings and this computation, I infer that Camille could assimilate tasks with two, but not three, levels of units, which was consistent with her activities throughout the teaching experiment.

Making a one-degree angle. To investigate Camille's meaning for a one-degree angle, I asked Camille how she would make an angle with a measure of one degree. Camille explained that the angle would be "really small" and "there wouldn't be that much space inside." As she talked, Camille drew the angle shown in Figure 5.114 below. Camille's description indicated that her figurative degree scheme was activated as she described a one-degree angle as indicating a small angular magnitude.

Figure 5.114. Camille draws a one-degree angle.

I pressed Camille further asking how she might check if her angle was indeed a one-degree angle; Camille mentioned using a protractor and did not describe any other method for checking when I pressed for alternative methods. I asked Camille to draw a right angle and asked how the two angles would be related to one another. Camille explained, “I could do ninety of these [one-degree angles] in here [the right angle] and it would still be one big right angle.”

Camille did not spontaneously describe a method for producing one-degree angles or verifying a given angle had a measure of one degree. However, when I prompted Camille to relate a one-degree angle to a right angle, Camille demonstrated an extensive quantitative meaning for one degree that leveraged iteration. Camille’s response indicated an awareness a right angle could be exhausted via 90 iterations of a one-degree angle.

Estimation. During the final interview, I asked Camille to estimate the measure of two angles, both of which were 120° angles from my perspective; the orientation of the angle in each task was different. When estimating the measure of the angle in standard orientation (i.e., with one side horizontal), Camille initially estimated that the angle had a measure of 100°. From her description and gestures, I infer Camille inserted a right angle in standard position and then estimated amount by which the given angle exceeded the right angle to be 10°. As she described her thinking, Camille revised her estimate of the measure of the excess angle to be 20° and later 30°. Camille settled 120° for the measure of the given angle adding 90° and 30°. Camille settled on 30° for the measure of the excess angle because she viewed this angle as congruent to the thirty-degree angle from the previous units coordinating task. Camille’s activities in this task indicate a templated figurative degree scheme was activated. Camille inserted a right-angle referent and visually estimated the amount by which the given angle exceeded a right angle. Camille compared the excess angle to a re-presentation of the 30° angle from a previous task and gave no indication of measuring the right angle using the excess angle.

When estimating the measure of the angle in non-standard orientation (i.e., neither side horizontal or vertical), Camille hesitantly estimated the angle to have a measure of more than 180°. Camille explained that she wanted to insert a right angle in standard position, but the sides of the given angle were "not really straight," which made it more difficult for her "to imagine it." Camille's response on this estimation task indicated her scheme for measuring angles in degrees was still orientation dependent.

Contraindication of a ratio quantification of angularity. In the closing minutes of the final interview session, I asked Camille to solve two tasks involving a circle and a central angle. In these tasks, I provided measurements for the intercepted arc and the circumference and asked Camille to determine the measure of the blue central angle. Figure 5.115 gives an example of one such task wherein the circumference is 30cm and the length of the intercepted arc is 5cm.

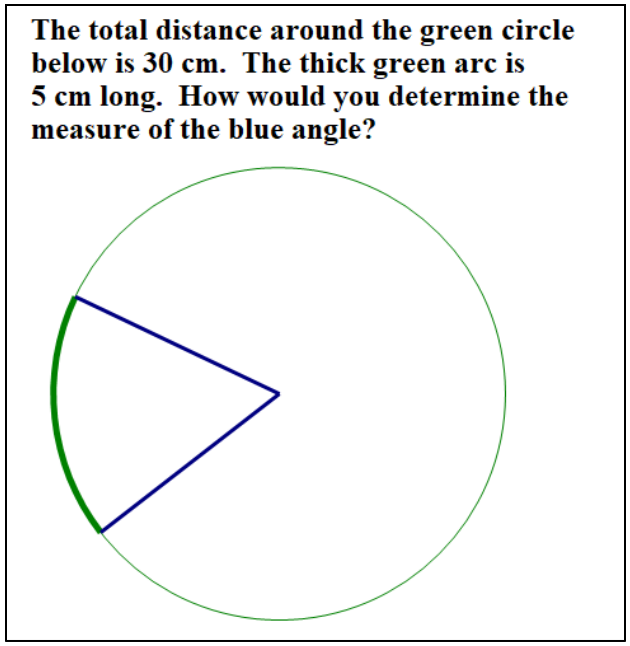


Figure 5.115. An example of a task involving circular quantities.

In each of these tasks, Camille described she would determine the measure of the central angle by adding the arc length and the circumference and then subtracting this sum from 360°. From Camille's response, I infer that she established a goal of decreasing a full angle, which was likely salient for Camille due to the image of a circle, by some amount to determine the measure of the blue angle. I infer that Camille added the measurements of the arc length and circumference because she did not view the arc length as being included in the circumference. Camille's activities on these tasks suggested that her templated figurative degree scheme was activated and contraindicated she had established a ratio quantification of angularity.

Summary of Camille's final interview session. In her final interview session, Camille's activities confirmed that she had established schemes for angular congruence, comparison, and repetition. Camille's activities on the angular splitting task contraindicated the splitting operation. In angular units coordinating tasks, Camille used a procedural scheme involving numerical division to determine the measures of angles within right and full angles. Camille did not provide a method for producing a one-degree angle; however, when I asked her how a one-degree angle and a right angle would be related, Camille indicated that a right angle would contain 90 one-degree angles. On estimation tasks, Camille inserted a right-angle referent, but also indicated an orientation dependent conception of degree measures. Camille's activities on a task involving a circle and a central angle contraindicated that Camille had quantified angularity as ratio.

Part 4: Kacie's Last Three Sessions

After I decided to separate Kacie and Camille in the teaching experiment, Kacie participated in two additional teaching sessions and one final interview session. It was my intention for Kacie to be paired with Bertin for her remaining teaching sessions; however, due to differences in the students' extracurricular activities, she could only be paired with Bertin on one occasion. As such, Kacie's last two teaching sessions consisted of one paired teaching session with Bertin and one solo teaching session.

To this point in the teaching experiment with Kacie, I had worked almost exclusively to explore and foster angular iteration and partitioning. At this time, I confidently attributed angular iteration and iteration of composite angular units to Kacie; however, I was acutely aware that Kacie's partitioning operations—her operations for simultaneously producing units within units—lagged behind her operations for iteration—building units from units. Kacie demonstrated she could assimilate a given partitioning in TPRA and TPP to her equisegmenting degree scheme; in other words, when provided with a partitioned plane or right angle, Kacie could compute the measure of one of the angles in the partitioning. Yet, Kacie never partitioned an angular unit herself and instead always engaged in equisegmenting activity—exhausting an angular whole through the sequential production of equiangular parts.

In Kacie's final two teaching sessions, I had three major goals. First, I wanted to closely examine Kacie's ways of reasoning on planar covering tasks to see whether Kacie might engage in partitioning. Second, I wanted to examine whether Kacie assimilated reflex angles in the absence of an immediate experience involving a sweeping laser. Third, I wanted to explore whether Kacie had quantified (or might quantify) angle measure as a ratio or rate.

Different from extensive quantifications of angle measure, ratio and rate quantifications involve an individual conceptualizing angle measure as a composite quantity formed through a multiplicative comparison of non-angular circular quantities (e.g. arc length or sector area). An individual has quantified angle measure as ratio if she measures a central angle through multiplicatively comparing two non-angular circular quantities in a *particular* circle. For example, an individual may note that a given central angle subtended by a central angle and circle's circumference. An individual has quantified angle measure as ratio if she recognizes the multiplicative comparison is invariant across circles of any radius.

Kacie's April 5th Session with Bertin

In the only paired teaching session of Kacie and Bertin, I asked the pair to solve planar recursive partitioning tasks, planar covering tasks, and tasks involving circles. Both students could solve planar recursive partitioning tasks and planar covering tasks using their existing ways of reasoning. For example, I asked the pair to find the measures of a blue angle if it took 10 red angles to sweep out the entire plane and it took 6 blue angles to sweep out one red angle. Bertin explained, “if you divide 360 by ten that’ll give you the degree of one of the [red] angles.” Kacie continued where Bertin left off, explaining, “the blue one would be six…because if one of the red angles equals thirty-six degrees...six times six is thirty-six.” Because Kacie and Bertin had not previously worked together during teaching sessions, I elected to start the session with tasks I was confident both students could solve to help the students relax and become accustomed to their new partnership.

Though both students could solve these tasks, Bertin was able to more flexibly shift between three-level-of-unit structures than Kacie. For example, after the students determined that 36 ten-degree angles would be needed to sweep out the entire page, I asked the students to repeat the task for a 5-degree angle. Kacie computed the result via the long division 360÷5. In contrast, Bertin indicated inverse proportional reasoning and explained that it should be double the previous result.

In the section below, I briefly discuss the pair’s interaction on one task I designed to examine whether Kacie or Bertin had constructed angle measure as ratio; this was Kacie’s first experience in the teaching experiment explicitly involving a circular context and an angle.

Evidence of an anticipatory scheme for determining degrees. To examine whether Kacie (and Bertin) had quantified angularity as a ratio involving circular lengths, I presented the pair with red angle whose vertex was positioned at the center of a green circle (Figure 5.116). I provided no measurements or labels and asked, “How could you use the circle to measure the angle?” After staring at the screen for approximately four seconds, Kacie and Bertin described a strategy for determining the measure of the red angle. Kacie suggested seeing “how many times the angle needed to go around before it made a whole circle.” Bertin continued after Kacie explaining, “and then you divide three sixty by that number.” Both students agreed this would give the measure of the red angle.

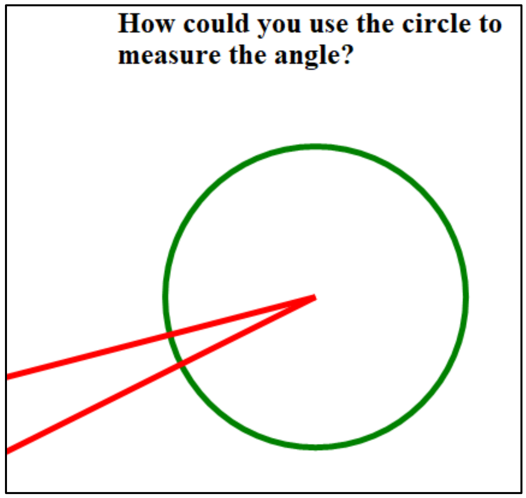


Figure 5.116. How could you use the circle to measure the angle?

The strategy Kacie and Bertin described for determining the measure of the red angle involved dividing 360 by the number of iterations of the red angle required to exhaust the plane. Because the students described the process they would use to determine the measure of the red angle without actually carrying out this process, I consider the students to have demonstrated an anticipatory equisegmenting degree scheme utilizing a full angle template.[70] Although the students' provided an eloquent solution, their responses gave no indication of a ratio quantification of angularity. As such, I pressed the students for a different measurement process and stipulated they (a) could measure anything they wanted along the circle but (b) could not move or make

[70] Bertin had developed a more sophisticated scheme involving the splitting operation, which can be viewed as an accommodation of the equisegmenting degree scheme (see Hardison, 2017 for some of the angular operations Bertin constructed during the teaching experiment).

copies of the angle. My intention was to see if the students might consider measuring the circumference as well as the length of the intercepted arc.

After sitting silently for fifteen seconds, Kacie suggested, “it probably has something to do with like the diameter and all that stuff. I don’t know.” Kacie went on to suggest measuring the radius or calculating the radius from the circumference, but she was unsure how she might use any of these measurements to find the measure of the angle. After listening to Kacie’s suggestions, Bertin explained he did not have any additional ideas to contribute. Neither Kacie nor Bertin spontaneously decided to measure the length of intercepted arc along the circle or the circle’s circumference, which contraindicated either had quantified angle measure as ratio.

Kacie’s April 21st Session

In Kacie’s final teaching session, I asked Kacie to engage in planar recursive partitioning and planar covering tasks to examine whether Kacie would implement the splitting operation. To examine whether Kacie would assimilate reflex angles, I presented Kacie with an angular identification task wherein she identified as many angles as she could within a given figure. Finally, to examine whether Kacie might quantify angularity as a ratio or rate, I asked Kacie to solve tasks involving angles with vertices positioned at the center of circles. As always, Kacie was a polite and willing participant in this teaching session; however, she seemed a bit downhearted, and I suspect she may have been dealing with challenges outside of the teaching experiment.

Contraindication of plane splitting. At the start of Kacie’s final teaching session, I asked her to consider a scenario involving two angles: three sweeps of first angle would sweep out the entire plane, and two sweeps of a second angle would sweep out one of the first angles. I asked Kacie to determine the degree measure of the second angle. Kacie quickly computed 360÷3 = 120 and 120÷2 = 60 using long division before asserting, "the second angle would be sixty degrees." When I asked Kacie to explain why it made sense to use division for this problem, Kacie explained, "since there was three of the first angle…it would have to be a number that if you multiplied it by three it would equal three hundred and sixty, so I divided three hundred and sixty by three."

Kacie had clearly established a numerical scheme for determining the degree measure of an angle fit into the plane (or another angle) a whole number of times. Kacie's explanation for why division was an appropriate operation foregrounded the iterative nature of Kacie's reasoning. Kacie said she was looking for a number so that three times that number was 360. Kacie recognized division as a numerical process which shortcut the need for trial-and-error multiplication. However, Kacie did not provide a quantitative interpretation for the division (e.g., creating three equal shares from 360°). From her explanation, there is little evidence Kacie mentally partitioned a full rotation into three equal parts. Instead, Kacie's explanation suggested equisegmenting the measure of a full angle by an unknown value, and Kacie determined this value through numerical division.

To examine whether Kacie held in mind a three-level-of-units structure as she reasoned about the planar recursive partitioning task, I asked Kacie how many sixty degree angles would be needed to sweep out the entire page. Kacie responded, "six," and explained, "because if there's two in one of the first angles and … it took three times for the first angle to go around, I just multiplied three and two and got six." Kacie's explanation suggested she remained aware of the three-level-of-units structure—a full angle composed of three units of two—throughout her calculations, though Kacie did not explicitly indicate a simultaneous insertion of two units into three units. When I asked Kacie to explain why she multiplied, she began to draw the situation. For clarity, I refer to the first angle as a red angle and the second angle as a black angle.

