

COLLEGE STUDENTS' KNOWLEDGE OF FUNCTIONS AS AFFECTED  
BY INSTRUCTION USING THE RULE OF THREE

by

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Under the Direction of Nicholas Oppong

ABSTRACT

This study investigated college students' knowledge of functions before and after instruction using the Rule of Three. The Rule of Three is the dictum that functions should be taught symbolically, graphically and numerically, and as many examples as possible should combine all three views. O'Callaghan (1998) and Markovits, Eylon, and Bruckheimer's (1986) frameworks for examining components of the concept of function were used to investigate the students' knowledge.

Four students were chosen based on pretest scores, written responses, and scheduling conflicts. Data were collected from sorting and translating tasks, three interviews and two tests, and an ongoing comparative analysis was used (Glaser & Strauss, 1967).

Initially, the students were tied to their calculators and used them extensively. The completion of any task depended on the students' skill with the calculator. After instruction, the students were less reliant on technology. Calculators were now used to confirm solutions acquired using a combination of procedures.

INDEX WORDS: Rule of Three, Functions, Non-functions, Calculus, Modeling, Translating, Reifying, Recognizing, Sorting, Tools

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## DEDICATION

In loving memory of

my father, Lucious Hartfield; my grandfather and grandmother, Samuel and Mattie Taylor; my uncles: Bill Hartfield, Cal Hartfield, Jack Hartfield, Tom Hartfield, Ezekiel Strother, and Napoleon Eagle; my aunts: Daisy Eagle and Gussie Hartfield; my cousins: Sylvester Eagle, Otis Taylor, Uretta Spencer, Mildred Blocker, Tyrone Taylor; and my friends: Jean Kirkland; Amy Stephens; Tilman McDaniel Sr. and Junior Chilsom.

Most of my love ones passed away at some point during my time  
in graduate school, but I know they are  
celebrating with me in spirit.

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## CHAPTER 1

### INTRODUCTION

Many mathematicians have asserted that functions play a fundamental and essential role in mathematics and are critical throughout the entire spectrum of mathematics from high school through college (O’Callaghan, 1998). Since the concept of function is present in every branch of mathematics, it should be an important unifying idea central to the organizing of secondary and higher mathematics curricula (National Counsel of Teachers of Mathematics, 1989; O’Callaghan, 1998). In addition, functions should provide the medium through which mathematics courses are developed and taught (Haimes, 1996).

The belief that the concept of function should be a focal point in mathematics or in its instruction is not a recent or unique idea. During the late nineteenth and early twentieth centuries, several mathematics educators concluded there was a need for a greater emphasis on the concept of function in school mathematics (Breslich, 1928; Hedrick, 1922; Townsend, 1915). The German mathematician Felix Klein in 1904 (cited in Hamley, 1934, p. 52) alluded to functions as the “soul” of mathematics and said the elementary treatment of functions should be in the regular course material of all types of high schools.

During the 1920s numerous recommendations suggested the concept of function should be emphasized in every area of high school mathematics to strengthen the efforts to unify the mathematics curriculum (Breslich, 1928; Hedrick, 1922). In addition,

Breslich (1928) said that since mathematics deals with functional relationships between quantities, it would be difficult to have any reliable understanding or an appreciation of mathematics without functional thinking. Hedrick (1922) also stated that functions or functional relations, in one form or another, are present on each page of every textbook on mathematics unless we suppress or overlook them. Although efforts have been made to understand why many students regularly do not understand the concept of function, the issue is unresolved in spite of its historical and current prominence in mathematics (Even, 1996; Tall & Vinner, 1981; Vinner, 1991; Vinner & Dreyfus, 1989).

One useful area of investigation in helping students understand the concept of function is instruction using a new statement for the Rule of Three from the Harvard Consortium, a reform calculus project funded by the National Science Foundation. The current statement or dictum for the Rule of Three is that functions should be viewed or taught not only symbolically (analytic/algebraic), as is typical in traditional calculus courses, but also graphically (geometric) and numerically (tabular), and that as many examples and problems as possible should combine all three views (Hughes-Hallett, Gleason, Flaith, Lock, Gordon, Lomen, Lovelock, McCallum, Quinney, Osgood, Pasquale, Tecosky-Feldman, Thrash, Tucker, & Bretscher, 1998). Historically, the Rule of Three commonly referred to the rule for finding the unknown fourth term in a proportion or equation given any three of the four terms (Hughes-Hallett et al., 1998).

Many mathematicians believe the calculus reform movement began with the Tulane Conference in 1986 (Douglas, 1987) and the subsequent Washington Conference (Douglas, 1987). The consensus among the mathematics educators and mathematicians who attended the conferences was that several problems were associated with the

teaching of traditional calculus (Tucker & Leitzel, 1995). Of the several problems expressed by the conference participants, some were: (1) very few students were prepared for calculus; (2) the percentage of students who passed calculus was low; (3) calculus was being used as an unwarranted gatekeeper in too many disciplines that did not require the use or application of this knowledge; (4) technology applicable to the teaching of calculus was not being utilized, or it was being utilized inefficiently; and (5) few students had any understanding of the fundamental calculus concepts (Douglas, 1987; Smith, 1994; Tucker & Leitzel, 1995).

Of the problems expressed by the participants at the Tulane and Washington conferences, I was most concerned with the opinion that very few students are prepared for calculus. One factor leading to the successful completion of calculus may be knowledge of the concept of function, or lack thereof, held by college students. Wheeler (1982) argued that both College Algebra and Precalculus, the main curriculum content of which is the topic of functions, are traditionally taught for their importance as a tool needed to handle the mathematics that is to come later—calculus. Most of the examples in which calculus concepts are seen and understood in terms of objects and processes have to do with functions (Dubinsky, 1992), and functions are the mathematical tools used to describe relationships among variable quantities. They are at the heart of mathematics (Thompson, 1985). In addition, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) called for the concept of function to be the main organizing standard for secondary and *higher mathematics curricula* (O’Callaghan, 1998).



### *Rationale*

The purpose of this study was to investigate college students' knowledge of the concept of function, before and after instruction, when functions are presented using Rule of Three. Since the use of different instructional formats or presentations (like the Rule of Three) along with a function approach to teaching introductory concepts in calculus is comparatively new, there is a need for studies concerning the impact of different instructional formats in the classroom (Haimes, 1996).

According to Cooney and Wilson (1993), more research has been conducted on the knowledge of concept of function than on how it evolves among college students. For instance, some research has answered questions such as: Can college students define a function? Can college students identify functions? If so, what types of functions are students the most knowledgeable? What types of functions are problematic? Several studies (Even, 1996; Haimes, 1996; Markovits, Eylon, & Bruckheimer, 1986; Vinner, 1990) investigated students' and pre-service teachers' knowledge of functions from mathematics classes. The results of the research by Markovits et al. are that many pupils do not hold a modern definition of a function. In addition, they added that many students appear to view most functions as linear functions and their conceptual knowledge seemed to trail far behind their procedural knowledge. Vinner and Dreyfus (1989) found the results of the research by Markovits et al. (1986) to hold even when students could provide a correct modern definition. Even (1996) stated that many students expect graphs of functions to be "reasonable" smooth curves and functions to be represented by a formula or equation over their entire domain.

Additionally, not many years ago, the results from research commonly defined students' subject matter knowledge in quantitative terms: by the number of courses taken in college, scores on standardized tests, or course grades rather than by qualitative descriptions (Ball, 1990). Taking several mathematics courses in a given area or obtaining good final grades in mathematics courses did not necessarily imply that conceptual knowledge rather than procedural knowledge was obtained or understood (Selden & Selden, 1992). The measures were problematic since they may not have represented the students' knowledge of the subject area, or knowledge and understanding of facts, concepts, or theorems (Even, 1996).

### *Research Questions*

A qualitative basis for research is provided by the quantitative nature of past studies regarding pre-service teachers' and students' knowledge of functions (Ball, 1990), along with the call for a deeper understanding of how their knowledge is affected by using different instructional formats (Haimes, 1996). Therefore, the Rule of Three may allow students to increase or modify their knowledge of functions. In order to increase or modify students' knowledge, we must first explore the nature of their knowledge of functions. The research questions were asked before and after instruction using the Rule of Three:

1. What type of relations in symbolic, graphical or numerical form do college students recognize or identify as functions before and after instruction using the Rule of Three? What characteristics or tools, do college students use to classify symbolic, graphical or numerical representations of relations as functions before and after instruction using the Rule of Three?

2. Before and after instruction using the Rule of Three, what types of functions are college students able to translate from one representation to another?  
  
What characteristics or tools do college students use to match or translate from one representation to another before and after instruction using the Rule of Three?
3. Given a set of functions in symbolic, graphical or numerical form, what new functions do college students construct or create, before and after instruction using the Rule of Three?

### *Theoretical Background and Definitions*

The theoretical framework for this study evolved from a modification of O’Callaghan’s (1998) framework by combining the two concepts of recognition and classification—from the framework of Markovits et al (1986) that O’Callaghan used but did not state in his framework— into one concept, recognizing, and adding it to O’Callaghan’s framework. O’Callaghan’s framework evolved from Kaput’s (1989) sources on meaning in mathematics. Kaput designated sources on meaning in mathematics as divided into two paired categories. The first category is called referential extension and is composed of (1) translations between mathematical representation systems and (2) translations between mathematical and non-mathematical systems, such as natural languages. The second category, consolidation, contains: (1) pattern and syntax learning through transforming and operating within a representation system and (2) building conceptual entities through reifying actions and procedures. O’Callaghan reformulated Kaput’s sources of meaning in mathematics to obtain the components, *modeling*, *translating*, and *reifying*, into terms specific to functions. He then studied students’ understanding of the concept of functions

by examining the components of the concept of function, which he called competencies, and used them to describe functional knowledge.

- *Modeling* is the ability to represent raw data, usually in numerical or tabular form, using functions to form either a symbolic or graphical representation of the relationship or of the data. For example, suppose an individual's checking account was five dollars overdrawn for January, was empty at the end of February, but had a surplus of three dollars for March. There was a surplus of four dollars for April and a surplus of three dollars for May. It was empty for June and overdrawn five dollars once again for July. What function describes the checking account balances for the first seven months of the year?

The capacity to represent a problem situation using functions is an important component in students' understanding of the function concept (O'Callaghan, 1998). Selden and Selden (1992) noted that modeling real-world situations helps one organize the physical world and is one of the most frequent uses of functions. The view of functions as a suitable tool in this regard is referred to by Sierpiska (1992) as a *sine qua non* for understanding the function concept.

- *Translating* is the ability to move from one representation of a function to another or one context to another. For example, consider the function represented as table of values depicted in table1.

Table 1: The numerical representation of a function

$x$	-2	3	-1	.5	$-\sqrt{3}$
$y$	3	8	0	-0.75	2

Now determine its symbolic and graphical representation. Verstappen (1982) referred to the three-core representation system—symbolic, numerical, and graphical— as the algebraic arithmetic when moving from one depiction or representation of a function to another.

- *Reifying* is defined as the construction or creation of a mental object from what was initially perceived as processes, examples, definitions, or procedures. Through reifying, new functions can be constructed or created from a given set of functions by mathematical processes such as transformations (which may be compositions, adding, subtracting, multiplying, or dividing of functions), differentiation, or integration. For example, given the functions  $f(x) = x^2 + x$  and  $g(x) = x + 2$ , a student may be able to construct the new functions  $h(x) = (x^2 + x)(x + 2)$  or  $s(x) = (x + 2)^2 + (x + 2)$  by multiplying  $f(x)$  times  $g(x)$  and using the composition of functions,  $f(g(x))$ .

Markovits et al. (1986) said in their theoretical framework that an understanding of the concept of function has many aspects, such as (I) the knowledge to recognize and classify functions and non-functions, and (II) the knowledge to use the concept in different contexts along with the ability or knowledge to transfer information from one context to another in mathematics. Markovits et al. subdivided aspect (I) above into components (a) and (b) below. Similarly, aspect (II) above was subdivided into components (c) through (g) below. To determine students' knowledge of functions, Markovits et al. investigated components of the function concept—(a) through (g) below—and analyzed the participants' responses to determine their understanding of the concept of functions. The components are as follows:

- (a) the ability to identify relations as functions or non-functions;

- (b) the ability to classify similar functions and similar non-functions;
- (c) the ability to state examples of functions and non-functions;
- (d) the ability to transfer from one representation to another;
- (e) the ability to state the domain and range of functions and non-functions and find the pre-images and images;
- (f) the ability to state a definition of a function; and
- (g) the ability to give or construct an example of a function satisfying some constraints.

The term *ability* is used by Markovits et al. and O'Callaghan and is defined below.

- *Ability* is defined to be knowledge and procedural skills, such as transformations, along with other procedures that allow students to operate within a mathematical representation system (Markovits et al. 1986; O'Callaghan, 1998). More succinctly, ability can be thought of as the skills and knowledge necessary to perform a task. Sierpinska (1992) said that a certain amount of algebraic awareness, procedural skills, and knowledge of algebraic methods is necessary for studying functions.

Recognition and classification are concepts from the framework of the research by Markovits et al. (1986) that were used also by O'Callaghan (1998) in his research but not explicitly or formally stated in his theoretical framework. Recognition and classification, which are already specific to functions, were combined in the present study and renamed *Recognizing*, and formally stated as part of O'Callaghan's model.

- *Recognizing*, based on concept definitions and concept images, is defined to be the ability to identify or classify relations, in their various forms, as functions and non-functions, as well as state their names and the corresponding names of their graphs

when appropriate. Also included is the ability to state or give examples of functions and non-functions.

Concept definitions are comparable to conventional mathematical definitions, and concept images consist of all the cognitive structures that are associated with a given concept, including mental pictures, properties and processes (Tall & Vinner, 1981). For example, all the mental pictures, sounds, and smells that come to one's conscience when asked to imagine a fire engine or a landfill are concept images. One can never know a student's concept image for a specific topic. However, if an answer was given repeatedly, I assumed that it represented one of the student's concept images for that topic. Thus, concept images created by the student might contain factors that conflict and interact with concept definition, and these conflicts and interactions might not lead to a harmonious or correct match, with each influencing or shaping the other and often yielding misconceptions (Vinner, 1991, 1992). This inconsistent behavior, described by Vinner and Dreyfus (1989), is a specific case of the compartmentalization phenomenon, and it occurs when a person has conflicting schemes in his or her cognitive structure (Even, 1996).

For example, let us first consider the vertical line test—the concept that if a vertical line intersects the graph of an equation more than once, the graph is not a physical representation of a function. Suppose a student knows this concept. Now assume that the student thinks the graphs of all *quadratic equations* are parabolas (which is true and may be tied to their concept image) and that all parabolas, not just the ones opening up or down, represent functions (which is not true but again may be tied to the concept image) and classifies the graph of the quadratic equation,  $x = y^2$ , as a parabola and hence

a function. The knowledge about the vertical line test and the knowledge that all parabolas are representative of functions may each be contained in its own compartment, with neither perspective influencing or overriding the other (Vinner & Dreyfus, 1989).

My theoretical framework is a modification of O'Callaghan's (1998) framework and contains the components *modeling*, *translating*, *reifying*, and *recognizing*.

Recognition and classification are not the only points of intersection between the frameworks of Markovits et al. (1986) and O'Callaghan. Another point of similarity is between Markovits et al.'s component of ability to transfer from one representation to another and O'Callaghan's translating component. These two components have similar definitions. Still another point where the frameworks are related is O'Callaghan's reifying component and Markovits et al.'s component of the ability to give or construct an example of a function satisfying some constraints. These are similar because they both deal with the construction of mental objects or functions.

The first two statements below are definitions of a function taken from a traditional textbook (before the calculus reform movement of 1986) and reform calculus textbook (after the calculus reform movement), respectively. The last two statements are definitions of algebraic and transcendental functions, respectively. Without defining an algebraic function, some authors have chosen to classify it as an elementary function along with the trigonometric, inverse trigonometric, exponential, and logarithmic function, and any simple combination of these functions (Taylor & Mann, 1972). Other authors have chosen to define an algebraic function as any function that can be constructed from polynomial functions using the operations, additions, subtraction, multiplication, division, and taking roots (Lial & Miller, 1989). I chose to use a definition



for an algebraic function that is different from either of the two previous ones.

- A function  $f$  from a set  $A$  to a set  $B$  is a well-defined subset of ordered pairs of the Cartesian product of  $A$  and  $B$ , such that for every  $a \in A$  there is exactly one  $b \in B$  such that  $(a, b) \in f$ . Since the definition of a function relies on the concept of a set, it is natural to call the set of all first elements of a function the *domain* and the set of all second elements the *range* (Stewart, 1991, p. 28).
- A function is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the domain of the function, and the set of resulting output numbers is called the range of the function (Hughes-Hallett, et al., 1998, p. 2).
- An algebraic function is the ratio of two or more functions where at least one of the functions is not a polynomial (Stewart, 1991, p. 43). For example, both  $f(x) = \frac{|x|}{x-1}$  and  $g(x) = \frac{\sqrt{x-1}}{x^2}$  are algebraic functions.
- Transcendental functions are logarithmic, exponential and trigonometric functions (Stewart, 1991, p. 43).

In conclusion, the components of my framework evolved from O'Callaghan's (1998) applying Kaput's (1989) theory about the sources of meaning in mathematics to function concept. The component of *recognizing*, which O'Callaghan used but did not include in his framework, corresponds to the sources Kaput categorized as referential extension, along with *modeling* and *translating*, whereas *reifying* corresponds to Kaput's sources in his consolidation category. O'Callaghan stated that his framework contains the sources of meaning for the function concept or describes the component of a conceptual

knowledge of functions. He also said that an important feature of his model is the close correlation between these behaviors and the abilities associated with function, as described in NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics*.

### *Outline of the Study*

I have attempted to outline the nature of my study in the introduction by discussing its background and rationale, along with delineating the research questions and definitions used with my theoretical framework. A review of related literature is presented next, followed by the methodology, which is organized into sections that discuss the design of data collection and the analysis, including a description of the analysis.

## CHAPTER 2

### REVIEW OF LITERATURE

The purpose of this study was to investigate the knowledge of the concept of function held by college students before and after instruction using the Rule of Three, the dictum from the Harvard Consortium. Along with my framework, I reviewed other literature pertaining to different perspectives on functions and functional knowledge, teaching mathematics, and the historical aspect of functions.

#### *Alternative Theoretical Frameworks*

Some mathematics educators have offered alternative theoretical frameworks on viewing functions and functional knowledge. Several have focused on the dual nature of the concept of function, noting that it can be viewed as either a process or an object (O'Callaghan, 1998). Sfard and Thompson (1994) described a theory of reification involving a transition from an operational (process-oriented) conception to a structural (object-oriented) vision of a concept, in particular, that of a function. Dubinsky and Harel (1992) expressed a similar perspective in their discussion of action versus process conceptions of functions, and Sierpiska (1992) stressed the idea that an object conception is required for an understanding of the notion of a function.

Schwarz and Dreyfus (1998) noted another framework for considering functional knowledge. They asserted that acquisition of the concept of function is intimately linked to action on objects and the conservation of invariants (properties of functions) under actions. Confrey and Smith (1991) presented a somewhat analogous framework for

viewing and reviewing functions. It involved the study of prototypes (families of functions like polynomials and transcendental), multiple representations, and transformations.

### *Teaching of Mathematics*

Learning is part of everyday life. Typically the process of acquiring knowledge in an educational context involves two key elements: the educator who actively imparts knowledge and the student who actively gains knowledge. Both elements are equally essential for successful learning. Thus, mathematics teaching is engaging in mathematics with students in order to enrich their understanding (Wittmann, 1984).

Conventional approaches to the teaching of preliminary concepts in mathematics have focused on the development of the variable concept through interpretations and generalizations of mathematical expressions and statements (Haimes, 1996). These approaches have stressed the development of algorithms and of manipulation and simplification skills for factoring and solving polynomials, rational expressions, and equations. Ultimately, students are introduced to the concept of a function, albeit in a formal setting (Haimes, 1996), while the teaching of functions in algebra classes has tended to emphasize structural rather than procedural interpretations (Kieran, 1992). In more contemporary curriculum developments, the focus has changed, with the concept of *variable* being developed by having students manipulate functions, and functions are introduced by way of a modern definition (Haimes, 1996).

A recommendation of the Calculus Reform Movement was that instruction in calculus should center on the conceptual understanding of functions, using a combination of different modes of instruction along with encouraging an inclusive spirit of

cooperation (Tucker & Leitzel, 1995). In most calculus reform courses, the lecture is being partially or totally replaced by multiple classroom and laboratory activities. The key word here is activity: active involvement of the students as learners and participants, while also emphasizing discovery, sharing, and invention (Smith, 1994).

One program in widespread use that has been developed to improve instruction in calculus is the Professional Development Program (PDP). This program was first implemented by Uri Treisman at the University of California at Berkeley (Taylor, 1996).

Taylor stated:

Recently at the University of California, Uri Treisman conducted a study of factors that determined success of calculus students at the Berkeley campus. He observed that race, sex, and socioeconomic status ...[were] not critical factors. The critical factor was whether the students participated in study groups. He observed that whereas Asian students tended to work together in study groups, black students tended to study alone. He was able to get black students to work together through forming the Black Honors Calculus Society, and members' grades increased dramatically. (p. 47)

Some of the components of PDP are as follows: (1) a workshop environment using small study groups, (2) faculty leadership for these groups, (3) challenging mathematical materials to facilitate learning in a non-threatening, non-remedial manner, and (4) peer support networks. The workshops supplement and complement the usual three to four hours of classroom time. An important component of PDP is time spent in workshops under the supervision or guidance of faculty (Taylor, 1996). One of the most important components of PDP seems to be time spent on tasks under faculty supervision with peer support and feedback.

### *Historical Perspectives of the Concept of Function*

Around the beginning of the twentieth century, many mathematics educators said that the concept of function should receive greater emphasis in school mathematics (Breslich, 1928; Hedrick, 1922; Townsend, 1915). In an effort to unify the secondary mathematics curriculum, the National Committee on Mathematical Requirements of the Mathematical Association of America recommended in 1921 (Hedrick, 1922) that the concept of a function be the unifying principle of secondary mathematics. Another consideration that gave impetus to the importance of functions in mathematics curriculums in the first half of the twentieth century was the opinion that functions occurred prominently in the real world (Hedrick, 1922). Equally important, there were no constraints in the definition of a function that prevented the presentation of applicable, real-world examples. The various recommendations began to show results, although the results were infrequent and uneven (Schorling, 1936).

Bulter and Wren (1941), Schorling (1936), and Young (1927) each had a mathematics methods textbook that encouraged teachers to stress functions as the unifying principle of school mathematics. Some textbooks published in the 1930s, one in particular by Betz, tried to unify a course around functions. A companion Algebra II textbook (Welchons & Krickenberg, 1949b) did provide some emphasis on functions.

The 1950s and 1960s brought recommendations stressing mathematical structure while defining functions in terms of a set of ordered pairs. In 1959 the College Entrance Examination Board (CEEB) recommended that increased attention and importance be given to functions in high school mathematics, and the central theme of functions continued as a unifying idea. The CEEB even recommended a separate course on

functions. Between 1960 and 1967 the School Mathematics Study Group (SMSG) produced a series of high school textbooks designed to implement the CEEB's 1959 recommendations. The SMSG (1965) designed one textbook, particularly, as a high school function course. A dominant theme of the SMSG materials was that all mathematics should be considered as a unified whole (SMSG, 1985).

The 1970s saw the end of the new math era, as indicated by the report of the National Advisory Committee on Mathematics Education (NACOME, 1975) and the beginnings of numerous criticisms of college and high school mathematics along with several calls for reforms of their respective curriculums. The signals for reforms in the teaching and learning of mathematics persisted, with proclamations from the National Council of Teachers of Mathematics (1980) and the National Commission on Excellence in Education (1983). In fact, nearly a dozen additional major reports on U. S. schools emerged in 1983 alone, and the educational reform movement continued to gain momentum in the mid-1980s, beginning with the Reagan administration's report *A Nation at Risk* by the National Commission on Excellence in Education (1983). The common thread throughout these reports was the demand for a national commitment to excellence in education (Mathematical Association of America, 1981) with at least one proclamation calling for the concept of function to be central theme for the organizing of secondary and higher mathematics curricula (NCTM, 1989).

While mathematicians and mathematics educators were calling for a greater emphasis on the concept of function during the late nineteenth century and the first part of the twentieth century, the definition of the function concept was changing (Kleiner, 1989). In the middle of the nineteenth century, the work of the mathematician, Dirichlet,

illustrated the idea that functions could include arbitrary correspondences. Yet many prominent mathematicians in the early 1900s (Lebesgue, Baire, and Borel) thought of functions as dependent relations or supported the requirement of a correspondence between variables in the function definition as some type of rule or law. The acceptance of a definition of functions that included arbitrary correspondences was gradual (Kleiner, 1989). A standard definition of a function early in the twentieth century was: “If for each value of a variable  $x$  there is determined a definite value or set of values of another variable  $y$ , then  $y$  is called a function of  $x$  for those values of  $x$ ” (Townsend, 1915).

There was little change between the accepted definition of a function in the early 1900s and the accepted definition of the 1950s. However, the definition was refined somewhat to include an increased emphasis on the concept of a set, the acceptance of functions as arbitrary correspondences, and the requirement that each value of the independent variable has a unique image as a dependent variable. This characteristic was missing from Townsend’s definition. Definitions of functions in the early part of the twentieth century usually centered on dependence and correspondence with an emphasis on formulas and graphs. Later, refined definitions in the late 1950s referred to functions as a set of ordered pairs. The contemporary definitions of relations and functions represented a fundamental change in the way people thought about functions. Defining the concept of a relation as a set of ordered pairs gave a previously vague term a precise mathematical meaning (May & Van Engen, 1959).

The concept of a function as a set of ordered pairs for which certain conditions were satisfied gave a new generality and precision to the meaning of a function. It was more precise to refer to a function as a set of ordered pairs represented by a table, a



graph, a rule, or a description. For example, the function symbolized by the formula  $y = x^2$  could be more accurately referred to as the set of ordered pairs of the form  $\{(x, y): y = x^2\}$ . Since the definition of a function relied on the concept of a set, it was natural to call the set of all first elements of a function the domain and the set of all second elements the range. These ideas gave rise to the modern definition of a function: A function  $f$  from  $A$  to  $B$  (in two-dimensions) is a well-defined subset of ordered pairs of the Cartesian product of  $A$  and  $B$ , such that for every  $a \in A$  there is exactly one  $b \in B$  such that  $(a, b) \in f$  (Cooney and Wilson, 1993).