Kacie drew an estimate for the red angle, which was nearly a right angle from my perspective, and partitioned it into two parts to account for two black angles (Figure 5.117 left). Kacie then repeated the red angle and partitioned this repetition to account for two more black angles (Figure 5.117 center). Kacie repeated the red angle a third time, but this time she partitioned two angles: (a) the repetition of the red angle and (b) the remainder of the plane unaccounted for by three repetitions of the red angle, which was approximately congruent to the red angle. After producing the drawing shown at right in Figure 5.117 which contained a total of eight black angles, Kacie began to count the number of black angles and then remarked, “well, that’s not a good representation…it was more in fours instead of like three times all the way around.”

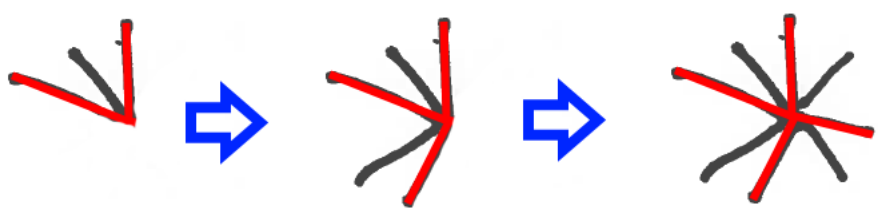


Figure 5.117. Kacie equisegments the plane with partitioned red angles.

Kacie was engaged in an equisegmenting activity. Rather than partitioning the plane, which would require Kacie to simultaneously project three angular units, Kacie sequentially produced united copies and intended for three united copies to exhaust the plane. Although Kacie clearly demonstrated she could structure the various units in a numerical sense, she was not immediately able to coordinate the corresponding angular units.

Kacie began a second attempt at representing the angular situation on paper and stopped suddenly remarking, “no, I can’t draw that…it kind of confuses me.” Kacie then produced a 4-partitioned plane and tried to draw the 3-partioned plane again. This time, Kacie drew a circle which she fragmented into three non-congruent angular sections (Figure 5.118). I asked Kacie again if it would be possible to draw the 3-partitioned plane and our conversation is described in Excerpt 5.22 below.

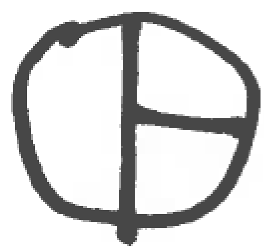


Figure 5.118. Kacie fragments a truncated plane into three non-congruent angular sections.

Excerpt 5.22. Kacie does not produce a 3-partitioned plane.
T: So would there be a way to draw the three case?
K: [6 sec] Like I can see it in my head, but I don’t know how to draw it. [5 sec].
T: Tell me more about that. What are you imagining?
K: [4 sec] Well, a circle and then lines inside of it splitting it in three different parts, but I just don’t know how to draw that. [7 sec] So, yeah.
T: Do you think it’s possible to draw?
K: [3 sec] Yeah, because I think I’ve seen it before. I just don’t know how to do it.

After her attempts to produce a 3-partitioned plane, I asked Kacie to produce a 6-partitioned plane. Kacie fragmented the plane to produce 6 non-congruent angular sections (Figure 5.119). Kacie's implementation of equisegmenting and fragmenting operations contraindicated the splitting operation.

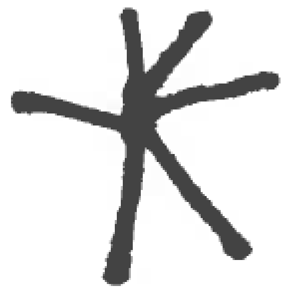


Figure 5.119. Kacie fragments the plane into 6 non-congruent angular sections.

Identifying angles. In previous teaching sessions while working within laser contexts, Kacie indicated she could assimilate reflex angles. To investigate if Kacie could assimilate reflex angles without an immediate experience involving a sweeping laser, I posed an angle identification task to Kacie. In this task, I presented Kacie with an image of four distinct segments sharing a common endpoint (Figure 5.120).[71] I asked Kacie how many angles were in the picture and instructed her to identify as many as she could.

[71] Segments were not labeled in the version presented to Kacie.

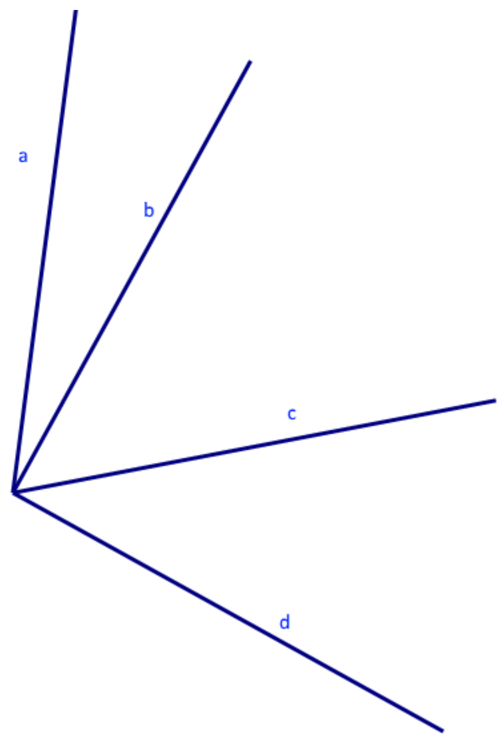


Figure 5.120. Image for the angle identification task.

Immediately, Kacie responded, “three,” and identified the three smallest convex angles, which were bounded the following pairs of sides: (a, b), (b, c), and (c, d). When I asked Kacie if it would be possible to find anymore, Kacie considered the largest convex angle, which was bounded by (a,d), and then remarked, “this is like a brain teaser.” Kacie then systematically identified all six ways to pair distinct segments—(a, b), (a, c), (a, d), (b, c), (b, d), and (c, d)—as she counted from one to six. For each pair of segments, Kacie dragged her finger through the interior of a convex angle as she moved from the first segment in the pair to the second. As such, I consider Kacie to have identified six convex angles in this task. After Kacie identified these six angles, I asked Kacie again if it would be possible to find any more angles. Kacie said it was not possible “because you’ve used all the lines with all the other lines.” She elaborated explaining it would be inappropriate to count (c, a) because it had already been counted as (a, c).

On the angle identification task, Kacie did not identify any reflex angles and gave no explicit indication of considering a sweeping ray as she had done in previous teaching sessions. Kacie explicitly did not count different orderings of the same pair of sides as distinct, which suggests she assigned no orientation (e.g., clockwise or counterclockwise) to the angles she identified. This lack of orientation also contraindicates Kacie was considering sweeping imagery as she engaged with this task. Although Kacie had attended to reflex angles and orientation in rotational contexts, these elements were not brought forth in her way of reasoning in this non-rotational context.

Working with circular quantities. In her previous teaching session with Bertin, Kacie's activities on a single circular task suggested she had not yet quantified angularity as ratio because she did not spontaneously consider using arc length and circumference to measure a given central angle. One of my major goals for Kacie's final teaching session was to engender her quantification of angle measure as ratio and, if possible, rate. At this point in the teaching experiment, I was hopeful Kacie might develop a ratio quantification of angle measure because of her capacity for multiplicatively structuring units; I was skeptical Kacie might construct a rate quantification because she did not exhibit a well-established scheme for proportional reasoning at the onset of the teaching experiment.

To engender Kacie's construction of angle measure as ratio, I developed a series of tasks involving three circular quantities: measure of a central angle, length of the intercepted arc, and the circumference of the circle.[72] In these tasks, I provided measures for two of the three quantities and asked that Kacie determine the measure of the third

[72] I use *circular quantities* to refer to attributes an individual might measure when considering a situation involving a central angle and a circle. Other circular quantities include a circle's radius or diameter, as well as the areas of the circle, sector, or arc segment.

quantity. My intention in providing the measures of two quantities was to focus Kacie's attention on other quantities in the circular context in hopes she would multiplicatively structure the situation in determining the measure of the unknown quantity. In my discussion of Kacie's activities in the following sections, I refer to these tasks involving circular quantities as *CQ tasks*. To indicate the measurements given in each task, I use ordered triple notation, (Circumference, Arc length, Angle measure), and replace the names of the two circular quantities with numerical measurements provided in the prompt; I replace the third unknown quantity with a three-letter abbreviation. For example, I refer to Kacie's first circular quantity task (Figure 5.121) as CQ(23.54cm, Arc, 28.93°) since I provided Kacie with measurements for the circumference (23.54 cm) and central angle (28.93°); Kacie's goal in this task was to use the given measurements to determine the length of the intercepted arc.

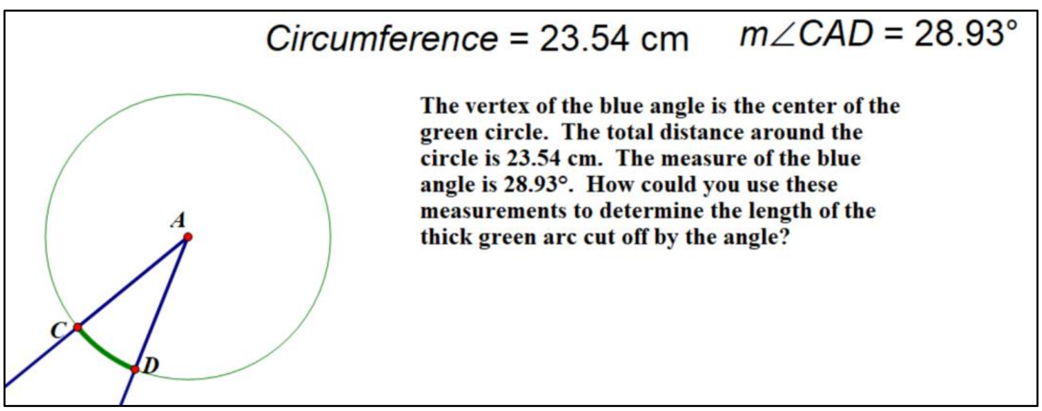


Figure 5.121. CQ(23.54cm, Arc, 28.93°)

Kacie assimilates CQ tasks as additive structures. After reading the prompt for CQ(23.54cm, Arc, 28.93°) twice, Kacie sat in thought for twelve seconds before remarking, "well, you're going to have to subtract something from the circumference to get the length of the green arc… I don't know what you would do with the angle." After another 20 seconds of silent thought, Kacie explained that she was considering subtracting the measures of the circumference and the angle. Kacie expressed some doubt about this method, hedging, "I don't know if that's right." Nevertheless, she computed the difference of the angle measure and the circumference by hand. Just before beginning the subtraction, 28.93 – 23.54, Kacie casually remarked, "we'll put this [28.93] on top," which signaled Kacie was not constructing a quantitative difference in a normative sense. After arriving at a numerical difference of 5.39, Kacie justified her computation as described in Excerpt 5.23.

> Excerpt 5.23. Kacie assimilates CQ(23.54cm, Arc, 28.93°) as an additive structure.
>
> K: I subtracted the circumference from the measure of the blue angle, and [3 sec] I got five point thirty nine. And I guess I did that because the green arc is kind of like a chunk out of the circle, so it wouldn't make the circle bigger. Or wait – [3 sec] if you – you're like losing that part of the circle, so the – [6 sec] so you would subtract because you're losing it and not like regain – like adding it. Um, so that's why I subtracted.
>
> T: Okay. And so that number that you've gotten as the result of your subtraction, like what does that tell you in this – what does that mean in terms of the situation?
>
> K: I think that's like – oh. That would be – it – well – it's the distance between C and D (indicating the intercepted arc), if that's what you're asking. [5 sec]
>
> T: And can you tell me one more time how you know that it's the distance between C and D?
>
> K: Because the green arc was cut off by the angle and that's where the angle hit's the circle [pointing back and forth between C and D]. And so C and D, the distance between them would be that [pointing to 5.39, the result of her subtraction].

From her remarks, I infer Kacie viewed the green arc as embedded within the circumference and established a goal of reducing the circumference by some amount to obtain the length of the intercepted arc. Wishing to reduce the circumference, Kacie assimilated the situation as an occasion for subtraction. I infer Kacie chose to use the measurement of the central angle in her numerical calculations because this was the only other number provided in the prompt. In her calculation, Kacie subtracted the circumference from the angle measure, which I suspect was intentional to avoid obtaining a negative result for the length of the arc. Kacie assimilated the situation to a numerical operation and interpreted the computation in terms of the situation; however Kacie did not instantiate the quantitative operations corresponding to the numerical operations she enacted.

Kacie's speech as she worked toward a solution indicated the novelty of the CQ task for Kacie. In particular, she demonstrated multiple expressions of doubt, frequent pauses, and numerous restarts in speech throughout the exchange. These features of Kacie's speech indicated her uncertainty about the solution she was describing. Because of the prevalence of these features of speech, I suspect Kacie assimilated inconsistencies in her own reasoning—Kacie was perturbed. However, she had no readily available means of restoring her cognitive equilibrium.[73]

Following Kacie's activities in CQ(23.54cm, Arc, 28.93°), I posed three additional CQ tasks: CQ(24cm, Arc, 30°), CQ(23.54cm, 9.03cm, Ang), and CQ(23.54cm, 5.79cm, Ang). Kacie assimilated each of these tasks to an additive numerical structure. For CQ(24cm, Arc, 30°), Kacie subtracted the measures for the circumference and central angle and asserted the arc would measure 6cm. For the two CQ tasks involving unknown angle measure, Kacie computed the sum of the arc length and circumference rather than the difference. The reason Kacie chose to compute a sum is unclear to me as Kacie did not elaborate her rationale for this calculation. Kacie may have changed the numerical operation because she was asked to determine the measure of a different quantity (i.e., angle measure instead of arc length). Alternatively, Kacie's choice in operation may have

[73] As mentioned in Chapter 4, neither Kacie nor Camille exhibited well established schemes for reasoning proportionally.

been impacted by her templated figurative degree scheme. In CQ(23.54cm, 9.03cm, Ang), for example, an obtuse angle was perceptually available; as such, Kacie may have chosen to add in order to produce the largest numerical value possible using the additive structure to which she had assimilated the task.

Until this point on the CQ tasks, I had assumed a primarily passive role to allow opportunities for Kacie to spontaneously modify her reasoning without my intervention. Although Kacie appeared perturbed as she solved these four CQ tasks, she did not make any discernable modifications. After Kacie asserted the measure of the central angle in CQ(23.54cm, 5.79cm, Ang) was 29.33°, I took a more active role, hoping to engender Kacie's assimilation of the CQ tasks to a multiplicative structure.

I intentionally designed the fourth CQ task, CQ(23.54cm, 5.79cm, Ang), to provide a perceptually available central angle Kacie might assimilate as a right angle (Figure 5.122). In order to draw Kacie's attention away from the numerical measurements and potentially activate her extensive angular operations, I ask Kacie to consider the number of iterations of the central angle would be required to exhaust the plane. Our interaction is described in Excerpt 5.24.