Because the function concept was rarely included in most textbooks from the late nineteenth century to the middle twentieth century, it was seldom taught. Breslich (1928) stated that many instructors were skeptical about teaching the function concept in high school mathematics. He said some teachers insisted that functions should be saved for higher mathematics courses. Yet, Wells and Hart (1929) introduced in their textbook the function concept using a sequence of examples such as the following: The money earned by a man who is being paid 50 cents per hour depends upon the number of hours he works.

The Algebra II textbook by Welchons and Krickenberg (1949) asked students to evaluate  $f(2)$  if  $f(x) = x^2 + 3x - 4$ . By the 1950s, functions were beginning to be taught and used more in terms of symbols and ordered pairs. The Algebra I textbook by Dolciai, Berman, and Wooton (1963) placed a strong emphasis on the graphs of functions, identifying the domains and ranges of functions, and determining whether relations are functions. From the 1960s through the 1970s, instruction of the function concept in school mathematics became more prominent. Functions were still basically viewed

symbolically, yet graphical representations had begun to emerge. In the 1980s, the teaching of functions began to incorporate many, if not all, of the recent recommendations because of the impetus of the reform movement in the mathematics curriculum, in particular, the Calculus Reform Movement.

## CHAPTER 3

### METHODOLOGY

#### *Pilot Study*

I conducted a study during spring semester 1998 titled, *College Students' Subject-Matter Knowledge of Quadratic Functions*. My study investigated (1) college students' knowledge of functions, particularly quadratic functions; and (2) the affect of their college major had on their knowledge. Approximately 62% of the 35 students from my MAT 191 calculus class volunteered for the study completed a questionnaire (Appendix E) and a test (Appendix F). The use of graphing calculators was permitted on the test and approximately 33% of the students passed with a grade of 70% or better.

One question on the questionnaire asked the students to state whether or not they were mathematics majors. The results from this question were used to divide the students into two disjoint groups, majors and non-majors. The students were also asked to list the number of mathematics courses taken in high school and college. The criteria used for the selection of each participant was the combined number of mathematics courses taken in high school and college, and the availability for the study rather than the test scores. Scheduling conflicts eliminate 9 students of the 22 students, five from the majors and four from the non-majors. The 2 students with the most extensive mathematics background were selected from each group, majors and non-majors, to be representatives of their classmates and their test scores were a 66% and 59% for the majors and a 63% and 55% for the non-majors, respectively.

Data was gathered from the test, sorting activities and interviews. The sorting activities consisted of the students distributing the 30 functions from the test into the matrix below by assigned numbers. See figure 1. A comparative analysis was conducted on data obtained from the majors and non-majors. Of the four students selected, two withdrew from the study near its midpoint. Fortuitously, only one student withdrew from each group, which did not prevent the intended form of analysis. The remaining students had test scores of 59% and 55%.

Four one-hour audiotaped interviews, one concerning the test, one dealing specifically with quadratic functions (Appendix G) and two dealing with sorting, were obtained from each of the remaining students. A comparative analysis of the data showed that neither the mathematics major nor the non-major had a complete or accurate concept of functions, particular quadratic functions, or non-functions. In addition, the ability to distinguish between functions and non-functions was displayed to a greater degree, through the use of the vertical line test and graphing calculator, than the ability to identify different types of functions by name.

	Quadratic	Non-quadratic
Functions		
Non-Functions		

Figure 1. Pilot Study Sorting Matrix

Upon reflecting on the pilot study, only focusing on the number of courses taken in high school and college in the pilot study as a main criterion while ignoring the test scores may have limited the scope of the research. Students with better test scores and the same or less extensive mathematics backgrounds may have given different results. In addition, they may have been more committed to the study. I also realized that some criteria like the number of mathematics courses taken might need to be waved or modified to best serve goals of the study.

### *Selection of Participants*

Founded in 1833, Mercer University is a private university located in Macon, Georgia. All participants were chosen from Calculus I (MAT 191), a reform calculus class, held during spring semester 2001 in the College of Liberal Arts at Mercer University. Calculus I has been a part of the general education requirement for years, at Mercer University. It has served mathematics majors but also students from a wide variety of disciplines. The members of the MAT 191 calculus classes were commonly freshmen students, and they were frequently diverse with respect to race, ethnicity, sex, and mathematical background.

The University of Georgia and Mercer University's Internal Review Boards granted permission for human subjects research during the spring semester of 2001. This MAT 191 calculus class contained 29 students, and all students were asked to participate voluntarily in the study. Based on the percentage of students who completed consent forms from a pilot study conducted during the last few weeks of spring semester 1998, I expected about 62 percent to volunteer for the study, approximately 17 or 18 students out of the 29. However, there were 22 volunteers. Reflecting on the results of the pilot study,

my goal was not to choose four students who would be representative of the class, but to select students with strong academic background, especially in mathematics, who also performed acceptable on the pretest.

The pretest for this study was a modification of the test given during the pilot study. Adding graphical and numerical representations of relations and a request for the definition of a function as well as requests for examples of functions and non-functions altered it. Participants were given the option of taking the pretest in one 2-hour testing period or two 1-hour testing periods. Everyone opted for the one 2-hour testing period. Similar to the test given during the pilot study, graphing calculators were permitted, and all tests were administered on Mercer's campus. Unlike the pilot study, I chose a grade of 70% as a minimum cutoff score because I was as interested in studying students who performed satisfactory on the pretest (Appendix A) as well as those who had an extensive mathematics background.

Because approximately 33 percent of the participants from the pilot study made grades of 70% or higher on the test during spring semester 1998, I expected to narrow the 22 volunteers to approximately 6 or 7 students. The actual number of students who obtained a score of at least 70% on the pretest was 7. Next, I narrowed my selection to 4 potential participants by reviewing the following: (1) how informative and elaborate their responses were to the questionnaire; (2) how they supported or justified their pretest answers; (3) scheduling conflicts for interviews; and (4) a willingness and commitment to continue with the study. Using the pilot study as a guide, criterion (4) above was as important as any other was because it was used to insure that students would be less likely to drop from the study. Again using the pilot study as template, I reserved the

option waving a criterion and selecting a student whose pretest score was less than 70% if he or she met all of the other requirements. After applying the minimum score of 70% and the other four criteria, I had selected only two participants. Therefore, I waved the minimum score of 70% and used only the four remaining criteria to select the last two participants. All four selected participants were traditional college students, 18 to 22 year-olds who met the above four criteria. The four chosen students were Eddie, Kristin, Whitney, and Buren and their pretest scores were 31, 83, 64, and 83, respectively.

### *Participants*

**Eddie.** Eddie was classified as a sophomore, and he was a single male age 20 from the state of Georgia. He was a good student whose major field of study was accounting and minor was finance. Eddie was a member of the basketball team and attended Mercer full-time. He had graduated from high school in June 1998. Eddie took Geometry, Trigonometry, Calculus and Algebra II in high school along with the honor courses: Physics, Chemistry, Biology, English, Calculus, and History. He had an A average in high school and an A cumulative grade point average in college. Eddie described his experiences with mathematics as “having thoroughly enjoyed math my whole life.” Calculus I (MAT 191) was his first college mathematics course, and he claimed that he had a good background for calculus. Eddie was always on time to class, consistently participated in class, and faithfully attempted the homework. His only absences from class were out-of-town basketball games. Even though Eddie’s pretest score was less than 70%, he met all the other criteria. Yet, he was selected in part based on the courses taken in high school and his high school average which was used as a means to compensate for a score of 31%.

**Kristin.** Kristin was a native Georgian majoring in Spanish with a minor in International Business. She was 20 years of age, a sophomore, and single. She was a good student and took the following mathematics courses in high school: Advanced Algebra, Trigonometry, Geometry, and AP Calculus. Other honor courses taken in high school were Physical Science (9, 10, and 11), English Literature, AP United States History, and AP Government. Kristin had an A average in high school and an A cumulative average in college. She stated that she had a good background for MAT 191 and that this was also her first college mathematics course. Kristin described a bad experience with mathematics from high school as “taking the AP Calculus [examination] and realizing I actually had no clue how to do calculus.” She said the bad experience was caused by “limited class time and crammed instruction due to block scheduling.” Kristin was selected because her pretest score was 83% and she met all the remaining criteria.

**Whitney.** Whitney was 20 years of age, resided in Georgia, and was a good student who had not yet decided on a major or minor. She took the following honors courses in high school: Geometry, Algebra II, English I, II, and III, Chemistry, and Biology. She also took Trigonometry, AP Calculus, Chemistry, and History. Whitney had an A average in high school and an “A” cumulative average in college. She had mixed experiences concerning mathematics in high school. Some of her courses were taught well, and she learned the subject matter. In other courses, she was able to memorize the information and perform well on tests because the students were actually told or given the material that was going to be on the tests. In still other courses, she made some of her lowest mathematics grades because “the teacher did not explain any of the concepts beyond working problems on the board and assigning problems to be worked with



partners.” Having met all the criteria except the minimum score of 70%, Whitney, like Eddie, was selected in part based on her high school average and the courses taken.

**Buren.** Buren was a sophomore whose major was Biology. His minor was Spanish. He resided in Georgia and was a 20-year-old single male. The courses he took in high school were Algebra I, Geometry, Algebra II, and Gifted Precalculus. The honor courses taken were Senior English, Physical Science, Biology, Spanish, Chemistry, and AP Biology. Buren had an A average in high school and a B+ cumulative average in college. He had good experiences with mathematics in high school. He said, “My high school experiences ... have been good. I’ve made A’s and B’s in all of my mathematics courses. There were a few concepts that I did not understand, but I received help from my teachers!” Buren, like Kristin, was selected because of he had a pretest score of 83% and he met all the other criteria.

### *Data Collection*

Classes for Calculus I met three days a week. On the first day of classes, a Monday, the students were asked to participate in the study and were given human subjects consent forms and a questionnaire to complete (Appendix D). The consent forms and questionnaires were turned in the same day and a pretest was scheduled for the next day at 10:00 a.m., 1:00 p.m., and 4:00 p.m. No one took the test scheduled for 4:00 p.m. and all 22 pretests were graded by Wednesday, the second day of classes. The seven participants who obtained pretest scores of at least 70% were reduced to four participants by Friday, the third day of classes. Data collection continued on Friday and Saturday with the pretest interviews. Monday and Tuesday of the second week of classes were devoted to the sorting activities. The interviews for the sorting activities were conducted

Wednesday and Thursday of the same week. The remainder of the week through Saturday was reserved for translating tasks. The following Monday and Tuesday of the third week were devoted to translating interviews. The remainder of the week was reserved for any missed interviews, sorting activities or translating tasks.

First class day, Monday, January 8, 1998, the students were asked to participate in the study. They were given consent forms and questionnaires to be completed and turned in the same day.
Non-class day, Tuesday, January 9 <sup>th</sup> at 10:00 a.m. and 1:00 p.m., the pretest was given. I began grading the pretests.
Second class day, Wednesday, January 10 <sup>th</sup> , I finished grading the pretests and reduced to the number of participants from 22 to 7 using the 70% minimum score.
Non-class day Thursday, January 11 <sup>th</sup> , I initially narrowed the number of participants from 7 to 2 since only 2 students satisfied all criteria. The number of participants was increased to 4 after the criterion of 70% was waved.
The third class day Friday, January 12 <sup>th</sup> through Saturday, January 13 <sup>th</sup> was scheduled for pretest interviews.
Fourth class day Monday, January 15 <sup>th</sup> through Tuesday, January 16 <sup>th</sup> was scheduled for sorting activities.
Fifth class day Wednesday, January 17 <sup>th</sup> through Thursday, January 18 <sup>th</sup> was scheduled for sorting interviews.
Sixth class day Friday, January 19 <sup>th</sup> through Saturday, January 20 <sup>th</sup> was scheduled translating tasks.
Monday January 22 <sup>nd</sup> and Tuesday 23 <sup>rd</sup> were scheduled for translating interviews.
Wednesday January 24 <sup>th</sup> through Saturday 27 <sup>th</sup> were scheduled for missed interviews, sorting activities or translating tasks.
Monday, January 29 <sup>th</sup> through Wednesday February 7 <sup>th</sup> was devoted to studying functions using the Rule of Three.
Thursday February 8 <sup>th</sup> was scheduled for the posttest. All four posttest were graded.
Friday February 9 <sup>th</sup> through Saturday 10 <sup>th</sup> were scheduled for posttest interviews.
Monday February 12 <sup>th</sup> through Tuesday 13 <sup>th</sup> were scheduled for sorting activities.
Wednesday February 14 <sup>th</sup> through Thursday 15 <sup>th</sup> were scheduled for sorting interviews.
Friday February 16 <sup>th</sup> through Saturday 17 <sup>th</sup> were scheduled for translating task.
Monday February 19 <sup>th</sup> through Tuesday 20 <sup>th</sup> were scheduled for translating interviews.
Wednesday February 21 <sup>st</sup> through Saturday 24 <sup>th</sup> were scheduled for any missed interviews, sorting activities or translating tasks.

Figure 2: Time Line

The fourth week and a half of the fifth week of classes were used to studying functions, using the Rule of Three. Starting with Thursday of the fifth week of classes, data collection began again in the form of a posttest scheduled from 10:00 a.m. until 12 noon. All four posttests were graded that day and the corresponding interviews began Friday and continued through Saturday.

The sorting activities began Monday and continued through Tuesday of the sixth week. Wednesday through Thursday was scheduled for sorting interviews. The remainder of the week, Friday and Saturday, was reserved for translating tasks. The translating interview began the following Monday and concluded that Tuesday of the seventh week. The remainder of the week was scheduled for any missed sorting activities, interviews or translating tasks. The study covered nearly the first seven weeks of Calculus I during the spring semester of 2001 while the actual time devoted to studying functions using the Rule of Three was a week and a half.

Before instruction using the Rule of Three to study and review functions, all four participants were asked the questions in the interview protocol (Appendix B). Led by the students' responses, the interview protocol oftentimes deviated from the one in the appendix. The participants were asked different but similar questions before and after instruction using the Rule of Three. All interviews were audiotaped and transcribed. Data were also collected from field notes taken by me and written responses by the participants during interviews, translating tasks and sorting activities. The data collected before and after instruction using the Rule of Three were compared and analyzed to obtain a qualitative narration that was directly tied to my research questions on the concept of function.

### *Data Analysis*

After an initial collection and examination of the data, categories or classes were constructed (Owens, 1982). While categorizing data, I used an ongoing comparative analysis throughout the study to find patterns within and across categories as opposed to a final end-result analysis (Glaser & Strauss, 1967; Goetz & LeCompte, 1984; Patton, 1990).

Equally important, an ongoing comparative analysis provided direction to the collection of data by indicating what should be sought, checked, or rechecked, and how to broaden the scope of the data collection itself (Owens, 1982). In the early stages of the study, most of the time and effort was consumed gathering data, and little time was given to analysis. In the latter stages of the study, this trend was reversed. In addition, I was prepared to look for unanticipated points of view arising from the ongoing analysis on the data (Owens, 1982), and I continually scanned the data for patterns or changes in the students' knowledge through their written or oral responses (Goetz & LeCompte, 1984).

I used audiotaped interviews, along with notes and written responses from the interviews, as a means of enrichment, clarity, and information and as indicators of changes in the students' knowledge (Goetz & LeCompte, 1984). The main objective of this qualitative analysis was to identify, organize, categorize, and compare explanations, ideas, and meanings common to the participants that were contained in the data into an exposition that was understandable and tied to theory and previous research. The identified, organized, and categorized explanations, ideas, and meanings constituted my units of analysis. Connections and relationships among and between categories were investigated through a continual process and analyzed with respect to current and

previous data. Conclusions were based on the analyzed units of analysis, were based on theory, and were supported by the data.

A fair question may be the following: “Did I capture what I was supposed to measure?” I tried to minimize this potential problem by collecting the same data through a triangulation process with multiple sources of data such as: (1) pretest; (2) interviews (and any written work given during the interviews); (3) sorting activities; and (4) posttest. This method of data collection helped to insure the validity of my research and demonstrated that the propositions generated, refined, or tested matched what occurs in human life (LeCompte & Preissle, 1993).

To what extent were my findings applicable across groups (Patton, 1990)? I sought to minimize the non-applicability of my findings across groups by stating the dependence, if any, of the data on the social setting, my social relationship with the participants, the participants themselves, or any possible interactions. I attempted to ensure applicability to other settings or groups of students by examining functions that occurred in traditional and reform calculus textbooks. The use of, the Rule of Three in the study was also applicable to other groups or settings. I do not think I had any instruments or constructs that were specific or unique to the participants. I tried to minimize this potential problem by collecting data that were as objective as possible.

One aspect of reliability refers to the extent to which studies can be replicated. I identified and attempted to handle five problems that may have allowed the reliability of the data to be enhanced: (1) researcher status position; (2) informant choices; (3) social situations and conditions; (4) analytic constructs and premises; (5) and methods of data collection and analysis (LeCompte & Preissle, 1993). Informant choices, social situations

and conditions, and method of data collection were discussed under the heading, *Selection of Participants and Data Collection*. All methods of analysis, assumptions, theories, and choices of terminology were plainly stated. Definitions of concepts were made clear and sufficiently lacking in idiosyncrasies. All observations were recorded mechanically, notes were composed on location, and I explained how materials from various sources were integrated into the study. Moreover, I identified and discussed the data analysis processes. I thought these procedures increased the potential for subsequent researchers to reasonably reconstruct my original analytic strategies.

Since all encounters, except the pretest and posttest, with the students were taped, an audio-log was maintained throughout the study of all contacts in which verbatim accounts of participant conversations and descriptions were used. The audio-log allowed me to have what Guba and Lincoln (1981) called as an “audit trail” for qualitative research. My audit trail allowed the documentation of the nature of decisions in the research plan and the data, theory, and reasoning on which the decisions were based. The audit trail usually permitted two important things. First, it allowed for an examination of the procedures of the study, either in progress or after the fact, in order to verify the study’s validity by independent, external auditors. Finally, the audit trail made it possible for reproduction of the study, which in turn provided a check for reliability.

#### *Researcher’s Role*

First, I clearly identified my role and status within the group that was investigated and any biases or perceptions I thought were caused by my professional status-- a college mathematics instructor. I had three roles during this study: a collector of data, a researcher, and a direct observer. All roles were arranged with the participants and

seemed to have had a minimal effect on their classroom or laboratory behavior as I sought to establish and maintain a friendly, informal environment that was also impartial to the essential concerns of my research. However, my role of a direct observer spanned the spectrum from that of a quiet, passive, uninvolved observer to that of an actively, involved participant.

#### *Researcher's Bias*

Biases based on my teaching experience, theoretical framework, previous research, literature, and reading were clearly and fully documented. First and foremost, I was the instructor to the participants. Since I subjectively interpreted and analyzed what I thought the participant meant when he or she spoke or wrote, I attempted to clearly differentiate between what was said or written, the symbols used, and my own interpretations. I maintained a log of my interpretations, along with the taped interviews, direct observations, and class presentations, which enabled me to remain vigilant regarding my subjectivity.

#### *Limitations*

As with most research, there were limitations with this study. The first limitation was the limited review time to study functions using the Rule of Three. Next, I was only able to describe the participants' knowledge by their responses or answers while they were engaged in tasks. Therefore, a complete description of their knowledge was difficult.

## CHAPTER 4

### RESULTS

#### *Instruction Before Using the Rule of Three*

This chapter reports data concerning college students' knowledge of functions before and after instruction, using the Rule of Three. Before instruction using the Rule of Three, data was collected from a pretest, sorting activities, and interviews. After instruction, using the Rule of Three, data was collected from a posttest, sorting activities, and interviews. After initially collecting data, categories were constructed based on the way the information was gathered (Owens, 1982). The two main categories were *Instruction Before Using the Rule of Three* and *Instruction Using the Rule of Three*. The partition seemed to be a natural way to divide the data. Subcategories constructed for the category, *Instruction Before Using the Rule of Three*, were: (1) Definition of a Function; (2) Examples of Symbolic, Graphical, and Numerical Representation; (3) Examples of Non-functions; (4) Classification of Relations as Functions; (5) Recognizing Functions; (6) Translations; (7) All Three Representation; (8) The Easiest Translations; (9) The Most Difficult Translations; (10) Creating New Functions; (11) Creating Functions from Non-functions; (12) Sorting Cards; (13) and Tools. Some of the subcategories were partitioned, once again. For instance, the category, Recognizing Functions, was subdivided into the smaller categories, Symbolic, Numerical or Graphical, while the category, Sorting Cards was split into categories based on the students' names for their partition of the cards. These same subcategories and smaller categories were used once



again for the category, *Instruction Using the Rule of Three*, since the data was collected in a similar manner.

I began the reporting of the data with the students' pretest scores and their Definition of a Function, and ending with subcategory, Tools. The data collected in the part of the study designated, *Instruction Using the Rule of Three*, was reported in a similar manner. As the data collection phase of my research lessened and the analysis process increased, some of O'Callaghan's (1998) and Markovits et al.'s components of the concept of function called competencies that were used by them to describe functional knowledge emerged from the student's data. To be more specific, evidence of the students' functional knowledge, and the strategies used by them to recognize, classify, and translate from one representation of a function to another surfaced.

The pretest scores for Eddie, Kristin, Whitney, and Buren were 31, 83, 64, and 83, respectively. The students' definitions of functions are given after the bulleted statements. The following bulleted statements below are definitions of a function that were taken from a traditional and reform calculus textbook, respectively, and were stated originally in chapter one. All functional definitions given by students were compared to one of the textbook definitions and each other. Neither definition was given to any of the students before the pretest or the first interview, and I had no expectations concerning the type of definitions the students would give. In addition, I did not expect the students to give an exact rendering of either definition, traditional or reform.

- A function  $f$  from a set  $A$  to set  $B$  is a well-defined subset of ordered pairs of the Cartesian product of  $A$  and  $B$ , such that for every  $a \in A$  there is exactly one  $b \in B$  such that  $(a, b) \in f$ . Since the definition of a function relies on the concept of a set, it is

natural to call the set of all first elements of a function the domain and the set of all second elements the range (Stewart, 1991).

- A function is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the domain of the function and the set of resulting output numbers is called the range of the function (Hughes-Hallett et al., 1998, p.2).

### *Definition of a Function*

Eddie: A function is an algebraic representation of an equation.

Eddie's definition of a function did not look as if it corresponded to either of the textbooks' definitions. It seemed to be incomplete because he did not utilize any of the words like "rule," or the phrase like "a set of ordered pairs," mentioned in the textbooks, and he did not include the functional components, the domain and range. Yet, Eddie demonstrated some knowledge of the domain and range of a function when he said, "Two different y-values in the range can't go with (be associated with) the same x-value in the domain." Later, during the same interviews, Eddie also said that every function has a domain and a range. The Domain is the numbers placed in (a function) and the range is the numbers gotten out.

When asked to define the term "algebraic representation," Eddie did not give a definition but, instead, gave the equations,  $2x + 3y = 12$  and  $y^2 = x$ , as examples of algebraic representations and functions. Although one of Eddie's examples,  $2x + 3y = 12$ , was a symbolic representation of a function, the other one,  $y^2 = x$ , was not. Initially, based on his responses, it appeared that when Eddie used the phrase "algebraic representation," he was referring to equations that were just as likely to be a

function as not. He seemed to think that any equation was an algebraic representation and hence a function.

Later, Eddie appeared to realize that his definition was probably incomplete when he solved  $y^2 = 4$  for  $y$  and got  $y = 2$  or  $-2$ . Then he said, “Something is wrong because one  $x$ -value yielded two  $y$ -values” which contradicted his earlier statement that two different  $y$ -values cannot have the same  $x$ -value.

Kristin: A function is a relation in which there are no two coordinates with the same  $x$ -coordinates and different  $y$ -coordinates.

There were no similarities between Eddie and Kristin’s definition. Kristin’s definition seemed to parallel the definition from the reform textbook more so than the definition from the traditional text. Nevertheless, when compared with the definition from the reform textbook, Kristin’s definition, like Eddie’s, did not contain two components of a function, its domain and range. Kristin used the word “relation” in her definition of a function, but failed to define it on the pretest. During the interviews, Kristin defined a relation as only “a set,” but she did not give a meaning for “a set” when asked. Kristin’s use of the word “relation” in her definition appeared to mirror the application of the word “rule” in the reform textbook’s definition—that of a tool use to take certain numbers as inputs and allocate to each a distinct output number. Kristin said a “relation” is the apparatus (tool) that assigns to each  $x$ -value the different or unique  $y$ -value.

However, like Eddie’s definition, Kristin’s appeared to be incomplete.

Whitney: A function is a special type of relation in which for every unique value of  $x$ , there is a unique value of  $y$  corresponding to it.

Whitney's definitions seemed to parallel Kristin's as well as the definition from the reform textbook more so than the definition from the traditional text. Since Whitney's definition was similar to Kristin's, she also failed to include the domain and range in her definition. Like Kristin, Whitney also used the word "relation" in her definition, failed to define it on the pretest, and did not offer a definition for it during the interviews. Like Eddie and Kristin's definitions, Whitney's also seemed incomplete.

Buren: A function is a set of ordered pairs whose coordinates cannot have the same  $x$ -coordinate and different  $y$ -coordinates. A function also has a domain and a range. The domain and range as the set of all input numbers and the set of all output numbers.

Buren's definition of a function also seemed to parallel the definition from the reform textbook similar to Whitney and Kristin's definition. Resembling the reform textbook's definition, but contrasting with Whitney and Kristin's definition, was Buren's inclusion of the functional components, domain and range along with their meanings, in his definition. In addition, he was also proficient in obtaining the corresponding  $y$ -value in the range for a specific  $x$ -value in the domain and obtaining an  $x$ -value in the domain for a given  $y$ -value in the range.

Furthermore, for the symbolic representation of a linear function like  $2x + 3y = 12$ , Buren was able to state the domain and range—the set of all real numbers. Buren's knowledge of the domain and range of functions seemed to be different from Whitney's and Kristin's, who did not mention the domain and range, and it appeared to be more than just basic information but was it linked to linear functions? When supplied with the symbolic representations of the rational and radical functions (non-polynomial

functions),  $y = \frac{x}{x-1}$  and  $y = \sqrt{2x-4}$ , Buren continued to provide the set of real numbers

as the domain and range. Even when furnished with the symbolic representations of quadratic and constant functions,  $y = x^2 + 1$  and  $y = 2$ , Buren persisted in giving the set of real numbers as the range. He seemed to be unable to supply an answer other than the set of real numbers for the domain or range of any function presented to him. So I asked Buren if he actually thought that most, if not all, functions had as their domains and ranges—the set of real numbers. He said, “No, but the functions that I don’t know [the domain or range], I guess the set of real numbers. I am familiar enough with functions to know that this (the set of real numbers) is not a bad guess.” Buren’s comments were informative because he seemed to make educated guesses when his knowledge failed him. I was still curious concerning Buren’s knowledge of the domain and range, so on the spur of the moment, I created the following equations (functions):  $y = 2x + 4$ ,

$$y = x^2 + 1, y = 2, y = \frac{x}{x-1}, y = \sqrt{2x-4}, \text{ and } y = \log x.$$

I chose the first three functions because I thought that he would be familiar with them and the last three functions because I thought Buren may have less experience working with them. I also wanted functions (the first three) whose domain was the set of all real numbers and functions (the last three) whose domain was not the set of all real numbers because I desired to know what his response would be for each type. Buren stated that the domain and range of each function was the set of all real numbers. At least for the latter three functions, Buren appeared to have given an educated guess.