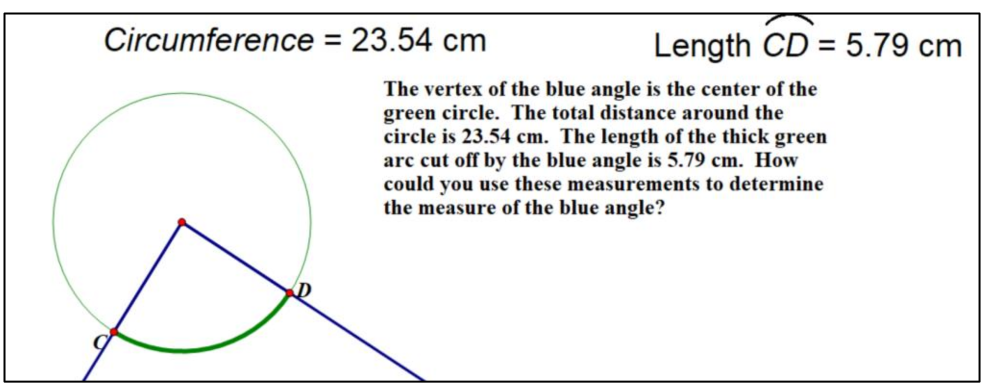


Figure 5.122. CQ(23.54cm, 5.79cm, Ang)

Excerpt 5.24. Kacie assimilates a right angle in CQ(23.54cm, 5.79cm, Ang)

T: About how many copies of that angle do you think you would need to make up the whole circle?
K: [5 sec] Um [3 sec] maybe [10 sec] four. [15 sec] Four or five.
T: Four or five? So what do you think about that?
K: Um, well looking at it now it kind of looks – well just looking at like the angle, it kind of looks like a ninety-degrees angle. But that's not what I got. So, [whispering] I don't know.

As illustrated in Excerpt 5.24, my prompting engendered Kacie to verbalize a source of her perturbation. Ignoring the provided numerical measurements and after iterating the central angle to exhaust the plane, Kacie assimilated the central angle in CQ(23.54cm, 5.79cm, Ang) as a right angle. Kacie had permanently assigned a right angle a measure of ninety-degrees; however, Kacie was aware this was different from the numerical measurement, 29.33°, she had obtained via addition.

Kacie assimilates CQ tasks as multiplicative structures. At the conclusion of Excerpt 5.24, Kacie was in a state of disequilibrium she did not appear immediately able to resolve. Because Kacie had not solved tasks involving circular arcs in prior teaching sessions, I knew very little about Kacie's quantification of arc length. The previous CQ tasks involved arc lengths that were, from my perspective, composite units. As such, Kacie would have needed to assimilate the arc length and circumference to a three-level-of-unit structure to quantitatively solve the task. For example, in CQ(23.54cm, 5.79cm, Ang), Kacie needed to view the circumference in terms of arcs and arcs in terms of centimeters. I hypothesized that Kacie might be able to solve a CQ task wherein the arc length did not need to be taken as a composite unit. In hopes of supporting the restoration of Kacie's cognitive equilibrium, I asked Kacie to solve CQ(12in, 1in, Ang), which we drew on paper as shown in Figure 5.123. Kacie's reasoning on this task is described in Excerpt 5.25 below.

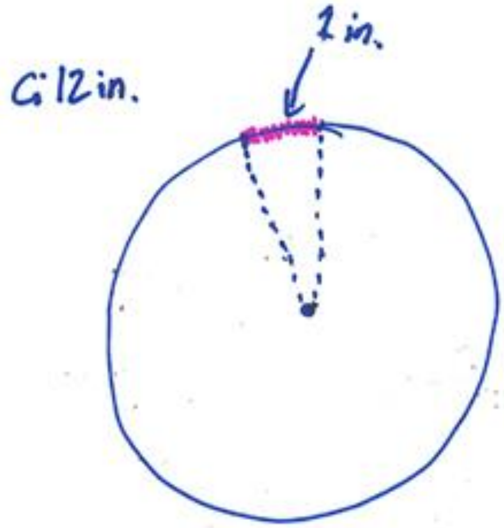


Figure 5.123. CQ(12in, 1in, Ang).

Excerpt 5.25. Progress toward a ratio quantification for Kacie.

T: What do you think the measure of that angle would be? And so we're saying again that this arc length here, that this would be one inch, right? [labels arc as 1 in.]

K: Um. [14 sec] I want to say like thirty degrees.

T: Okay. Tell me about that.

K: Because since this part [the arc] is one inch and that's like the – where the angle ends I guess. And since the whole circle is twelve inches, I did, um, twelve – [computes using long division] three hundred and sixty divided by twelve and that would be thirty. And since there's twelve of those angles in the circle, it would be thirty degrees because twelve thirty-degree angles would make it all the way around.

T: Oh. So how did you know that it would take twelve thirty-degree angles to make it all the way around?

K: Well, since – if this [arc] was actually like an inch and it takes twelve of these [arcs] to get all the way around, then it would have to take, um – [8 sec] hmm. Well I knew that it wasn't like one degree because it didn't take three hundred and sixty to get around. So [6 sec] I just did that [points to long division].

T: Mm hmm. So how – can you tell me about um –

K: Oh. Oh. So I did – I understand what you're saying now. Cause I did this [long division] because since this takes twelve times to go around

T: What takes twelve times to go around?

K: And the – the angle and the one inch. And since if that [points to "1in." label] took twelve times to go around, the angle would have to go around twelve times too. And so whatever – however many degrees this [angle] was, if you multiplied that by twelve it would have to give you three hundred and sixty.

In the 14 seconds of silence before Kacie initially asserted the angle measured "like thirty degrees," Kacie made significant temporary modifications to her way of operating in CQ tasks. From her explanation, I infer Kacie iterated the arc around the circle and, in doing so, assimilated the circumference as a composite unit composed of twelve unit arcs. For each iteration of the arc, Kacie inserted and united a copy of the central angle. Thus, Kacie mentally united the arc and angle to form a single object, which I call an *arc-angle*. One verbal indication Kacie united these quantities and simultaneously subjected them to iteration was her conversational coupling of "the angle and the one inch" as she responded to my query, "what takes twelve times to go around?"

Having constructed an arc-angle, Kacie's operations on the arc were simultaneously implemented on the angle. At left in Figure 5.124, the solid red arc represents the given arc which Kacie iterated to constitute the circle as a 12-unit composite, which is indicated by the segmented, dotted red circle. The black arched arrow emanating from the solid red arc is intended to represent this iteration. Kacie's construction of the arc-angle is represented by the sector shown to the right of the segmented circle, which is the unification of the given blue angle and red arc into a single multiplicative object. An open arrow points right from the arc-angle toward a second circle, which represents the simultaneous dual operations Kacie implemented as she iterated the arc angle—when iterating the arc to exhaust the circle Kacie simultaneously iterated the angle to exhaust the plane and produced a 12-partitioned plane.

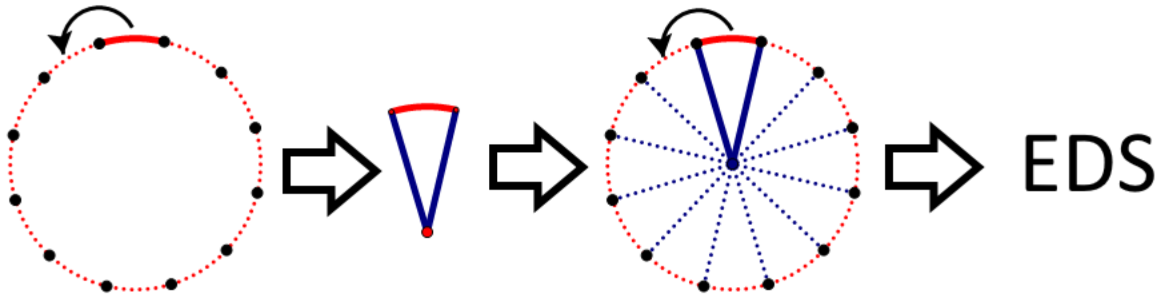


Figure 5.124. A model of Kacie's assimilation of CQ(12in, 1in, Ang) to the equisegmenting degree scheme.

Having simultaneously constituted the circle and the (truncated) plane as 12-unit composites of arcs and angles respectively, Kacie assimilated the situation to her equisegmenting degree scheme. In Figure 5.124 above, this assimilation is represented by the open arrow pointing toward the abbreviation EDS. Kacie determined the measure of each angle in the 12-partioned plane by computing 360°÷12 = 30°.

Kacie's initial difficulty in explaining how she knew twelve of the given angle would "make it all the way around" indicated she was not immediately aware of the operations she had enacted. However, by the end of the excerpt, Kacie demonstrated an awareness of her way of operating. Kacie's activities suggested a ratio quantification of angularity was within her zone of potential construction. A ratio quantification involves measuring an angle through creating a multiplicative comparison between two non-angular quantities, which Kacie instantiated in this task with the length of the intercepted arc and circle's circumference.

To encourage Kacie's new way of operating, I presented a sequence of follow up tasks, which ultimately involved a total of four concentric circles that I drew on the same paper as CQ(12in, 1in, Ang) as shown in Figure 5.125. First, I drew a new brown circle, labeled the circumference as "6in.," and asked Kacie how long the intercepted arc would be in inches. Kacie quickly responded, "it'd be half and inch," and explained, "because six is half of twelve and half of one is point five or half." In this case, Kacie indicated some sense of proportionality, which I suspect was occasioned by the geometric similarity of the figures and the familiar scale factor, one-half. Kacie did not appear to use the central angle or its measure in her reasoning.

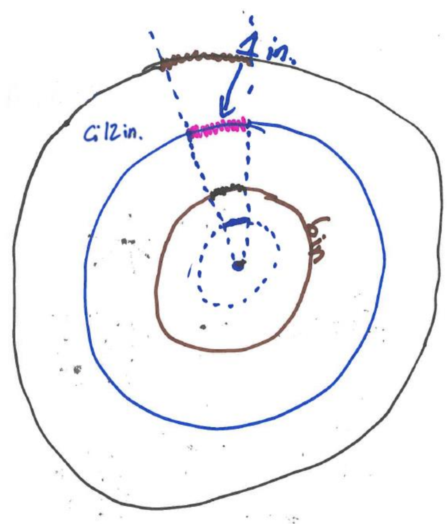


Figure 5.125. Concentric circles for CQ tasks.

Following Kacie's assertion that the intercepted brown arc measured half an inch, I asked Kacie what the degree measure of the angle would be. Kacie sat in thought for nearly 30 seconds before responding:

> This [brown arc] would still have to go around twelve times. Because if the measurement of it, which is half an inch, if you multiplied that by six – wait. If you multiplied that by twelve it'd give you six [pointing to the "6in" label for the circumference]. So it would still have to go around twelve times to get to six inches. So [3 sec] um [5 sec] the angle would still be thirty degrees.

As in CQ(12in, 1in, Ang), Kacie indicated she determined the number of angular iterations required to exhaust the plane through a multiplicative comparison of the intercepted arc length and circumference. Kacie was in the process of establishing a ratio quantification of angularity. That Kacie sat in thought for nearly 30 seconds prior to responding to this task was significant. The pause and her subsequent response indicated Kacie needed to enact the mental operations on the new arc length and circumference. She was not aware the multiplicative relationship would be invariant across all circles, which contraindicated a rate quantification of angle measure.

Further contraindications of a rate quantification for arc length. Following the tasks involving the brown circle, I drew a larger black circle (the largest circle in Figure 5.125 above) and asked Kacie to consider what fraction of this circle was cut off by the central angle. Kacie asked, "how big is the entire circle?" In the previous task, Kacie had reasoned about the measure of the intercepted arc in terms of the measure of the circumference. From her question, I infer Kacie was unable to determine the measure of the arc without a specified value for the circumference, which indicated she was unaware of the invariant fractional relationship between the lengths of arcs intercepted by the central angle and the circles' circumferences.

In the case of the black circle, Kacie initially responded that the arc would account for 1/24 of the circle. Kacie explained she viewed the black circle as approximately double the length of the blue (12-inch) circle, which suggested she needed to consider particular values to ascertain the fractional amount of the circle cut off by the central angle. Her response, one twenty-fourth, resulted from doubling the circumference without doubling the arc length. Later, Kacie self-corrected explaining the arc length would be one-twelfth of the circumference because "if [the black circle] is double the blue circle then that measurement [the black arc length] would also have to double."

Following Kacie's explanation, I asked her to consider the fraction of the circle accounted for by the intercepted arc if the circumference of the black circle was one and a half times the circumference of the blue circle. Kacie tried to reason in terms of mental calculations, though she did not specify precisely what she was calculating, and ultimately concluded the arc would account for one-eleventh of the black circle, which again contraindicated a rate quantification of angularity.

Finally, I drew a smaller dashed circle inside the brown circle, shaded the intercepted arc, and asked Kacie how many arcs of that size would be needed to make up the entire circle. As the bell rang, Kacie explained, "I'm going to go with six." Kacie explained she viewed the dashed circle as half the length of the brown circle, whose circumference was 6 inches. Although Kacie accounted for the changes in the circumferences of the circle she did not account for the changes in the lengths of the arcs. Through iterating an arc-angle, Kacie could determine an unknown angle measure when provided with arc length and circumference; however, Kacie did not iterate an arc-angle to determine the unknown length of an intercepted arc.

Summary of Kacie's April 21st session. In her April 21st session, Kacie indicated she had not constructed the splitting operation and was reliant on equisegmenting and fragmenting operations. Kacie systematically identified all convex angles in the angle identification task, but did not assimilate any reflex angles. Kacie initially assimilated CQ tasks as additive structures, but later assimilated the tasks as multiplicative structures. Through forming and iterating an arc-angle, Kacie assimilated CQ tasks involving unknown angle measure to her equisegmenting degree scheme, which indicated she was on the verge of quantifying angularity as ratio. For CQ tasks involving concentric circles, Kacie reenacted operations on the arc-angle for different circles and the corresponding calculations, which contraindicated a rate quantification of angularity. When length of the intercepted arc and circumference were not provided Kacie substituted particular numerical measures to determine the fractional amount of the circle's circumference accounted for by the intercepted arc, which indicated she was not aware of the invariance of this ratio across all circles.