Another difference between Buren’s definition and the definitions of Whitney and Kristin was his use of the words “a set of ordered pairs” in his definition of a function rather than the word, “relation.” Nevertheless, when asked to define “a set of ordered pairs,” he said it was a relation and a relation was the tool that assigned each  $x$ -value its

$y$ -value. Although Buren did not use the word “rule” in his definition of a function, the phrase, “a set of ordered pairs,” seemed to serve the same purpose. Buren’s definition appeared to be the more complete than Whitney’s and Kristin’s because he was the only student so far to mention and define the domain and range of a function.

Most of the students’ definitions seemed to resemble the definition from the reform textbook. Although neither student used the word “rule” in their definition of a function, the words, “relation or apparatus” or the phrase, “a set of ordered pairs” appeared in most definitions and seemed to have been employed in their definitions as the word “rule” was used in the textbook definition. Yet, their definitions appeared to be incomplete since all students, except Buren, did not discuss the domain or range. Later during the interviews, the students demonstrated knowledge of the domain and range of a function when they said the  $x$ - and  $y$ -axes corresponded to its domain and range respectively. It also seemed that every student had some knowledge about the components of a function such as the domain and range, but did not include them in their definitions. Again, insight about the students’ knowledge concerning domains and ranges was gained because it seemed they knew more about the domains and ranges of functions than their definitions indicated.

#### *Examples of Symbolic, Graphical, and Numerical Representations*

The only representations given by students were those of polynomial functions. Eddie, Kristin, Whitney, and Buren gave  $y = 2x + 2$ ,  $f(x) = x^2$ ,  $f(x) = x^2$ , and  $y = x^2 + 2$  as their symbolic representations, respectively. The students verified that their symbolic representations were functions by identifying them by names like, linear or quadratic functions. Furthermore, the ability to identify symbolic representations by

name seemed to demonstrate some level of competency in the recognizing component of the framework by the students. They used vertical line test to confirm that the graphical representations were functions, and numerical representations were verified as functions by insuring that no  $x$ -value was associated with more than one  $y$ -value.

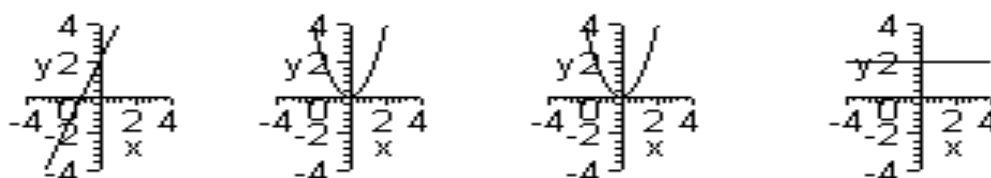


Figure 3: Eddie, Kristin, Whitney and Buren's graphical representations

Table 2: Eddie, Kristin, Whitney and Buren tables

$\begin{array}{c c c c } x & 0 & 1 & 2 \\ \hline y & 2 & 4 & 6 \end{array}$	$\begin{array}{c c c c } x & 0 & 1 & 3 \\ \hline y & 0 & 1 & 9 \end{array}$	$\begin{array}{c c c c } x & 0 & 1 & 3 \\ \hline y & 0 & 1 & 9 \end{array}$	$\begin{array}{c c c c } x & 3 & 1 & 4 \\ \hline y & 0 & 2 & 18 \end{array}$
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All students, except Buren, gave a numerical and a graphical representation that corresponded to their symbolic representation of a function. Eddie, Whitney and Kristin used their symbolic representation along with the graphing calculator, to obtain the corresponding numerical and graphical representations. Perhaps, without knowing it, the students had engaged in the translation of functions, a task that was to come later. In addition, the students' ability to give examples of functions seemingly demonstrated some level of competency in the recognizing component of the framework. However, it must be noted that the students seemed more comfortable or knowledgeable with polynomials and only gave them as examples.

Initially, Buren gave three representations—symbolic (quadratic), graphical (constant), and numerical (no particular type of function)—that were all different from each other. Buren was also asked to give representations that corresponded to each other, like the other students, to see if he was able to perform this task and not rely on memory, and he was successful. Buren, like the other students, had prematurely engaged in the translation of functions. Furthermore, the fact that Buren was able to give three different representations that did not correspond to each other as well as three representations that did correspond may be indicative of his level of knowledge concerning functions which seemed to be different from that of the other students.

Every student correctly identified each symbolic representation by name as either a linear, constant, or quadratic again displaying competency in the recognizing component. Graphical representations were verified as functions through the use of the vertical line test, and numerical representations were verified as functions by insuring that no  $x$ -value was associated with two different  $y$ -values.

On the initial examination, the two methods—the vertical line test, and the procedure of insuring that no  $x$ -value was associated with two different  $y$ -values—used by the students to verify graphical and numerical representations as functions appeared to be different. However, the method used by the students to verify numerical representations as functions was worded very similar to their statement of the vertical line test. This same method, insuring that no  $x$ -value was associated with two different  $y$ -values, is the process the students said the vertical line test allowed them to apply to graphs. Although these two methods, one for verifying graphical representations as functions and the other for verifying numerical representations as functions, were



different in their names or wording, they were nearly the same in their application by the students. Yet, the students said they were applying different procedures or processes.

### *Examples of Non-functions*

Eddie and Whitney gave a numerical representation (table of values) for their example of a non-function.

Table 3: Eddie and Whitney's Tables

$x$	0	1	3	0
$y$	0	1	9	4

$x$	10	1	13	1
$y$	2	7	8	2

Kristin did not give an example of a non-function even though there were symbolic, graphical, and numerical examples of non-functions on the pretest that she had correctly identified previously as non-functions. Amazingly, no one used any examples of a non-function from the pretest. Buren gave the symbolic representation of the non-function,  $y = \pm\sqrt{x}$ , which was a modified version of  $y^2 = x$  from the pretest.

Table 4: Buren's numerical representation of a non-function

1	1	1	1
1	2	3	4

Instead of insuring that no  $x$ -value was associated with two different  $y$ -values, this time, Eddie and Whitney intentionally linked an  $x$ -value in their numerical representations (tables of values) with more than one  $y$ -value to verify that their representation was a non-function. Buren's numerical representation is depicted in table 4 above. Buren said he used the constant function,  $f(x) = 1$ , then switched the domain and

range and took a subset of each. Buren was also asked to give a numerical representation like the other students to ensure that he was capable of perform this task. Once again he was successful.

### *Classification of Relations as Functions and Non-functions*

#### *Numerical*

By insuring that no  $x$ -value was associated with two or more different  $y$ -values, the students correctly identified each numerical representation of a non-function as a non-function and again exhibited the recognizing component. Similarly, each student, except Whitney, accurately classified each numerical representation of a function as function. Whitney identified the function, shown in table 4, as a non-function. When asked her representation was a non-function, she said, “Clearly it is a function. I just made a mistake.” The students did not appear to have much difficulty distinguishing between numerical representations of functions and non-functions. If they were careful, it seemed that the students could have probably correctly recognized all numerical representations of functions as functions and recognized all non-functions as non-functions.

Table 5: Whitney’s initial misclassification of the function as a non-function

$x$	-2	3	-1	.5	$-\sqrt{3}$
$y$	3	8	0	-0.75	2

#### *Symbolic*

Most students correctly identified all symbolic representations of non-functions as non-functions, except  $y^2 = 3x$ , which was classified as a function by Kristin. Although, each student successfully classified at least 86% of the symbolic representations of

functions as functions, all students, except Buren, had difficulty identifying the symbolic representations of functions when the symbolic representations (equations) were not already solved for the variable  $y$  in terms of the variable  $x$ . When distinguishing between functions and non-functions, the students looked for symbolic representations of the form  $y = \text{something}$  and classified them as functions, whether they were functions or not. On the other hand, the symbolic representations of functions like  $y^3 = x^2 + x + 1$  and  $2xy = 1$  were usually designated as non-functions if an answer was given at all. The fact that  $y$  was on the same side of the equation as  $x$ , or that  $y$  was on a different side of the equation but raised to an odd integer power was problematic for each student except Buren. He was the only student to solve symbolic representations like  $y^2 = x$ ,  $y^3 = x^2 + x + 1$  and  $2xy = 1$  for the variable  $y$ . Buren got  $y = \pm\sqrt{x}$ ,  $y = \sqrt[3]{x^2 + x + 1}$ , and  $y = \frac{1}{2x}$ , respectively. He recognized and justified the altered three representations as a non-function, a function and another function, respectively, and was even able to recognize  $y = \sqrt[3]{x^2 + x + 1}$  as a radical function but could offered no name for  $y = \frac{1}{2x}$ , a rational function. Yet, the students would sometimes correctly recognize the representations of functions in one part of the study and then fail to recognize the same representations in another part. For instance, early in the study, Buren identified  $x = 5$  as a non-function, in fact he even classified  $x = 5$  as the equation of a vertical line. However, he later identified  $x = 5$  as a constant function during an interview. Incorrect answers that seemed to contradict earlier correct responses and the inability to identify rational

functions like  $y = \frac{1}{2x}$  were some of the mistakes that frequently occurred in the first part of the study, instruction before using the Rule of Three.

### *Graphical*

The students had used and referenced the vertical line test several times by this part of the study, and no one had trouble distinguishing between the graphical representations of functions and non-functions. Each student applied it correctly to the graphs of functions and non-functions. All graphical representations of functions were identified as functions and similarly for non-functions.

### *Recognizing Functions*

Distinguishing between functions and non-function regardless of the representations (symbolic, graphical, and numerical) used seemed to be an easier task for the students than recognizing functions based solely on their representations. This apparent fact may have been indicated by the students' lower identification rates. When making a distinction between functions and non-functions, Eddie correctly identified 86% of the functions as functions, both Kristin and Whitney identified 93%, and Buren identified 100%. However, given only functions to recognize, Eddie accurately identified only 29% of the symbolic representations, Kristin identified 43%, Whitney identified 36%, and Buren identified 50%.

One reason may have been that the tools or strategies used previously to distinguish between functions and non-functions were no longer applicable, like the vertical line test or the procedure of insuring that no  $x$ -value was associated with two different  $y$ -values. Since all the functions were already written as  $y = \text{something}$  or  $f(x) = \text{something}$ , there was no longer a need to solve any equations for the variable  $y$ . It

seemed that the students probably relied more on their knowledge when recognizing functions than when differentiating between functions and non-functions. Having to rely more on their knowledge appeared to have made the task of recognizing functions more difficult.

In addition, each student seemed to have a “core set” of functions that he or she was able to recognize correctly using either representation. The “core set” was the largest set of functions that the students were able to recognize using all three representations. In other words, it was the set of functions obtained by intersecting the sets of functions recognized in symbolic, numerical and graphical form.

Eddie and Whitney had core sets that contained five and four functions individually. Whitney’s core set contained linear, quadratic, absolute value, tangent and the identity functions. Eddie’s core set contained the linear, quadratic, tangent, and constant functions. Kristin and Buren had the largest core sets of functions containing six and seven functions, respectively. Kristin core set contained the linear, quadratic, constant, absolute value, tangent, and identity functions. Buren’s core set contained the linear, quadratic, absolute value, radical, tangent, constant, and the identity functions.

The majority of functions contained in any student’s core set were polynomials and the number of function in the students’ core set appeared to correspond with their pretest scores. The students with the largest number of functions in their core sets were the ones with the highest pretest scores, 83, achieved by Kristin and Buren. Eddie and Whitney had the lower pretest scores of 64 and 31 and fewer number of functions in their core sets. Regardless of the representation presented to the students to identify, there was little fluctuation in the number of functions contained in their core sets. For example, if a

student was able to recognize the symbolic representation of a cubic function, he or she was usually able to identify the other two representations.

### *Symbolic*

The only symbolic representations of functions recognized by all students were the linear, quadratic and tangent functions. However, the students could not explain how they correctly identified the linear function but could not recognize the equation,  $y = x$ , as a special type of linear function or as a 1<sup>st</sup> degree polynomial. The constant function was often confused with the identity function or not identified at all. The fact that there is no variable on the right hand side of constant functions of the form,  $f(x) = c$ , seem to compound the uncertainty the students may have experienced with this type of function. Only Kristin and Buren correctly recognized both the constant and identity functions. Eddie recognized both the constant and identity functions as constant functions while Whitney classified the identity function but did not recognize the constant function. Excluding the linear, quadratic, and tangent functions, the only other functions recognized by a majority of the students were the identity and absolute value functions. Using their knowledge of the characteristics of the symbolic representations, at least three-fourths of the students classified the following functions: linear, quadratic, tangent, identity, and absolute value.

The quadratic, tangent and absolute value function were written as  $f(x) = x^2 - 2$ ,  $f(x) = \tan(x)$  and  $f(x) = -|x| + 3$ . The absolute value symbols enabled the students to identify the absolute value function and word “tan” helped them recognize the tangent function. In addition, they also recognized the general form of linear and quadratic functions function and the identity function as being a special case of the linear function.

Yet, not one student used the word “log” to classify the logarithmic function,  $f(x) = -\log x$ . Their knowledge of functional characteristics for symbolic representations seemed to be centered more on polynomial, trigonometric and absolute values functions.

The functions that presented classification problems were usually non-polynomials although the constant, cubic and identity functions sometimes presented identification problems for some of the students. A few of the non-polynomial functions that the students consistently did not recognize were the rational and algebraic functions. Each student incorrectly classified the rational function as the algebraic function and the algebraic function as the rational. Since both functions were written as ratios, the students seemed to be unable to distinguish between the rational and the algebraic function. In addition to the rational and algebraic functions, the other functions that every student missed were the greatest integer, exponential, logarithmic, cubic, segmented and radical functions. These functions were frequently not classified at all. Based on the interviews, it appeared that the students were relying on their knowledge to enable them to identify the symbolic representations of functions, but their knowledge seemed to fail them when referencing information about non-polynomial functions or polynomials of third degree or higher.

### *Numerical*

The numerical representations of functions correctly recognized by each student were similar to the functions that he or she recognized in symbolic form. There were little if any deviation from this pattern. Recognizing functions not identified earlier, based only on numerical representations, seemed to be a difficult task for the students who often

relied on their interpretations of their hand-made graphs. The hand-made graphs were often inaccurate probably because the students often plotted few points. The inaccurate graphs likely produced misclassifications and the misclassifications may have prevented students from recognizing additional functions outside of his or her core set. Eddie, Kristin, Whitney, and Buren correctly identified only 21%, 36%, 29%, and 50% of the numerical representations.

No student attempted to graph the table of values for the tangent function. Each appeared to be unable to recognize the tangent function from the table of values when the  $x$ -values in the table were originally given in radians. Once the change from radians to degrees was made, no one experienced trouble classifying the tangent function. It seems the numerical form of the  $x$ -coordinates, radians or degrees, influenced the students' ability to identify logarithmic functions in numerical form. It is likely that when trying to identify trigonometric functions from numerical representation, the students looked for  $x$ -values given in degrees.

### *Graphical*

Once again, each student recognized only the functions that he or she identified earlier in symbolic and numerical representations. Previous knowledge appeared to be a factor in performing this task because it seemed the students knew the appearances of the graphical representations of the functions in their core set. For example, discussing the functions in his core set, Eddie said that the graph of the linear function was a diagonal line, the graph of a quadratic function was a parabola, the graph of a constant function was a horizontal line, and the graph of a tangent function was the graph that had an infinite number of vertical asymptotes. The other students' performance in this task was



similar to Eddie's. Eddie's recognition of the graphical representations of the functions was, once again, consistent with his recognition of symbolic and numerical representations of functions. For a third time, he correctly identified the constant, linear, and quadratic functions, but he also added the identification of the tangent function. Kristin, Whitney and Buren also roughly identified the same types of graphical representations of functions that they had correctly identified before for symbolic and numerical representations.

When discussing the graphical representations of functions, the students used the fact that the graph of a tangent function has an infinite number of vertical asymptotes. This knowledge probably prevented them from confusing it with the graphs of rational or algebraic functions, which contained only a finite number of asymptotes. Yet, the students were unable to correctly recognize the rational or algebraic functions presented to them using any representation. Finally, if the students were able to recognize a function using one representation, they were usually able to recognize it using another representation and these functions were almost always the functions in their core sets.

### *Translating*

The data for this discussion were gathered from the sorting activities. The students were given a stack of 42 cards that could be partitioned into three stacks, consisting of fourteen cards each. One stack had only symbolic representations of functions, another had only graphical representations and the last stack had only numerical representations. The fourteen cards in each stack depict the functions: constant, linear, quadratic, cubic (third degree polynomials), logarithmic, exponential, rational,

radical, segmented (piece-wise), absolute value, greatest integer, the identity, algebraic and trigonometric (tangent).

One card depicting a function in one representation (symbolic, graphical, or numerical) will be given to the participant and he or she will be initially asked to match it only to one of its other two representations chosen by the investigator. If the participant is not successful, the activity begins anew and another card depicting the same initial representation (not function) is given to the participant who will be asked to match the card again to one of its other two remaining representations chosen by the investigator. If the participant is successful in matching the first two initial representations, he or she will immediately be also asked to match these two representations with its last remaining representation.

Translating from one representation of a function to another appeared to be an easier task than the task of recognizing individual functions in which the students usually relied on their knowledge. As Eddie said, “Identification is hard because it requires that you know something about the function. Matching (translating) is easier.” Using their knowledge, calculators and a process of elimination, the students were able to translate from one representation to another representation for some functions that they could not previously identify using any representation. In other words, the students could sometimes translate from one representation to another for functions not contain in their core sets. For the individual student and a given translation, the functions translated from one representation to another could vary from the ones contained in their core sets to all functions. As a reminder, the “core” set was the largest set of functions consistently

recognized by the students using all three representations during the part of the study titled, *The Classifications of Functions*.

As the students translated functions from one representation to another, a new “core” set of functions that contained most, if not all, of the same functions as the original core sets emerged. This new “core” set for translating functions shall be called the “expanded core set.” For the six translations: (1) symbolic to graphical; (2) symbolic to numerical; (3) numerical to symbolic; (4) numerical to graphical; (5) graphical to symbolic; and (6) graphical to numerical, the “expanded core set” will be the largest set of functions correctly translated from one representation to another for each of the above translations. In other words, it is the set that is obtained from the intersection of all of the six translations.

Eddie’s expanded core set contained the linear, quadratic, identity and constant functions. Kristin’s expanded core set contained the linear, quadratic, constant, absolute value, cubic and identity functions. Whitney’s expanded core set contained the linear, quadratic, absolute value, constant and the identity functions. Buren’s expanded core set contained the linear, quadratic, constant, radical, absolute value, cubic, and the identity functions.

A pattern that first seemed to emerge with the task of classifying functions and continued through the task of translating functions was the reliance by the students on their knowledge of the functions contained in their core sets for difficult tasks and later on a reliance on their expanded core sets for difficult translations. The more difficult the task, the greater the reliance on the core sets and expanded core sets appeared to be. However, similar to recognizing different representation of functions earlier, the students

were frequently able to translate functions that were not contained in their expanded core sets. The types of functions usually translated that were not contained in the expanded core sets were polynomials, zero degree (constant function), first degree (identity function), and third degree (cubic function).

### *Symbolic to Graphical*

The students used their calculator to obtain the graphs from the symbolic representation. The number of functions each student was able to translate from symbolic to graphical to representation exceeded the number of functions he or she had in his or her expanded core sets. In fact, all students were able to translate from symbolic to graphical for every function except the algebraic and segmented (piece-wise) functions or about 86%. The algebraic and segmented functions were difficult translations for the students because their knowledge or process of elimination or calculators seemed to have provided little help. The students could not graph the algebraic or segmented function on their calculators, so they resorted to hand-made graphs that were usually inaccurate because only a few points were plotted. As Eddie said, “I didn’t know how to graph this function...so I could not match them with their other forms.” So far in the study, regardless of the task, algebraic, segmented were usually problematic for the students.

### *Symbolic to Numerical and Numerical to Symbolic*

Translating from symbolic to numerical representation or from numerical to symbolic representation did not appear to be a difficult task since every student did so successfully for every function. When translating from symbolic to numerical representation, the students initially chose a couple of numerical representations then used their calculators to substitute the values from the numerical representations into the

symbolic representations. The students modified the process when translating from numerical to symbolic form. Given a numerical representation, the students chose two or three symbolic representations as possible matches and then substituted the values into the symbolic representations until a match was obtained. In performing either translation, the students chose probable symbolic or numerical matches and then substituted values into the symbolic representations. The difference between the processes used to translate from symbolic to numerical representation and from numerical to symbolic representation seemed to be the representations initially chosen as possible matches. The use of the calculator appeared to have made task easy and seemed to have minimized the reliance on their expanded core sets.

### *Numerical to Graphical*

For each of the students, the task of translating from numerical to graphical representation seemed to be more demanding than translating from either symbolic to numerical, or from numerical to symbolic representation, or from symbolic to graphical. One likely reason seemed to have been the use of inaccurate hand-made graphs. When constructing graphs, the students usually plotted three points or less from the numerical representations and tried to identify the corresponding graphical representation. For example, Eddie only plotted the ordered pairs, (0, 1), (1, 3), and (2, 11), taken from the table (numerical representation) of  $y = x^3 - x + 1$  and displayed in table 6 to graph it.

Table 6: The table Eddie used to graph  $y = x^3 - x + 1$

$x$	0	1	2	3	-2	-3	-1
$y$	1	3	11	31	-9	-29	-1

The graphical representation is depicted in figure 4. However plotting his chosen three points, Eddie obtained the graph show in figure 5 which he recognized as part of a parabola.

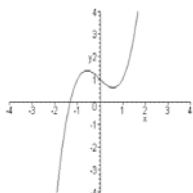


Figure 4: The actual graph of  $y = x^3 + x + 1$

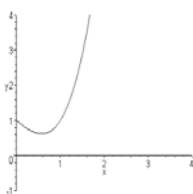


Figure 5: The graph Eddie obtained

Eddie's misclassification may have been the reason he was unable to translate from the numerical representation depicted in table 6 to its graphical representation shown in figure 4. Only one student, Buren, translated more than 50% of the functions from numerical to graphical representation with Eddie, Kristin, and Whitney translating four, six, and five functions, respectively.

The appearance that the task of translating from numerical to graphical representation seemed more demanding for the students than any previous translations was supported by the fact that each student was usually able to translate from numerical to graphical representation only for functions in his or her expanded core set of functions

and no additional functions. It appeared that demanding tasks caused the students to rely on their knowledge about functions, and these functions were frequently the ones located in their “core” or “expanded core” sets. The set of functions that Eddie actually translated from numerical to graphical representation were the linear, quadratic, identity and constant functions. As mentioned earlier, these functions became his expanded core set. His “core” set of functions, the linear, quadratic, tangent, and constant functions, were nearly the same functions as the ones contained in his expanded core set. The tangent function, a member of Eddie’s core set, was not contained in his expanded set while the expanded set contained the identity function which was not contained in his core set.

Kristin translated the linear, quadratic, constant, absolute value, cubic and identity functions from numerical to graphical representation and these functions, as stated previously, were designated her expanded core set. Her “core” set contained the linear, quadratic, constant, absolute value, tangent, and identity functions. Similar to the relationship between Eddie’s core and expanded core sets, Kristin’s core and expanded core sets overlapped at every function except the cubic and the tangent functions which were contained exclusively in the expanded core and core sets, respectively.

Whitney translated the linear, quadratic, absolute value, constant and the identity functions from numerical to graphical representation, and these were designated her expanded core set. Her “core” set for classifying functions contained linear, quadratic, absolute value, tangent and the identity functions. The only functions contained in either Whitney’s core or expanded core sets that was not contained in both sets were the constant and tangent functions, and this pattern paralleled the relationship between

Eddie's and Kristin's core and expanded core sets. The constant function was a member of Whitney's expanded core set, while the tangent function was a member of her core set.

Buren translated from numerical to graphical form for the linear, quadratic, constant, radical, absolute value, cubic, and the identity functions. These functions formed his expanded core set, while his core set contained the linear, quadratic, absolute value, radical, tangent, constant, and the identity functions. Buren's expanded set contained one function, the cubic, which was not a member of his core set, whereas the tangent function was not an element of the expanded core set. Buren was the only student whose expanded core set of functions contained the same number of functions as his core set.

Previously, when recognizing functions, the representation that seemed to cause the most recognition problems was numerical which showed the trouble the students encountered with the modeling component of the framework. Similarly, the students experience trouble translating from numerical to graphical representation, which again reflected on their ability to model numerical data. Translating from numerical to graphical representation also seem to be more challenging than translating for any previous representation, especially from numerical to symbolic, which could be accomplished by substituting values into the symbolic representations until a match was found. Translating from numerical to graphical representation also seemed to be problematic because the students probably had to recognize the numerical representation and this task was already deemed difficult.

Yet, being able to identify the function was no guarantee that the students would be able to translate it from one representation to another. For example, the numerical



representation of the tangent function was contained in each student's core set, and it may have been the easiest representation to identify since the  $x$ -values were given in degrees, but it was not translated from numerical to graphical representation by any student.

### *Graphical to Symbolic*

Translating from graphical to symbolic representation seemed to be an easier task than translating from numerical to graphical representation with Eddie, Kristin, Whitney and Buren correctly translating 36%, 50%, 43% and 50% of the functions, respectively. Although the task of translating from graphical to symbolic representation appeared to be an easier task than numerical to graphical, all students, except Buren, were only able to translate from graphical to symbolic representation for only one additional function that was not contained in the expanded core set.

The functions translated by Eddie from graphical to symbolic representation were the linear, quadratic, constant, tangent, and identity functions. This set of functions represents his core set of functions with the identity function added or his expanded core set with the addition of the tangent function. Kristin translated the linear, quadratic, constant, absolute value, cubic, identity, and the tangent functions. The functions translated from graphical to symbolic representation represented her core set with the cubic function added or her expanded core set with the tangent function added. Whitney translated the linear, quadratic, constant, absolute value, identity, and the tangent functions. These functions represent her core set with the constant function added and her expanded core set with the addition of the tangent function. Buren translated the linear, quadratic, constant, absolute value, cubic, tangent and the identity function. The seven functions Buren translated from graphical to symbolic representation were the same

number of functions contained in both his core set and expanded core set of functions even though some of the functions were different. His expanded core set contained the cubic function while the core set contained the radical function.

### *Graphical to Numerical*

Based on the interviews and observation, the students seemed to translate from graphical to numerical representation by approximating points on the graph and then trying to find corresponding point in the tables of values (numerical representation). Every student, except Buren, was able to translate from graphical to numerical representation for at least one more function than they were able to translate from graphical to symbolic representation. Eddie added the absolute value function and the exponential function to the set of functions he was able to successfully translate from graphical to symbolic. Kristin added the exponential function to the set of functions she translated from graphical to symbolic representation. Whitney added the logarithmic and cubic to the set of functions so that she was able to translate graphical to symbolic representation. Buren was unable to add any additional functions when translating from graphical to numerical representation which may have prevented him from outperforming his fellow students for one of the few times.

### *All Three Representations*

Translating functions using all three representations in any order appeared to be a less difficult task than sometimes translating from one representation to another. The set of functions that each student was able to translate using all three representations, symbolic, graphical, and numerical in any order, was frequently larger than their expanded core sets. Previously, while translating from one representation to another, the

students were sometimes able to translate more functions than those contained in their expanded core sets and this pattern continued when all three representation were used.

Eddie translated the following functions, linear, quadratic, constant, tangent and the identity functions. These functions represented his expanded core set which contained the linear, quadratic, identity and constant functions with the addition of the tangent function. Kristin translated the linear, quadratic, constant, absolute value, tangent, identity, and cubic functions. These functions represented her expanded core set with the addition of the tangent function.

Kristin and Eddie were each able to translate, using all three representation, for only one more function than the number contained in their expanded core sets. This one function was the tangent functions, which both students said they had become very familiar. Whitney translated the liner, quadratic, constant, absolute value, tangent functions and these functions would be identical to the ones contained in her expanded core set if the identity function was substituted for the tangent function. She was the only student who translated the same number of functions (not same functions) as those contained in her expanded core set, but Whitney also did not translate for every function in her expanded core set, namely, the identity function.

Buren translated the linear, quadratic, constant, absolute value, tangent and the identity function, and his expanded core set contained the linear, quadratic, constant, radical, absolute value, cubic, and the identity functions. Buren, like Whitney, did not translate for every function, contained in his expanded core set. Whitney did not translate the identity function and Buren did not translate the radical and cubic functions. Yet, like

Eddie and Kristin, they probably have become familiar with all representations of the tangent function because each student translated it also.

Another fact that seemed to be more visible was the types of functions contained in the core and expanded core sets. Every student was able to translate the linear, quadratic, constant, and tangent functions using all three representations. Most, if not all, of these functions were contained in each student's core sets, and, expanded core sets and with the exception of the tangent function, all were polynomial functions. Based on the analysis of the data so far, it seemed that polynomial functions were the types of functions that the students had the most experience or knowledge.

### *The Easiest Translations*

As a reminder, the easiest translations as ranked by the students using all three representations were: (1) symbolic to graphical to numerical; (2) symbolic to numerical to graphical; and (3) numerical to symbolic to graphical. The students actually translated from one representation to another and then to another representation for only (3) the numerical to symbolic to graphical translation—the one considered the most difficult of the three easiest translations. Translating from one representation to another and then to another may have been one reason the students found the last translation (3) to be the most difficult of the three translations. Another possible reason was the fact that the students had to translate from a numerical representation to a symbolic representation as part of the entire three-fold translation. The translation from numerical to symbolic required modeling which the students were unsuccessful with earlier.

For the easiest translation, (1), the students used their calculators to translate from symbolic to graphical representation and then use the calculators again to substitute

values from the tables (numerical representation) into the symbolic representations until a match was obtained. The actual order of translation was symbolic to graphical and then numerical to symbolic. The non-linear order of translation along with the use of technology may be the reason translation (1) was ranked the easiest.

As it was with translation (1), the order of translation for (2) was non-linear also. The students used their calculators to first translate from numerical to symbolic using the same procedure displayed in translation (1). Then, the calculator was employed to find the graph of the symbolic representation. The numerical to symbolic translation of the entire three-part translation required modeling again and was probably the reason the students ranked translation (2) as the second easiest translation.

### *The Most Difficult Translations*

The three most difficult translations as ranked by the students were: (1) numerical to graphical to symbolic; (2) graphical to numerical to symbolic; and (3) graphical to symbolic to numerical. These translations, (1), (2) and (3) were originally designated by the letters (c), (b), and (a), respectively, in Chapter Four under the heading, The Most Difficult Translations. The students had trouble recognizing functions represented by numerical and graphical representations. They also had trouble translating functions from numerical representations to almost any other representations except symbolic which was usually accomplished through a process of elimination and the aid of the calculator. In addition, the numerical to graphical translation required modeling which was another task with which the students were commonly unsuccessful. Therefore, it was not surprising that the translations the students deemed to be the most difficult, using all three representations, had either a numerical to graphical translation or a graphical to numerical

as part of the entire translation. Amazingly, the students did not attempt to use any of the non-linear translations for the above translations (1) and (2) as they did successfully for translation (3). For translation (3), the students started with the symbolic representation and used their calculators to obtain the graph. Then they translated from numerical to symbolic. Once again, the use of technology and the non-linear translation may have made (3) the easiest of the most difficult translations.

### *Creating New Functions*

When creating new functions using symbolic representations, most students frequently chose polynomials as their building blocks. Eddie, Kristin, Whitney, and Buren chose  $y = 2x + 2$ ;  $y = 4x - 3$  and  $y = \sqrt[4]{x-1}$ ;  $f(x) = 4x$  and  $g(x) = 4x - 3$ ; and  $f(x) = 4x$  and  $g(x) = 2x^2 + 12x + 12$ , respectively.

Table 7: Eddie and Kristin's tables

$\left  \begin{array}{c c c c c} x & -2 & 7 & 9 & \\ \hline y & 3 & 5 & 8 & \end{array} \right $	$\left  \begin{array}{c c c c c} x & 0 & 1 & -1 & -2 \\ \hline y & \Pi & \Pi & \Pi & \Pi \end{array} \right $	$\left  \begin{array}{c c c c c} x & 0 & 3 & -3 & -5 \\ \hline y & 2 & 8 & -4 & -8 \end{array} \right $
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Table 8: Whitney and Buren's tables

$\left  \begin{array}{c c c c c} x & 0 & 1 & -1 & -2 \\ \hline y & \Pi & \Pi & \Pi & \Pi \end{array} \right $	$\left  \begin{array}{c c c c c} x & 0 & 3 & -3 & -5 \\ \hline y & 2 & 8 & -4 & -8 \end{array} \right $	$\left  \begin{array}{c c c c c} x & 1 & 2 & -1 & 5 \\ \hline y & 4 & 8 & -4 & 2 \end{array} \right $	$\left  \begin{array}{c c c c c} x & 0 & 1 & -1 & -2 \\ \hline y & 12 & 26 & 2 & -4 \end{array} \right $
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The linear and quadratic functions (1<sup>st</sup> and 2<sup>nd</sup> degree polynomials) used by the students to create new functions were the same type of functions, polynomials, that the students were commonly successful with in earlier tasks. In spite of diversity of operations and functions afforded them, the students only created polynomials.

Since one of the requirements for the tasks of constructing new function was the verification of the new function as a function, the construction of new functions from the graphical representations of linear and quadratic functions was commonly accomplished through vertical and horizontal shifts in the  $x$ - $y$  plane or reflections about the  $x$ - or  $y$ -axes.

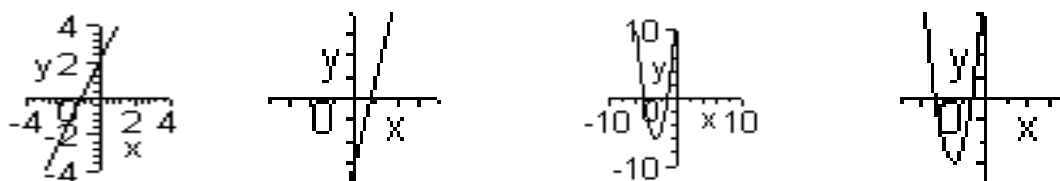


Figure 6: Eddie, Kristin, Whitney, and Buren's graphs

The verification requirement, accomplished through the application of the vertical line test, seemed to have directed the students to use no rotations (90 or 270 degrees) when constructing new functions from graphical representations, especially the graphs of parabolas that originally opened up or opened down. Although the students knew that the graphical representations of parabolas that opened to the right or to the left were not functions, they did not seem to know that the rotations (90, 180, 270, or 360 degrees) of the graphs of diagonal lines yielded other diagonal lines which still would pass the vertical line test. For instance, when given the graph of  $f(x) = x$ , each student was able to graph its reflection about the  $x$ - and  $y$ -axes, but no one seemed to know that a reflection of  $f(x) = x$  about the  $x$ -axis could be equivalent to a clockwise rotation of 90 degrees and the reflections about the  $y$ -axis could be equivalent to a counter-clockwise rotation of 90 degrees. The students did not seem to know that for some graphs, reflections and rotations were equivalent. The application of the vertical line test also seemed to have hindered the construction of new graphs from two different graphs because no one

seemed to know how to join the two graphs so that the vertical line test would be satisfied. Therefore, no new graphs were created by combining two or more graphs.

The procedures used by the students to construct numerical representations of functions were somewhat dissimilar to the procedure employed to construct graphical representations. First, all the students, except Eddie, chose more than one numerical representation and did not restrict their choices to polynomials. Having experienced trouble earlier recognizing numerical representations of functions such as constant, linear, quadratic, etc, may have made the students' preference for choosing polynomials unattainable and increased the diversity of their selections. Adding, subtracting, or multiplying the corresponding  $x$ -values, the corresponding  $y$ -values or both modified their representations. These procedures allowed the students to construct numerical representations of functions that commonly passed the vertical line test. Other procedures employed were switching the  $x$ - and  $y$ -values, multiplying all the  $y$ -values in a numerical representation by the same specified constant, and adding the same specified constant to each  $y$ -value in a numerical representation. Although the students could state why their new "creation" was a function, usually they not identify what type of function it was and did not seem to realize that the last set of procedures sometimes allowed their new numerical representations to be vertical or horizontal shifts, or reflections of the original numerical representations which the same procedures use to construct new graphical representations of functions.

### *Creating Functions from Non-functions*

When presented with the numerical representation of a non-function, each student simply found the case(s) where one  $x$ -value was associated with more than one  $y$ -value



and changed the succeeding  $x$ -values. Although this procedure could be seen as an application of the vertical line test, not one student designated it to be. There seemed to be an inability to recognize seemingly different, yet similar, procedures when they were applied in dissimilar contexts—the vertical line test and the procedure of insuring that no  $x$ -value was associated with more than one  $y$ -value—as parallel procedures.

An interesting approach was used to convert the graphs of non-functions into functions. All students, except Buren, simply deleted all portions of the graphs that did not pass the vertical line test. The portions deleted may have been the upper or lower half of the graph.

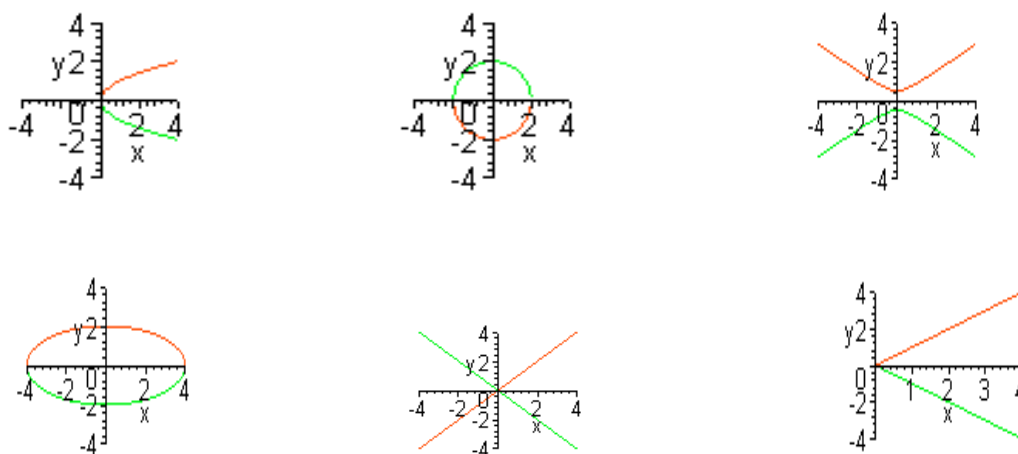


Figure 7: Non-functions used by the students to create functions

Buren performed the same procedure but somewhat differently. He obtained equations for the numerical representations then took the square root of both sides and finally dropped the minus signs. Buren's procedure did not appear to be more efficient but it did seem to be more complicated. The students also seemed to recognize that a symbolic representation (an equation) that contained the variable  $y$  raised to an even

power was frequently not a function. Attempts were made, sometimes correctly, other times incorrectly, to remove the even exponent from the variable  $y$ . For example, Eddie and Whitney changed an equation like  $y^2 = 3x$  to  $y = \sqrt{3x}$  without the intermediate step,  $y = \pm\sqrt{3x}$ , because they did not appear to know there was an intermediate step.

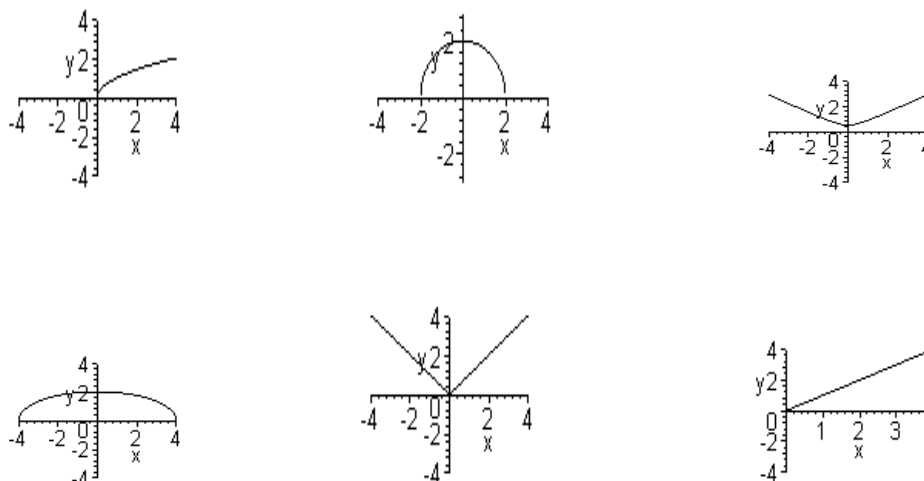


Figure 8: Functions created by the students

Omitting this intermediate step may be the reason why some students viewed  $y = \sqrt{3x}$  as equivalent to  $y^2 = 3x$  and since  $y = \sqrt{3x}$  was a function,  $y^2 = 3x$  was a function too. Not surprisingly, Eddie and Whitney sometimes attained correct answers even though their steps may have been incorrect. Similar to Eddie and Whitney, Buren would also take the square root of both sides of the equations, but he would then delete the negative sign. For example, to change  $y^2 = 3x$  into a function, Buren would take the square root of both sides to get  $y = \pm\sqrt{3x}$  and then deleted the negative sign to obtain  $y = \sqrt{3x}$ . Kristin's procedure was similar to Buren's in that she attained the square root

of both sides too but instead of deleting the negative sign, she took the absolute value of both sides.

Most students also seemed to know that an equation similar to  $x = c$ , where “c” is some fixed constant, prevented it from being a function. However, based on the interviews, no one seemed to know that  $x = c$  was the general equation of a vertical line which does not pass the vertical line test. Usually when an equation similar to  $x = c$  occurred, the  $x$  variable was frequently changed to a  $y$ , and the equation  $y = c$  was attained. However, the students would sometimes in a contradictorily manner recognize  $x = c$  a non-function in performing one task and then later recognize it as a constant function in another. They seemed to have had a short memory span for recognizing characteristics that would have allowed them to distinguish between relations that appeared to be similar like  $x = c$  and  $y = c$ .

### *Sorting Cards*

The students were given a new (different) stack of cards depicting the symbolic, graphical and numerical representations of the constant, linear, quadratic, cubic (third degree polynomials), logarithmic, exponential, rational, radical, segmented (piece-wise), absolute value, greatest integer, the identity, algebraic and trigonometric (tangent) after the *Translations* part of the study. They were asked to sort the cards into (1) two different categories, (2) three different categories, and (3) four different categories where the cards in each category shared similar characteristics. They were also asked to minimize (not prevent) the likelihood of a card being correctly placed in more than one category. No other restrictions were imposed on this task, and the students were encouraged to choose

their own individual categorical titles. If a student wanted to use more categories than specified for a given sorting task, he or she was allowed to use additional categories.

### *Eddie*

When Eddie divided the cards into categories, *Powers* and *Non-Powers*, he did not include any representation of the constant, linear, or identity functions in the category, *Powers*. Eddie said, “*Powers* means any function with an exponent. When Eddie said powers, I thought he meant functions with integer powers like polynomials—and he did—but he also included exponential functions and segmented functions. I asked Eddie why he included exponential and segmented functions in the categories, *Powers (Exponential)*, and he said, “that function (exponential) is a number raised to a power” but gave no reason for the inclusion of segmented functions. When I pointed to the symbolic representations of the constant, linear or identity functions, Eddie said, “I don’t see any ... powers. They (the linear, identity and constant functions) weren’t included because they have no powers.” Like Buren, if no powers were written, it seemed that the students did not perceive their existence.

Yet, Eddie placed the radical function in the category, *Powers*, and it, like the constant, linear, and identity functions, had no visible exponents. During one of the interviews, a reason was requested, but no justification was given for placing the radical function in the category, *Powers*. Later, he placed the segment function in the category, *Powers*, because “it seemed to fit.” This was the only explanation Eddie gave, but examining the segment function revealed that one of its components was a radical function, which could be rewritten without the radical symbol as a function raised to a fractional exponent. This fact may have been the reason for its categorization by him. The

rational function, which was a ratio of a constant and linear function and had no visible exponents, was also sort in to the Power's category while the linear, identity, and constant functions contrastingly were not. Some of the functions Eddie sorted in the category, *Powers*, clearly had numerical or variable exponents. Other functions sorted into *Powers* did not. Furthermore, he did not appear to consistently apply the same logic to every function to determine its category. Like Buren, Eddie did not acknowledge or recognize exponents whose values were the integers one or zero.

Using the categories, *Powers*, *Algebraic*, and *Linear* to sort the cards into three stacks, Eddie did not change any of the functions or their representations that were previously contained in the *Powers* category. *Algebraic* was the title given for algebraic, logarithmic, and absolute value functions. I thought his "definition" for *Algebraic* had the potential to be informative because Eddie had not identified earlier in the study algebraic, logarithmic, and absolute value functions. According to his definition, Eddie's *Algebraic* category correctly contained all representations of the logarithmic and absolute value functions but no algebraic functions. When presented with this dilemma, Eddie said that he still did not know the correct definition of an algebraic function and simply made a guest in this case. The *Linear* category was not defined and appeared to be "default" category because it contained most of the representation of the remaining functions. However, using the *Linear* category as a default category still did not prevent Eddie from leaving all representations of the constant and greatest integer functions as well as the numerical representation of the identity function without a category. His explanation was that they did not seem to fit any of the categories.

The titles *Powers*, *Algebraic*, *Linear* and *Arithmetic* were used to sort the cards into four categories. Eddie defined *Arithmetic* to be segmented and absolute value functions. However, the number of functions in category remained the same or its category retained a subset of its original set of functions. The functions that were not retained in each of the categories were added to the new category, *Arithmetic*, which Eddie appeared to use as a new default category. The symbolic, graphical, and numerical representations of the rational, quadratic, cubic, radical and exponential functions were placed in the *Powers (Exponential)* category. Except for some polynomials, Eddie was consistent with the placement of functions into the *Powers* category because every function fitted his definition or criteria or placed into the category during one of the earlier partition of the cards. The symbolic, graphical, and numerical representations of the logarithmic and trigonometric (tangent) functions were once again sorted into the *Algebraic* category, while the absolute value function was deleted. The functions contained in the *Linear* category remained unchanged and contained all the representations of the linear function and the symbolic and graphical representations of the identity function. The *Arithmetic* category contained the symbolic, graphical, and numerical representations of the absolute value, constant, greatest integer and segmented functions, the numerical representation of the identity function, which previously had no categories. It seemed Eddie removed some functions from some categories if he was unsure of their status and placed them into his latest default category, *Arithmetic*.

*Kristin*

Kristin divided the cards into two stacks using the same titles, *Polynomials* and *Non-Polynomials*, as had Whitney. Amazingly, she sorted the exact same representations

of the same functions into the two categories, as did Whitney. All the representations of the cubic and quadratic functions were placed in *Polynomials*, and the remaining representations of the other functions were placed in *Non-Polynomials*. Like Eddie before her, Kristin did not seem to view the constant, linear, or identity functions as polynomials. To determine if Kristin's reasoning actually paralleled Eddie's, I asked her why the constant, linear, and identity function were not included in the category, *Polynomials*. Kristin said, "They (constant, linear, and identity function) are not polynomials because there are no powers." Whitney and Buren made similar comments about the constant, linear, and identity functions later on in the study. There seemed to be insufficient knowledge about the degrees of polynomials that every student appeared to have, and not viewing the constant, linear, or identity functions as polynomials was probably one of them.

Kristin partitioned the cards into four rather than three stacks and used the titles *Polynomials*, *Non-Polynomials*, *Trigonometric*, and *Non-Trigonometric*. Since the students were allowed to choose any categorical titles, I made no comments to Kristin about my thoughts concerning her four titles. If the titles were grouped in pairs like (*Polynomials* and *Non-Polynomials*) and (*Trigonometric*, and *Non-Trigonometric*), the categories in each pair are mutually exclusive and all inclusive. Therefore, it seemed that at least two of the four titles should have been changed or replaced. For example, she should have replace *Non-Polynomials* and *Non-Trigonometric* with different categorical titles. Since any of the functions could be viewed as a polynomial or a non-polynomial and similarly viewed as a trigonometric function or non-trigonometric function. I also

made no comments concerning her titles because I was curious how her categories would partition the cards and thought the result would be informative.

Only the representations of the tangent function were placed in *Trigonometric*. The category, *Polynomials*, contained the same functions as it did previously. The category, *Non-Polynomials*, retained all the representations of the linear function, the symbolic representations of the logarithmic and segmented function, and the numerical representations of the radical, absolute value, exponential, logarithmic, segmented, and rational functions. Based on the way she sorted the cards, it appeared that Kristin did not view the categories, *Polynomials* and *Non-Polynomials*, as mutually exclusive and all-inclusive. Yet, every representation of any function that was placed into the category, *Non-Polynomials*, except the linear function, was correctly sorted. In the category, *Non-Trigonometric*, Kristin correctly sorted the all representation of the constant function, identity, algebraic, and greatest integer functions, the graphical representations of the segmented, logarithmic, rational, radical, absolute value, exponential, and the symbolic representations of the exponential, radical, rational, and absolute value functions. The categories, *Non-Polynomials* and *Non-Trigonometric*, did not contain all representations of every function placed in them. Not placing all representations of the same function into the same category may have made this task similar to the matching task with which Kristin and the other students experienced difficulty earlier in the study.

When Kristin sorted cards into four stacks again, she used titles similar in nature to the ones used by Whitney earlier. Kristin's titles were: (1) *Domain: All Real Numbers*; (2) *Domain: All Real Numbers except the Number One*; (3) *Domain: All Positive Real Numbers*; and (4) *Domain: All Real Numbers Greater Than or Equal to the Number Two*.



The only functions sorted into the categories, *Domain: All Real Numbers except the Number One*, and *Domain: All Positive Real Numbers* were the representations of the rational and logarithmic functions respectively. The radical function was placed into the category, *Domain: All Real Numbers Greater Than or Equal to the Number Two*, and the representations of the algebraic function was sorted into the category, *Domain: All Real Numbers not Equal to Zero*. It seemed that Kristin was beginning to acquire information, about the domain, from repeated exposure to certain functions that she could not recognize earlier in the study like the rational and algebraic functions.

All representations of the cubic, linear, quadratic, exponential, absolute value, segmented, constant and trigonometric (tangent) functions were sorted into the category, *Domain: All Real Numbers*. Since the cubic, linear, quadratic, exponential, absolute value, segmented, constant functions have a domain that is the set of all real numbers, each was correctly sorted. The lone error was the tangent function. Equally informative was the fact that Kristin included all representations of every function in the category. However, as with Buren, Eddie, and Whitney, Kristin still sometimes confused one representation of one function with the corresponding representation of another function—like the numerical representations of the constant and linear functions.

#### *Whitney*

Whitney used the categories, *Polynomials* and *Non-Polynomials*, when sorting the cards into two stacks. The *Polynomials* category only contained the representations of the cubic and quadratic functions—two functions with visible exponents. Once again as with Eddie and Buren, it was informative to see what type of functions Whitney did not include in the category, *Polynomials*. Similar to Buren's *Polynomial* and Eddie's *Powers*

categories, Whitney did not place any representation of the constant, linear, or identity function into her category, *Polynomials*. As with Buren and Eddie, Whitney seemed to recognize symbolic representations as polynomials only if an exponent was explicitly written. With the exception of the constant, linear, and identity functions, all the remaining functions were sorted correctly in the category, *Non-Polynomials*.

When sorting the cards into three categories, Whitney appeared to borrow terms from the study and used the titles *Graphical*, *Symbolic*, and *Numeric* and sorted every representation of every function correctly except for the symbolic representation of the segmented function which was sorted into the *Numeric* category. I assumed her error was a careless mistake rather than error in her logic or a lack of knowledge, and my assumption was confirmed during the interviews when she was given a chance to change the cards in each of the three categories. Whitney, without speaking, placed the symbolic representation of the segmented function into the *Symbolic* category.

Whitney sorted the cards again into three stacks and used the titles, *Second Quadrant Only*, *Third and Fourth Quadrant Only*, and *Fourth Quadrant*. Her justification for this second partition was, “I felt like I fudged a little by using the titles, *Graphical*, *Symbolic*, and *Numeric*, the first time.” The functions were divided into three stacks based on graphical location of the function in the Cartesian plane, i.e. the quadrants that contained the symbolic representations of the functions.