Kacie's Final Interview

In Kacie's final interview, Kacie gave strong indications she had maintained schemes she constructed relatively early in the teaching experiment including angular congruence via superimposition, angular comparison via superimposition, and angular iteration. In the following sections I discuss Kacie's activities on angular splitting, angular units coordinating, estimation, and making a one-degree angle tasks. As I discuss these tasks, I briefly compare Kacie's activities to similar tasks throughout teaching experiment to illustrate stable elements in Kacie's ways of operating with angles as well as major modifications in her way of operating.

Kacie equisegments on the angular splitting task. When I asked Kacie how she would make her pair of chopsticks so a given pair was five times as open as her pair, she responded:

> Well, I would probably just start with a pair of chopsticks that was smaller than yours, like less open. And then I'd see if I did like five of those, it would equal yours. And if it didn't, I would adjust the measurement of the chopsticks until it did.

From her explanation, I infer Kacie established a goal of producing five congruent angular units within the given angular model. As in previous angular splitting tasks, Kacie's explanation indicated a sequential production of these units. Absent from her explanation was any indication of simultaneously injecting five units into the interior of the angle model. Kacie described how she would produce a partition through her activity; however, Kacie gave no indication of using a simultaneous insertion of five units to guide her initial estimate for the desired angle. As such, I infer Kacie needed to equisegment the given angle in order to produce the desired angle. Kacie did not need to engage in physical actions as she explained how she would solve the task, which indicates that she was reflectively aware of her way of operating. As such, the block quotation above is one of the clearest indications of Kacie's equisegmenting activity across the teaching experiment.

Kacie assimilates units coordinating tasks to her equisegmenting degree scheme.

> Imagine two angles: a green angle and a blue angle. It takes 18 green angles to sweep out the entire plane. It takes 3 green angles to sweep out one blue angle. What are the measures of the blue and green angles?

Figure 5.126. An angular units coordinating task from the final interview session.

Kacie assimilated the units coordinating task in Figure 5.126 above to her equisegmenting degree scheme. She determined the measure of the green angle by computing $360^\circ \div 18 = 20^\circ$ using long division. To determine the measure of the blue angle, Kacie initially computed $20^\circ \div 3 = 6.6^\circ$; she spontaneously self-corrected and, by multiplying 20° times three, determined that the blue angle would have a measure of 60°. When I asked Kacie how she knew to use division, Kacie responded:

> Well if it takes eighteen green angles to sweep out the entire plane, it's going to equal three hundred and sixty no matter what because the entire plane is three hundred and sixty degrees. And so – so something times eighteen has to equal three sixty because it has to add up to three sixty. And so instead of going through a bunch of numbers multiplied by eighteen, I just divided three hundred and sixty by eighteen and got twenty.

As she had done throughout the teaching experiment, Kacie used numerical division because it was the computational inverse of multiplication. Kacie did not indicate a quantitative meaning for partitive division (e.g., distributing 360 degrees across 18 angles).

Imagine two angles: a yellow angle and a red angle. It takes four yellow angles to sweep out the entire plane. It takes 3 red angles to sweep out one yellow angle. What are the measures of the yellow and red angles?

Figure 5.127. A planar recursive partitioning task from the final interview session.

On a second units coordinating task involving planar recursive partitioning (Figure 5.127), Kacie immediately assimilated the yellow angle, which exhausted the plane in four sweeps, as a 90-degree angle. This immediate recognition again confirmed her re-presentable template for a 4-partioned plane. Kacie determined the measure of her red angle using her equisegmenting degree scheme: "the red angle is thirty because it takes three to sweep out one ninety, and thirty plus thirty plus thirty is ninety."

Following Kacie's determination of the degree measures for both angles, I asked Kacie to draw a picture of the situation. She began by drawing a right angle, which she initially bisected commenting, "oh, that's four." From Kacie's comment, I infer that she was imagining bisecting each of the 45-degree angles she had created. I pressed Kacie to try again, and she quickly produced a 4-partitioned plane to account for the yellow angles. As she considered the red angles, Kacie described her thinking:

> And then the three red angles, I don't really know how to draw those because it's like if you did this is one [segments the right angle into two non-congruent parts as shown in Figure 5.128 left]. Then you have two, but then if you add – oh wait, just kidding. [Segments the larger part of the right angle into two approximately congruent parts as shown at right in Figure 5.128.] Yeah, like that.

Figure 5.128. Kacie equisegments a right angle into three equiangular parts.

I infer that Kacie established a goal of producing three equiangular parts within the right angle when I asked her to draw the situation. However, Kacie was initially unsure how to physically accomplish this goal, which indicated she did not simultaneously project three equiangular parts into the right angle. Instead, she inserted congruent parts sequentially and thus was engaged in equisegmenting behavior. Kacie's surprise as she produced three approximately congruent angular parts within the right angle is indicated by her comment, "oh wait, just kidding."

Following Kacie's drawing of the situation, I wanted to see if Kacie maintained a three-level-of-units structure throughout her activities on the task. So, I removed the drawing from Kacie's visual field and asked Kacie how many sweeps of the red angle would be needed to exhaust the plane. Kacie quickly responded, "twelve," and justified the result by verbally describing inserting three red angles into each of the four yellow angles. Thus, Kacie indicated that she maintained an awareness of three levels of units throughout this task.

Making an angle with a measure of one degree. When I asked Kacie how she would make an angle with a measure of one degree, Kacie thought for 18 seconds before remarking, “that’s a good question. I don’t really know.” I infer that Kacie did not produce a one-degree angle through splitting an angular template of known measure (e.g., the plane or a right angle) precisely because she had not constructed the splitting operation. After another 14 seconds of silence, I asked Kacie how she would check if a given angle had a measure of one degree. Kacie explained that she would repeat the angle to exhaust the plane:

> Since it’s a one-degree angle there’s three hundred and sixty of them because it has to add up to three hundred and sixty degrees. So if that angle went around, until it got back to where it started, three hundred and sixty times, then it would be a one-degree angle.

Kacie demonstrated a quantitative meaning for a one-degree angle relied on iteration. She knew that a one degree iterated 360 times would exhaust the plane. This operative conception of verifying a given angle had a measure of one degree was markedly different than Kacie’s January 25th response to the same prompt wherein she suggested checking via “a formula or something.”

Estimation. On the two angle estimation tasks, Kacie reasoned using her templated figurative degree scheme. When provided with a 120° angle in non-standard orientation, Kacie inserted a straight angle and estimated the angle had a measure of 110°. Kacie explained, “since it’s a little less than one eighty, um, I said one ten.” For the 120° angle in non-standard orientation, Kacie estimated the angle had a measure of 120°. She inserted a right angle and noted that amount by which the given angle exceeded the right angle “looked about right for it to equal one twenty.” In both these estimation tasks, Kacie leveraged properties of order on angular templates for which she had assigned permanent measures.

CQ(22.83cm, 3.47cm, Ang). Recall that Kacie first solved CQ tasks in her final teaching session where she initially assimilated CQ tasks as additive structures and later indicated she was on the verge of constructing a ratio quantification of angularity. In the final interview session, I presented Kacie with CQ(22.83cm., 3.47cm, Ang), which is shown in Figure 5.129. As she had done in her previous teaching session, Kacie initially assimilated the CQ task to an additive structure as is indicated in Excerpt 5.26.

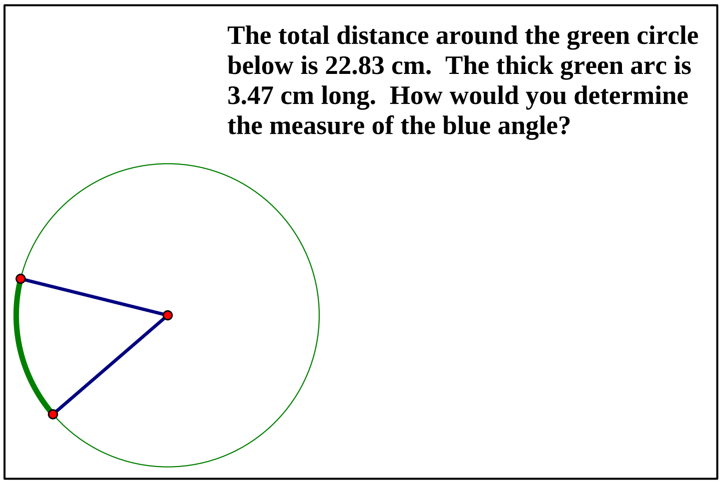


Figure 5.129. CQ(22.83cm., 3.47cm, Ang)

Excerpt 5.26. Kacie solves CQ(22.83cm, 3.47cm, Ang). (Onset)

T: How would you determine the measure of the blue angle?
K: Um, [11 sec]. You could subtract, um, the three point four seven and the twenty two point eighty three. And that might give you your measurement. Because that's what that angle is like that um – well, no. Just kidding. Um [16 sec]. Yeah. I guess you could subtract.

In the onset of Excerpt 5.26, Kacie's initial eleven second pause indicated that she had not yet established a scheme for determining angle measure through a multiplicative comparison of circular lengths. Kacie indicated she wanted to subtract the measures of the arc length and circumference to ascertain the measure of the central angle. Kacie seemed dissatisfied with this method, which is indicated by her sudden rejection of the method ("well, no. Just kidding"). In the subsequent 16 second pause, I infer that Kacie was searching through her existing operations to find a viable alternative to the subtraction she seemed disinclined to enact. Unable to find a viable alternative in the present moment, Kacie reluctantly settled on subtraction remarking, "I guess you could subtract."

To understand why Kacie had settled on subtraction as the appropriate operation, I asked Kacie what that subtraction would tell her. In the following moments, which are described in the continuation of Excerpt 5.26 below, Kacie rejected the subtraction and the additive structure to which she had originally assimilated the task.

> Excerpt 5.26. Kacie solves CQ(22.83cm, 3.47cm, Ang). (Continuation)
> T: And what would that subtraction tell you?
> K: Um [4 sec]. No! Wait. You could do twenty two point eight three divided by – wait, no. Yeah. Divided by three point four seven and, um, that would give you the number of times the angle would go around the circle. And then you could do that number – uh, well three hundred sixty divided by that number and then that would give you the measurement of the angle.
> T: Can explain why that works?
> K: Because the three point four seven is the green arc, and that's the – like the edge of the, um, angle. And if you want to find out how many – what the measure is, you want to find out how many times it can go around cause that's kind of easier to do than just like – I don't know, but it seems easy to me. And so, then you could do that divided by – well twenty two point eighty three divided by three point forty seven, and then that would give you a number of how many times the blue – the green arc could go around the circle. And then that would give you how many times the blue angle would need to go to the circle to reach its – back to its starting point. And then if you did three hundred and sixty divided by the number of times the blue angle needed to go around it would give you the measurement of the circle – or the angle.

Kacie spontaneously implemented operations similar to those she had enacted in CQ tasks from her final teaching session. Kacie indicated she assimilated the CQ task in terms of a multiplicative structure in that she assumed the circle could be constituted in terms of units of the intercepted arc. This multiplicative structure is indicated by Kacie's call to divide the circumference by the arc length, which she interpreted as "how many times…the green arc could go around the circle." Because Kacie also interpreted this quotient as the number of times the angle would need to be iterated to exhaust the plane, I consider Kacie to have mentally formed an arc-angle by uniting the central angle and the intercepted arc. At this point, Kacie assimilated the situation to her equisegmenting degree scheme and explained she would determine the measure of the angle by dividing 360 by the quotient of the circumference and arc length.

The solution Kacie described in the continuation of Excerpt 5.26 was markedly different from her solution to a tasks in the previous teaching session (e.g., CQ(12in, 1in, Ang) and CQ(6in, .5in, Ang)). Here, Kacie described the calculations she would need to enact to determine the measure of the central angle *without actually computing* particular values, and she provided quantitative interpretations for these calculations. As such, I consider Kacie to have made a reflected abstraction in that she was aware of how she would solve the task without carrying out the solution process.

As indicated by the onset of Excerpt 5.26, Kacie had not established a scheme for operating in CQ tasks involving unknown angle measure. Due to her limited experience with these CQ tasks in the teaching experiment, she had not instantiated these operations regularly enough to construct a characteristic way of operating in these situations. However, because of Kacie was reflectively aware of the operations she would carry out to solve CQ(22.83cm, 3.47cm, Ang) in this final interview session, I am confident that

Kacie would have constructed a stable ratio quantification of angle measure rather quickly if the academic calendar had allowed for additional teaching sessions.

Summary of Kacie's final interview. In her final interview session, Kacie's activities indicated she had constructed schemes for angular congruence, angular comparison, and angular iteration. Kacie's activities on the angular splitting task contraindicated the splitting operation; instead, Kacie relied on her equisegmenting operation. Kacie assimilated angular units coordinating tasks to her equisegmenting degree scheme and angular estimation tasks to her templated figurative degree scheme. Kacie did not provide a method for producing a one-degree angle, but she described a process for checking via iteration to see if a given angle had a measure of one-degree. Kacie's activities on CQ tasks confirmed a ratio quantification of angularity was within her zone of potential construction; however, Kacie indicated her solution was novel, which contraindicated a ratio quantification of angularity as an established way of operating.

CHAPTER 6

CONCLUSIONS & IMPLICATIONS

In this concluding chapter, I have two major aims: (a) to summarize the findings I presented in chapters 4 and 5, and (b) to discuss implications of these findings for teaching and research.

Summary of Findings

In the Chapters 4 and 5, I presented a detailed analysis of the students' activities throughout the teaching experiment and the mathematical operations and schemes I abstracted from my analysis of these activities. In this section, return to the purpose and research questions I set forth in the introductory chapter. The overarching purpose of this study was to examine how high school students understood angle measure. At the onset of this dissertation, I outlined three specific research questions:

1. What motions and operations serve in students' constructions of angularity?
2. What are students' quantifications of angularity at the onset of the teaching experiment, and how do these quantifications change throughout the teaching experiment?
3. How do students' quantifications of angularity compare in rotational and non-rotational contexts?

I present this section in two parts. First, I outline the conceptual structures I used in modeling the Camille's and Kacie's quantifications of angularity throughout the study. In this discussion, I address my first research question by describing motions and operations that serve in the construction of angularity, which I abstracted from my analysis. Second, I discuss the progress of the students throughout the teaching experiment in terms of these schemes and operations and, in doing so, I return to the remaining two research questions I set forth in the introductory chapter.

Summary of Operations and Schemes

Motions involved in the construction of angularity. An awareness of angularity involves motion of some kind through the interior of what an observer would call an angular object. These motions may be enacted physically or mentally. In my analysis of the students' activities, I have presented three such motions: radial sweep, re-presented opening, and segment sweep (Figure 6.1). A radial sweep involves the rotation a single ray (or segment), whose endpoint is fixed at the vertex of the angle, through an angle's interior (Figure 6.1 right). Re-presented opening involves imagining two distinct rays (or segments) opening from a closed position (Figure 6.1 left). Radial sweep differs from re-presented opening in that the former motion involves a single rotating ray while the later involves two distinct rays opening from a closed configuration. Segment sweep involves a linear segment, with one endpoint on each side of the angle, moving through the interior of the angle (Figure 6.1 center); the length of the segment increases at it is imagined moving away from the vertex.

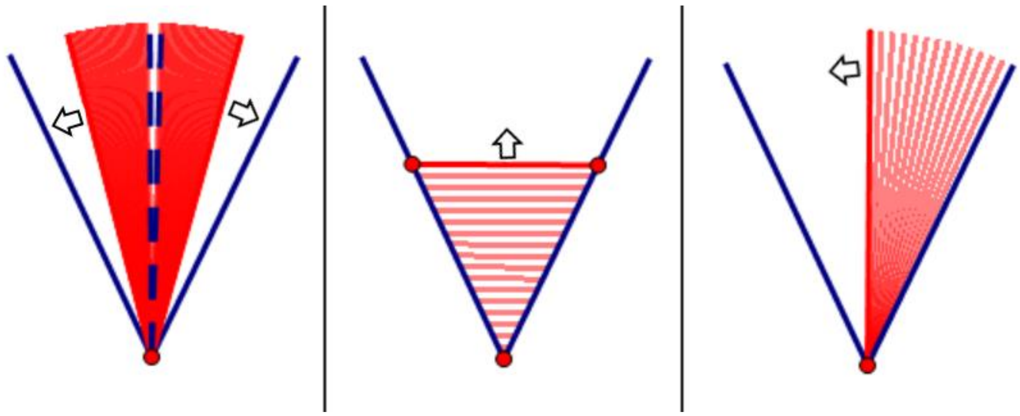


Figure 6.1. Re-presented opening (left), segment sweep (center), and radial sweep (right)

If an individual conceives the interior of an angle model as having an infinite area, all three of these motions can be viewed as being bounded in one sense and unbounded in another. In the case of re-presented opening and radial sweep, the motion is bounded temporally in that the motion has a beginning and an end; the extent of angularity is indicated by the duration of the motion. Re-presented opening and radial sweep are unbounded in a spatial sense, in that the length of the ray(s) moving through the interior of the angle is potentially infinite. In contrast, segment sweep is bounded spatially in that the segment moving through the interior of the angle is always of finite length; however, segment sweep can be thought of as temporally unbounded: the motion of the sweeping segment starts at the vertex and can be imagined continuing indefinitely through the interior of the angle. In the case of segment sweep, the extent of angularity is indicated by the rate at which the segment grows as it moves away from the vertex and is, therefore, an intensive quantity (Piaget, 1965) resulting from a coordination of the length of the sweeping segment and this segment's distance from the vertex of the angle.

Length altering and angularity preserving operations. In my analysis, I identified five operations students used to alter lengths in angular contexts while preserving angularity: experiential dilation, truncation, hyper-truncation, elongation, hyper-elongation. Experiential dilation involves imagining the perceptual length of an angle model varying as if one was moving directly toward or away from the object. In the case of truncation and elongation, the sides of an angle model are mentally shortened or extended, respectively, to a particular length (e.g., the length of another angle model). For hyper-truncation and hyper-elongation, the sides of angle model are mentally shortened or extended indefinitely; thus, hyper-truncation and hyper-elongation render an angle model's sides either infinitesimal segments or infinite rays, respectively. Experiential dilation is distinct from the other four length altering operations in that differences in length are conceived through an imagined change in perspective rather than intentional modifications to the length of the angle model.

Congruence and comparison. When one or more motions serving in the constitution of angularity have been abstracted, individuals can take the interior of an angle as an object of reflection without deliberately attending to the motions described above. Having made such an abstraction, individuals can subject the interior of an angle to further action or operation when they assimilate a situation involving an awareness of indefinite angularity. Two examples of these schemes involve enacted or imagined superimposition: angular congruence via superimposition scheme and angular comparison via superimposition scheme. These schemes are fundamental in establishing angularity as a gross quantity.

In the case of the congruence scheme, if an individual establishes a goal of producing an angle congruent to another angle (e.g., setting a pair of chopsticks to have the same openness as another pair), the individual may superimpose the vertex and sides of one angle model atop the other. The result of the angular congruence via superimposition scheme is a second angle congruent to the first. In the case of the comparison scheme, an individual may establish a goal of determining which of two angles indicates a greater extent of angularity. After engaging in the mental or physical superimposition, the individual becomes aware of a difference in angularity as the interior of one angle is contained within the interior of the other. The result of the angular comparison via superimposition scheme is an awareness of which angle indicates a greater extent of angularity. When the angle models involved in the situation of these schemes for congruence and comparison involve sides of different lengths, the activity of either scheme necessarily involves at least one of the length altering operations outlined in a previous section.

The span operating scheme. A span operating scheme is a pseudo-angular scheme because its activity entails operations on the span of an angle model (i.e., the distance between the endpoints of an angle model's sides) rather than operations enacted on the interior of an angle model. For example, an individual may compare two angle models by comparing their spans. A span operating scheme may also include extensive operations (e.g., iteration or partitioning), which are also enacted on the span of an angle model.

Producing units from units. Angle as extensive quantity arises when an individual produces units from other units. In my analysis, I have described a variety of such operations, which I summarize here. Angular repetition involves the physical action of producing a whole-number multiple of a given angle. In other words, angular repetition involves the production of new angle by uniting adjacent copies of angular units. For example, an individual might repeatedly trace and join three copies of a given angle to produce an angle three times as open as the given angle (See Figure 6.2). If the individual could unite copies of an angle in re-presentation, then the individual would be engaged in angular iteration.

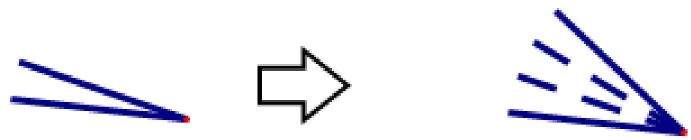


Figure 6.2. Producing an angle three times as open as a given angle.

In this dissertation, I have distinguished between two different operations for breaking angular units into same-sized parts: equisegmenting and equipartitioning. Equisegmenting involves the sequential production of equiangular parts within an angular whole. Equipartitioning involves the simultaneous insertion of equiangular parts within an angular whole.

If an individual develops an awareness that partitioning and iterating are inverse operations, these operations may be united into a single operation, splitting. An individual who has constructed the splitting operation would be able to solve a task stated in the language of iteration through partitioning. For example, if asked to produce an angle so that a given angle was four times as open as the desired angle, an angular splitter might mentally partition the given angle into four congruent parts and disembed one of the parts to produce the desired angle. Such an individual would be aware that four copies of the desired angle constituted the given angle and that the given angle broken into four equiangular parts would contain four instantiations of the desired angle.

The operations outlined above produce composite angular units, which individuals might take as input for further action or operation. For example, Figure 6.2

above illustrates a three-unit angular composite resulting from angular iteration. If an individual could take this angular unit of three as input for iteration while maintaining an awareness of the constituent units, the individual can iterate composite units. Operations on composite units produce three-level-of-unit structures.

Re-presentable templates and planar coverings. A re-presentable template is a mental image that can be brought forth in visualized imagination. Examples of re-presentable templates for angles include a right angle, a straight angle, and a full angle; the interiors of these angles are a quarter-plane, half-plane, and full-plane, respectively. If an individual's re-presentable angular template is limited to a single orientation, then the template is orientation dependent.

A planar covering is set of adjacent angles where the interiors are disjoint and the union of the interiors is the plane. An *n*-partitioned plane is a planar covering formed by *n* congruent, disjoint, adjacent angles.[74] In the teaching experiment, I found students constructed re-presentable templates for 2- and 4-partitioned planes.

Schemes involving degrees. A figurative degree scheme (FDS) emerges when an individual coordinates her number sequence, in an ordinal sense, with relative extents of angularity, which is a modification of schemes for angular congruence and comparison. Upon the construction of an FDS, an individual can assign numerical measures to particular angles such that the numerical measures and relative extents of angularity are consistently ordered. In other words, if angle *A* is less open than angle *B*, an individual having constructed the FDS might assign measures of *m* and *n* degrees to angles *A* and *B*,

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[74] An individual need not operatively partition to produce a partitioned plane. Instead of imagining a simultaneous insertion of units, an individual may produce a partitioned plane through the sequential production of equiangular parts (i.e., via the equisegmenting operation).

respectively, with $m < n$. I emphasize these are *numerical* measures and not operative measures, which arise through extensive quantitative operations.

If an individual permanently assigns a numerical measure in degrees to a re-presentable angular template, then she has constructed a templated figurative degree scheme (TFDS). For example, she may designate a right angle as a 90-degree angle.[75] Upon constructing a TFDS with a 90-degree designated right-angle template, an individual would impute a numerical measure less than 90 degrees to an angle less open than a right angle and a numerical measure greater than 90 degrees to an angle more open than a right angle. An individual who has constructed the FDS or TFDS would reason about a one-degree angle in comparative terms (e.g., "a small angle").

The equisegmenting degree scheme (EDS) emerges when an individual conceives the degree-designated template involved in a TFDS as a composite angular unit via the equisegmenting operation. For an individual who has constructed the EDS, a one-degree angle is an angle that needs to be iterated *n* times in order to constitute an *n*-degree designated re-presentable template. Thus, a crucial distinction between the EDS and the FDS is that degrees are operative units of angular measure in the former and figurative units in the later. If an individual having constructed the EDS is asked to consider an *n*-degree designated template broken into *m* equiangular parts, she may determine the measure of the equiangular parts by searching for a value *p* so that *n* is the result of *m* iterations of *p*. For example, an individual having constructed the EDS might consider the measure of each angle in a 3-partitioned right angle as having a measure of 30 degrees because 30° + 30° + 30° = 90° or because 3*30° = 90°. Because the EDS does not entail equipartitioning operations, an individual who has constructed the EDS may require

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[75] A TFDS may have more than one degree-designated re-presentable angular template.

multiple attempts to produce a drawing of three equiangular parts contained in a right angle precisely because she imagines the insertion of the parts sequentially rather than simultaneously.

Summary of Students' Progress in the Teaching Experiment

In the previous section, I outlined the schemes and operations I abstracted from my analysis of Kacie's and Camille's activities throughout the teaching experiment. In this section, I first present a summary of each students' progress throughout the teaching experiment in terms of these schemes and operations; in doing so, I address my second research question: what are students' quantifications of angularity at the onset of the teaching experiment, and how do these quantifications change throughout the teaching experiment? Following this discussion, I consider significant differences in the students' quantifications of angularity. To close the section, I address my third research question by comparing students' quantifications of angularity in rotational and non-rotational contexts.

Camille's progress in the teaching experiment. I structure my discussion of Camille's quantification of angularity and modifications to this quantification in four parts. First, I discuss inferences about Camille's extensive quantitative operations and quantification of angularity indicated in the initial interviews (Chapter 4). Then, I present three sections summarizing the modifications Camille made in her quantification of angularity throughout the teaching experiment, which correspond to Parts 1, 2, and 3 of Chapter 5.

Camille's initial interview sessions. During her initial interview sessions, Camille's activities indicated that she could assimilate situations with two levels of units and produce a third level of unit in activity. In the case of linear material, I inferred Camille could produce composite units through repeating lengths of string; however, Camille had not yet constructed the splitting operation. Camille demonstrated an awareness of angularity via radial sweep in the laser context. Camille attended to the interior of angle models on tasks involving chopsticks and indicated the angular congruence via superimposition scheme. For example, in Camille's initial interview she set short chopsticks to have the same openness as long chopsticks by superimposing the former atop the latter so the chopsticks would have "the same like space."

At the onset of the teaching experiment, Camille did not produce units from other angular units through iteration or partitioning. Instead, Camille used bursts of cadenced motions to open or close angle models on angular multiple and splitting tasks, respectively. These cadenced motions were a novel application of Camille's number concepts in an unfamiliar context.

Camille's quantification in Part 1. In Part 1 of the teaching experiment, Camille indicated two motions as she compared relative extents of angularity in non-rotational contexts (i.e., situations involving chopsticks): re-presented opening and segment sweep. Additionally, Camille indicated the truncation, elongation, and experiential dilation operations. On November 12th for example, Camille compared the openness of two sets of chopsticks by mentally truncating the longer pair to the same length as the shorter pair and comparing the duration of motions required to open each pair of chopsticks (i.e., re-presented opening). As a second example, Camille indicated an awareness of angularity via segment sweep as she compared the openness of her chopsticks to Kacie's chopsticks in the November 12th teaching session. To justify that Kacie's chopsticks were more open that her chopsticks, Camille explained, “mine’s starting off really small and getting bigger…and that one’s just like…open really big” as she separated her fingers moving over her chopsticks (See Figure 5.4) and repeated a similar gesture with her hand in midair as she discussed Kacie’s chopsticks. Camille developed the angular comparison via superimposition during Part 1 on the teaching experiment as well.

Camille’s use of superimposition for congruence and comparison indicated she had constructed a comparative quantification of angularity (i.e., a gross quantification). During Part 1, Camille did not demonstrate she could form composite angular units. For example, when asked to set a pair of chopsticks to be twice as open as a given pair of chopsticks, Camille established congruence via superimposition and then opened her chopsticks in two bursts, just as she had done during her initial interview.

In her paired teaching sessions with Kacie in Part 1, Camille generally assumed a more assertive role in teaching sessions than Kacie, though there were occasions where Camille imitated Kacie. For example, Camille considered comparing the spans of angle models after observing Kacie comparing the spans. Also, Camille imitated Kacie’s use of hyper-truncation to compare the openness of angle models.

Camille’s quantification in Part 2. In Part 2 of the teaching experiment, Camille made substantial progress in constructing extensive angular operations. On January 11th, Camille indicated angular repetition by drawing an angle model two and three times as open as a given angle model. This change in Camille’s reasoning on angular multiple tasks can be explained, at least in part, by her observation and imitation of Kacie’s activities on an angular measurement task within the same teaching session. On this measurement task, Camille attempted to imitate Kacie’s strategy for measuring the openness of one angle model using another angle model. In her imitation, Camille did not produce an exhaustive segmentation of the interior of the angle model to be measured; however, this activity of counting instantiations of one angle within another may have occasioned angular repetition in Camille.