The functions sorted under the title *Second Quadrant Only* were all the representations (symbolic, graphical, and numerical) of the logarithmic, exponential, constant, segmented (piece-wise) trigonometric (tangent) and radical functions. The cards sorted under the title, *Third and Fourth Quadrant Only*, were all representations of the

rational, absolute value, cubic, linear, and identity functions. The cards sorted under the title, *Fourth Quadrant*, were all the representations of the quadratic function. All of the functions were sorted incorrectly into their respective categories and one function, the greatest integer, was not sorted into any category. For example considering the functions sorted in the category, *Second Quadrant Only*, neither the logarithmic, exponential, constant, segmented trigonometric or radical functions had a graph that was located entirely within the second quadrant. After noticing what I thought appeared to be inconsistencies, I asked Whitney if she wanted to modify the set of functions sorted under any categorical title. She said, “No.” I also noticed that Whitney was not using her calculator and when I offered her mine, she replied, “I have one” but she still did not use it. I tried to get Whitney to graph a few of the functions to get a clear picture of the graphical representations which may have alerted her that either she needed to rearrange some functions or re-title her categories.

However, I still was intrigued that she had correctly grouped all representations of some of the functions into the different categories since she was unable to match all three representations of some of these functions previously. Had Whitney attained the ability to match all representations of functions missed in earlier tasks together? During one of the interviews, I discovered that she had become more familiar with the different types of functions and their representations, but Whitney was still confusing certain representations of one function with the representations of another. For instance, she confused the graphs of the exponential and logarithmic functions. Yet, it seemed that Whitney was gaining the ability to collectively group all representations of certain

functions together in a given category even though she was still unable to match all individual representations of a given function together.

The categories used for sorting the functions into four stacks were based on the domains and ranges of the functions. All representations of the cubic function and the symbolic and graphical representations of the linear function were sorted into the category, *Domain: All Real Numbers and Range: All Real Numbers*. All representations of the trigonometric (tangent), segmented (piece-wise), exponential, and constant functions were sorted into the category, *Domain: All Real Numbers and Range: All Positive Real Numbers*. The category, *Domain: All Positive Real Numbers and Range: All Real Numbers*, contained only representations of the logarithmic function. The category, *Restricted Domains or Ranges*, contained all representations of the rational, absolute value, quadratic, and radical functions. The category, *Restricted Domains or Ranges*, defined functions whose domains or ranges or both were not the set of all real numbers.

Whitney sorted some of the representations of functions correctly. Some she did not sort correctly, and some representations she did not sort at all into a category. Observing the functions not placed into categories were as informative as the functions included. Previously, Whitney was not able to recognize the identity function as another linear function. This oversight may have prevented her from including it in the category (or any other category), *Domain: All Real Numbers and Range: All Real Numbers*, as she did with the symbolic and graphical representations of the linear function. The greatest integer function which did not appear to fit any of Whitney's four categories was left without a category probably because she could not state its range, but she did state the

domain to be the set of all real numbers. Her actions seemed consistent with earlier partitions of the cards since the greatest integer had yet to be sorted into any category.

All the representations of the logarithmic function were sorted correctly into the category, *Domain: All Positive Real Numbers and Range: All Real Numbers*. Earlier when sorting the cards into three stacks, Whitney placed all representations of the logarithmic function into the category, *Second Quadrant Only*, which would seem to prevent the domain from being the set of all real numbers. Often times, Whitney and the other students would give responses in one part of the study that would be invalidated or contradicted by responses in a later but related part of the study.

Of the functions sorted into the category, *Domain: All Real Numbers and Range: All Positive Real Numbers*, only the exponential function was placed correctly. Yet, even this placement of the exponential function conflicted with its earlier placement in the category, *Second Quadrant Only*, which would prevent the domain from being the set of all real numbers once again. The next two functions, segmented and constant, sorted into the category, *Domain: All Real Numbers and Range: All Positive Real Numbers*, had a domain that was the set of all real numbers, but neither function had a range that was the set of all positive real numbers. The tangent function was the remaining function incorrectly placed in the category because it had neither a domain that was the set of all real numbers or a range that was the set of all positive real numbers. The last category, *Restricted Domains or Ranges*, contained all the representations of the quadratic and absolute value functions, which were accurately placed in the category by Whitney. Although both functions have a domain that is the set of all real numbers, neither function has a range that is the set of all real numbers. Yet, Whitney did not know the range of the

absolute value function or how to determine the range of a quadratic function when she placed the functions in the category. She justified their selection by saying, “I don’t think either has a range that is all real numbers.” I felt Whitney knew more information about the ranges of the quadratic and absolute value functions but was unable or unsure about expressing it. It appeared that her knowledge was often used to make informed guesses.

### *Buren*

The capacity to sort cards seemed to rest of the students’ ability to recognize them and this fact may have present the opportunity for earlier errors to be repeated because the same or similar skills, abilities or knowledge were probably required again. When sorting the cards into two categories, *Transcendental* and *Non-Transcendental*, Buren clearly defined each category and sorted all the functions (each representation) correctly, according to his definitions except for the representations of algebraic function, which he misidentified as the representations of a rational function and sorted into the category, *Non-Transcendental*. Buren and the other students would repeatedly misidentify the algebraic function as a rational function as they did earlier when recognizing functions.

The headings, *Very Little Learned-Middle School*, *Most Learned-High School*, and *All Learned-College*, were used when Buren sorted the cards into three categories. Since Buren’s headings were well defined but not mathematically based, it was difficult to analyze this date. However, it was informative to see which cards were placed in each category. For instance, Buren claimed that he was introduced to the numerical representation of the segmented (piece-wise) in middle school before he ever saw the symbolic and graphical representation in high school. I mentioned to him that the order of presentation for some of the functions did not seem to follow any curriculum with which

I was acquainted. Still, he made no changes in any of his first category. Since Buren took precalculus in high school and in college it is likely that he was exposed to at least some of the symbolic representations of the other functions for which he claimed to have seen in the categories, *Most Learned-High School* and *All Learned-College*.

The last titles used for sorting the cards into four categories were *Polynomial*, *Rational*, *Trigonometric*, and *Other*. *Other* would contain all representations of functions that were not polynomial, rational, or trigonometric. What was informative about Buren's partition of the cards into four categories was the type of functions that were missing from the category *Polynomial* and included in the category *Rational*. No representation of the constant, linear or identity functions was sorted into *Polynomial*. Previously in other tasks, Buren correctly identified the constant and linear functions, but he nor the other students were able to recognize the constant and linear functions could also be viewed as zero and first degree polynomials.

Buren, upon seeing no written exponent for these functions, did not recognize them to be zero or first degree polynomials. However, if a quadratic function were given to him, Buren was always able to classify the functions as a polynomial. It seemed that the students' ability to recognize a function as a polynomial was based on whether or not an integer exponent was explicitly written somewhere in the function. Even though being able to recognize constants, linear, and the identity functions as polynomials functions was not a goal of this study, the students' inability to do so and their lack of explanations was intriguing.

Still another pattern that repeated itself was the misclassification of the algebraic and rational functions. As he did previously when recognizing functions and again when

sorting cards into two categories, Buren continued to identify the algebraic function as a rational function as did many of the other students and these two patterns— the misclassification of the algebraic and rational functions, and the inability to recognize the constant, linear, and identity functions as polynomials— continued with Eddie.

### *Tools*

#### *Experience*

Experience was as a tool that all students said that they used in every task. For certain concepts or topics covered in this study, their experience or knowledge could range the spectrum from sufficient to incomplete to almost non-existent. If their experience or knowledge was sufficient, the students usually gave a correct answer or an appropriate response. If their experience or knowledge was incomplete or almost non-existent, they used their experience as tool to “weed out” or eliminate possibilities or made informed guesses. In the part of the study before the students were given instruction using the Rule of Three, each student seemed to acquire some information about the functions, their representations domains and ranges and were becoming more knowledgeable with the tasks presented them. The acquisition of knowledge did not appear to be consistent for any student or any particular type of function. Each student would often be able to answer questions correctly in one task and miss similar questions in related tasks or give responses that would conflict with or contradict previous answers.

#### *Vertical Line Test*

Each had prior experience with the vertical line test and used it as a tool to distinguish between the graphs of functions and non-functions. Equally important, the students understood why the vertical line test worked—that one  $x$ -value could not have



two different  $y$ -values. Yet, the vertical line test's effectiveness or applicability seemed tied to the students' ability to produce accurate graphs using the graphing calculator and to a lesser extent by hand. The one-step-task of distinguishing between functions and non-function when the graphs were provided became more troublesome when the students had to construct the graphs for themselves and then apply the vertical line test. The one-step-task became a more difficult two-step task especially for functions like the greatest integer and segmented, and non-functions like circles and ellipses.

### *Definition of a Function*

Even for students whose definitions were inaccurate or incomplete, the definition of a function was a tool that every student used almost exclusively to decide whether or not numerical representations (tables of values) were functions or non-functions. More specifically, they examined the table of values to make sure that any given  $x$ -value was associated with only one  $y$ -value, not seeming to realize that this procedure was similar to their application of the vertical line test. Amazingly, imprecise or partial definitions did not prevent students from almost always distinguishing between the numerical representations of functions and non-functions correctly.

### *Graphing Calculator*

The graphing calculator was used to construct graphs and then in conjunction with the vertical line test to determine whether or not the graphical representations were functions. It was also used to help determine the domains and ranges of relations by producing graphs. This was an endeavor with which the students were rarely successful, but they became more successful as their ability to interpret the graphs increased.

The use of the calculator as a graphing tool was only limited by the students' ability to graph various functions. During the interview, I discovered some the reasons for some of the students' graphing problems. For example, the students did not know how to enter the absolute value symbols when graphing absolute value functions or how to graph a version of the greatest integer function. In addition, parentheses were rarely used or used incorrectly (not balanced) in graphing algebraic, rational, or logarithmic functions. For instance, the *Texas Instruments'* graphing calculator (TI-82) interprets and graphs the symbolic representations,  $y = \log x + 1$  and  $y = \log(x + 1)$  differently. The students would attempt to graph a function similar to  $y = \log(x + 1)$  but would enter  $y = \log x + 1$  into their calculators and get a graph that did not match the numerical or graphical representations on the cards. Common error like these appeared to affected their ability to recognize functions which in turn may have affected their ability to translate functions which seemed to have affected their ability to sort cards. Some graphs were also not obtained because the students used the subtraction sign “-” when they intended to use the negation sign “(-)” and received an error message. These graphing errors may have seemed small and insignificant when pondered individually, but when considered together they were significant and some times appeared to have a “domino effect” in that they may have prevented the students from arriving at a correct recognition or translation.

### *Solving for the Variable Y*

The procedure of solving for the variable  $y$  in an equation to determine whether or not the representation was that of a function was only used by Buren and he applied it when distinguishing between functions and non-functions. When he solved an equation

for the variable  $y$ , Buren was looking to see if he obtained “plus and minus” signs on one side of the equation. If so, he stated that the equation was not the symbolic representation of a function because one  $x$ -value would be associated with two  $y$ -values. By solving for the variable  $y$ , it appeared that Buren was applying a version of his definition of a function or the vertical line test because the main idea contained in both concepts was to make sure that no  $x$ -value was associated with more than one  $y$ -value.

### *Plotting Points*

Plotting points was a procedure used by the students to construct graphs. The vertical line test was then used to determine whether or not the representation was that of a function. Plotting points was not used as an alternative to the graphing calculator but in lieu of the graphing calculator—because of their inexperience with this graphing technology. The functions commonly graphed by plotting points were frequently, but not exclusively, the algebraic, absolute value, greatest integer, piece-wise (segmented) and functions given in table (numerical) form. These functions were the ones the students consistently had trouble recognizing, translating and sorting.

When constructing graphs, the students rarely plotted more than three to four points. Their position was that only three to four points were needed to construct an accurate graph. Kristin echoed the other students’ remarks by saying, “You only need to plot a couple of points until you see a pattern or have a picture of the graph.” The students also plotted points in their attempt to translate from graphical to numerical representation. They approximated points on the graphs and tried to find similar points ( $x$ - and  $y$ -values) in the table of values. If matching points were found, they said that they had obtained a match.

### *Instruction Using the Rule of Three*

Along with instruction using the Rule of Three, the students were instructed how to solve equations for the variable  $y$ , if possible, to help distinguish between functions and non-functions. They were also instructed in the use of their graphing calculators. For example, the students were instructed how to enter an absolute value and a greatest integer function, advised to use and be careful with parentheses and to avoid mistaking the subtraction symbol, “-” for the negation sign “(-).” It was also suggested to the students that hand-made graphs containing more than three plotted points would probably be more informative, especially for functions where their knowledge may be limited.

The definitions below were given to the students before the posttest and the following interviews. I had no expectations concerning the type of definitions the students would give because the definitions were not designated as either traditional or reform. The posttest scores for Eddie, Kristin, Whitney, and Buren were 84, 91, 86, and 92, respectively.

- A function  $f$  from a set  $A$  to set  $B$  is a well-defined subset of ordered pairs of the Cartesian product of  $A$  and  $B$ , such that for every  $a \in A$  there is exactly one  $b \in B$  such that  $(a, b) \in f$ . Since the definition of a function relies on the concept of a set, it is natural to call the set of all first elements of a function the *domain* and the set of all second elements the *range* (Stewart, 1991).
- A definition of a function from a reform calculus textbook, “A function is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the domain of the function and the set of resulting

output numbers is called the range of the function” (Hughes-Hallett, et al., 1998, p.

2). The following statements represent the students’ definitions of a function.

*Definition of a Function*

Eddie: A function is a set of ordered pairs where no two ordered pairs have the same  $x$ -coordinates and different  $y$ -coordinates. The  $x$ -values are the domain and the  $y$ -values are the range.

Kristin: A function is a well-defined relation in which each  $x$ -coordinate only has one unique  $y$ -coordinate. The set of all  $x$ -coordinates is the domain and the set of  $y$ -coordinates is the range.

Whitney: A function is a special type of relation in which no two ordered pairs has the same  $x$ -values and different  $y$ -values. The domain the set of all possible values for which the function is defined. The range is the set of values that are the images of the values of the domain under the function. In other words, the range depends on the domain and the function.

Buren: A function is a rule that assigns to each input value  $x$  in the domain a unique value  $y$  in the range. The domain is the set of all input values and the range is the set of all output values.

Although neither of the students’ definitions, except for Buren’s, talked about or defined the word “rule” similar to their non-mention of the word “rule” in their earlier definitions, the students’ definition of a function did talk about a set of ordered pairs or a relation rather than a “function  $f$  from a set  $A$  to set  $B$ ” or “a well-defined subset or ordered pairs of the Cartesian product of  $A$  and  $B$ .” Therefore, their definitions of a function appeared to be more similar to the definition from the reformed textbook than the one from the traditional textbook. Equally important, each student’s definition, when compared with his or her previous one, seemed more complete since it stated: (1) the domain and range along with defining them, and (2) that no two ordered pairs can have the same  $x$ -coordinate and different  $y$ -coordinates. Previously, before instruction using the Rule of Three, Buren’s definition was the only one to even mention—but not define-

the domain and range of a function. Finally, the students seemed to prefer the definition of a function from the reform textbooks for reasons probably similar to Buren's statement, "It was easier to understand and memorize."

*Examples of Symbolic, Graphical, and Numerical Representations*

As the students had done earlier, everyone was able to give a symbolic, graphical, and numerical representation of a function and verified that their examples were representative of functions through the vertical line test, their definitions, or by recognition. Previously, the students only gave representations of constant, linear, and quadratic functions. This time, the students gave representations of non-polynomial functions or polynomials of degree three or higher.

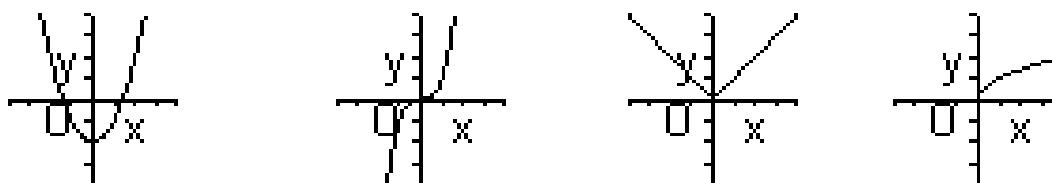


Figure 9: Eddie, Kristin, Whitney and Buren's graphs

Table 9: Eddie, Kristin, Whitney and Buren's numerical representations

$\begin{array}{c c c c c} x & 1 & 2 & -1 & -2 \\ \hline y & -1 & 2 & 1 & 2 \end{array}$	$\begin{array}{c c c c c} x & 0 & 1 & 3 & -1 \\ \hline y & 0 & 1 & 27 & -1 \end{array}$	$\begin{array}{c c c c c} x & 3 & 1 & 0 & -1 & -2 \\ \hline y & 3 & 1 & 0 & -1 & 2 \end{array}$	$\begin{array}{c c c c c} x & 0 & 1 & 2 & 3 & 4 \\ \hline y & 0 & 1 & \sqrt{2} & \sqrt{3} & 4 \end{array}$
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The representations given were those of the cubic, absolute value, and radical functions which may suggest that the students had broadened or diversified their view of functions since no student gave a representation that he or she had used previously in the first part of the study call, *Instruction Before Using the Rule of Three*. Eddie, Kristin,

Whitney, and Buren gave  $f(x) = x^2 - 2$ ,  $f(x) = x^3$ ,  $f(x) = |x|$ , and  $y = \sqrt{x}$  for their symbolic representations.

### *Examples of Non-functions*

Before instruction using the Rule of Three, Eddie and Whitney only gave numerical examples of non-functions while Kristin did not give an example. Buren was the only student who gave a symbolic representation. This time, everyone gave symbolic representations rather than numerical of non-functions and these representations were similar to the ones given on the posttest. Eddie, Kristin, Whitney, and Buren gave  $2x^2 + 2y^2 = 18$ ,  $2x^2 + 2y^2 = 18$ ,  $y^2 = x$ , and  $x^2 - y^2 = 1$  as examples of non-functions.

Non-functions were recognized by name as circles, parabolas, and hyperbolas, and it was shown that the graphs did not pass the vertical line test. Even as the students' ability to recognize the representations of functions and non-functions grew, they still seemed to gravitate more toward symbolic representations rather than numerical or graphical. A reason for their affinity for symbolic representations, to use their own words, may have been the repeated exposure to equations more so than graphs or tables throughout high school and college along a certain level of comfort that the exposure to equations brought.

### *Classification of Relations as Functions and Non-functions*

#### *Numerical*

As they did formerly, the students identified each table of values (numerical representations) correctly as a non-function by "making sure no  $x$ -coordinate had two different  $y$ -coordinate." Their ability to recognize tables as functions and non-functions

seemed to be unaffected by instruction using the Rule of Three. Their ability was nearly flawless before and after instruction.

### *Symbolic*

Previously, most of the students could not recognize functions that were not solved for the variable  $y$  in terms of  $x$  like  $y^3 = x^2 + x + 1$  more so than non-functions. After instruction, no student had trouble deciding whether or not relations of the form,  $y^3 = x^2 + x + 1$ , or,  $y^2 = x^2 + x + 1$ , were functions or non-functions. As Buren usually did earlier, this time they all solved equations of the form,  $y^3 = x^2 + x + 1$ , or  $y^2 = x^2 + x + 1$  for  $y$ . If the solved equation had only a single sign, it was designated a function. If there were more than one sign, the equation was called a non-function.

The non-functions that most of the students did not recognize, even after some prodding, were relations of the form,  $|y| = x$  and  $x = c$  where “c” was some number. The relation,  $|y| = x$ , was frequently confused with the function,  $y = |x|$ , while the relation,  $x = 3$ , was viewed as the function,  $y = 3$ . Based on their responses during the interviews, it seemed the similarities between  $y = |x|$  and  $|y| = x$ , and  $x = 3$  and  $y = 3$  were too great to enable the students to distinguish between the corresponding equations. Eddie and Whitney misidentified both  $x = 3$  and  $|y| = x$  as functions. Kristin correctly recognized  $x = 3$  as a non-function. Only Buren recognized  $|y| = x$  as a non-function, but he also misidentified  $x = 3$  as a function.

The students always seemed to encounter problems when functions like  $|x| = y$  and  $[[x]] = y$  had their  $x$  and  $y$  variables switched to get non-functions similar to  $|y| = x$



and  $[[y]] = x$ . It seemed that similar characteristics among relations, or among relations and functions, or among functions were a factor that contributed to misidentifications.

### *Graphical*

Similar to their performance with numerical representations, the students were unerring when distinguishing between functions and non-functions using the vertical line test. Once again, the students were able to distinguish relations as functions and non-functions that they were unable to recognize by name when it was appropriate. For example, they were able to recognize to graph of an algebraic function as a function but could not state its name.

### *Recognizing Functions*

#### *Symbolic*

Every student seemed to have performed better than he or she had previously before instruction using the Rule of Three, and each increased the number of functions that he or she was able to correctly recognize with Eddie's 71% being the lowest. However, the functions that were commonly not recognized by the students were the same ones incorrectly identified earlier in the study—the algebraic and rational functions with Eddie once again also misclassifying the identity function.

In spite of instruction, some functions were not recognized in both parts of this study. Functions that may have been viewed by the students to be similar in appearance to each other, like the rational and algebraic functions, were commonly confused with each other as were the functions and non-functions like  $y = |x|$  and  $|y| = x$ , and  $x = 3$  and  $y = 3$ , respectively. Similarity seemed to have been a factor that contributed to one function being recognized as another function or a function being classified as a non-

function. Still, each student increased the number of functions contained in his or her “core set.” As a reminder, the “core set” was the largest set of functions that the students were able to recognize using all three representations.

Previously, Eddie’s core set contained the linear, quadratic, tangent, and constant functions. His core set after instruction contained the constant, linear, quadratic, absolute value, cubic, logarithmic, exponential, radical, the identity function and cosine functions which was an increase of 150%. The extra functions, the cubic, logarithmic, exponential, radical, the identity function and cosine functions, had characteristics that seemed to have made each of them more recognizable and somewhat easy additions after instruction. An exponent with the value three may have allowed Eddie to identify the cubic function while the words *log* and *cos* seemed to have facilitated the identification of the logarithmic and cosine functions. The form of the exponential function,  $y = b^x$ , may have aided in its classification since there were no other functions with similar characteristics that could have been confused with it. The radical symbol of the radical function also seemed to be an aid in its identification. Finally, Eddie seemed to have just memorized the form of the identity function,  $y = x$ , and no longer confused it with constant functions,  $y = c$ , or vertical lines,  $x = c$ .

Equally informative were the functions not added to his core set—the rational, segmented, algebraic and greatest integer functions. Earlier in the study, the rational, segmented, algebraic and greatest integer functions caused problems during the task of recognition, translation, and modeling (which was a subset of translating). These same functions were simply avoided altogether by Eddie and the other students during the task of constructing functions. Even after instruction, the students had trouble becoming

comfortable with functions written as ratios like the rational and algebraic functions. Function with complicated symbolism like the greatest integer,  $y = [[x]]$ , or more than one part, like the segmented, also appeared unaffected by instruction.

Previously, before instruction, Kristin's core set contained the linear, quadratic, constant, absolute value, tangent, and identity functions. After instruction, her core set contained the constant, linear, quadratic, absolute value, cubic, logarithmic, exponential, radical, identity, cosine and rational functions, which was an increase of 83.33%. The functions added to and omitted from Kristin's core set were similar to the functions added and omitted to the core sets of Eddie with the same probable reasons for their addition and omission with the exception of the rational function. Kristin was one of the first students who seemed to have acquired the ability to recognize rational functions and it was added to her core set. As she said, "This [function] is a ratio of polynomials and I can now identify polynomials."

Earlier in the study before instruction, Whitney's core set contained linear, quadratic, absolute value, tangent and the identity functions. After instruction, the following functions were added to her core set: cubic, constant, logarithmic, exponential, and identity function. These additional functions increased the number of functions in her core set by 100% and gave her the same core set as Eddie's. As with Eddie, the functions added had characteristics that probably made them more recognizable, while the functions not added, like the algebraic, segmented, rational and greatest integer, were the same functions that had been problematic for Whitney and the other students throughout the study to this point, even after instruction.

Buren's core set contained the linear, quadratic, absolute value, radical, tangent, constant, and the identity functions before instruction earlier in the study. After instruction, his core set had expanded to contain every function except the greatest integer, algebraic, and segmented functions. Buren's core set increased by 57%, and the additional functions gave him the same core set as Kristin's. After instruction, like Kristin, Buren also seemed to be able to recognize rational functions. Nevertheless, like the other students, the functions added and omitted by Buren paralleled the ones added and omitted by the other students, probably for similar reasons.

There may have been a connection between the posttest scores and the increase in the number of functions contained in the core sets. The pretest and posttest scores of Eddie, Kristin, Whitney, and Buren were {31, 84}, {83, 91}, {64, 86}, and {83, 92}, respectively. As the students' posttest scores increased, so did the number of functions contained in their core set. The percent increases ranged from a high of 150% for Eddie to a low of 57% for Buren. This result may have indicated that instruction using the Rule of Three afforded the greatest help to students with the lowest pretest scores. For a mathematics class, it could mean that instruction using the Rule of Three would be more helpful to the students with weak or incomplete backgrounds.

However, even for students who scored reasonably well (a score of 80% or better) on the pretest, there was still an increase in the number of functions contained in their core sets. It seemed that instruction using the Rule of Three was beneficial to all the students but more so to particular students. Yet there were certain types of functions that instruction using the Rule of Three did not seem to help, at least initially. These functions were usually rational, segmented, algebraic, and greatest integer. Only the two students

with the highest pretest scores, Kristin and Buren, eventually assimilated the rational function. As mentioned before, functions that had complicated symbolism or more than one part or component or appeared similar to other functions or relations were consistently problematic regardless of the task or instruction using the Rule of Three.

### *Numerical*

The students also increased the number of numerical representations of functions that they were able to correctly recognize with 71% being the lowest percent. Numerical representations not recognized were usually the same type of functions that the students had trouble recognizing in symbolic form. Rational functions were often misidentified as either algebraic or segmented functions, and sometime the rational, algebraic or segmented functions were not classified at all. The greatest function was not identified by any student similar to its non-classification earlier in the study before instruction.

### *Graphical*

The students appeared to be more successful identifying graphical representations than symbolic or numerical representations with no student recognizing less than 86% of the representations correctly. The fact was insightful and maybe contradictory because every student said that he or she was more knowledgeable concerning symbolic representations than any other type. Yet, the minimum accuracy rate for recognizing symbolic was only 71%.

In addition to the frequent misclassification of algebraic functions as segmented functions, the exponential function was sometimes confused with the logarithmic function. In theory, exponential and logarithmic functions are inverse functions of each other, and their graphs may have appeared similar to the students. Once again, similarity

seemed to be a characteristic that may have induced the students to confuse one function with another through out the study.