Establishing angular repetition was an indication Camille could form composite angular units when supported by her physical actions on perceptually available angle models. When Camille began to establish composite angular units through repetition, she did not initially take these composite units as input for producing new units (see January 19th teaching session). In other words, although Camille could assimilate angular material as an occasion for repeating, she did not initially apply repetition to an angular composite unit to produce three levels of angular units in activity, even though she produced three levels of linear units in activity during her initial interview session. Later in Part 2, Camille did indicate she could coordinate three levels of angular units in activity by deliberately monitoring repetitions of composite angular units (see February 29th teaching session) through counting.

In Part 2, Camille discussed her prior experiences with angle measure (within and beyond school mathematics) and, in particular, described (a) the attribute being measured and (b) what it meant for an angle to have a measure of one degree. Camille indicated that angle measure might refer to multiple attributes including side length, orientation, and openness (See January 11th & 19th sessions). Camille did not demonstrate an operative conception of a one-degree angle, and, for a one-degree angle, drew an angle with short sides whose measure exceeded 45°, from my perspective. On February 22nd, Camille was unsure how many parts full and right angles should be partitioned to produce one-degree angles.

For Camille, degrees were not exclusively a unit for measuring angularity. Instead, degrees were conflated with other attributes like side length and orientation. However, Camille's conception of angularity was *not* conflated with these attributes. In both Parts 1 and 2 of the teaching experiment, Camille regularly manipulated the side lengths and orientation of angle models while maintaining an awareness of invariant angularity. For example, in Camille's February 1st session, she indicated changing side lengths and orientation would not impact the openness of angle models in GSP. So, while Camille's conception of degrees was problematic from my perspective, her conception of angularity was not.

Also in Part 2, Camille indicated she had established a re-presentable template for a right angle; however, she had not designated this template as having a measure of 90°. As I demonstrated in my analysis of her February 22nd teaching session, a 90° angle was different from a right angle for Camille. Camille indicated she had established a 360° designated full-angle template; but Camille had yet to establish a re-presentable template for a 4-partitioned plane.

Throughout Part 2 of the teaching experiment, Camille adopted a more passive role during paired teaching sessions than Kacie. Camille often relied on Kacie to respond first to tasks and also imitated her actions. In several instances, Camille's imitation was productive in that her reflection upon the acts engendered the construction of new ways of operating (e.g., angular repetition was engendered in part by the angular measurement task). In other instances, Camille's deferential regard for Kacie became problematic in that Camille began to view her mathematical capabilities in the teaching experiment as inadequate (see February 29th session).

Camille's final sessions (Part 3). Throughout her final three sessions, I observed Camille's reasoning to be constrained by two persistent obstacles. First, Camille could assimilate tasks with two, but not three, levels of units. Second, Camille's conception of degrees as a unit of angular measure remained conflated with orientation, which I attribute to her prior experiences and her orientation-dependent right-angle template.

In Part 2, Camille had assimilated an angle model with perpendicular sides in nonstandard orientation as a right angle in her February 22nd paired teaching session with Kacie; throughout Part 3, however, Camille consistently indicated that her right-angle template was orientation dependent. Furthermore, throughout the teaching sessions in Part 3 Camille indicated she had not permanently designated a right angle as having a measure of 90°.

In interactions where Camille had established the measure of a right angle as being 90°, Camille demonstrated she could determine the measures of the constituent angles in an *n*-partitioned right angle through the computation 90 ÷ n. Furthermore, Camille established a meaning for a one-degree angle that relied on the equisegmenting operation; specifically, Camille knew that 90 iterations of a one-degree angle would be needed to exhaust a right angle.

In her final interview session, Camille indicated a 90-degree designated right angle template, though her conception of degrees was still orientation dependent. Camille did not establish a re-presentable template for a 4-partioned plane throughout Part 3 of the teaching experiment. However, Camille could establish a 4-partitoned plane through repetitions of four right angles. On at least one occasion (see March 31st session), Camille indicated she could structure a full angle as four right angles each containing 90 one-degree angles by equisegmenting.

Kacie's progress in the teaching experiment. I structure my discussion of Kacie's quantification of angularity throughout the teaching experiment in four parts: Kacie's initial interview sessions (Chapter 4), sessions in Part 1 (Chapter 5), sessions in Part 2 (Chapter 5), and Kacie's final sessions (Part 4 of Chapter 5).

Kacie's initial interview sessions. At the onset of the teaching experiment, I inferred from Kacie's activities involving linear material that she could coordinate three levels of units in assimilation and that she had constructed the splitting operation. In her initial interview, Kacie indicated an awareness of angularity via radial sweep in the laser context and an awareness of angularity via re-presented opening in the chopstick context. For example, when solving a task that involved re-presenting two hidden angle models, Kacie's justification and gestures suggested she imagined closing one angle model to obtain the configuration for the second angle model.

Although Kacie exhibited an awareness of angularity, she did not implement extensive quantitative operations on the interior of angle models to produce composite angular units in subsequent tasks with chopsticks during the initial interview. Throughout these tasks, Kacie indicated a span operating scheme, which she implemented when solving angular tasks designed to investigate congruence, comparison, multiples, and splitting. For example, when asked to set a pair of chopsticks to be four times as open as a given pair of chopsticks, Kacie set her chopsticks so the span contained four iterations of the span of the given chopsticks.

Kacie's quantification of angularity in Part 1. In Part 1 of the teaching experiment, Kacie maintained a relatively passive disposition by deferring to Camille, and by imitating and extending her actions. For example, I attribute Kacie's attention to the interior of the angle models for congruence and comparison to her observation of Camille's activities in paired sessions. Additionally, Kacie demonstrated the hyper-truncation operation only after Camille used a truncation as part of her justification for comparing the angularity of two angle models. Kacie developed schemes for congruence and comparison reliant on superimposition and demonstrated truncation elongation, hyper-elongation operations as well.

Beyond Camille's influence, I attribute much of Kacie's progress in Part 1 to her experimentation with various attributes of angle models. The span of chopsticks was salient for Kacie, and she considered how the span and spanned area were related to the openness of the chopsticks. In particular, Kacie established a generalization involving side length, span, and openness, stating that, for two pairs of chopsticks that were equal in length, the openness could be compared via comparing the spans (see November 17th and December 3rd sessions).

In Part 1, Kacie did not develop a scheme for angular repetition in that she did not repeat an angle model *n* times to produce an angle *n* times as open as a given angle. After distinguishing between span length, spanned area, and openness, Kacie continued to operate on the span of angle model when setting chopsticks in angular multiple tasks. However, Kacie checked her result by repeating one angle model into another angle model; I attribute this development in Kacie's reasoning to the distinction she developed between openness and span length. In her December 3rd session, Kacie began to set chopsticks in angular multiple tasks by repeating the span obliquely, which is accounted for by her reflection on repeating adjacent copies of an angle model to check the estimates she made on angular multiple tasks.

Kacie's quantification of angularity in Part 2. In Part 2 of the teaching experiment, Kacie demonstrated that she could measure the angularity of an angle, *A*, with a less open angle, *B*, by exhaustively segmenting *A* with units of *B*. After producing such an exhaustive segmentation, Kacie modified her way of operating in angular multiple tasks. After reflecting on her oblique span repetition, Kacie developed angular repetition (see January 11th session) and began forming composite angular units.

Kacie indicated she could insert rotational motion into non-rotational angle contexts when prompted, but did not insert rotational motion spontaneously. Kacie's domain for openness included closed, convex, and straight angles. In contrast, Kacie's domain for amount of sweep in rotational contexts included reflex and non-reflex angles; additionally, Kacie established a notion of conjugate angles in rotational contexts (see February 29th session).

After establishing a scheme for angular repetition, Kacie indicated she could mentally copy and adjoin composite angular units to form planar coverings; in other words, Kacie could segment the plane using composite angular units and produce three-level-of-unit structures. Kacie indicated she had constructed re-presentable templates for a right angle, as well as 2- and 4-partitioned planes. Kacie did not indicate partitioning or splitting operations in angular contexts.

When asked to consider the attribute being measured when one discusses angle measure, Kacie drew arcs near the vertex of an angle model and explained the angle measure would change depending on where the arc was drawn, which indicated a conflation of angularity and other attributes (e.g., arc length, sector area). Kacie described degrees as a measure of an angles openness, which I attribute to her experiences in the teaching experiment. Kacie did not initially exhibit an operative conception of a one-degree angle and instead reasoned about a one-degree angle comparatively (e.g., "a very small angle). Kacie demonstrated a templated figurative degree scheme (TFDS), which was initially reliant on a 90-degree designated right-angle template and later included a 360-degree designated full angle template. Ultimately, Kacie assimilated right-angles as composite angular units containing 90 one-degree angles (see February 22nd teaching session).

Combining her TFDS with her equisegmenting operations, Kacie developed the equisegmenting degree scheme (EDS), which was facilitated by her ability to coordinate three levels of units produced through iteration. When an *n*-partitioned plane or *m*-partitioned right angle was provided or suggested, Kacie could determine that the measure of one of the constituent angles was $360 \div n$ or $90 \div m$, respectively. Kacie's numerical division in these contexts was occasioned by equisegmenting full or right angles via composite units. To establish the measure of the constituent angles in an *n*-partitioned plane, for example, Kacie searched for a degree measure, *d*, so that *n* repetitions of *d* would result in 360°. Kacie had not established the equipartitioning operation in angular contexts and did not interpret these contexts as partitive division. As such, Kacie's division was a numerical curtailment resulting from her quantitative operations and she could interpret this numerical division quantitatively, even though she had not established angular equipartitioning. Although Kacie could the determine the measure of an angle in degrees when a partitioned familiar template was suggested, Kacie had difficulty representing these partitioned templates precisely because she had not constructed partitioning operations in angular contexts.

In her interactions with Camille in Part 2, Kacie assumed a more assertive role in teaching sessions than Camille. In general, Kacie responded to tasks much more quickly and confidently than she did in Part 1, which I attribute to her assimilation of angular material to her extant extensive operations. Kacie was always patient with Camille, who experienced obstacles more frequently in Part 2. Toward the end of Part 2, I inferred Kacie was exercising deliberate restraint to allow Camille opportunities to solve tasks independently.

Kacie's quantification of angularity in Part 4. In Part 4, Kacie indicated schemes for angular congruence, comparison, iteration, and equisegmenting. Additionally, Kacie indicated an operative conception of a one-degree angle reliant upon segmenting a full angle; Kacie explained 360 iterations of a one-degree angle would exhaust the plane in her final teaching session. Throughout Part 4, Kacie's activities indicated she could coordinate three levels of units in assimilation and contraindicated she had constructed partitioning and splitting operations in angular contexts.

Kacie did not establish a circular quantification of angularity (i.e., ratio or rate quantifications), though she made significant strides toward a ratio quantification. Kacie initially assimilated tasks involving circular quantities (i.e., CQ tasks) as additive structures (see April 21st session); for example, Kacie subtracted the measure of a central angle from the circumference of a circle to obtain the length of the intercepted arc. When asked to focus on iteration in these tasks during her last teaching session, Kacie united the central angle and the intercepted arc, forming an arc-angle, and subsequently interpreted CQ tasks as multiplicative structures. By enacting operations on the arc-angle, Kacie simultaneously implemented operations on the central angle and the intercepted arc. For example, Kacie reasoned that since 12 one-inch arcs would be needed to exhaust a circle with circumference 12 inches, 12 copies of the subtended angle would be necessary to exhaust the plane. Kacie then assimilated this partitioned plane to her equisegmenting degree scheme to determine the measure of the central angle, computing $360° \div 12 = 30°$.

Kacie's activities indicated her extensive angular operations rendered a ratio quantification of angularity within her zone of potential construction. In her final interview session, Kacie spontaneously instantiated a multiplicative comparison of an intercepted arc length and circumference and used this ratio to determine the measure of the central angle; however, the solution was novel for Kacie, which indicated that she not established a scheme for solving CQ tasks at the conclusion of the teaching experiment. With additional opportunities for reflection on her activities in these kinds of tasks, I am confident Kacie would have established a ratio quantification of angularity. Kacie had not yet developed well established schemes for proportional reasoning, which I hypothesize would be an obstacle for her quantifying angularity as rate.

Looking across Kacie's and Camille's quantifications of angularity. Throughout the teaching experiment, each student developed a more sophisticated quantification of angularity than she indicated at the beginning of the teaching experiment. Neither Camille nor Kacie formed composite angular units through extensive angular operations during her initial interview. Both students constructed extensive quantifications of angularity during the study. Here, I briefly compare the students' quantifications of angularity.

At the onset of the teaching experiment, I inferred Kacie could coordinate three levels of units in assimilation based on her activities with linear material; in contrast, Camille could coordinate two levels of units in assimilation and three levels of units in activity. In my analysis of students' activities, the levels of units each student could coordinate in angular contexts never exceeded the level of units she could coordinate in linear contexts. Although it took time for students to activate their extensive quantitative operations, Kacie and Camille ultimately coordinated three and two levels of units, respectively, in assimilation in angular contexts.

From the initial interviews, I inferred Kacie had constructed the splitting operation while Camille had not. Throughout the teaching experiment, I was never able to attribute angular partitioning or splitting to Kacie. There are two possible explanations for this discrepancy: either Kacie had not constructed the splitting operation or Kacie never applied her splitting operation in angular contexts despite have constructed the operation. Both explanations are plausible. In the initial interview, Kacie needed to engage in sensorimotor activity to produce the split string, which, in retrospect, may have indicated she engaged in equisegmenting rather than splitting. On the other hand, if Kacie had constructed the splitting operation, the sequential element of the sweeping motion used in many tasks may have engendered equisegmenting rather than partitioning or splitting. Whether Kacie had constructed the splitting operation remains an open question, though based on her difficulties drawing angles in these tasks I can say her operations for producing units within units were not as developed as her operations for building units from units.[76]

In Camille's final interview session, I inferred Camille engaged in angular partitioning, an operation I was not able to attribute to Kacie throughout the teaching experiment. Although further corroboration would be needed to attribute angular partitioning to Camille generally, this suggests Camille may have constructed at least one operation I did not attribute to Kacie.

In general, Kacie's quantification was more sophisticated in that she could solve a wider range of problems than could Camille. Three resources I attributed to Kacie, and not to Camille, account for the differences in Kacie's and Camille's activities throughout the teaching experiment: (a) coordinating three levels of units in assimilation, (b) establishing re-presentable angular templates that were not orientation dependent, and (c) interpreting degrees as a unit for measuring angularity alone.

Comparing students' quantifications of angularity in rotational and non-rotational contexts. Throughout the teaching experiment, students worked on tasks in rotational and non-rotational angle contexts. In rotational contexts (e.g., rotating laser), we referred to angularity as an amount of rotation or an amount of sweep. In non-rotational contexts (e.g., hinged chopsticks), we referred to angularity as openness. Kacie and Camille each indicated she could apply extensive quantitative operations in each context; however, both students had different domains of angularity differed across rotational and non-rotational contexts.

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[76] My attention to whether Kacie had constructed the splitting operation was occasioned by accounting for differences in Kacie's and Bertin's ways of operating during retrospective analysis.

In non-rotational contexts, Kacie's and Camille's domains for openness did not include reflex angles. Both students indicated closed and straight angles were the least and most open chopstick configurations, respectively. Throughout the duration of the study, neither Kacie nor Camille spontaneously inserted rotational imagery into a non-rotational angle model, though each student indicated she could insert rotational motion (i.e., a radial sweep) when prompted. Throughout the teaching experiment, both students interpreted the openness as indicating a convex angle. In rotational contexts, both students' domain for amount of sweep included convex and concave angles. Kacie and Camille did not spontaneously demonstrate a consideration of oriented angles in either context, nor did either student consider angles exceeding a full angle during the teaching experiment.

Implications and Future Research Directions

In this section, I discuss implications for mathematics teaching and future research directions considering the findings of this study, which I summarized in the previous section.

Implications and Recommendations for Teaching

Acknowledging and celebrating students' quantifications of angularity. The major guiding hypothesis for this study was that operations constitutive of students' quantifications of length can be used in the quantification of angularity. The findings presented in this dissertation support this hypothesis. I have shown that students can develop non-circular quantifications of angularity involving extensive quantitative operations (e.g., iteration and segmenting). To my knowledge, the study reported in this dissertation is the first to examine such quantifications of angularity. These extensive quantifications of angularity should be acknowledged, encouraged, and celebrated in mathematics classrooms.

In the introductory chapter of this dissertation, I presented the Grade 4 standards for angle measure outlined in the CCSMM:

> 5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
>   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An arc that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
>   b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees. (pp. 31–32).

Mathematics educators must not take standards like the ones above as the only criteria for assessing students' understandings of angle measure. Kacie and Camille each developed productive ways of reasoning about angularity that are not recognized by this fourth-grade standard. Because extensive quantifications of angularity emerge from the same operations used in quantifying length, a focus on engendering extensive angular operations may be a more appropriate starting point for angle measure instruction than attempting to engender circular quantifications of angularity.

Contexts and tasks. The contexts and tasks used in this study are a resource for mathematics teachers intending to foster students' quantifications of angularity. In this study, I used two contexts, hinged chopsticks and rotating lasers. These contexts involve objects that might be familiar to students from their experiences outside of school, and these objects can easily be brought into mathematics classrooms for students to physically manipulate. Experiences manipulating these angle models—opening hinged chopsticks from a closed position and enacting turns with a laser—are important if students are to later internalize these motions. In school mathematics, angles are often introduced in polygonal contexts where there is little opportunity for angle to vary. In the contexts I used in this study, variation in angularity is possible, and this variation is critical for fostering productive quantifications of angularity. In addition to variation, the laser and chopstick contexts also target two different facets of the angle concept: angle as a single turning ray and angle as two distinct rays sharing a common endpoint.

I designed the tasks in this study to investigate and engender students' implementation of extensive quantitative operations in angular contexts. Instead of introducing angle measure with protractors and degrees, I recommend teachers first ask students to solve angular multiple and measurement tasks involving an arbitrary unit angle stated in terms of openness or amount of sweep. I encourage mathematics teachers to use and adapt these tasks to engender extensive their students' extensive quantifications of angularity.

The importance of acknowledging other attributes in angular contexts. Students often conflate other attributes with angularity (Baya'a, Daher, & Mahagna, 2017; Clements et al., 1996; Crompton, 2017; Devichi & Munier, 2013; Keiser, 2000; 2004; Lehrer, Jenkins, & Osana, 1998; Owens, 1996; Thompson, 2013). Kacie and Camille each attended to attributes other than angularity at the onset of the teaching experiment. Span length and spanned area were particularly salient attributes of angle models for Kacie. In Part 1 of the teaching experiment, I found that naming, varying, and comparing these attributes was productive for helping Kacie to disentangle angularity from these attributes. In Part 2 of the teaching experiment, both students examined which attributes varied as they manipulated angle models in GSP (see discussion of FPA and FPLA in February 1st and February 8th teaching sessions, respectively). Therefore, I encourage teachers to engage in discussions about the attributes that are salient for students and to focus on distinctions between attributes as opposed to ignoring or discarding particular attributes.

Instruction regarding degrees. One of Camille's greatest challenges in the teaching experiment involved degrees as a unit of angular measure. For Camille, degrees were conflated with attributes other than angularity, most notably orientation. To be clear, I do not consider Camille to have conflated angularity with orientation at the conclusion of the teaching experiment. Throughout the teaching experiment, Camille regularity reoriented angle models to achieve a variety of goals (e.g., congruence, comparison, production of angular multiples). Camille's challenge was not distinguishing openness or amount of sweep from orientation in the teaching experiment. Instead, Camille's difficulties stemmed from her prior experiences with degrees. Degrees were a "unit" associated with multiple attributes including length, openness, and orientation.

Because of Camille's difficulties with degrees, I recommend teachers maintain a focus on a contextualized angular attribute when introducing degrees as a unit of angular measure. For example, teachers might ask students to imagine a pair of chopsticks 1/90th as open as a square corner or to imagine a sweep so small that it would take 360 sweeps of that size to sweep out the entire page. Using technology to show the insertion of 90 one-degree angles into a right angle may be a productive activity for students (see Part 3 of Chapter 5).

Degrees as a unit of angular measure are often introduced via rotational imagery in terms of a 360-unit angular composite as in the CCSSM. This is a suitable approach for defining degrees if students are considering angles in rotational contexts. However, students' experiences with angles in mathematics classroom often involve non-rotational contexts. Throughout my teaching experiment, students did not spontaneously insert rotational motion into non-rotational angle contexts, which is consistent with prior research (Clements, et al., 1996; Crompton, 2013; Lehrer, Jenkins, & Osana, 1998; Mitchelmore, 1998; Mitchelmore & White, 1995; 1998; 2000). Moreover, I found that students' domain for angularity in non-rotational contexts involving openness was limited to non-reflex angles.

If (a) students do not spontaneously insert rotational motion into non-rotational angle contexts and (b) do not recognize angles exceeding a straight angle in non-rotational contexts, introducing degrees by defining a full angle as a 360-unit composite is entirely inappropriate for non-rotational angle contexts. Therefore, I propose an alternative approach for developing degrees in non-rotational contexts. Right angles are a prototypical angle for many students (Devichi & Munier, 2013; Matos, 1999); in this dissertation, I have shown that students do develop re-presentable templates for right angles. Therefore, in non-rotational angle contexts I recommend introducing students to degrees by positing a right angle as a 90-unit angular composite.

I do not share the sentiments of Mitchelmore & White (1998), who recommended that angle as turn be removed entirely from the elementary mathematics curriculum. Instead, I suggest that early instruction in angle measure should involve both rotational and non-rotational contexts and different approaches for introducing degrees are merited depending on the context. Once students have had opportunities for implementing extensive quantitative operations in both contexts, teachers might work to engender students' recognition of similarities across these two contexts by drawing pictures to represent rotations or inserting rotational imagery into non-rotational angle models (see February 8th teaching session for example).

Potential Directions for Future Research

In this dissertation, I presented models of students' quantifications of angularity. It is my intention that the models I have developed in this dissertation inform future mathematics teaching and research. Because this is the first study that has examined students' extensive quantifications of angularity, much work remains to be done in this area. Future research is needed to refine the models I have presented and to examine what quantifications are possible with students who have constructed a set of extensive quantitative operations different from Camille or Kacie. In particular, I encourage studies investigating students' non-circular quantifications of angularity at the elementary level and studies for engendering circular quantifications at the high school level. In the sections that follow, I provide some specific suggestions for future investigations.

Units coordination and fraction schemes. A fundamental distinction between Kacie's and Camille's quantifications of angularity was that Kacie could assimilate tasks with three levels of units, whereas Camille could assimilate tasks with two, but not three, levels of units. According to Steffe (2017), only about 10% of third-grade students will have constructed operations that produce three levels of units like Kacie. About 45% will have constructed operations that produce two levels of units like Camille. The remaining

45% of third-grade students will have constructed operations for producing only one level of unit; future research should investigate how these students reason about angularity.[77]

Beyond levels of units coordination, future studies in angularity should incorporate an analysis of students' fractions schemes, which have been described by Steffe & Olive (2010). As a result of my study, I consider operative conceptions of degrees to involve positing a familiar template (e.g. right or full angle) as a composite unit and subjecting this template to subsequent operations. For example, making a seven-degree angle with extensive quantitative operations involves breaking a right angle into ninety equiangular parts and uniting seven of these parts. I hypothesize that the angular templates students can take as composite units along with the fraction schemes they have constructed will determine, to a certain extent, the kinds of angles students can make or measure. The previous description of producing a seven-degree angle involves the operations of a partitive fraction scheme. In contrast, if a student can posit a right angle as a 90-unit composite but not a full angle, then producing a 91° angle requires the same operations that produce improper fractions (i.e., the iterative fraction scheme). Specifically, producing a 91° angle would involve partitioning a right angle into 90 equiangular parts and uniting 91 parts of that size. For students who can operate on a right angle and not a full angle, producing obtuse angles is equivalent to producing improper fractions. For this reason, it may be that obtuse angles are harder to students to measure than acute angles, which potentially accounts for Matos (1999) finding that, for fourth and fifth graders, "obtuse angles were not as good exemplars of the category of angles as acute or right angles" (p. 177). Future research is needed to investigate the relationship between students' fraction schemes and quantifications of angularity.

[77] Recall that angle measure is introduced in Grade 4 in the CCSSM.

Engendering circular quantifications of angularity. Neither Kacie nor Camille developed circular quantifications of angularity during the study, though a ratio quantification of angularity was within Kacie's zone of potential construction by the end of the teaching experiment. As such, I have provided some evidence extensive quantifications of angularity can support the construction of circular quantifications of angularity. Clearly, the construction of ratio and rate quantifications of angularity are non-trivial and additional research is needed to investigate the mental operations and schemes supporting these quantifications. I hypothesize that proportional reasoning, which neither Camille nor Kacie indicated at the onset of the teaching experiment, is critical for fostering these circular quantifications of angularity.

In addition to proportional reasoning, I hypothesize quantifying angle measure as rate can be supported by a fourth motion through the interior of an angle—an arc sweep. An arc sweep involves imagining the interior of an angle being swept out by a circular arc bounded by the sides of an angle and where the circle containing the arc is centered at the vertex of the angle (Figure 6.3). Neither Camille nor Kacie indicated accounting for motion through the interior of an angle via arc sweep. I hypothesize such a motion through the interior of an angle will support students to develop quantifications of angularity that entail generalized multiplicative comparisons of circular lengths holding across a class of circles. Through such a motion, the plane can be conceived as a collection of concentric circles and the angle conceived as a collection of arcs (i.e., an angle of arcs). Future research is needed to investigate the roles of proportional reasoning and the motion of an arc sweeping through the interior of an angle.

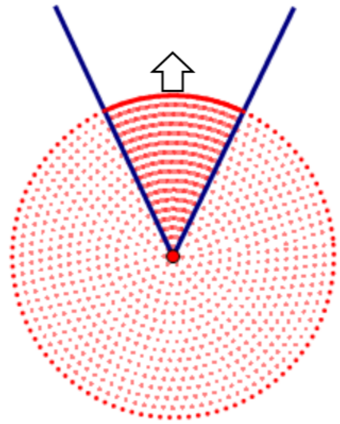

Figure 6.3. Arc sweep through the interior of an angle.

Future investigations that focus on engendering circular quantifications of angularity should include a review of literature regarding students' quantifications of curved lengths, particularly arcs. While students' quantifications of curved lengths are likely related to their quantifications of length in linear context, it cannot be taken for granted that students will necessarily assimilate each of these contexts to the same operations. For example, if a student can use iteration to produce a segment five times as long as a given segment, the student may not necessarily use iteration to produce an arc five times as long as a given arc. An analysis of students' quantification of curved lengths will likely play a critical role in understanding how students come to quantify angularity as ratio or rate. Additionally, I can see no reason to limit circular quantifications of angularity to involving multiplicative comparisons of lengths. Future research should therefore investigate quantifications of angularity that involve multiplicative comparisons involving the area of a sector and the area of a circle.

The role of motion in quantifications of angularity. In this dissertation, I abstracted three such motions (radial sweep, segment sweep, and re-presented opening) in my analysis of students' activities and hypothesized the utility of a fourth motion (arc sweep). Extant research has primarily focused on engendering the insertion of rotational motion (i.e., radial sweep) into angular contexts. How prevalent are each of these motions in students' construction of angularity? Are there other motions that students use to account for the interior of an angle? Which motions do students insert spontaneously? As I have noted, prior studies have shown students do not spontaneously insert rotational motion into non-rotational angle contexts, which leads me to believe that if students spontaneously insert motion into angle contexts they are either inserting span sweep or re-presented opening. The former seems more likely given the static nature of angles in early instruction on angle measure, which may account for conflations between linear attributes (e.g., side length and span length) and angularity. Further investigations into the role of motion in students' quantifications of angularity are needed, particularly those that focus on engendering re-presented opening and arc sweep. Furthermore, I conjecture generalized quantifications of angularity, meaning quantifications that are context independent, entail a recognition that different motions through the interior of an angle still account for the same magnitude of angularity.

The role of length altering and angularity preserving operations. In this study, I presented five operations students used to alter the lengths of angle models while preserving angularity: experiential dilation, truncation, elongation, hyper-truncation, and hyper-elongation. I hypothesize that these operations play a critical role in early quantifications of angularity (i.e., gross and extensive quantifications) precisely because experiential angle models are always of finite length. Furthermore, I conjecture truncation operations develop more naturally than the elongation operation. Future studies should examine the development of these operations in young students who have not received classroom instruction on angle measure, as well as the affordances and limitations each of these operations occasion in students' quantifications of angularity.