### *Translating*

#### *Symbolic to Graphical*

Earlier in the study before instruction, the students were successful translating from symbolic to graphical representation, while only missing functions that they could not graph by hand or by using their calculators, which was the main tool used for this task. Later in the study, the students had received instruction in the use of their graphing calculators, and no one made an error translating from symbolic to graphical representation with everyone achieving a 100%. This result seem to indicate the help given by the calculator because the 100% accuracy rate was obtained even though students could still not identify, by name, some of the symbolic, graphical, numerical representations.

#### *Symbolic to Numerical and Numerical to Symbolic*

Through the use of the graphing calculator, every student seemed to translate correctly from symbolic to numerical representation and numerical to graphical representation which was an example of modeling. Once again, everyone obtained 100%. Given a symbolic representation of a function first, the students selected several numerical representations (table of values) as possible matches.  $X$ -values from the table of values were substituted into the symbolic representations in an effort to match the  $y$ -values in the table. If several ordered pairs ( $x$ - and  $y$ -values) obtained in this manner matched ordered pairs in the table, a correct translation from symbolic to numerical was said to have occurred. If neither of their chosen possible numerical representations gave a

match with the given symbolic representation, each student discarded the previous numerical representations and chose another numerical representation that may be a match with the given symbolic representation. As I observed the students, this procedure seemed to be more than just “trial and error” or “a process of elimination” because it appeared that the students were trying to match symbolic representations with probable numerical representations. To confirm or deny my suspicions, I asked each student what procedure he or she used when translating from symbolic to numerical representation. Their similar comments can be stated by Buren who said, “I choose the tables (numerical representations) most likely to match my equation. If this table of values does not match, I choose another one that I think will match. I continue (with this process) until I get a match.”

If the students were given a numerical representation first then the process used by them to model the numerical data was similar to the one employed to translate from symbolic to graphical representations. The students chose several symbolic representations as likely symbolic matches then placed  $x$ -values from the numerical representations into the symbolic representations until a match was obtained. The graphing calculator, once again, proved useful and facilitated the task of translating from numerical to graphical representation.

### *Numerical to Graphical*

Not overlooking the fact that every student increased the number of functions they were able to translate from numerical to graphical representation using hand-made graphs, there were still a few functions that continued to cause problems for the students, even in this part of the study. They were the rational, algebraic, greatest integer and

segmented functions. The students seemed not to be able to distinguish between the rational and algebraic functions beyond the fact that they were both written as ratios, and they commonly identified the segmented only by one of its components. The greatest integer function was not translated at all. The students' seemingly had a limited scope of the greatest integer, rational, algebraic, and segmented functions, along with the fact that the rational and algebraic looked like each other, made tasks dealing with these functions difficult or unsuccessful. Nonetheless, Eddie and Whitney correctly translated 79% of the function while Kristin and Buren translated 86%.

Another reason why the students may have experienced trouble with the algebraic, greatest integer, rational, and segmented functions was the use of hand-made graphs, even though they had received instruction in the use of the graphing calculator. The students stated that they found some of the procedures too long, time consuming, or tedious for graphing certain functions and routinely resorted to hand-made graphs. The students plotted between seven and twelve points when constructing graphs with Eddie and Whitney frequently located at the upper end of the spectrum and Kristin and Buren commonly situated at the middle or the lower end. Although the hand-made graphs were oftentimes only partially correct, when used in conjunction with the students' knowledge, the graphs were accurate enough to allow the students to translate from one representation to another, especially numerical to graphical translations except for four functions mentioned above.

#### *Graphical to Symbolic*

Of the rational, algebraic, and segmented functions, the students could translate from graphical to symbolic for only the rational function because even Eddie and



Whitney had come to recognize its graph as well as some general characteristics for the graphs of rational functions similar to the ones used in this study. These characteristics were usually vertical and horizontal asymptotes and a graph that appeared to be somewhat hyperbolic in shape. However, as for algebraic and segmented functions, no general characteristic seemed to have been acquired because their graphs were still being confused with each other, or they were not being recognized as the graphs of any functions.

### *Graphical to Numerical*

The algebraic and segmented functions seem to be the only functions that were problematic once again for the same reasons stated earlier above. Equally important was the discovery that no student actually translated from a graphical to a numerical representation without first translating to a symbolic representation as an intermediate step. The process used was graphical to symbolic to numerical and may have been the reason why the students were more successful translating from “graphical to numerical” in the study after instruction. Earlier in the study, the students tried to translate from graphical to numerical by approximating points on the graphs and then matching the points to the ones in the table of values (numerical representations) which they found to be difficult task. However, translating from graphical to symbolic and then to numerical may have allowed the students to use any successes of the previous two tasks—(1) translating from graphical to symbolic and (2) translating from symbolic to numerical. As a reminder, every student correctly translated from symbolic to numerical for each function while the only functions missed in translating from graphical to symbolic were the segmented and algebraic functions.

Before instruction using the Rule of Three, the students had “expanded core sets” for the translating tasks which were defined to be the largest set of functions correctly translated from one representation to another for each of the above translations. In other words, it was the set obtained from the intersection of all of the six translations. The expanded core set will be defined the same after instruction using the Rule of Three.

Previously, Eddie’s expanded core set contained the linear, quadratic, identity and constant functions while Whitney’s expanded core set contained the linear, quadratic, absolute value, constant and the identity functions. After instruction, Eddie’s and Whitney’s expanded core sets contained the linear, constant, quadratic, cubic, absolute value, greatest integer, logarithmic, exponential, rational, radical, and cosine functions. Their expanded core sets, after instruction, showed an increase of 175% for Eddie and 83% for Whitney.

Earlier in the study, Kristin’s expanded core set contained the linear, quadratic, constant, absolute value, cubic and identity functions, and Buren’s contained the linear, quadratic, constant, radical, absolute value, cubic, and the identity functions. Later, after instruction, Kristin’s and Buren’s expanded core sets contained the linear, constant, quadratic, cubic, absolute value, greatest integer, logarithmic, exponential, rational, radical, cosine and rational functions. Kristin and Buren’s expanded core sets showed an increase of 100% and 71%, respectively.

The functions added to and omitted from each student’s expanded core set were similar in nature to the functions added to and omitted from each student’s “core sets.” Each function appeared to have characteristics that made them more recognizable. For example, the students seem to recognize the form of polynomials and no longer had

trouble with the cubic or identity functions. Words like *log* and *cos* aided the recognition of the logarithmic and cosine functions. The form of the exponential function now seemed to be easily recognizable. The radical symbol of the radical function, the double brackets of the greatest integer function, and absolute value symbols of the absolute value function seemed to be an aid in their recognition.

The functions not added to anyone's expanded core set—the segmented and algebraic functions—remained problematic. As the study progressed, other functions that had been problematic for the students like the rational and greatest integer functions had slowly but surely been assimilated. So there was hope that the students would eventually obtain some sort of mastery over the segmented and algebraic functions before data collection ended.

It also should be noted that each student's expanded core set was at least as large as his or her "core set." This fact was true before instruction and after instruction using the Rule of Three. As a reminder, the core set was defined and based on the students' abilities to recognize functions. The fact that expanded core set was as large or larger than the "core set" seemed to confirm earlier analysis and the students' comments concerning the effort required to perform the tasks of recognizing and translating functions. It was easier to translate functions than to recognize them, which may have been a reason for the larger expanded core sets. In addition, the larger expanded core sets may be indicative of the students learning or acquiring knowledge about the functions. The larger expanded core set could also be pinpointing their ability to use procedures like a process of elimination or tools such as the graphing calculator rather than rely solely on their knowledge.

### *All Three Representations*

#### *Symbolic to Graphical to Numerical*

Every student was able to translate from symbolic to graphical to numerical representation for each function. However, their comments showed that this fact might not have been the actual procedure used. During the interviews, I discovered that once the students were given a symbolic representation, he or she used the calculator to obtain its graph. Then, the calculator was used again to obtain the table of values (numerical representation) from—the symbolic representations—rather than from the graphical representations. Therefore, the order of translation was really from (1) symbolic to graphical, then (2) back to symbolic and from there to numerical.

#### *Symbolic to Numerical to Graphical*

The students' written work once again supplied results that Eddie, Kristin, Whitney, and Buren correctly translated from symbolic to numerical to graphical representations for every function. However during the interviews, each student said that he or she first translated from symbolic to numerical representation and—starting over again—he or she next translated from symbolic to graphical representation rather than completing the process by translating from numerical to graphical. The complete process was (1) symbolic to numerical and then (2) symbolic to graphical.

#### *Numerical to Graphical to Symbolic*

After instruction using the Rule of Three, the rational, greatest integer, algebraic, and segmented functions were problematic in the numerical to graphical to symbolic translation. They appeared to be the only such thought-provoking functions in this task. The rational, greatest integer, algebraic, and segmented functions that Eddie and Whitney

were unable to translate from numerical to graphical to symbolic were the same type of functions they were unable to match when only translating from numerical to graphical. Similarly, Kristin and Buren were not able to translate from numerical to graphical to symbolic representation for the algebraic and segmented functions, and these were the same functions that were problematic earlier for the numerical to graphical representation. It should also be noted that no student previously was able to translate from graphical to symbolic representation for the algebraic and segmented functions, and this fact was demonstrated again.

In fact, functions that were demanding in one task were usually taxing in other tasks for similar reasons. For the rational, segmented, greatest integer and algebraic function the reasons may have been an inability to construct accurate graphs by hand or using the calculator, or the non-recognition of functional characteristics. The students were thought to have some level of mastery or familiarity over the functions contained in their expanded core sets. However, some functions like the greatest integer and the rational function seem to be only transient members of the students' expanded core sets—figuratively removing themselves and reasserting their problematic natures for certain tasks like translating from numerical to graphical to symbolic representation.

#### *Numerical to Symbolic to Graphical*

Translations for which the students were successful earlier, they were frequently successful later on in other translations or in other tasks. No student encountered trouble translating from numerical to symbolic to graphical representation for any function, probably because every student correctly translated earlier from numerical to symbolic for each function after instruction. Once the task of translating from numerical to

symbolic was completed, the task of translating from symbolic to graphical was accomplished through the use of the graphing calculator. As they did before, every student had a 100% correction rate when translating from symbolic to graphical representation.

#### *Graphical to Numerical to Symbolic*

After instruction, the students experienced trouble when translating from graphical to numerical for the segmented and algebraic functions, and these functions were again problematic when translating from graphical to numerical to symbolic— but only for the graphical to numerical part— of the entire translation. No problems were encountered now or earlier when translating for numerical to symbolic part of the whole translation. Finally, if a particular task was difficult, it or similar tasks were usually difficult when encountered again.

#### *Graphical to Symbolic to Numerical*

The students were able to translate from graphical to symbolic to numerical representation for every function except the algebraic and segmented functions. The algebraic and segmented functions, along with the rational function, were the same ones that caused the students problems earlier when translating from graphical to symbolic representation.

With few exceptions namely the sometime transient members, the set of functions the students were able to translate using all three representations were their expanded core sets. In other word, there appeared to be little difference between translating using two representations and translating using all three representations. Eddie and Whitney were able to translate using all three representations for every function except for the

algebraic, segmented (piece-wise), and rational functions. As mentioned earlier, the rational function was sometimes a transient member of the expanded core set for both of them.

Kristin and Buren were able to translate using all three representations core for every function except the algebraic and segmented (piece-wise) functions. These functions were precisely the same ones contained in their expanded core sets.

### *The Easiest Translations*

I was surprised that the students chose the numerical to symbolic to graphical translation as the easiest, because I thought they would choose the symbolic to graphical to numerical translation, even though they had 100% accurate rate for both translations. The latter translation, initially, was more accessible to the graphing calculator, which they used at every opportunity. However, the students did not really translate from symbolic to graphical to numerical; they actually translated from symbolic to numerical and then went back and translated from symbolic to graphical. This extra step place in symbolic to graphical to numerical translation by the students may have been a reason they chose the former translation as the easiest.

Although the students ranked the numerical to symbolic to graphical translation as the easiest, I was amazed, after asking them to do so, that no one ranked the other two translations—symbolic to graphical to numerical and symbolic to numerical to graphical—as second and third. Once the numerical to symbolic to graphical translation was chosen as the easiest, I thought that the symbolic to graphical to numerical translation would be ranked second because everyone had 100% success rate for it, and the symbolic to numerical to graphical translation would be ranked third because the

students' success rate for it was less than 100%. The latter two translations were ranked the same. Furthermore, when prodded, no one offered an explanation.

As I searched for a possible reason, I examined the parts of the two translations that the students had been exposed to in earlier tasks rather than the entire translations. From the symbolic to graphical to numerical translation, the students had trouble, previously, translating from graphical to numerical especially for algebraic and segmented functions. However for the entire translation, translating from symbolic to numerical, and then going back and translating from symbolic to graphical as mentioned earlier circumvented this problem area. This may have been how the students were able to obtain 100% success rate for the entire translation when part of it was problematic. For the entire symbolic to numerical to graphical translation, the students also experience trouble earlier translating from numerical to graphical for rational, segmented, and algebraic functions. The problem areas of these two translations may have been a reason why the students ranked them equally.

### *The Most Difficult Translations*

After instruction, when translating, using only two of the three representations, symbolic, graphical, and numerical, the students encountered difficulty when the graphical representation was either one of the representations. Not surprisingly, the students chose a translation where graphical was one of the first two representations. However, I was surprised that they listed the graphical to numerical to symbolic translation as the most difficult translation. Ranked second was numerical to graphical to symbolic translation, and ranked third was graphical to symbolic to numerical translation.



Based on their preceding work, I thought the students would choose the numerical to graphical to symbolic translation as the most difficult because they had trouble: (1) recognizing numerical representations; (2) translating from numerical to graphical; and (3) experienced trouble one again translating from graphical to symbolic. Yet, for the graphical to numerical to symbolic translation, the students had 100% correction rate translating from—numerical to symbolic—and only experience trouble translating from—graphical to numerical. This result may have been the reason the students chose the graphical to numerical to symbolic translation as the most difficult when only one part of the entire translation was troublesome.

### *Creating New Functions*

#### *Symbolic*

Even after instruction, the students chose only polynomials as their building blocks when creating new symbolic representations of functions. Buren, like Kristin had done before instruction using the Rule of Three, was the only student to choose a non-polynomial. All functions created before instruction using the Rule of Three were polynomials, even if some of building block functions were not. After instruction, the students constructed rational, logarithmic, radical, and absolute value functions by taking the ratio, log, square root and absolute value of polynomials. They seemed to have broaden their perspective on functions by creating a more diverse group than the one created before instruction. The students also seemed to diversify their methodologies for creating new functions by using a combination of procedures or processes. Finally, they were more likely to verify that the new function was actually a function by recognizing it by name rather than relying on vertical line test as they did previously.

### *Graphical*

Before instruction when the students constructed graphical representations of functions, they chose only linear or quadratic functions and created new functions by vertical shifts or reflections about the  $x$ - or  $y$ -axes. After instruction, linear and quadratic functions were once again the main building blocks with Kristin and Whitney adding cubic and exponential functions, respectively. Whitney's process for creating new functions seemed more diverse than they were before instruction. Although she still used reflections about the  $x$ - and  $y$ -axes she also use vertical shifts. Kristin along with Eddie constructed new functions from cubic and quadratic functions by restricting the domain and graphing the function over the restricted domain. No one attempted this process of constructing new functions earlier in the study and this process may be indicative of a more complete view of the relation between a function's domain, range and its graph.

As he often did before instruction, Buren once set himself apart from the other by the process he used to create graphical representations of functions. He did not use vertical or horizontal shifts or reflections about the  $x$ - or  $y$ -axes. Instead, Buren chose the graphs of a horizontal and diagonal line and translated them into symbolic representations. Then he combined the equations through addition and subtraction to obtain the first two new functions. The last new function was created by multiplying the linear equation by the constant, "3.14." To finish the process, Buren then translated the symbolic representation back to graphical representations. It appeared that Buren had taken his exposure to task of translating and use it in another endeavor. This procedure was uniquely Buren's. However, it was a common occurrence for the students to use a

combination of procedures or processes acquired in one task in another endeavor after instruction using the Rule of Three.

### *Numerical*

As they did throughout the entire study, the students still consistently experienced trouble recognizing, translating and modeling numerical representations more than any other type. Even after instruction using the Rule of Three and the fact that the students frequently used multiple procedures on many task, there seem to be something uniquely difficult about working with numerical data without the use statistics or statistical programs, both of which were beyond the scope of this research. For the most part, no student was able to recognize the numerical representation chosen by name (constant, linear, etc.) used as his or her building blocks. Yet, they had no trouble creating new functions using addition, subtraction, or multiplication and verifying that the new functions were actually functions, but they frequently could not recognize what type of functions they had created.

### *Creating Functions from Non-functions*

Before instruction, the students encountered minor trouble converting non-functions into functions, even though their methods were not always correct. After instruction, only two types of non-functions were problematic,  $x = c$  and  $|y| = x$ . As they had done throughout the study, the students would call  $x = c$  a function in one task and a non-function. I can only assert that maybe they could only retain differences between  $x = c$  and  $y = c$  for a short time or did not spend enough time on the equations to learn how to distinguish  $x = c$  from  $y = c$  permanently. It also seemed that equations that appeared to

be similar were oftentimes confused with one another. For instance,  $|y| = x$  was often misidentified as  $y = |x|$ .

### *Sorting Cards*

#### *Eddie*

Eddie correctly sorted all the cards into two stacks, using the titles, *Polynomials* and *Non-Polynomials*, when he had earlier said, before instruction, that the constant, linear, and identity functions “had no powers” and did not recognize them as polynomials. This time, Eddie not only recognized the constant, linear, and identity functions along with the quadratic and cubic as polynomials but designate them by their degrees. The titles, *Polynomials*, *Transcendental*, and *Others*, were used to sort the cards into three stacks with the categories, *Transcendental*, and *Others*, replacing the category *Non-Polynomials* used previously. Similar to Buren, Eddie seemed to use material from the review and again sorted each card correctly. When sorting the cards into four categories, Eddie, once more, appeared to subdivide a prior used category, *Others*, into *Others and Rational*. Amazingly, only the rational function was placed into the category, *Rational*, since this type of function was consistently problematic for the students in both parts of the study, along with the segmented and algebraic functions, which were also sorted correctly. In addition, being able to sort all the functions accurately may be indicative of another improvement in Eddie’s ability to recognize functions since he used the calculator much less than Buren in sorting the cards.

#### *Kristin*

When sorting the cards into stacks using the titles, *Domain: All Reals*, and *Domain: Not All Reals*, Kristin did not perform any work or use her calculator to help her

find domains as Whitney had done earlier. She seemed to be relying solely on memory, and she sorted each function correctly. Yet, there were several functions for which Kristin seemed not to know the domain but was usually able to obtain them using her calculator or information from the review. It seemed that sorting the functions into the categories, *Domain: All Reals*, and *Domain: Not All Reals*, was an easier task actually stating the domains.

Like the other students before her, Kristin also appeared to borrow concepts, limits and domain, from the class and review lectures when she sorted the cards into the three categories, *Limits Over All Reals*, *Limits Over Its Domain*, *Limits at Some Points in Its Domain*. Based on observations and interviews, Kristin did not directly evaluate the limit of the functions at various points to sort them. Instead, she seemed to use her knowledge about the domains of functions and the calculator to obtain their graphs. Her stated goal was to determine where the functions were continuous. She may have used the following statement, which was mentioned in class on several occasions, “If a function is continuous at a point, then the limit exists at that point.” A graph was obtained for every function, and all the functions that had unbroken graphs over the real numbers were placed in *Limits Over All Reals*. Functions with uninterrupted graphs over their domains were sorted into *Limits Over Its Domain*, and the greatest integer function was the only one placed in the category, *Limits at Some Points in Its Domain*. Using her knowledge of the domains in conjunction with examining the graphs may have allowed Kristin to sort every function correctly.

When Kristin sorted the cards into four categories, she used the titles, (1) *Increasing Over the Reals*; (2) *Decreasing Over the Reals*; (3) *Increasing and*

*Decreasing Over the Reals*; (4) *Constant Over the Reals*. Again, she used the calculator to acquire graphs in addition to her knowledge about the domains of the functions to sort the cards. The categories where functions were incorrectly placed were *Decreasing Over the Reals*, and *Constant Over the Reals*. First, the categories may have been more fittingly named as, *Decreasing Over Its Domain* and *Constant Over Its Domain*, since only one function, the constant, sorted into either category had a domain that was the set of all real numbers. Second, every function placed into the category, *Decreasing Over the Reals*, was actually increasing over its domain. Instead of examining the graph of a function by viewing it from “left to right,” Kristin may have reversed this process and did not appear anymore adept at examining graphs than Buren was earlier. While both the constant and algebraic functions were sorted in *Constant Over the Reals*, only the constant function was constant over the reals. It is possible that Kristin mistakenly placed the algebraic function in *Constant Over the Reals* because the graph of the algebraic function was composed of one horizontal line segment, located in the first quadrant and another horizontal line segment in the third quadrant.

#### *Whitney*

Whitney used a version of the calculus concept, continuity, when she sorted the cards into stacks of two, three, and four. This concept may have come from the class discussions since no student had used it in any prior to this task, before or after instruction. *Continuous Over the Reals* and *Discontinuous Over the Reals* were used to divide the cards into two stacks. Like Buren before her, Whitney used the calculator, extensively, to determine the set of numbers over which a function was continuous by examining the graphs. All functions placed into the category, *Continuous Over the Reals*,

were sorted correctly and similarly for *Discontinuous Over the Reals*. The category, *Discontinuous Over the Reals*, seemed to lack selectivity. Although it contained all functions that were discontinuous over the entire set of real numbers, it also contained functions that were continuous, but only over their domains, which was just subsets of the real numbers. However, even these functions were correctly placed in *Discontinuous Over the Reals* based on her definition and the instructions given for creating the categories. It appeared that the category, *Discontinuous Over the Reals*, could have been more discriminating even though the functions were correctly sorted into it.

When Whitney sorted the cards into three stacks, she used the titles, *Continuous Over the Reals*, *Continuous Over Its Domain*, and *Discontinuous Over Its Domain*. The last two titles seemed to be a refinement of the title, *Discontinuous Over the Reals*, used earlier and afforded Whitney more selectivity than was displayed formerly. All the functions previously placed in the category, *Continuous Over the Reals*, remained the same and the only function sorted into the category, *Discontinuous Over Its Domain*, was the greatest integer function. Since the greatest integer function is only discontinuous over its domain but only at all integers, the category may have been more appropriately called *Discontinuous Over the Integers*.

Whitney may have remembered some domains of the functions from the review earlier and appeared to use procedures from the review to obtain others. However, she still used the calculator to obtain the graphs of the functions, but now the graphs were examined to find or confirm domains. All remaining functions, those not placed into either *Continuous Over the Reals*, or *Discontinuous Over Its Domain*, were correctly sorted into the category, *Continuous Over Its Domain*.

Instead of sorting the cards into four stacks, Whitney divided them into five stacks and said she simply made a mistake. She was not asked to resort the cards because her “mistake” may have proved to be insightful. The titles used were *Continuous Over the Reals*, *Continuous Over Its Domain*, *Jump Discontinuity*, *Infinite Discontinuity*, and *Missing Point Discontinuity*. Based on her titles, Whitney like Eddie and Buren, appeared to use material from the review or topics from the class. Again, the functions previously placed in the category, *Continuous Over the Reals*, did not change. However, the rational function was removed from *Continuous Over Its Domain* and placed in *Infinite Discontinuity*, even though it could have been placed in either category. Nevertheless, Whitney’s selection process seemed to indicate that functions kept in *Continuous Over Its Domain* category were those functions that were actually continuous over their domains but also had no vertical asymptotes. This appeared to be the criteria that prevented the rational function from remaining. However, it did not stop the logarithmic function from staying when the  $y$ -axis was a vertical asymptote. Yet, comparing the graphs of logarithmic and the rational function revealed more information. The calculator explicitly displayed the vertical asymptotes of the rational function, but the vertical asymptote of the logarithmic function, which was the  $y$ -axis, was only displayed as the  $y$ -axis. Whitney may not have realized that the  $y$ -axis and the vertical asymptote were actually the same, in this case, and this may be the reason why one function was kept and another was deleted. All representations of the greatest integer function was placed in *Jump Discontinuity* which could be seen as a refinement of the previously used category, *Discontinuous Over Its Domain* and gave Whitney a category that only contained the representations of one function. No functions were placed into *Missing Point*



*Discontinuity*, so in essence, Whitney just had four categories and all functions other than the logarithmic were sorted correctly.

### *Buren*

Buren used the titles, *Polynomials* and *Non-Polynomials*, when sorting the cards depicting the symbolic, numerical, and graphical representations of functions into two stacks. When sorting the cards before instruction into four categories, *Polynomial* was one of the categories used, and it only contained the representations of the quadratic and cubic functions because their degrees were plainly written. After instruction, *Polynomials* contained all representations of the constant, linear, identity, quadratic, and cubic functions. Buren now appeared to be able to recognize the general form of polynomial functions, along with the fact that its degree need not always be explicitly written, which was the reason he gave earlier for not classifying the constant, linear, and identity as polynomials. All the remaining functions were correctly sorted into the category, *Non-Polynomials*.

When the concept of function was reviewed during the study, functional characteristics like even and odd were also covered. The titles, *Even* (symmetrical to y-axis), *Odd* (symmetrical to the origin), and *Neither* (neither even or odd), used by Buren may have borrowed from the review. It is also possible that Buren had seen these topics before in an earlier mathematics course.

Using the graphing calculator and the conditions,  $f(x) = f(-x)$  and  $f(-x) = -f(x)$ , that an equation must satisfy to be classified as even or odd, respectively, Buren placed all the functions correctly into each category except one, the cosine function. Even though he had obtained an accurate graph, Buren fail to recognize that the graph was

symmetrical to the  $y$ -axis. Buren classified the cosine function as an odd function possibly because he made errors when applying one or both conditions,  $f(x) = f(-x)$  and  $f(-x) = -f(x)$ . Buren wrote  $\cos(x) = \cos(-x) = -\cos(x)$  and obtained  $\cos(x) = -\cos(x)$  which he claimed was even.