Exploring other angular contexts. In this study, I examined two different angular attributes: openness and amount of sweep/rotation. Further research is needed to investigate students' quantifications of other contextualized variations of angularity (e.g., tilt, bend, sharpness). Prior research has shown students are more likely to recognize angles in some contexts than others (e.g., Mitchelmore & White, 2000). As such, it is reasonable to hypothesize students may quantify angularity differently in these contexts. In particular, students may develop different referents and domains for angularity in different contexts.

Additional research directions and closing remarks. There are many additional possibilities for future investigations in students' quantifications of angularity. I close this dissertation by briefly listing some additional directions. In this study, I developed tasks to engender and investigate students' multiplicative structuring of units (e.g., iteration); future studies may examine the additive structures students produce when reasoning in angular contexts. In this study, I did not thoroughly investigate students' construction of angular sums and differences, for example, which would be appropriate foci for future investigations.

Future studies aimed at developing a trigonometry rooted in students' quantifications of angularity will need to attend to a variety of additional components. What conceptions of trigonometry are possible if students have not yet established circular quantifications of angularity? Nether Kacie nor Camille spontaneously considered signed angles throughout this study. What experiences are critical for students to extend their domains of angularity to include negative measures? Studies including angles measured in radians will also need to attend to students' conceptions of pi, though this has received at least some attention in existing literature (e.g., Akkoc, 2008).

The future research directions I have outlined in the sections above are not intended to be an exhaustive agenda for future research on angularity. These directions stemmed from questions that I have about quantifying angularity after conducting my teaching experiment. Future studies will likely lead researchers to new hypotheses and additional research questions.

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APPENDIX A

INITIAL INTERVIEW PROTOCOL

Follow-up questions will be asked as students engage with each task to better understand how the students might be thinking. (How do you know? Why did you stop there? How would you check to see if you were exactly right? Etc.)

Students will have the following materials available for their use during the interview if they choose to use them: markers, paper, wikki stixx, loose chopsticks, transparencies, scissors, a ruler, and tape. (Should a student attempt to use the ruler to measure in task in which I've not asked them to measure, I'll ask them if they can do it without the ruler.)

If at any point during the interview a student mentions *angle, angle measure, or degrees*, I'll probe the student to better understand what those words mean to him/her.

1. Multiple
   Can you make a piece of string six times as long as my piece of string?

2. Bending & Straightening [Yarn.gsp]
   a. [Page 1 of yarn.gsp] Here's a piece of red yarn and a blue ball of yarn. Can you pull out a piece of blue yarn that's the same length as the red piece?

   b. [Page 2 of yarn.gsp] Here we have a pink piece of yarn and a blue piece of yarn. How would you figure out how many blue pieces you would need to make the pink piece?

3. Splitting
   This is my piece of string. My piece of string is five times as long as your piece of string.
   a. Can you make your string?
   b. My string is 360 mm long, how long is your string in millimeters?
   c. What fraction of my string is your string?

4. Units coordinating
   a. (4 pencils ≅ 1 string; 7 wikkis ≅ 1 pencil)
      Here's a pencil. Measure the length of this string in pencil lengths. How many pencil lengths long is it? Measure the pencil with this wikki. How many wikkis long is the pencil? Now, how many wikkis long would the string be?

b. (1 strip ≅ 18 pipe cleaners; 3 pipe cleaners ≅ 1 chopstick)
Here's a string. Measure the length of this strip in pipe cleaners. How many pipe cleaners long is the string? Measure the chopstick in pipe cleaners. How many pipe cleaners long is the chopstick? How many chopsticks long is the strip?

5. Proportionality
[Have the students measure the dimensions of the picture] This picture is 3 inches tall and 4 inches wide. I made prints of the picture in various sizes. [I have the prints available for students to view.]

    a. This print is 18 inches tall, how wide is it in inches?
    b. This print is 1 inch tall, how wide is it in inches?
    c. This print is 4 inches tall, how wide is it in inches?

    [If a student struggles with part a, ask: This print is 6 inches tall, how wide is it in inches?]

[For the angular contexts, two pairs of chopsticks of different lengths will be used initially. If a student explains that it's not possible to accomplish a particular task with the chopsticks because of differences in length, the task may be repeated with two pairs of chopsticks that are the same length. The pair of chopsticks that the student is not manipulating will be fixed (taped down on a piece of paper, or potentially hot-glued in a fixed position ahead of time) so that the student doesn't change the openness mid-task.]

6. Re-present and compare
In a moment, I'm going to show you two pairs of chopsticks. I'm going to give you a few seconds to look at the chopsticks and then I'm going to cover them up. Here are the chopsticks. Take a moment and get a picture of them in your mind. Now I'm going to cover up the pairs of chopsticks. Can you draw both pairs of chopsticks for me? When you're satisfied with your drawing let me know. Tell me about your drawing.

7. Congruence [I have long chopsticks; the student has short chopsticks]
    a. My chopsticks are open some amount. Can you set your chopsticks to have the same openness?

    b. [Follow up with this version later in the interview if the student sets the chopsticks using the linear distance between the endpoints in the previous task.] I took this picture of these chopsticks the other day. Can you set the chopsticks so that they have the same openness as when I took the picture?

8. Comparison [I have long chopsticks; the student has short chopsticks]
This is my pair of chopsticks. Can you set your pair of chopsticks to be more/less open than my pair of chopsticks? How do you know it's more/less open?

9. Multiple [I have long chopsticks; the student has short chopsticks]
   Can you set your pair of chopsticks to be four times as open as my pair of chopsticks?

10. Splitting [I have short chopsticks; the student has long chopsticks.]
    This is my pair of chopsticks. Can you set your pair of chopsticks so that my chopsticks are five times as open as your chopsticks? [This is tough, be prepared to reduce to three.]

11. Path swept out by a laser [laser.gsp]

[Have a laser pointer for the student to handle first.] Here's a sketch that shows a laser pointer. I'm going to hide the beam. When I click sweep, the laser pointer is going to move. While the laser pointer is moving, I want you to imagine where the laser beam would be shining if it were on. After the laser pointer is done moving, I want you to color in all the parts of the screen that the laser beam would have hit if it had been turned on. (Consider this follow-up: If the laser started where it is now and repeated the same movement, what parts of the screen would be hit by the laser beam then? Color those parts of the screen)
An alternative to this task is place the laser pointer on a piece of paper, turn it, and ask the student to color the areas that would have been hit by the laser.

12. Recursive Partitioning
    Pretend this strip is a long piece of candy. [Cover up the candy.] You have to share it with a total of four people. You take your piece and then, you run into more people. You now share your piece with a total of five people. You now get to eat one of those little pieces. What amount of all of the candy is your piece?

APPENDIX B

FINAL INTERVIEW PROTOCOL

13. Congruence [I have long chopsticks; the student has short chopsticks]
    a. My chopsticks are open some amount. Can you set your chopsticks to have the same openness?

14. Comparison [I have long chopsticks; the student has short chopsticks]
    This is my pair of chopsticks. Can you set your pair of chopsticks to be more/less open than my pair of chopsticks? How do you know it's more/less open?

15. Multiple [I have long chopsticks; the student has short chopsticks]
    Can you set your pair of chopsticks to be four times as open as my pair of chopsticks?

16. Splitting [I have short chopsticks; the student has long chopsticks.]
    This is my pair of chopsticks. How would you set your pair of chopsticks so that my chopsticks are five times as open as your chopsticks?

17. One degree
    a. How would you make an angle with a measure of one degree? (Would there be any other ways to make a one-degree angle?)
    b. Suppose I handed you an angle and said that it had a measure of one degree. How would you check?

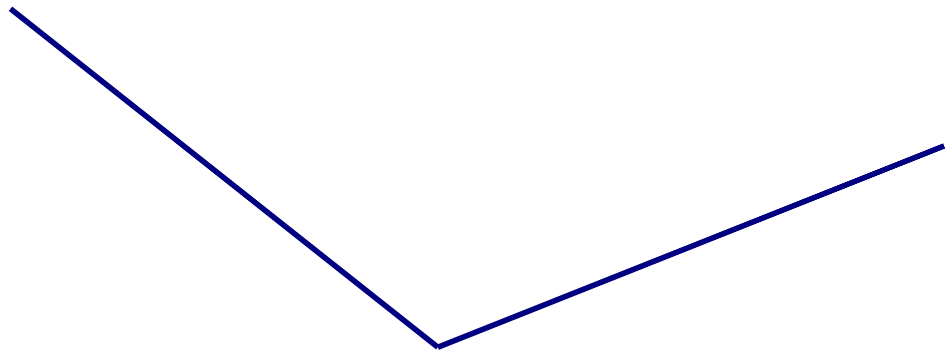
18. Imagine a purple angle. It takes five sweeps of the purple angle to sweep out a right angle. What is the measure of the purple angle? [78]

19. Imagine two angles: a yellow angle and a red angle. It takes four yellow angles to sweep out the entire plane. It takes 3 red angles to sweep out one yellow angle. What are the measures of the yellow and red angles?

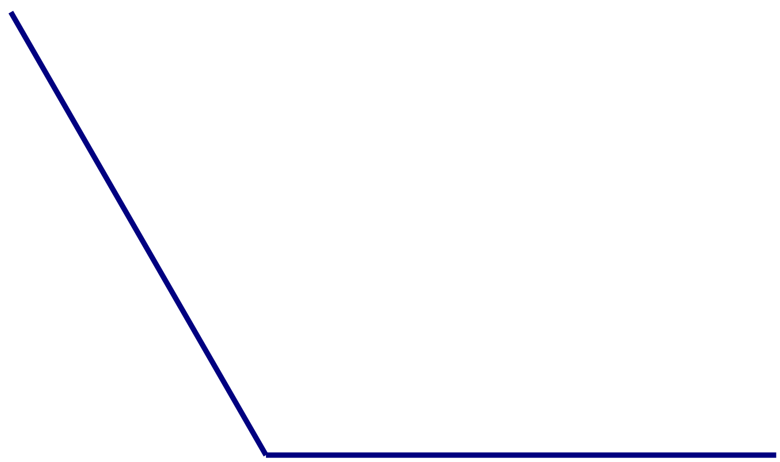
20. Imagine two angles: a green angle and a blue angle. It takes 18 green angles to sweep out the entire plane. It takes 3 green angles to sweep out one blue angle. What are the measures of the blue and green angles?

[78] This task was presented only to Camille.

9. Estimate the measure of the angle below.

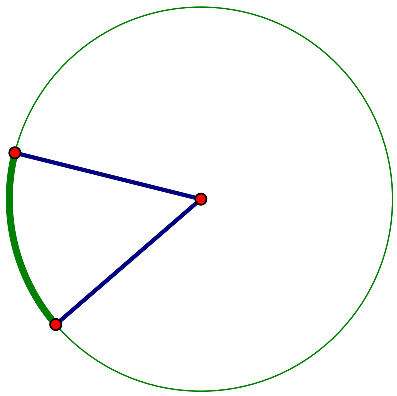


10. Estimate the measure of the angle below.



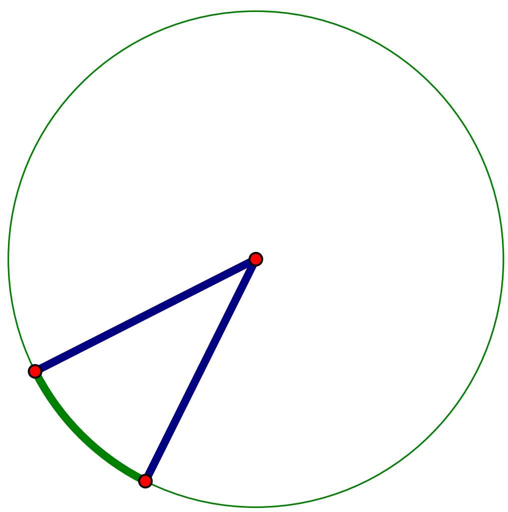
11.

The total distance around the green circle below is 22.83 cm. The thick green arc is 3.47 cm long. How would you determine the measure of the blue angle?



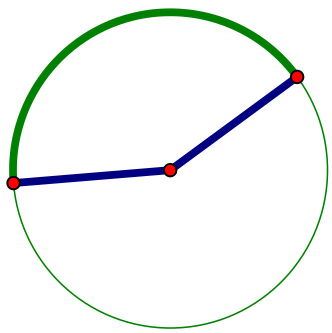
12.

The measure of the blue angle is 36.61°. The total distance around the circle is 26.10 cm . How would you determine the length of the solid green arc?



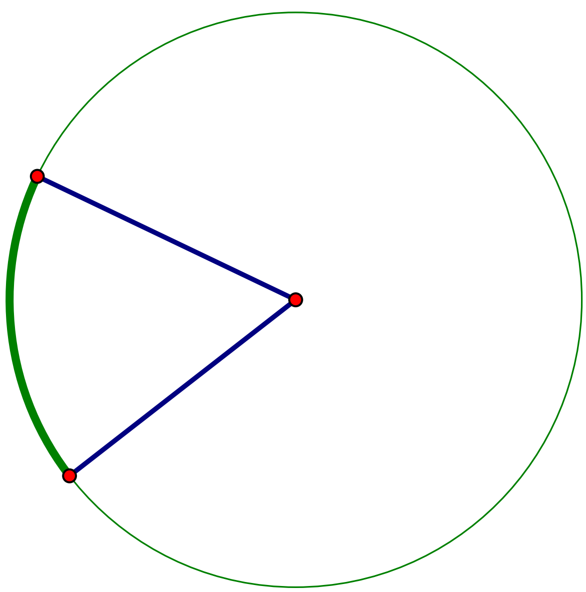
13.

**The measure of the blue angle is 148.50°.
The length of the thick green arc is 2.65 cm.
How would you determine the total
distance around the circle?**



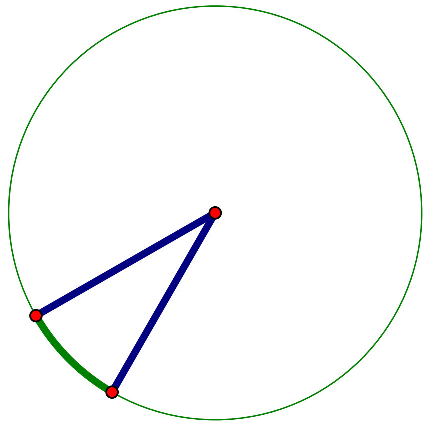
14.

**The total distance around the green circle
below is 30 cm. The thick green arc is
5 cm long. How would you determine the
measure of the blue angle?**



15.

The measure of the blue angle is 30°. The total distance around the circle is 24 cm . How would you determine the length of the solid green arc?



16.

The measure of the blue angle is 120°. The length of the thick green arc is 6 cm. How would you determine the total distance around the circle?