Buren once again seemed to have borrowed information from the review when he used the titles, *Increasing Over Its Domain*; *Decreasing Over Its Domain*; *Increasing and Decreasing Over Its Domain*; and *Constant Over Its Domain*, for sorting the cards into four stacks. However, Buren was not as accurate when sorting the cards correctly as he was earlier when using the titles, *Even*, *Odd*, and *Neither*. Buren's failure to correctly inspect the graph of the cosine function for symmetry may have been a precursor for determining where a function was increasing and decreasing. It appeared that Buren's understanding of the functional characteristics, increasing and decreasing, were less complete than his understanding of the characteristics, even, odd, and neither or he misinterpreted the graphs. Buren inaccurately sorted the cosine function into the category, *Increasing Over Its Domain*, even though the calculator was still being used and accurate graphs were being obtained. Equally important, the class had been instructed in the use of the first derivative to determine where a function is increasing and decreasing. Yet, this was one the few times that, after instruction, Buren did not use a combination of procedures or processes in completing a task. Of his four categories, the cosine function seemed to fit best in the category, *Increasing and Decreasing Over Its Domain*, which is exactly what the graph of this function does. The logarithmic and exponential functions were both wrongly placed in the category, *Decreasing Over Its Domain*. A better fit for these two functions may have been the category, *Increasing Over*

*Its Domain.* The last remaining function, the constant function, was sorted correctly in the category, *Constant Over Its Domain.*

### *Tools*

#### *Experience*

Earlier in the study, the students' functional knowledge seemed incomplete and they rarely used more than one tool or procedure together which may have a reason for lower pretest scores and the lower success rate on most tasks. After instruction, the students seemed more knowledgeable concerning almost every endeavor connected with the study which may have been evident by higher scores on the posttests and greater success with most tasks. However, more than just memorized knowledge was displayed. For instance, the students appeared to use their knowledge in conjunction with other tools or procedures like the calculator, the vertical line test or plotting points to correctly complete tasks such as translating, sorting, or distinguishing between functions and non-functions. Equally important, the tools or procedures, regardless of their natures, were used more frequently with each other rather than individually.

#### *Vertical Line Test and the Definition of a Function*

There was almost no change in the application of the vertical line test. Before instruction, the vertical line test had been primarily used successfully by all students to decide whether or not the graph of relation was representative of a function or a non-function. This time it was used once again to distinguish between functions and non-functions but one difference was that it was used along with other tools or procedures. As they had done previously, the students also applied a written version of the vertical line

test to numerical representations—no  $x$ -values can be associated with two different  $y$ -values—which was similar to their definitions of a function.

### *Graphing Calculator and Plotting Points*

After instruction, the students were able to graph more functions than they were able to graph previously, like the absolute value function for example. Earlier in the study, the calculator was primarily used to produce graphs, and this was once again its purpose as was the procedure of plotting points. Nonetheless, the obtainment of graphs, be they calculator generated or hand-made, were no longer the only goal—but the information that could be obtained from it as an aid in completing other tasks. For instance, graphs were now utilized to help determine or confirm domains, to help sort cards, to help classify functions, and to facilitate translating functions from one representation to another. Even though the students were more successful in obtaining accurate graphs by plotting points, there still was the occasional incomplete graph. However, the possible detrimental effects an inaccurate graph may have had on a particular task seemed minimized by the students using the information from the graph in conjunction with other tools, concepts, or procedures. Using one tool with another tool or concept or procedure appeared to be a rare event before instruction in the study.

### *Solving for the Variable $Y$ and Substitution*

Before instruction when distinguishing between functions and non-functions, Buren was the only student to solve symbolic relations for the variable  $y$  in terms of  $x$  correctly. After instruction when given the task of creating functions from non-functions, everyone was able to solve for  $y$  in terms of  $x$  as an aid in distinguishing between functions and non-functions or as an aid creating functions from non-functions or as an

aid in obtaining graphs. The last usage was new for every student and it gave them the opportunity to obtain the graphs of functions and non-functions that they previously had been unable to acquire. Once the graphs were obtained, the students were often able to gather information about concepts like increasing and decreasing, continuity, limits, symmetry with respect to the origin or  $y$ -axis, and the domains and ranges.

## CHAPTER 5

### ANALYSIS

#### *The Graphing Calculator*

Before instruction using the Rule of Three, the students received no instruction in the use of their graphing calculators but appeared tied to them and used the calculators in nearly every task, even minor ones like evaluating  $f(2)$  for  $f(x) = 2x - 4$  or just graphing it. In addition, the root function key or trace key on the calculator was often used to find or approximate zeroes of an equation like  $f(x) = x^2 - 4x + 3$  that was factorable. It seemed the graphing calculator was viewed as a lifesaver and the completion of any task often depended on the student's skill or versatility with this form of technology. During the occasions when the calculator was not applicable or the knowledge of its use was lacking, the students appeared lost and rarely made any attempts to complete the given task.

Even though the graphing calculator was used extensively, the information gained from its applications to particular tasks was specific rather than general in nature. For example, when deciding whether or not the symbolic representation of the relation,  $f(x) = x^3 - x$ , was a function, the students obtained its graph using the calculator and applied the vertical line test. Yet, while the graph of  $f(x) = x^3 - x$  was still displayed on the calculator, the students did not examine it to determine the domain or range or to find where the function was increasing or decreasing. Often times the students appeared to be more focused on using the calculator to help with a certain task, like producing a graph,

without realizing it was also providing additional information about many other characteristics. The almost magical powers of the graphing calculator seemed to facilitate rather than eradicate the students' single-minded approach to tasks, regardless of their mathematical backgrounds or pretest scores.

Distinguishing between functions and non-function, translating from one representation of a function to another, evaluating functional values and creating new graphical representations of functions were the tasks that the graphing calculator was the most helpful. As Eddie said, "the calculator did all the work." Not surprisingly, these were the tasks with which the students were the most successful.

However, if a task required prior or additional knowledge in conjunction with technology, then the calculator was frequently less helpful. For example, the calculator was of little use assisting the students to identify different types of functions. As Eddie said, "Identification is hard because it requires that you know something about the function." The calculator was not beneficial in producing the graphs of algebraic, greatest integer, and piece-wise functions because prior knowledge was again required to enter these functions and this knowledge was often lacking. Similarly, it was also of little use in constructing the graphs of numerical representations for related reasons. The tasks that the students were the least successful were recognizing, identifying different types of functions, and producing hand-made graphs. Each task required assistance or knowledge beyond the calculator.

Along with instruction using the Rule of Three, the students were also instructed in the use of the graphing calculator. The instruction seemed to cause the students to use it more skillfully and differently. For example, they were able now able to graph

functions like the absolute value and greatest integer functions that they were previously unable to graph. The students no longer used the subtraction symbol, “-,” for the negation, “(-)”, which caused numerous calculation and graphical errors.

Amazingly, the instruction did not cause the students to increase their reliability on the calculator. In fact, its uses changed and decreased. Since instruction using the Rule of Three seemed to have altered or increased the students’ knowledge about functions, the graphing calculator was used more to confirm answers or assumptions rather than obtain them. It seemed that the students were able to view a more complete picture rather than focusing on one or two puzzle pieces as they did before. Now, the students completed more tasks algebraically or mentally and then transferred their thoughts to paper. When distinguishing between functions and non-functions, the students did not pick up their calculators during the completion of this task. For example, asked to state why  $y = x^2 - 8x + 2$  was a function, the students first identified it by name as a quadratic function. When pressed on the topic, they responded by hand drawing the graph of a parabola that opened upward and said that any parabola that opened up always passed the vertical line test. On another occasion, the students did not even bring their calculators with them during the task of recognizing functions because they said the calculators “would not help.” The calculator was still being used but sparingly, or used to complete only one part of a task, or used as a tool to get new answers from previous ones. For example when creating functions, the students used their calculators to graph some of the same functions as before but this time the functions were graphed only over a subset of its domain. As another example, when translating from numerical to symbolic to graphical representation, the graphing calculator was used only during the symbolic to



graphical part of the entire translation. They now said the numerical to symbolic part of the translation was easy. However, it was problematic before instruction. It now seemed that the graphing calculator was frequently being used only after other methods, processes or procedures failed or in conjunction with other concepts or procedures.

### *Concepts*

Before instruction using the Rule of Three, concepts, procedures, or processes like the vertical line test, definitions, limits, continuity were often used independently of each other, and when one procedure failed there did not seem to be a backup process or safety net. A couple of notable exceptions to this pattern were the use of the vertical line test in conjunction with the graphing calculator and using the calculator as an aid when translating.

Solving an equation for the variable  $y$  was also used independently of any other methods or technology if it was used at all. For example, the students commonly looked for equations of the form  $f(x) = \text{something}$  or  $y = \text{something}$  to distinguish between functions and non-functions. If equations like  $y^3 = x^2 + x + 1$  or  $y^2 = x$  were given the students frequently could not decide whether or not it was a function, as they appeared to rely solely on their knowledge. Unlike the other students, Buren was able to solve equations such as  $x^2 + y^2 = 25$  (a circle) for  $y$  in terms of  $x$ , getting  $y = \pm\sqrt{25 - x^2}$ . He said it was not a function because for every  $x$ -value there would be two  $y$ -values. The same conclusion could have been acquired by applying the vertical line test to the graph of  $x^2 + y^2 = 25$ . However, Buren was unable to graph  $x^2 + y^2 = 25$  even though he had obtained the correct form,  $y = \pm\sqrt{25 - x^2}$ , to enter into the calculator. He just did not

appear to know that  $y = \sqrt{25 - x^2}$  was the top half of the circle and  $y = -\sqrt{25 - x^2}$  was the bottom half or that entering both equations into the calculator would have allowed it to produce the graph of a circle.

As another example that processes were used independently, the students almost always resorted to hand-made graphs when graphing piece-wise functions. In nearly every case, they constructed graphs that were smooth and continuous without breaks or points of discontinuity even though that had received lectures covering limits, continuity and derivatives and had been tested on this material. Yet, the concept of limits, continuity or derivatives was not used in the completion of these tasks. In addition, knowledge about the domain and range of functions could also have been helpful when constructing hand-made graphs. Yet, the only time students used domain and range of functions was to sort cards. Furthermore, when constructing new graphical representations, the students often employed vertical shifts, horizontal shifts, and rotations. Yet, no one use a combination of any two of the three procedures.

Not only did the students frequently use concepts, procedures, and processes independently of each other, sometimes the reliance on one concept or functional characteristic seemed to be exaggerated. For example, the students correctly identified  $y = |x|$ ,  $y = \lceil x \rceil$ , and  $y = \tan x$  as an absolute value, greatest integer and tangent function, respectively. However,  $y = \frac{|x|}{x}$ ,  $y = \frac{\lceil x \rceil}{x-1}$ , and  $y = \frac{\tan x}{x}$  were also identified as an absolute value, greatest integer and tangent function. Knowledge about the domain of  $y = |x|$ ,  $y = \lceil x \rceil$ , and  $y = \tan x$  or functional characteristics about their graphs or the

fact that  $y = \frac{|x|}{x}$ ,  $y = \frac{[[x]]}{x-1}$ , and  $y = \frac{\tan x}{x}$  are all ratios while  $y = |x|$ ,  $y = [[x]]$ , and

$y = \tan x$  are not, never seemed to have been considered to help distinguish one type of function from another.

Finally, the students' ability to recognize functions appeared to rest primarily on their knowledge or memory of symbolic representations. Graphical and numerical characteristics as well as the concepts of continuity and differentiability never seemed to be employed in this task. For example, given  $y = x^3 - 9x$ , the students either identified it as a cubic function or not at all. No one made statements similar to the following:

$y = x^3 - 9x$  has whole number exponents and no denominator, it might be a polynomial.

$y = x^3 - 9x$  has a graph that is unbroken over the set of real numbers and it might be a polynomial. All the derivatives of  $y = x^3 - 9x$  exist and are continuous. It might be a polynomial.

After instruction using the Rule of Three, oftentimes if one procedure did not work, the students had several backups. Equally important, solutions were frequently acquired using a combination of procedures or concepts rather than one in which only one scheme was used. For example, the students would frequently verify that an equation was a function by identifying it by name, giving a hand drawing of its graph and even giving a table of values.

The students displayed originality in several of the translating tasks. When translating from numerical to graphical representation, which was previously almost always problematic, the students would first construct a rough hand-made graph and then use a process of elimination to select the correct graph. For example, as they translated

from graphical to numerical representation, the students first translated from graphical to symbolic representation, which was not required by the task. Then they translated from symbolic representation to numerical which was one the easiest translations. One constant that appeared to be intertwined with all the procedures and concepts was the students' knowledge, seemingly altered by instruction. As Eddie said, "Once I learned [what] most of the functions were, it was easier to identify, translate and sort them."

Finally, there was also less reliance on the vertical line test to distinguish between functions and non-function. As Buren did before instruction using the Rule of Three, the other students solved equations of the form,  $y^3 = x^2 + x + 1$  or  $y^2 = x$ , for the variable  $y$  to determine if it was a function. Then they used the fact from their definitions that for every  $x$ -value there can be only one  $y$ -value.

## CHAPTER 6

### SUMMARY AND CONCLUSIONS

This chapter summarizes the study, including its purpose, theoretical framework, methodology and findings. It also contains implications for teaching calculus and recommendations for future research.

#### *Purpose of the Study*

The purpose of this study was to investigate college students' knowledge of the concept of function before and after instruction using the Rule of Three in a reform calculus class. Several factors motivated me to conduct this study. First, the concept of function occurs in every branch of mathematics (NCTM, 1989; O'Callaghan, 1998), and most of the examples in which calculus concepts are seen and understood in terms of objects and processes have to do with functions (Dubinsky, 1992). Second, since the use of different instructional formats or presentations (like the Rule of Three) along with a function approach to teaching introductory concepts in calculus is relatively new, there is a need for research concerning its impact in the classroom (Haimes, 1996).

Although some research has been conducted on students' knowledge and understanding of the concept of function, more research needs to be directed on how it evolves among college students (Cooney & Wilson, 1993). Research investigating students' knowledge of functions (Even, 1996; Haimes, 1996; Markovits, Eylon, & Bruckheimer, 1986; Vinner, 1990) and research investigating students' understanding of the concept of functions by examining the components of the concept of function

(O’Callaghan’s, 1998) formed a foundation for this study. Of the problems expressed by the participants from Tulane Conference in 1986 (Douglas, 1987) and the subsequent Washington Conference (Douglas, 1987), the one I that captured my interest was the concern that very few students were prepared for calculus, passed calculus or had any understanding of the fundamental calculus concepts (Douglas, 1987; Smith, 1994; Tucker & Leitzel, 1995).

My study extends previous research by examining whether or not the dictum, The Rule of Three, used as an instructional approach, helps in understanding or the evolution of the concept of function among college students. The study also addresses why some functions may be more difficult for students to understand, classify, and translate.

### *Theoretical Framework*

I chose to adopt as my framework a union of O’Callaghan’s (1998) and Markovits et al’s (1986) framework for describing functional knowledge. O’Callaghan reformulated Kaput’s (1989) sources of meaning in mathematics, referential extension and consolidation, into terms specific to functions to obtain the components, *modeling*, *translating*, and *reifying*, which he then examined and used to describe functional knowledge.

Markovits et al. (1986) said that understanding functions had several aspects, such as (a) the knowledge to recognize and classify functions and non-functions, and (b) the knowledge to use the concept in different contexts along with the ability or knowledge to transfer information from one context to another in mathematics. Aspect (b) is similar to O’Callaghan’s definition of his *translating* component. I combined aspect (a), the knowledge to recognize and classify functions and non-functions, into the new

component, *recognizing*, and inserted it into O’Callaghan’s framework. O’Callaghan had used the recognizing component but did not formally state it in his framework. My theoretical framework contained the components of *recognizing*, *modeling*, *translating*, and *reifying*. *Recognizing* helped me describe the students’ ability to distinguish between functions and non-functions and to classify functions. *Modeling* allowed me to characterized the students’ capacity to represent numerical data by a symbolic or graphical representation. *Translating* helped me to portray the students’ ability to move from one representation, and *Reifying* allowed me to describe their ability to create functions.

### *Research Design*

This study was conducted throughout a 15-week semester during the spring semester of 2001. From the 29 students enrolled in a Mercer University calculus course, I selected 4 students. Each of the four participated in all phases of the data collection. Data for this study were collected through a pretest, a posttest, interviews, observations, and sorting activities. The data collection procedures were designed to give meaningful information concerning the students’ knowledge about functions and how their knowledge was altered during the study.

### *Instruments*

A pretest and questionnaire were given to all 29 calculus students enrolled in MAT 191. Based on pretest scores, written responses to the questionnaire and pretest, scheduling conflicts, and available time for interviews, four students were selected as participants for the study whose pretest scores ranged from 31 to 83. A total of six hour-long interviews were scheduled with each student. One interview was scheduled to

discuss the responses on the pretest, one for the posttest, one for the translating tasks (before and after instruction) and one for the sorting activities (before and after instruction). All interviews were audiotaped and transcribed, and field notes were taken while observing the students sorting cards.

### *Analysis*

An ongoing comparative analysis was used throughout the study to find patterns within and across categories as opposed to a final end-result analysis (Glaser & Strauss, 1967; Goetz & LeCompte, 1984; Patton, 1990). Early in the study, most of the time and effort were devoted to gathering data, and little time was given to analysis. In the latter stages of the study, this trend was reversed. After initially collecting and examining data, I constructed categories (Owens, 1982) such as functions classified by all students, functions classified by most students, or functions rarely classified by any students.

### *Summary of Findings*

#### *Before the Rule of Three*

My findings were based on an analysis of the data collected before and after instruction using the Rule of Three. The 14 functions the students were exposed to were as follows: constant, linear, quadratic, cubic (third degree polynomials), logarithmic, exponential, rational, radical, segmented (piece-wise), absolute value, greatest integer, the identity, algebraic and trigonometric (tangent or cosine).

Buren was the only student who gave a complete definition of a function. Whitney and Kristin gave incomplete but serviceable definitions while Eddie's was incorrect. The students tended to be somewhat knowledgeable about the definition of a



function and used their definition or a variation of it to distinguish between functions and non-functions.

### *Research Question Results*

*Question 1.* What type of relations in symbolic, graphical or numerical form do college students recognize or identify as functions before and after instruction using the Rule of Three? What characteristics, tools or tests, do college students use to classify symbolic, graphical or numerical representations of relations as functions before and after instruction using the Rule of Three?

Relations in graphical and numerical form were recognized as functions more frequently than were relations in symbolic form. Students used the vertical line test for graphs and made statements similar to “one  $x$ -value cannot be paired with two different  $y$ -values.” This statement seemed to be a rewording of most of the students’ definition of a function. Although Eddie’s definition was incorrect, he also used a version of the above statement when determining whether or not a table of values was a function. Even though the students had more success recognizing relations in graphical and numerical form as functions, they usually did not classify what type of function the relations were. For example, a graph was identified as representative of a function, but the type of function, say radical, was not given.

However, the set of functions that the students were able to identify in one representation was commonly the same set for another representation. Functions identified by the students tended to be polynomial, constant, linear, quadratic, with the occasional trigonometric, absolute value, or radical function. For symbolic and graphical representations, experience was used to recognize the types of functions whereas trial and

error was used to recognize numerical representations. The types of functions that all the students were able to identify, regardless of representation, were the linear and quadratic functions. Not surprisingly, linear and quadratic functions and, to a lesser extent, other polynomial functions were the functions the students said they had the most experience working with.

*Question 2.* Before and after instruction using the Rule of Three, what types of functions are college students able to translate from one representation to another, or match, or model? What characteristics or tools do college students use to match or translate from one representation to another before and after instruction using the Rule of Three?

The students' ability to translate from one representation to another depended on representations used. All the students translated from symbolic to graphical representation using their calculators for nearly every type of function except the algebraic and segmented functions. Utilizing their calculator again, all students were 100% successful translating from symbolic to numerical and numerical to symbolic for every function.

For each of the remaining translations, the largest set of functions that all the students translated correctly using only two representations was the one that contained the linear, quadratic, constant, and identity functions. In addition, the set containing linear, quadratic, constant, identity, and tangent functions was the largest set that each student translated correctly using all three representations. Regardless of the translation, the majority of the functions all the students consistently translated were polynomials and the tools or aids used in these endeavors were experience and the calculator.

*Questions 3.* Given a set of functions in symbolic, graphical and numerical form, what “new” functions do college students construct or create, before and after instruction using the Rule of Three, using any combination of the following operations?

The operations used to construct new symbolic representations of functions were usually addition, subtraction, and multiplication, and the functions commonly used as building blocks were usually polynomial, linear, or quadratic. The new functions constructed were almost always polynomial. Using the calculator, the students obtained the graphs of only one building-block polynomial function and used it to create new graphical representations. The new graphs were almost always reflections of the original graphs about the  $x$ - or  $y$ -axis, and their symbolic representations were again polynomials. More often than not, when the students created new numerical representations, they chose a table of values from the pretest or constructed a table from the symbolic representations used earlier. Addition, subtraction, and multiplication were the operations they used most of the time to create new functions.

*After the Rule of Three*

Unlike their earlier responses, every student gave a complete definition of a function, including the domain and range. They used their definitions, as they did before instruction, to help them distinguish between functions and non-functions. The students were again successful distinguishing between functions and non-functions in graphical and numerical form, but even more successful than they were before instruction. Every representation of a function was recognized as a function, and all graphical and numerical representations of non-functions were classified as non-functions. To a lesser extent, symbolic representations were sometimes problematic but for only two types of non-

functions, those of the form,  $x = c$  and  $|y| = x$ . Each student recognized every representation of a function as a function and again demonstrated that distinguishing between functions and non-functions was an easier task than recognizing functions. Yet, there was improvement by the students even in recognizing functions as each student increased the number and type of functions contained in his or her core set.

### *Research Question Results*

#### *Question 1*

First, the students moved beyond the point of only being able to classify mostly polynomial functions. Regardless of the representation to be classified—symbolic, graphical, or numerical—the success rate ranged from 71% to 100% on the 14 functions (constant, linear, quadratic, cubic (third degree polynomials), logarithmic, exponential, rational, radical, segmented (piece-wise), absolute value, greatest integer, the identity, algebraic and trigonometric (tangent or cosine)) investigated by the study. Similar to the results obtained before instruction using the Rule of Three, the identification of numerical representations were more difficult than symbolic or graphical representations with no one having a success rate higher than 79%. Graphical identifications were obtained through the use of the calculator, hand-made graphs and “educated guesses.” Experience was used to identify symbolic and graphical representations, with particular emphasis on symbolic representation. The students seemed to have learned the general form or characteristics of most functions. For example, if an equation contained just a single variable and there were only integer exponents for this particular variable, the equation was called a polynomial with a given degree. Functions with absolute value symbols, radical signs, double brackets, and the words *cos* or *log* were called absolute value,

radical, greatest integer, cosine and logarithmic functions, respectively. Similarly, the graphs of certain functions were memorized. For instance, graphs like a horizontal line, diagonal line, parabola, “a flight of sets” and one having a “v-shape” were deemed to be representative of a constant, linear, quadratic, greatest integer and an absolute value functions, respectively. As the students had done before instruction, if they recognized one representation of a particular function, they were frequently able to recognize one, if not both, of the other two representations. However, functions that were, in spite of everything, consistently problematic were the algebraic, rational, and segmented (piece-wise). The students claimed that the algebraic and segmented functions had no characteristics that were common to each function, respectively, regardless of the representations used. The students never seemed to recognize that the segmented function was partitioned into at least two parts and did not use this fact to help identify it. Instead, the segmented function was usually classified according to functions contained in one of its parts, if it was classified at all. The rational function was often mistaken as the algebraic function.

### *Question 2*

Even though the students’ success rate rose again for every translation, compared with their earlier performances before instruction, it still depended on the representations requested and the usefulness of the calculator. Using the calculator, every student translated from symbolic to graphical, symbolic to numerical, graphical to symbolic and numerical to symbolic for almost every function. As the usefulness of the calculator lessened, the students relied more on their knowledge and a couple of functions, the algebraic and the segmented function, became problematic for the numerical to graphical

and graphical to numerical translations. Because the graphs were provided, limiting the calculator's helpfulness, the algebraic and segmented functions became problem areas for the numerical to graphical and the graphical to numerical translations.

Since the students stated that they saw no graphical characteristics that were universal to all algebraic or segmented function, the translation from numerical to graphical or from graphical to numerical was difficult. The students resorted to handmade graphs for the above two translations, which sometimes allowed the rational function to join the algebraic and segmented functions as problematic.

### *Question 3*

As they did earlier in the study, the students chose mostly polynomials as their building blocks for creating new functions from symbolic representations. Although the operations of addition, subtraction, and multiplication were still used, other operations or processes like division, taking the square root, absolute value, and logarithm of function were also employed successfully. The functions created were no longer just polynomials but were radical, logarithmic, rational, and absolute value, to name a few, as the students diversified the types of functions created. Yet, no segmented or algebraic functions were constructed by anyone.

Before instruction, the students chose the graphs of polynomials when constructing new functions. After instruction, Whitney was the only student who did not. Her choice was an exponential function. All students except Whitney refrained from using previous methods to create new functions. Kristin and Eddie created new functions by restricting the domains and only viewed parts of the original graphs. Whitney created new functions by using reflections and vertical shifts but never together. Buren used only

vertical shifts. Similar to the symbolic representations, the new graphical representations were diverse group of graphs especially when compared to those produced before instruction using the Rule of Three.

### *Limitations*

There were limitations connected to my study. One limitation of the study may have been the nature of the students' mathematical backgrounds. All the students had taken AP or honors mathematics courses in high school and had some experience using the Texas Instrument TI-82 graphing calculator. This fact may have prevented their backgrounds from being general enough to apply my results to other groups of students. Students who had taken less mathematics in high school along with having more or less experience using the TI-82 may have produced different results. The results may also have been different if all the students would have had pretest scores of at least 70% because they were more successful on most tasks before and after instruction. In addition, the students were always willing to learn. They were enthusiastic about being part of the study and energetic. Their answers on tests or tasks were not always correct, but there never was a lack of effort. In contrast, students with dissimilar natures or personalities from the students in the study may have simply gone through the motions and performed differently on the tests and tasks.

Initially, I prodded the students during the first interview concerning only incorrect answers on the pretest before I realized my actions. After the first interview, the students were prodded throughout the remaining pretest interviews, the sorting activities and the posttest responses for every problem or task. Analysis of the pretest data may have been different without my actions. Although the students had some knowledge

using the graphing calculator, I also gave no additional instruction in its use before instruction because one of my goals was to ascertain what they could accomplish without any help or instruction.

### *Methodological Reflection*

My personal reflections on the research process are contained in this section. Although I initially wanted to choose only students who had pretest scores of at least 70%, it became apparent this criterion would not be met. I expected about 11 or 12 (approximately 33%) to meet or exceed this threshold. However, only 20% or 7 students attained a score of 70% or better. Kristin and Buren were chosen primarily on their pretest scores while Eddie and Whitney were chosen more so on their availability for interviews, a willingness to participate and continue in the study, and their responses on the pretests rather than their pretest scores. The selection process would have been easier if I had first determined the availability for interviews of the volunteers and next focused on their pretest responses, and then finally examined the pretest scores.

I wanted the interviews to have an informal nature and by doing so, I developed a long lasting relationship with the students that moved beyond that of teacher and student—to one of student and mentor—and finally to one based on mutual friendship. I have had similar relationships with students before, but they were rare. Earlier, I had informed the students that our relationship with each other was going to be based on mutual respect and trust. My relationship with the students put them at ease during the interviews and sorting activities and they seemed much more relaxed during the posttest, and this environment may have made them more responsive than what would be



expected. They seemed willing to bare their souls, and I never felt that they were hesitant or reluctant with their responses.

### *Recommendations for Future Research*

Before instruction using the Rule of Three, there are recommendations that I suggest for further research. We need to find out why the students were more knowledgeable or familiar with polynomials, particular linear and quadratic functions. My finding paralleled the results of Markovits et al. (1986), who found that many students appeared to view most functions as linear—but why? Was their familiarity developed because of repeated exposure to the symbolic representations of linear and quadratic functions in middle and high schools? Are there characteristics that enable the students to more readily identify linear and quadratic functions as opposed to other functions? Was the students' familiarity with linear and quadratic functions common to all students? Are constants, linear and the identity functions initially not viewed as polynomials common to all or most students? My analysis of the data led me to believe that it was because the exponents were not explicitly written, but could there have been another reason or reasons?

I also found that the students sometimes experienced trouble with constant functions but almost always with segmented (piece-wise) functions and its graphs. These findings were similar to the findings of Kieran (1992) and Even (1996), respectively, but why were constant and segmented functions problematic? Why do students expect the graphs of functions to be smooth, continuous flowing curves? What are some of the characteristics of the segmented, rational, algebraic functions that made them problematic, even after instruction using the Rule of Three? My analysis of the data led

me to believe that the students did not see any common characteristics for either the segmented, rational or algebraic functions individually and hence had trouble identifying any representation. Could other reasons, supported by research, be found, especially for a study that focused on these functions? Finally, why were equations that were not initially solved for  $y$  in terms of  $x$  more difficult to distinguish as functions or as non-functions?

### *Implications for Teaching*

Instruction using the Rule of Three to study functions seemed to allow each student to improve their performance on every task in the second part of the study when compared to their earlier performances. It seemed to have its most pronounced effect on the students with the lowest pretest scores. Along with instruction using the Rule of Three, the students also received instruction in the use of the calculator, and most graphical representations were usually provided by this technology. Even other members of the class said the instruction in how to use the calculator was helpful.

Reviewing functions appeared to be beneficial in several ways. Knowledge about the domains and graphs provided by the calculator appeared to help the students with the concepts of limits and continuity by providing likely points where the limits may not exist or points of discontinuity. The students became better at interpreting where graphs to determine where they were increasing and decreasing or to find local extremum. This endeavor became a way to confirm that the local maximum and minimum found algebraically through the use of the derivative was correct.

### *A Final Thought*

It seemed that the presentation of functions through instruction using the Rule of Three in conjunction with technology might be beneficial in the learning of mathematics,

particularly calculus. All students appeared to benefit from the instruction especially students who may be unprepared or have weak mathematics backgrounds. More studies are needed to answer the questions generated by this study because students' knowledge about functions are constantly changing and evolving, and the evolution may not follow a linear path to the goal set by an educator.

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## APPENDICES

## APPENDIX A

## PRETEST

Name \_\_\_\_\_

1. Define a function.

1b. Give an example of a symbolic, graphical, and numerical (tabular) representation of a function.

1c. Give an example of a non-function using any representation, symbolic, graphical, and numerical (tabular).

For each of the following relations, decide whether or not it represents a function. If the relation is a function, write the word “function” and give its name, domain, range, also give the name of its graph when applicable. If the equation is not a function, write the word “non-function” and describe or give an appropriate name for it. Discuss the reasons for your decisions.

2.  $y = 2x - 4$

3.  $y^2 = x$

4.  $x = 5$

5.  $y = x^2 - 8x + 2$

6.  $y = 7$

7.  $x^2 + y^2 = 25$

8.  $2x^2 - 4y^2 = 1$

9.  $y = x$

10.  $y^3 = x^2 + x + 1$

11.  $\frac{x^2}{9} + \frac{y^2}{16} = 1$

12.  $\left| \begin{array}{cc|cc|c|c} x & -2 & 3 & -1 & .5 & -\sqrt{3} \\ y & 3 & 8 & 0 & -.75 & 2 \end{array} \right|$

13.  $y = x^3 - x$

14.  $y = |x|$

15.  $\left| \begin{array}{c|c|c|c|c|c} x & -2 & 3 & 0 & .5 & 2 \\ y & -2 & 3 & 0 & .5 & 2 \end{array} \right|$

$$16. y = \lfloor \lfloor x \rfloor \rfloor$$

$$17. 2xy = 1$$

$$18. y = \log x$$

$$19. \left| \frac{x}{y} \right| \left| \frac{7}{12} \right| \left| \frac{3}{19} \right| \left| \frac{6}{60} \right| \left| \frac{5}{10} \right| \left| \frac{7}{21} \right|$$

$$20. y = \frac{1}{x}$$

$$21. y = \frac{x}{|x|}$$

$$22. y = \sqrt{2x - 3}$$

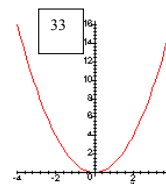
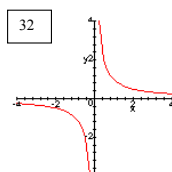
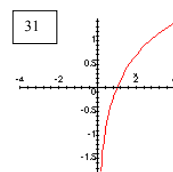
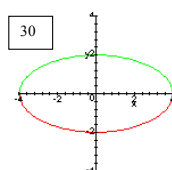
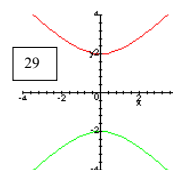
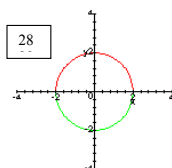
$$23. y = 6^x$$

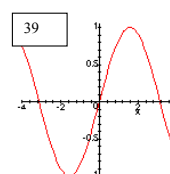
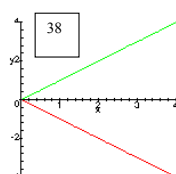
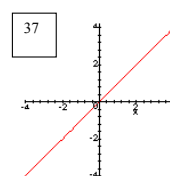
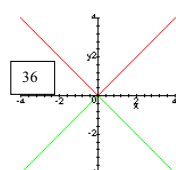
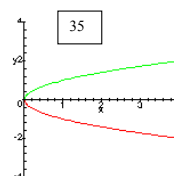
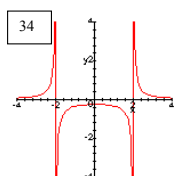
$$24. y = \sqrt{x}$$

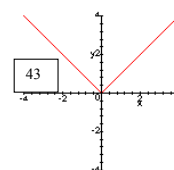
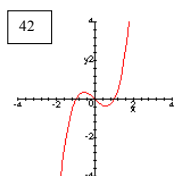
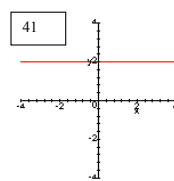
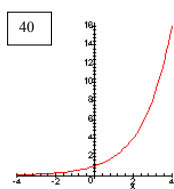
$$25. y = \tan x$$

$$26. y = \begin{cases} x+1 & \text{if } x \geq 0 \\ -x+1 & \text{if } x < 0 \end{cases}$$

$$27. y = \frac{x}{(x^2 + x + 1)}$$









## APPENDIX B

### INTERVIEW PROTOCOL

#### Interview I: Pretest and Posttest Problems Revisited

The purpose of Interview I is to give the participants an opportunity to discuss, rationalize, or explain their written responses to problems on the pretest and posttest. All problems for both tests will be discussed to investigate in more depth the participants' knowledge about functions and their symbolic, graphical, and numerical representations. During Interview I, problems 1 through 43 will be discussed. The participants will be reminded to think of this process as a conversation between friends and that I am interested in why or how they answered a problem with a particular response and their thoughts, knowledge, or reasoning that was the basis for their decision. The outline in Format I will be used in Interview I. The outline contains some possible supplemental questions based on the participants' original responses on the pretest. Interview I is intended to be friendly, informal, open-ended, and conversational. Therefore the actual interview format of Interview I may vary from this outline.

#### Format I: Pretest Discussion

Problems 1-2.

If the participant did not give a written response to either Problem 1 or 2 on the pretest/posttest, give him or her a second chance to respond. If the participant is still unable to create examples, proceed to Problems 3- 23. Return to Problems 1 and 2, if time permits, after reviewing Problems 3-23.

- Why did you give the following statement, . . . ., as the definition of a function? Have you ever used your definition of a function to determine if a relation was actually a function or non-function? If yes, elaborate (when, where, and how). If no, why not?
- Why did you select this particular symbolic, graphical, or numerical representation to be your example of a function? Can you give other representations of each form that are also functions?
- How do you know that each representation you gave is correct? If you are not sure that a particular representation is a function, what functional characteristics (or lack thereof) caused your uncertainty?

### Problems 3-23

Ask the participant to discuss the basis or reasoning for his or her responses on the pretest. For each of the first three bullets, ask each participant the two follow-up questions: What are the properties or characteristics of this relation or its representation that identified it as a function or non-function? Is one more important or useful than the others?

- How did you decide that relations in graphical form are function or non-functions?  
How did you decide that relations in numerical form are functions or non-functions?  
How did you decide that relations in symbolic form are functions or non-functions?
- Did the representation of the relations—symbolic, graphical, or numerical—influence your decision to decide that it was a function or non-function? If yes, how? If no, why not?
- Why is Problem \_\_\_\_ in symbolic form a function and Problem \_\_\_\_ in numerical form a non-function?

- Choose three non-functions from the pretest, one in symbolic form, one in graphical, and one in numerical form. How would you change each one to make it a function? How do you know that your modified relation is actually a function?
- If the vertical line test was mentioned on the pretest or during the interview, ask the participant to discuss it (how and why it works).
- Choose three of any pair of functions from the pretest: one pair in symbolic form, one pair in graphical, and one pair in numerical form. Employ each pair of functions to construct a new function with the same representation, using any combination of the following operations addition, subtraction, division, multiplication, and composition. Discuss your new function. How do you know it is a correct example for the given representation?

## Interview II: Sorting and Matching Activities

The purpose of this interview is to identify college students' knowledge about 14 functions— constant, linear, quadratic, cubic (third-degree polynomial), logarithmic, exponential, rational, radical, segmented (piece-wise), absolute value, greatest integer, identity, algebraic, and trigonometric (tangent or cosine)—along with their corresponding representations that are likely to be encountered in a calculus course. The students will be given a stack of 42 cards, with each card depicting 1 of the 14 functions in each of the following three representations: symbolic, graphical, or numerical. Since there will be two sorting activities, I shall construct at least two different stacks of the 42 cards. One symbolic depiction of the functions to be used is as follows:

$$f(x) = 2x + 2 \quad f(x) = x^3 + x + 1 \quad f(x) = \lfloor x \rfloor \quad f(x) = \cos(.5x) + 1$$

$$\begin{array}{llll}
 f(x) = -|x| + 3 & f(x) = \sqrt{x+2} & f(x) = x^2 - 2 & f(x) = (.5)^x \\
 f(x) = x & f(x) = \Pi & f(x) = -\log x & \\
 f(x) = \frac{.5}{x-1} & f(x) = \frac{x-1}{|x-1|} & f(x) = \begin{cases} \sqrt{(-x)} \text{ if } \dots x \leq 0 \\ \sqrt{(x)} \text{ if } \dots x > 0 \end{cases} & 
 \end{array}$$

### Task 1

The 42 cards will be shuffled to introduce randomness and given to each participant. He or she will then be given several minutes to skim through the cards. The participant will then be asked to sort the cards in a manner that seems to represent a natural or logical organization of the contents of the cards. After the cards have been sorted, the participant will be asked the following questions.

- Why did you sort the cards in this manner? (What criteria did you use to sort the cards?)
- Are you sure that all the cards in each category satisfy the criteria for it?
- How would you designate or name your categories?
- Can your categories be refined or further subdivided?
- Are there any cards that caused uncertainty in sorting them? Are there any cards that did not seem to belong to any of your categories? Are there any cards that could be placed in more than one category?

### Task 2

Ask the participant if there are other ways of sorting the cards into different categories? Give him or her a few minutes to think and respond. If his or her answer is yes, ask him or her to sort the cards and proceed again with the same five bullets above. If his or her answer is no, ask him or her to discuss his or her reason for this decision.

### Task 3

Give each participant the set of cards depicting the 14 graphical representations of functions and the set of cards depicting the 14 numerical representations of functions.

Then, ask him or her to place the set of 14 cards depicting graphical representations in a row or column on a table separate from the 14 cards depicting numerical representations.

Next, one card depicting a function in symbolic form will be given to each participant, and he or she will be initially asked to match it only to its corresponding graphical representation. If the participant is not successful, the activity begins anew, and another card depicting a symbolic representation is given to the participant, who will be asked to match the card to its corresponding graphical representation.

If the participant is successful in matching the symbolic representation with the graphical representation, he or she will immediately be also asked to match the two cards to their corresponding numerical representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting symbolic representations of functions: constant, linear, quadratic, cubic (third-degree polynomial), logarithmic, exponential, rational, radical, segmented (piece-wise), absolute value, greatest integer, identity, algebraic, and trigonometric (tangent or cosine).

### Tasks 4—8

The methodology for Task 4 through Task 8 is nearly identical to the methodology for Task 3. The only difference among the tasks is the order of the matching, and the first or base card to which all matches are to be made. The objective for Task 4 through Task 8 is to determine if some representations—for example, numerical and graphical—are easier

to match than others such as symbolic and graphical. I shall also try to determine if the order of match is important to making a correct match. For example, is it easier for a participant to match numerical with graphical or graphical with numerical? In Task 4, participants are initially asked to match a numerical representation with a graphical one. If he or she is not successful, the activity starts anew. If the participant is successful, he or she is asked to match the numerical and graphical to their corresponding symbolic representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting numerical representations of functions

#### Task 5

The participants are initially asked to match graphical representation with a symbolic one. If he or she is not successful, the activity starts anew. If the participant is successful, he or she is asked to match the graphical and symbolic representations to their corresponding numerical representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting graphical representations of functions.

#### Task 6

The participants are initially asked to match a numerical representation with a symbolic one. If he or she is not successful, the activity starts anew. If the participant is successful, he or she is asked to match the numerical and symbolic representations to their corresponding graphical representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting numerical representations of functions.

### Task 7

The participants are initially asked to match a graphical representation with a numerical one. If he or she is not successful, the activity starts anew. If the participant is successful, he or she is asked to match the graphical and numerical representations to their corresponding symbolic representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting graphical representations of functions.

### Task 8

The participants are initially asked to match a symbolic representation with a numerical one. If he or she is not successful, the activity starts anew. If the participant is successful, he or she is asked to match the symbolic and numerical representations to their corresponding graphical representation. Regardless of the outcome of the last matching task, the activity begins anew. This activity will continue until the participant has been given each of the 14 cards depicting symbolic representations of functions.

For each task, the following questions will be asked:

- How did you match this representation (symbolic, graphical, or numerical) with this representation (symbolic, graphical, or numerical) for this particular function? Were there any special characteristics about this function that helped you with your match?
- Why were you unable to match the first two representations with their corresponding third representation for this function?
- What characteristics helped you match all three representations for this function?
- Was the order in which you were asked to match the representations helpful? If yes, how? If no, why not? What order did you consider the least helpful?

- For which of the 14 functions do you understand the most or have the most knowledge? Have the participant discuss a few of his or her choices. Were these functions the easiest for you to match?
- For which of the 14 functions do you understand the least or have the smallest amount of knowledge? (Have the participant discuss a few of his or her choices). Were these functions difficult for you to match?

### Task 9

Task 9 will be a written task in which the participant will be asked to choose any two or more of the 14 different functions to create “new” functions in each of the following categories: symbolic, graphical, and numerical. He or she will be allowed to use any combinations of the following operations: addition, subtraction, division, multiplication, and composition. If the participant can only create “new” functions in only one or two categories, focus on those functions.

- What functions did you choose? Why did you choose these functions?

For every “new” function constructed in each category, symbolic, graphical, and numerical, the following questions will be asked:

- What type of new function did you create? How do you know that your new function is actually a function? Does it have a name?
- What operation or combination of operations did you use to create your new function? Why did you choose these operations or combination of operations?
- What is the domain and range of your new functions? How are you sure your information is correct? Discuss any other characteristics you know about your new function.



Is your new function similar to one of the fourteen functions or is it one that is not on the list? Where or in what context have you studied or worked with your new function before?

## APPENDIX C

## POSTTEST

Name \_\_\_\_\_

1. Define a function.

1b. Give an example of a symbolic, graphical, and numerical (tabular) representation of a function.

1c. Give an example of a non-function using any representation, symbolic, graphical, and numerical (tabular).

For each of the following relations, decide whether or not it represents a function. If the relation is a function, write the word “function” and give its name, domain, range and the name of its graph when applicable. If the equation is not a function, write the word “non-function” and describe or give an appropriate name for it. Discuss the reasons for your decisions.

2.  $x + 3y - 4 = 0$

3.  $4y^2 - x + 1 = 0$

4.  $2x = 8$

5.  $3x^2 + 4x + 12 + y = 0$

6.  $3y = 15$

7.  $2x^2 + 2y^2 - 32 = 0$

8.  $x^2 - y^2 - 1 = 0$

9.  $7y = 7x$

10.  $0 = 6x^2 + 9x + 12 + 3y^3$

11.  $\frac{2x^2}{9} + \frac{2y^2}{16} - 2 = 0$

12.  $\left| \begin{array}{cc|cc|c|c} x & -2 & 3 & -1 & .5 & -\sqrt{3} \\ y & 3 & 8 & 0 & -.75 & 2 \end{array} \right|$

13.  $0 = 12x^3 + 8x + 4y$

14.  $2y = 8|x|$

15.  $\left| \begin{array}{cc|cc|c|c} x & -2 & 3 & 0 & .5 & 2 \\ y & -2 & 3 & 0 & .5 & 2 \end{array} \right|$

$$16. y = [[x - 1]]$$

$$17. 3xy = 6$$

$$18. y = 2 \ln x$$

$$19. \left| \frac{x}{y} \right| \left| \frac{7}{12} \right| \left| \frac{3}{19} \right| \left| \frac{6}{60} \right| \left| \frac{5}{10} \right| \left| \frac{7}{21} \right|$$

$$20. y = \frac{1}{x}$$

$$21. y = \frac{|2x|}{x}$$

$$22. y = \sqrt{x-1}$$

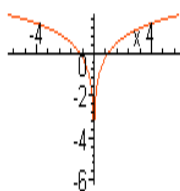
$$23. y = 2^x$$

$$24. y = \sqrt[3]{x-1}$$

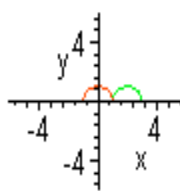
$$25. y = \cos x$$

$$26. y = \begin{cases} |2x| & \text{if } x \geq 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

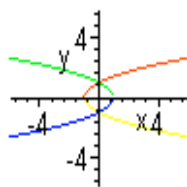
$$27. y = \frac{x}{x-1}$$



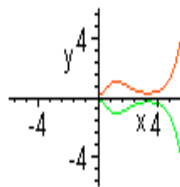
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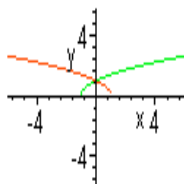
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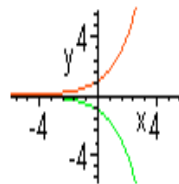
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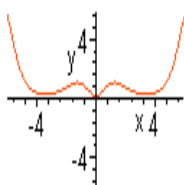
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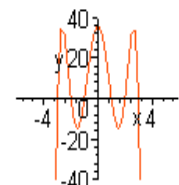
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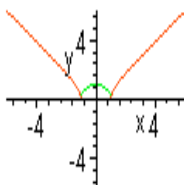
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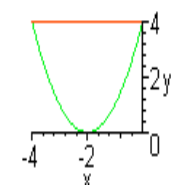
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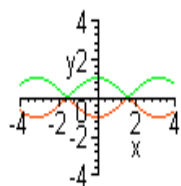
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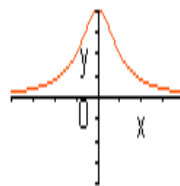
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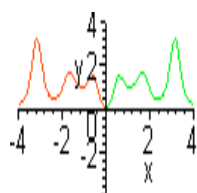
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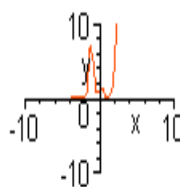
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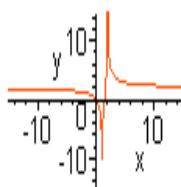
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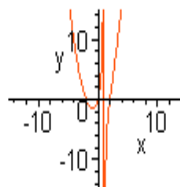
(40)



(41)



(42)



(43)

## APPENDIX D

## QUESTIONNAIRE

**TITLE: STUDENTS' KNOWLEDGE OF FUNCTIONS AS AFFECTED  
BY REFORM CALCULUS ENVIRONMENT**

Name \_\_\_\_\_

Please provide the following descriptive information. **Fill in black or circle an answer as requested.**

1. My gender is (1) male (2) female
2. My ethnicity is (1) Black (2) White-Non Hispanic (3) Native American (4) Asian American (5) Hispanic (6) Pacific Islander (7) Other.
3. My home address is located in the following state or country.
4. My age is
5. I am (1) married (2) single (3) divorced.
6. My classification is (1) freshman (2) sophomore (3) junior (4) senior.
7. My major and minor are respectively \_\_\_\_\_ and \_\_\_\_\_.
8. I understand mathematics (1) Usually (2) Sometimes (3) Rarely (4) Never.
9. The number of mathematics courses I have taken in high school is \_\_\_\_\_. Their names are:
10. The number of mathematics courses I have taken or will take in college is \_\_\_\_\_. Their names are:

Their course numbers are :



11. I understood mathematics in high school. (1) Usually (2) Sometimes (3) Rarely (4) Never.
12. I understand mathematics in college. (1) Usually (2) Sometimes (3) Rarely (4) Never.
13. Most of my high school teachers explained mathematics well enough for me to understand it. (1) Usually (2) Sometimes (3) Rarely (4) Never.
14. Most of my college instructors explained mathematics well enough for me to understand it. (1) Usually (2) Sometimes (3) Rarely (4) Never.
15. Regardless of the setting, high school or college, I have always had trouble understanding mathematics, regardless of my effort. (1) Usually (2) Sometimes (3) Rarely (4) Never.
16. Since college algebra is a prerequisite for calculus, I consider my algebraic background to be: (1) excellent (2) good (3) average (4) below average (5) poor.
17. I have been taught functions using multiple representations, symbolic, graphical, and numerical (1) Usually (2) Sometimes (3) Rarely (4) Never.
18. My high school grade point average was \_\_\_\_\_.
19. My current cumulative grade point average in college is \_\_\_\_\_.
20. The names of the honor courses I took in high school are:
21. In high school, I was usually taught mathematics from (1) traditional textbooks (2) reform textbooks (3) both types (4) I do not know the difference between the two types of textbooks
22. Describe your experiences (good and bad) with mathematics in high school and college.



## APPENDIX F: PILOT STUDY TEST

Name \_\_\_\_\_

Decide whether or not the following relations are functions. If a relation is a function identify it by name, give the domain, range and its graph. If a relation is not a function, state why and give its name if applicable.

1.  $y = x^2$

2.  $x^2 = -y$

3.  $S(x) = 120 + 1568x - 16x^2$

4.  $f(x) = 2x^2 - 4x - 1$

5.  $f(x) = x^2 + 2x - 2$

6.  $f(x) = [[x]]$

7.  $f(x) = x$

8.  $y = \begin{cases} x+1 & \text{if } \dots x \geq 0 \\ 1-x & \text{if } \dots x < 0 \end{cases}$

9.  $f(x) = 3$

10.  $y = x^3 + x^2$

11.  $f(x) = \sin x$

12.  $f(x) = 3\sqrt{x-1}$

13.  $2x + 3y = 12$

14.  $f(x) = \frac{x}{x^2 - 4}$

15.  $f(x) = 2^x$

16.  $f(x) = x^3 + x^2 + x - 4$

17.  $A = \{(1, 2), (3, 9), (6, 6), (5, 10), (7, 2)\}$

18.  $\left| \begin{array}{c|c|c|c|c} -2 & 3 & -1 & .5 & -\sqrt{3} \\ \hline 3 & 8 & 0 & -.75 & 2 \end{array} \right|$

19.  $f(x) = |x|$

20.  $2x^2 + 2y^2 = 18$

21.  $y^2 = x$

22.  $x^2 + y^2 + 2x + 6y = 6$

23.  $x^2 - y^2 = 1$

24.  $x = y^3 + y^2 + y$

25.  $x = 3$

26.  $xy + \frac{x}{y} + \frac{y}{x} = 3$

27.  $B = \{(2, 29), (8, 11), (3, -7), (4, 9), (2, 100)\}$

28.  $y^3 - xy = 2$

29.  $f(x) = \log x$

30.  $f(x) = \frac{\cos x}{x - 4}$

## APPENDIX G

## PILOT INTERVIEW QUESTIONS FOR QUADRATIC FUNCTIONS

1. In your own words, give the definition of a function. Give an example of a function.  
The words quadratic function means what to you?
8. Give an example of a quadratic function and its corresponding graph. Why is your example a quadratic function? Suppose the graph of quadratic function opens upward, what is the minimum point on the graph called? Conversely, if the quadratic function opens downward, what is the maximum point on the graph called?
9. Which of the following functions (give them the cards) are quadratics? Did the equation you chose cross the x-axis? If so, where? What is the maximum number of times your equation crosses the x-axis? Are you able to decide where the equation you chose crosses the x-axis without graphing it? If so, what is your procedure? Does your equation also cross the y-axis? If so, where?
10. I  
Is it possible for the graph of a quadratic function not to cross the x-axis? If yes, where is the graph of the quadratic function located? If not, why not? Is it possible for the graph of a quadratic function not to cross the y-axis? If yes, why? If no, why?
5. Is the inverse of  $y = 2x^2 + 6x + 4$  a function? What is the shape of the graph of the inverse  
 $y = 2x^2 + 6x + 4$ ?
6. If a quadratic function has complex solutions (imaginary solutions), what information does this fact give about its graph?
7. How many possible points of intersection exist between the graph of a quadratic function and the graph of a linear function? How many possible points of intersection exist between the graph of a quadratic function and the graph of a circle?